*"The nature of the infant is not just a new permutation - and - combination of elements contained in the natures of the parents. There is in the nature of the infant that which is utterly unknown in the natures of the parents."*

**– David Herbert Lawrence**

# 6

# Permutations, Combinations and Binomial Expansion

**Contents :**



Permutation and Combination are useful for solving many problems in our day to day life. e.g. 4 students are to be arranged on a bench of a particular class then in how many ways can a teacher arrange these 4 students ? A person has 6 friends and he wishes to invite only 2 friends to his family function then how many options that person has. Normally, we solve these problems with common sense. But for the mathematical solution for different problems of this type, we shall study certain principles and methods in this chapter.

First of all we shall study two types of fundamental principles of counting namely addition and multiplication with reference to permutation and combination.

**Fundamental principle of counting for addition :**

We shall study some examples to understand this principle. A restuarant serves 4 types of pizzas (say $P_1$, $P_2$, $P_3$, $P_4$) and 3 types of burgers (say $B_1$, $B_2$, $B_3$ ). If a person wants to order a pizza or a burger then he has total $4 + 3 = 7$ $\{P_1, P_2, P_3, P_4, B_1, B_2, B_3\}$ options. In the same way, a class teacher wishing to appoint a class representative from 40 boys and 20 girls studying in a class has 40 + 20 = 60 options. Thus, if there are $m$ distinct things in Group 1 and $n$ distinct things in Group 2 then selection of one thing from total things of both groups can be done in $m + n$ ways. This rule is called fundamental principle of counting for addition.

| | | |
|---|---|---|
| Selection of one thing from $m$ distinct things of Group 1 | OR | Selection of one thing from $n$ distinct things of Group 2 |
| ↓ | ↓ | ↓ |
| ***m* ways** | + | ***n* ways** |

It means the word 'OR' indicates addition.

**Note :** Fundamental principle of counting for addition can also be applicable to more than two groups.

**Fundamental Principle of Counting for Multiplication :**

If the first operation can be done in $m$ ways and second operation can be done in $n$ ways, then two operations together can be done in $m \times n$ ways. This rule is called fundamental principle of counting for multiplication. We shall understand this rule using the previous exmaple. If a person wants to order a pizza and a burger then he has total 4 × 3 = 12 $\{P_1B_1, P_1B_2, P_1B_3, P_2B_1, P_2B_2, P_2B_3, P_3B_1, P_3B_2, P_3B_3, P_4B_1, P_4B_2, P_4B_3\}$ options. In the same way, a class teacher wishing to appoint a boy and a girl as a class representative has 40 × 20 = 800 options.

| | | |
|---|---|---|
| Selection of one thing from $m$ distinct things of Group 1 | AND | Selection of one thing from $n$ distinct things of Group 2 |
| ↓ | ↓ | ↓ |
| ***m* ways** | × | ***n* ways** |

It means the word 'AND' indicates mulitplication.

**Note :** Fundamental principle of counting for multiplication can also be applicable to more than two groups.

## 6.1 Meaning of Permutation

Let us begin with some examples to understand permutation.

- How many different two digit numbers can be made using digits 2, 5 and 8 without repeating digits ?

If we make two digit numbers using given digits 2, 5, 8 then total 6 numbers 25, 28, 52, 58, 82, 85 can be made. It is necessary to note that the numbers 22, 55, 88 can not be formed as repetition is not allowed.

Now, we shall solve this problem using principle of counting. In two digit numbers, there are two places, unit and tens.

Arrange any one of the digits 2, 5 and 8

**Tens Place**

Now, excluding the digit placed in tens place, arrange any one digit here from the remaining two digits.

**Unit place**

Thus, there are three options 2, 5 and 8 for filling tens place. After filling tens place with any one digit there are only two options for filling unit place. That is total $3 \times 2 = 6$ numbers can be formed. Selecting digit here for unit place first and then selecting digit for tens place will not change the result.

**Note :** Permutations with repetition of items is not included in our syllabus.

- From 4 students A, B, C, D, one student is to be selected as captain and other student as vice captain of cricket team. How many alternative solutions are possible for this problem ?

If this problem is solved using common sense then following 12 different alternatives can be obtained :

| Altern-ative | Captain | Vice captain |
|---|---|---|
| 1 | A | B |
| 2 | A | C |
| 3 | A | D |
| 4 | B | A |
| 5 | B | C |
| 6 | B | D |
| 7 | C | A |
| 8 | C | B |
| 9 | C | D |
| 10 | D | A |
| 11 | D | B |
| 12 | D | C |

Now we shall solve this problem using priciple of counting.

Any one student can be made captain from 4 students A, B, C, D

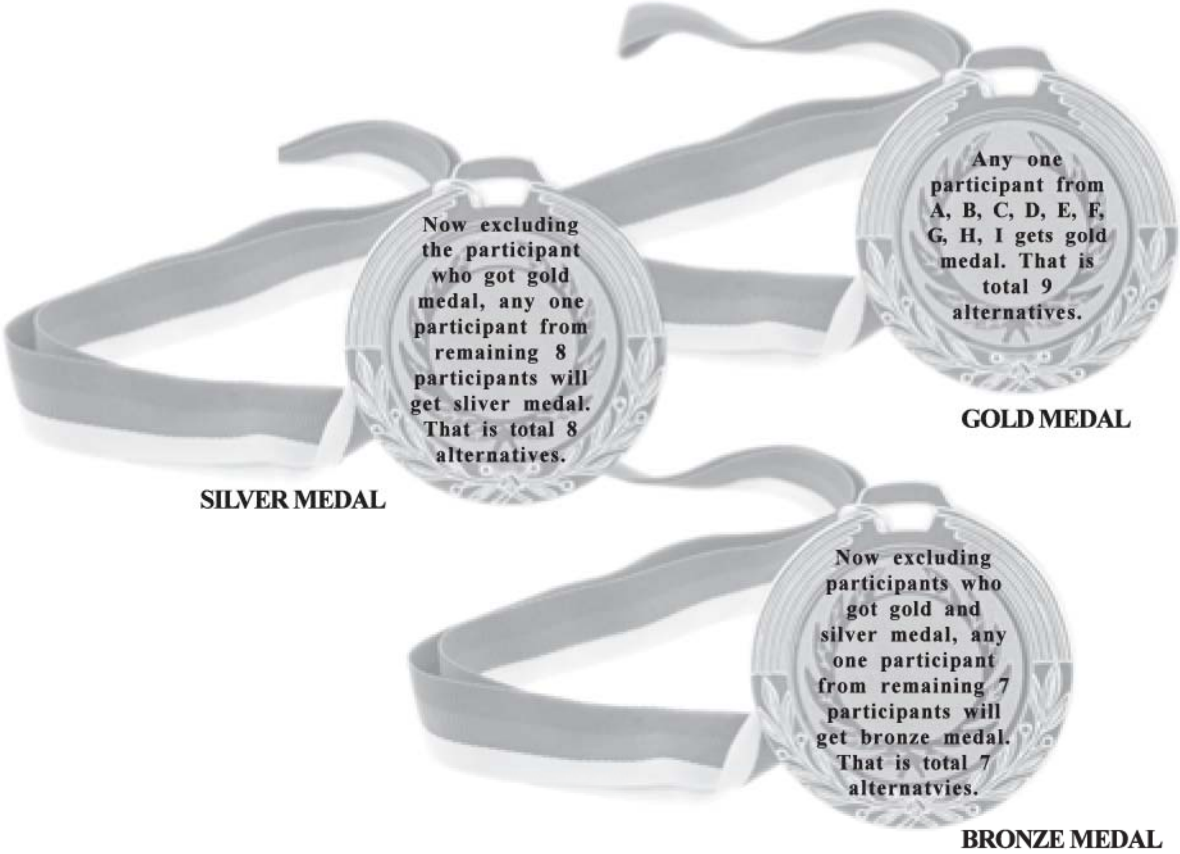
**Captain's place**

Now, excluding the student who is made captain, any one student can be made vice captain from remaining three students.

**Vice-Captain's Place**

Thus, there are four alternatives A, B, C and D for selection of a captain. There are three alternatives for selection of vice-captain after selecting captain. That is total $4 \times 3 = 12$ alternatives can be obtained.

- Nine participants A, B, C, D, E, F, G, H and I take part in a 100 meter running competition. Top three participants are given Gold, Silver and Bronze medals. Now, we have to find the number of alternatives for participants getting three medals.




Thus, according to fundamental priciple of counting, the first action (Gold medal) can be in 9 ways, the second action (Silver medal) can be in 8 ways and the third action (Bronze medal) can be in 7 ways. Thus, total number of alternatives of all the three actions together are $9 \times 8 \times 7 = 504$. The solution of this problem can also be expressed in following way :

| **Medal** | **Gold** | **Silver** | **Bronze** |
|---|---|---|---|
| **Alternatives of participants getting medal** | 9 | $9 - 1 = 8$ | $9 - 2 = 7$ |

According to the fundamental principle of counting, total alternatives are $= 9 \times 8 \times 7 = 504$

(It is important to note here that there is no tie between the participants)

If $r$ distinct things out of given $n$ distinct things are to be arranged in $r$ $(1 \leq r \leq n)$ different places, then each such arrangement is called a permutation. The total numer of such arrangements is denoted as $^nP_r$, $_nP_r$, $P(n, r)$, $P_r^n$ . We will use the notation $^nP_r$.

Thus, total permutations of n distinct things in r places will be $^nP_r$

Understand the next table defining $^nP_r$.

Suppose $r$ distinct things out of $n$ different things are to be arranged in $r$ places, i.e. we want to find $^nP_r$. This arrangement can be made as follows. (For easy understanding, compare the next table with the previous example of medals.)

| *r* Places | 1 | 2 | 3 | 4 | ... | *r* |
|---|---|---|---|---|---|---|
| **Possible ways for each place** | $n$ $= n - (1-1)$ | $n - 1$ $= n - (2-1)$ | $n - 2$ $= n - (3-1)$ | $n - 3$ $= n - (4-1)$ | ... | $n - r + 1$ $= n - (r-1)$ |

It is clear from the above table that for filling the first place there are $n$ ways available. Now, having filled the first place by any one thing, the second place can be filled by any one of the remaining $(n-1)$ things. Thus the total number of ways of filling the first two places is $n \times (n-1)$. Similarly the total number of ways of filling the first three places is $n \times (n-1) \times (n-2)$. Similarly, the total number of ways of filling all the $r$ places is $n \times (n-1) \times (n-2) \times (n-3) \times .... \times (n - r + 1)$. This is called total permutations of arranging $r$ things out of $n$ different things in $r$ places. So,

$$^nP_r = n \times (n-1) \times (n-2) \times (n-3) \times ... \times (n-r+1)$$

No. of ways of filling each place is one less than the no. of ways of filling previous place.

Factorial has a very important role in the study of permutations. First of all, let us understand the meaning of factorial. We have seen earlier that multiplication of consecutive numbers is involved while solving many problems. Factorial is used to express this multiplication in short. Factorial is denoted with symbol '!' or '$\lfloor\,$'. Factorial $n$ is denoted by $n!$ or $\lfloor n$. Now let us understand the meaning of $n!$.

The product of natural numbers from 1 to $n$ is $n!$, i.e.

$n! = 1 \times 2 \times 3 \times ... \times (n-1) \times n$

OR

$n! = n \times (n-1) \times ... \times 3 \times 2 \times 1$ (We will write this way for the sake of convenience)

For example,

$7! = 7 \times (7-1) \times ... \times 3 \times 2 \times 1$

$= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 5040$

$10! = 10 \times (10-1) \times ... \times 3 \times 2 \times 1$

$= 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 36,28,800$

Similarly,

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

$3! = 3 \times 2 \times 1 = 6$

$1! = 1$

$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 6 \times 5 \times 4 \times 3 \times 2!$

$= 6 \times 5 \times 4 \times 3!$

$= 6 \times 5 \times 4!$

$= 6 \times 5!$

See the utility...!

$\frac{6!}{4!} = \frac{6 \times 5 \times 4!}{4!} = 30$

Generalizing this, we get

$n! = n(n-1)(n-2) \times ... \times 3 \times 2 \times 1$

$= n(n-1)(n-2) \times ... \times 3 \times 2!$

$= n(n-1)(n-2) \times ... \times 3!$

$= n(n-1)(n-2)!$

$= n(n-1)!$

| |
|---|
| $n! = (n)(n-1)!$ |
| Putting $n = 1$ |
| $\therefore$ $1! = (1)(1-1)!$ |
| $\therefore$ $1 = 1 \times 0!$ |
| $\therefore$ $0! = 1$ |

Now, let us see some important results.

As we have seen earlier, ${}^nP_r = n(n-1)(n-2) ... (n-r+1)$. This result can also be written as follows :

$${}^nP_r = n(n-1)(n-2) ... (n-r+1) \times \frac{(n-r)(n-r-1) \times ... \times 3 \times 2 \times 1}{(n-r)(n-r-1) \times ... \times 3 \times 2 \times 1}$$

$$= \frac{n(n-1)(n-2) ... (n-r+1)(n-r)(n-r-1) \times ... \times 3 \times 2 \times 1}{(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

$\therefore$ ${}^nP_r = \frac{n!}{(n-r)!}$ where, $n > 0$, $r \geq 0$, $n \geq r$, $n$ is a positive integer and $r$ is non-negative integer.

The following results can be derived using this formula.

| **Some important results** | | | |
|---|---|---|---|
| ${}^nP_0 = 1$ | ${}^nP_n = n!$ | ${}^nP_1 = n$ | ${}^nP_{n-1} = n!$ |
| ${}^nP_r = \frac{n!}{(n-r)!}$<br>$\therefore$ ${}^nP_0 = \frac{n!}{(n-0)!}$<br>$= \frac{n!}{n!}$<br>$= 1$ | ${}^nP_r = \frac{n!}{(n-r)!}$<br>$\therefore$ ${}^nP_n = \frac{n!}{(n-n)!}$<br>$= \frac{n!}{0!}$<br>$= n!$ | ${}^nP_r = \frac{n!}{(n-r)!}$<br>$\therefore$ ${}^nP_1 = \frac{n!}{(n-1)!}$<br>$= \frac{n(n-1)!}{(n-1)!}$<br>$= n$ | ${}^nP_r = \frac{n!}{(n-r)!}$<br>$\therefore$ ${}^nP_{n-1} = \frac{n!}{[n-(n-1)]!}$<br>$= \frac{n!}{(n-n+1)!}$<br>$= n!$ |
| e.g. ${}^5P_0 = 1$<br>${}^{10}P_0 = 1$ | e.g. ${}^5P_5 = 5!$<br>${}^{10}P_{10} = 10!$ | e.g. ${}^5P_1 = 5$<br>${}^{10}P_1 = 10$ | e.g. ${}^5P_4 = 5!$<br>${}^{10}P_9 = 10!$ |

**Illustration 1 : Find the values of the following :**

**(1) ${}^8P_3$ (2) ${}^{60}P_2$ (3) ${}^7P_6$ (4) ${}^5P_5$**

(1) ${}^nP_r = \frac{n!}{(n-r)!}$

${}^8P_3 = \frac{8!}{(8-3)!}$

$= \frac{8!}{5!}$

$= \frac{8 \times 7 \times 6 \times 5!}{5!}$

$= 336$

**Alternative method :**

According to the definition of ${}^nP_r$,

${}^nP_r = n(n-1)(n-2) ..... (n-r+1)$

Put $n = 8$ and $r = 3$

$\therefore$ ${}^8P_3 = 8(8-1)(8-2)$

$= 8 \times 7 \times 6$

$= 336$

(2) According to the definition of ${}^nP_r$,

$${}^nP_2 = n\,(n-1)$$

$$\therefore \quad {}^{60}P_2 = 60\,(60-1)$$

$$= 60 \times 59$$

$$= 3540$$

(3) According to the definition of ${}^nP_r$,

$${}^nP_6 = n\,(n-1)\,(n-2)\,(n-3)\,(n-4)\,(n-5)$$

$$\therefore \quad {}^7P_6 = 7 \times 6 \times 5 \times 4 \times 3 \times 2$$

$$= 5040$$

(4) According to the definition of ${}^nP_r$,

$${}^nP_5 = n\,(n-1)\,(n-2)\,(n-3)\,(n-4)$$

$$\therefore \quad {}^5P_5 = 5 \times 4 \times 3 \times 2 \times 1$$

$$= 120$$

**Illustration 2 : If ${}^nP_3 = 720$ then find the value of *n*.**

According to the definition of ${}^nP_r$,

$${}^nP_3 = n\,(n-1)\,(n-2)$$

$$\therefore \quad n\,(n-1)\,(n-2) = 720$$

Now, instead of expansion we shall think about the three consecutive numbers whose multiplication is 720. Such numbers are 10, 9, 8. But we shall write it as follows :

$$\therefore \quad n\,(n-1)\,(n-2) = 10\,(10-1)\,(10-2)$$

$$\therefore \quad n = 10$$

**Illustration 3 : If ${}^7P_r = 42$ then find the value of *r*.**

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\therefore \quad {}^7P_r = \frac{7!}{(7-r)!}$$

$$\therefore \quad 42 = \frac{7!}{(7-r)!}$$

$$\therefore (7-r)! = \frac{7!}{42}$$

$$\therefore (7-r)! = \frac{7 \times 6 \times 5!}{42}$$

$$\therefore (7-r)! = 5!$$

$$\therefore \quad 7 - r = 5$$

$$\therefore \quad r = 2$$

**Alternative Method :**

$${}^7P_r = 42$$

$$\therefore \quad {}^7P_r = 7 \times 6$$

$$\therefore \quad {}^7P_r = {}^7P_2$$

$$\therefore \quad r = 2$$

**Illustration 4 : If $56 \times n! = 8!$ then find the value of $n$.**

$56 \times n! = 8 \times 7 \times 6!$

$\therefore \quad n! = 6!$

$\therefore \quad n = 6$

**Illustration 5 : If $\frac{n!}{2} = 60$ then find value the of $n$.**

$$\frac{n!}{2} = 60$$

$\therefore \quad n! = 120 \; (= 1 \times 2 \times 3 \times 4 \times 5)$

$\therefore \quad n! = 5!$

$\therefore \quad n = 5$

**Illustration 6 : If ${}^{(n+2)}P_3 : {}^{(n+1)}P_3 = 10 : 7$ then find $n$.**

$$\frac{{}^{(n+2)}P_3}{{}^{(n+1)}P_3} = \frac{10}{7}$$

$$\therefore \quad \frac{(n+2)\,(n+1)\,(n)}{(n+1)\,(n)\,(n-1)} = \frac{10}{7}$$

$$\therefore \quad \frac{n+2}{n-1} = \frac{10}{7}$$

$\therefore \quad 7(n+2) = 10(n-1)$

$\therefore \quad 7n + 14 = 10n - 10$

$\therefore \quad 3n = 24$

$\therefore \quad n = 8$

**Illustration 7 : In how many ways 3 members of a family; husband, wife and a daughter can be arranged in a row for a group photograph ?**

There are three members in this family. For a group photograph, they can be arranged in a row in ${}^3P_3$ ways.

$\therefore$ **Total permutations** $= {}^3P_3$

$= 3!$

$= 6$

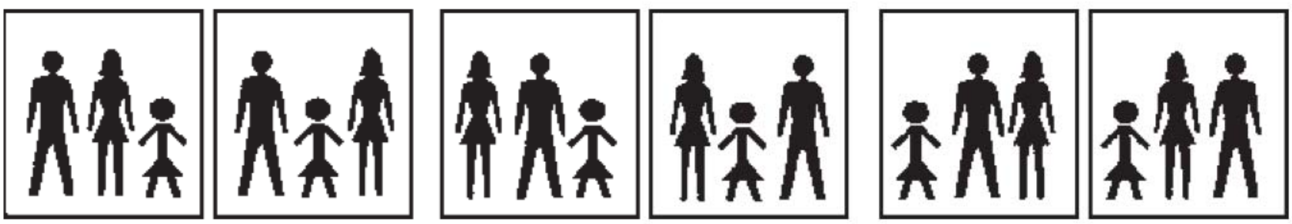


**Illustration 8 : Using all the digits 2, 5, 7, 8, four digit numbers are to be formed.**

**(1) How many such numbers can be made ?**

**(2) How many of them will be even numbers ?**

**(3) How many of them will be divisible by 5 ?**

**(4) How many of them will be greater than 5000 ?**

(1) Using all the digits 2, 5, 7, 8 i.e. 4 digits, total number of 4 digit numbers that can be formed is ${}^4P_4$.

$\therefore$ **Total permutations** $= {}^4P_4$

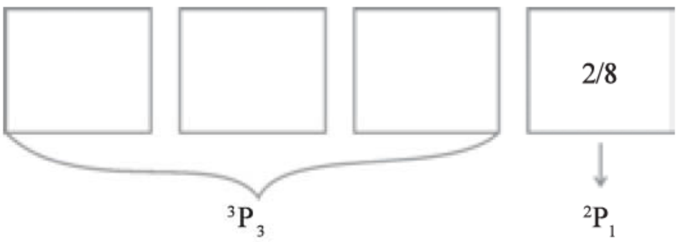
$= 4!$

$= 24$

| | | | |
|---|---|---|---|
| 2578 | 5278 | 7258 | 8257 |
| 2587 | 5287 | 7285 | 8275 |
| 2758 | 5728 | 7528 | 8527 |
| 2785 | 5782 | 7582 | 8572 |
| 2857 | 5827 | 7825 | 8725 |
| 2875 | 5872 | 7852 | 8752 |

Verify total permutations in each question with the arrangement in this table.

(2)




If even numbers are to be formed using all the digit 2, 5, 7, 8 then digit 2 or 8 should be in unit place. This arrangement can be done in ${}^2P_1$ ways. After arranging 2 or 8 in unit place, remaining 3 digits can be arranged in remaining 3 places in ${}^3P_3$ ways.

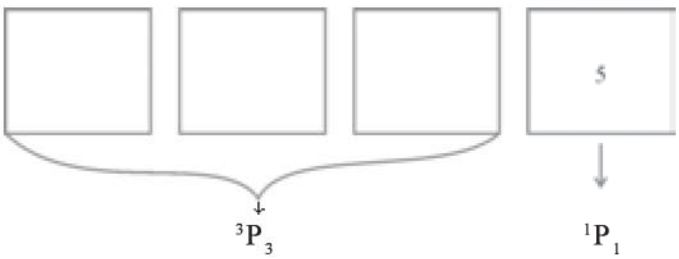
$\therefore$ **Total Permutations** $= {}^2P_1 \times {}^3P_3$

$= 2 \times 3!$

$= 2 \times 6$

$= 12$

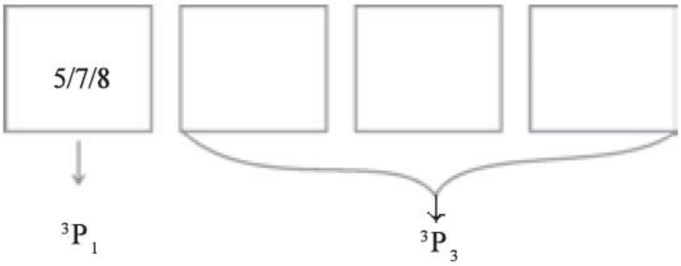
(3)




If numbers divisible by 5 are to be formed using all the digits 2, 5, 7, 8 then digit 5 should be in unit place. This arrangement can be done in ${}^1P_1$ ways. After arranging 5 in unit place, remaining 3 digits can be arranged in remaining 3 places in ${}^3P_3$ ways.

$\therefore$ **Total Permutations** $= {}^1P_1 \times {}^3P_3$
$= 1! \times 3!$
$= 1 \times 6$
$= 6$

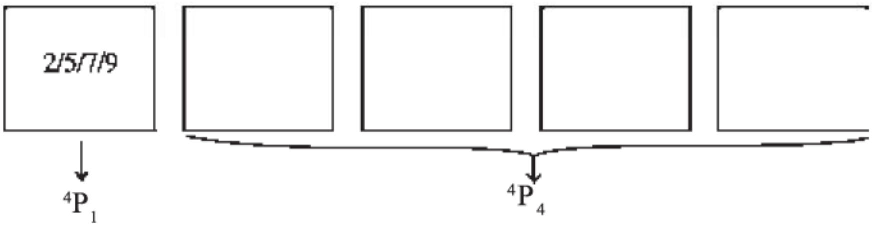
**(4)**




If numbers greater then 5000 are to be formed using all digits 2, 5, 7, 8 then digit 5, 7 or 8 should be in thousands place (first place). This arrangement can be done in ${}^3P_1$ ways. After arranging 5, 7 or 8 in thousands place, remaining 3 digits can be arranged in remaining 3 places in ${}^3P_3$ ways :

$\therefore$ **Total Permutations** $= {}^3P_1 \times {}^3P_3$
$= 3 \times 3!$
$= 3 \times 6$
$= 18$

**Illustration 9 : How many 5 digit numbers can be formed using all the digits 2, 5, 0, 7 and 9 ?**

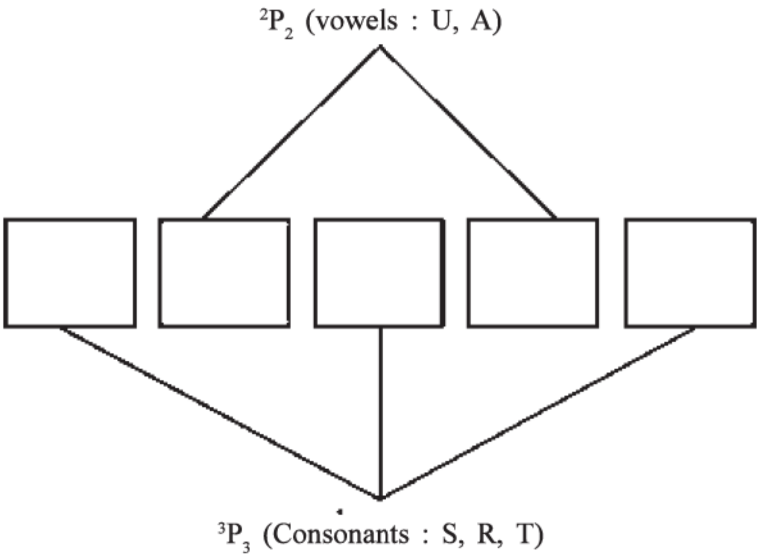



If 5 digit numbers are to be formed using all the digits 2, 5, 0, 7, 9 then digit 0 should not be in the first place. Therefore, excluding digit 0, one of the four digits can be placed in the first place in ${}^4P_1$ ways. Now, after arranging one digit from 2, 5, 7 or 9 in the first place, remaining 4 digits (including 0) can be arranged in remaining 4 places in ${}^4P_4$ ways.

> If zero is put in the first place, the number will be only 4 digit number.
> For example,
> 02579 = 2579
> 09752 = 9752

$\therefore$ **Total Permutations** $= {}^4P_1 \times {}^4P_4$
$= 4 \times 4!$
$= 4 \times 24$
$= 96$

**Illustration 10 : In how many ways can all the letters of the word SURAT be arranged such that vowels are at even places only ?**




There are two vowels U and A in word SURAT. Now, they can be arranged in even places i.e the 2nd and the 4th places in $^2P_2$ ways. Remaining 3 letters (consonants) can be arranged in remaining 3 places (odd places) in $^3P_3$ ways.

$\therefore$ **Total Permutations** $= {}^2P_2 \times {}^3P_3$

$= 2! \times 3!$

$= 2 \times 6$

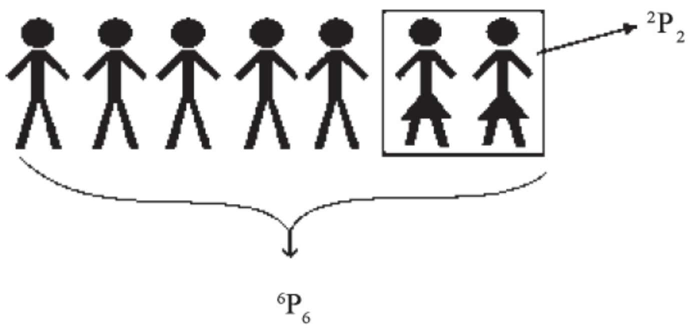
$= 12$

**Illustration 11 : In how many ways can 5 boys and 2 girls be arranged in a row such that,**

**(1) both the girls are together ?**

**(2) both the girls are not together ?**

(1)




Two girls are to be arranged together, so considering them as one person, total 6 persons can be arranged in $^6P_6$ ways and in each of these arrangements, two girls can be arranged among themselves in $^2P_2$ ways.

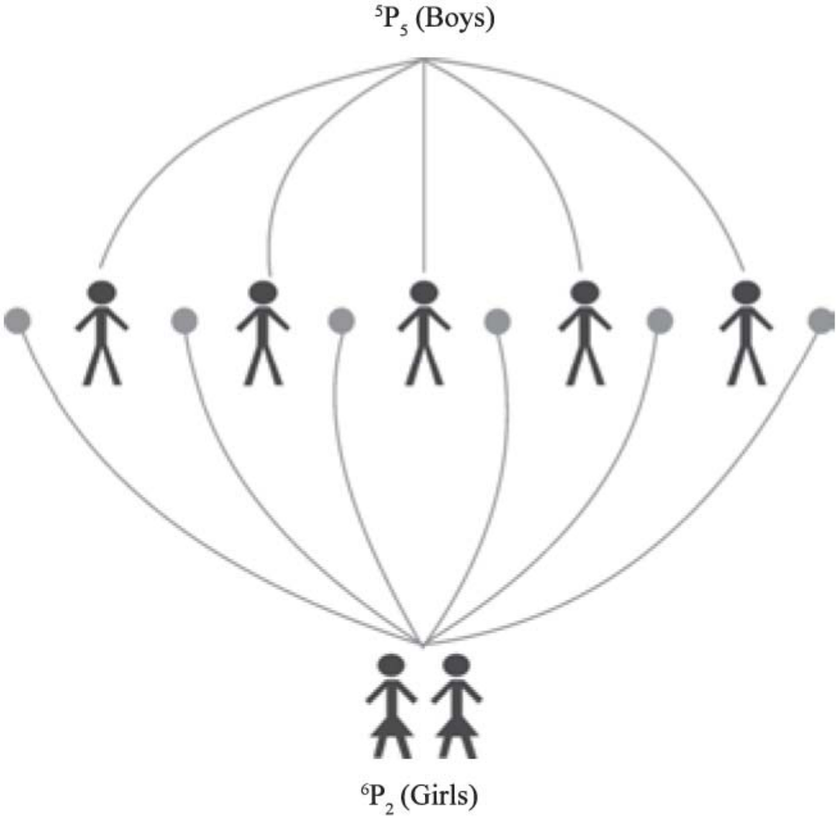
$\therefore$ **Total Permutations** $= {}^6P_6 \times {}^2P_2$

$= 6! \times 2!$

$= 720 \times 2$

$= 1440$

**(2)**




As the two girls are not to be arranged together, they can be arranged between 5 boys and on either sides. So, two girls can be placed in total 6 places in $^6P_2$ ways. Moreover 5 boys can be arranged in $^5P_5$ ways.

$\therefore$ **Total Permutations** $= {}^6P_2 \times {}^5P_5$

$= 6 \times 5 \times 5!$

$= 30 \times 120$

$= 3600$

**Illustration 12 : There are 3 different books of Gujarati, 4 different books of English and 4 different books of Sanskrit on a table. In how many ways these books can be arranged such that the books of each subject are together ?**

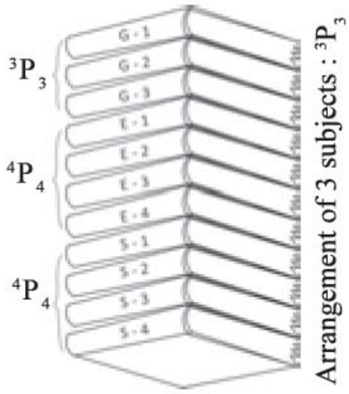
3 different books of Gujarati can be arranged in $^3P_3$ ways, 4 different books of English can be arranged in $^4P_4$ ways and 4 different books of Sanskrit can be arranged in $^4P_4$ ways. Moreover, arrangement of 3 subjects can be done in $^3P_3$ ways.

$\therefore$ **Total Permutations** $= {}^3P_3 \times {}^4P_4 \times {}^4P_4 \times {}^3P_3$

$= 3! \times 4! \times 4! \times 3!$

$= 6 \times 24 \times 24 \times 6$

$= 20{,}736$




**Activity**

Take your note books of Statistics, Accountancy and Economics. Also take your friend's notebooks of Statistics, Accountancy having his name on it. Now using your common sense, arrange all these notebooks one upon one such that notebooks of each subject remain together. Find out how many such different arrangements can be made ? Now obtain solution of the same problem using fundamental principle of counting and compare both the answers.

**Illustration 13 : In how many ways can 3 boys and 3 girls be arranged in a row such that boys and girls are alternately arranged ?**

Boys (B) and girls (G) are to be arranged alternately, so they can be arranged as shown alongside :

B G B G B G

*OR*

G B G B G B

If we start arrangement with boys then that arrangement can be done in $^3P_3$ ways ***OR*** if we start arrangement with girls then that arrangement can be done in $^3P_3$ ways.

$\therefore$ **Total Permutations** $= (^3P_3 \times {}^3P_3) + (^3P_3 \times {}^3P_3)$

$= (3! \times 3!) + (3! \times 3!)$

$= (6 \times 6) + (6 \times 6)$

$= 36 + 36$

$= 72$

**Illustration 14 : Arrangements are made using all the letters of the word YOUNG. If all these arrangements are arranged in the order of dictionary, what will be the rank of the word YOUNG ?**

There are 5 letters Y, O, U, N, G in the word YOUNG which can be arranged in $^5P_5 = 5! = 120$ ways. Now, we have to obtain the order of the word YOUNG from all 120 arrangements as per dictionary order.

Alphabetical order of all letters of the word YOUNG will be G, N, O, U, Y

Arrangements with G in the first place will be $^1P_1 \times {}^4P_4 = 24$.

Arrangements with N in the first place will be $^1P_1 \times {}^4P_4 = 24$.

Arrangements with O in the first place will be $^1P_1 \times {}^4P_4 = 24$.

Arrangements with U in the first place will be $^1P_1 \times {}^4P_4 = 24$.

Arrangements with Y in the first place and G in the second place will be $^1P_1 \times {}^1P_1 \times {}^3P_3 = 6$

Arrangements with Y in the first place and N in the second place will be $^1P_1 \times {}^1P_1 \times {}^3P_3 = 6$

Arrangements with Y in the first place, O in the second place and G in the third place will be $^1P_1 \times {}^1P_1 \times {}^1P_1 \times {}^2P_2 = 2$

Arrangements with Y in first place, O in second place and N in the third place will be $^1P_1 \times {}^1P_1 \times {}^1P_1 \times {}^2P_2 = 2$.

Arrangements with Y in first place, O in the second place, U in the third place and G in the fourth place will be $^1P_1 \times {}^1P_1 \times {}^1P_1 \times {}^1P_1 \times {}^1P_1 = 1$.

Thereafter, the word YOUNG comes which itself takes 1 position.

$\therefore$ Dictionary order of the word YOUNG = 24 + 24 + 24 + 24 + 6 + 6 + 2 + 2 + 1 + 1 = 114

To understand the above illustration, study the information in the table below :

| |
|---|
| If all the letters of the word TAB are used, $^3P_3 = 3! = 6$ arrangements can be made. These arrangement are TAB, TBA, ATB, ABT, BTA, BAT. These 6 arrangements arranged according to dictionary order are ABT, ATB, BAT, BTA, TAB, TBA. Thus, the order of the word TAB is 5. |

**Activity**

Using common sense obtain all arrangements using all the letters of the word ZERO. Arrange all these arrangements according to dictionary order and find the rank of the word ZERO. Now obtain solution of this problem as per calculations of above illustration and compare both the answers.

**Permutation of identical things**

When out of total N things, A things are identical and remaining things are different then total arrangements will be $\frac{N!}{A!}$. There are three letters (N = 3) in the word BEE of which E is repeated two times (A = 2). So, total arrangement of the word BEE will be $\frac{N!}{A!} = \frac{3!}{2!} = \frac{6}{2} = 3$. We can understand that only three arrangements BEE, EBE, EEB are possible. In broader sense, out of N total things, A things are identical of the first type, B things are identical of the second type, C things are identical of third type and remaining things are different then total number of permuntations of N things is $\frac{N!}{A!\,B!\,C!}$

**Illustration 15 : How many arrangements can be made using all the letters of the word CINCINNATI ?**

There are total 10 letters in the word CINCINNATI of which C is repeated 2 times, I is repeated 3 times and N is repeated 3 times.

$$\therefore \quad \text{Total arrangement} = \frac{10!}{2!\,3!\,3!}$$

$$= \frac{3628800}{2 \times 6 \times 6}$$

$$= 50{,}400$$

**EXERCISE 6.1**

1. Obtain the values of the following :

   (1) ${}^{10}P_3$ (2) ${}^{50}P_2$ (3) ${}^{8}P_7$ (4) ${}^{9}P_9$

2. If ${}^{n}P_3 = 990$ then find the value of $n$.

3. If ${}^{9}P_r = 3024$, find the value of $r$.

4. If $3 \cdot {}^{(n+3)}P_4 = 5 \cdot {}^{(n+2)}P_4$ then find the value of $n$.

5. In how many ways can 4 persons be arranged in a row ?

6. How many six digit numbers can be formed using all the digits 1, 2, 3, 0, 7, 9 ?

7. In how many ways can 5 boys and 3 girls be arranged in a row such that all the boys are together ?

**8.** There are 7 cages for 7 lions in a zoo. 3 cages out of 7 cages are so small that 3 out of 7 lions cannot fit in it. In how many ways can 7 lions be caged in 7 cages ?

**9.** Using all the digits 2, 3, 5, 8, 9, how many numbers greater than 50,000 can be formed ?

**10.** A person has 5 chocolates of different sizes. These chocolates are to be distributed among 5 children of different ages. If the biggest chocolate is to be given to the youngest child then in how many ways, 5 chocolates can be distributed among 5 children ?

**11.** How many total arrangements can be made using all letters of the following words ?

(1) STATISTICS (2) BOOKKEEPER (3) APPEARING

**12.** What is the ratio of number of arragements of all letters of the word ASHOK and GEETA ?

**13.** There are 5 seats in a car including the driver's seat. If 3 out of 10 members in a family know driving then in how many ways, 5 persons out of 10 members can be arranged in the car ?

**14.** If all the arrangements formed using all the letters of the following words are arragned in the order of dictionary then what will be the rank of that word ?

(1) PINTU (2) NURI (3) NIRAL (4) SUMAN

**15.** How many arrangements of the letters of the word SHLOKA can be made such that all vowels are together ?

**16.** 7 speakers A, B, C, D, E, F are invited to deliver a speech in a program. Speakers have to deliver speech one after the other. In how many ways, speeches of 7 speakers can be arranged if the speaker B has to deliver his speech immediately after the speaker A ?

*

## 6.2 Meaning of Combination

We have studied permutation which gives number of arrangements of distinct things. Now, we shall think about the number of ways to select some of the things from given distinct things. For example, Tanya has three friends Rutva, Kathan and Kirti. Now if these three friends are to be arranged in two places then it can be done in ${}^3P_2 = 3 \times 2 = 6$ ways as under :

| **Types of Arrangement** | **1** | **2** | **3** | **4** | **5** | **6** |
|---|---|---|---|---|---|---|
| **First Place** | Rutva | Rutva | Kathan | Kathan | Kirti | Kirti |
| **Second Place** | Kathan | Kirti | Rutva | Kirti | Rutva | Kathan |

But now we have to think about types of selection. Tanya wants to invite only two friends out of the above three friends for a function at her home. How many options does Tanya have to invite two friends out of three ? Will Tanya invite Rutva and Kirti ? or Rutva and Kathan ? or Kathan and Kirti ? Thus, only these three different ways can arise which are shown in the following table :

| **Selection ways** | **1** | **2** | **3** |
|---|---|---|---|
| **Selected friends** | Rutva and Kathan | Rutva and Kirti | Kathan and Kirti |

It should be noted here that the order is important in permutation so 'Rutva and Kathan' and 'Kathan and Rutva' are different with reference to permutation, whereas the order is not important in selection. That is, whether 'Rutva and Kathan' or 'Kathan and Rutva' come to Tanya's home, it means the same. Thus, selection of 2 friends out of 3 friends can be done in 3 ways. Each of these options of selection is called combination. Total number of such combinations is denoted by ${}^3C_2$ . Thus, it can be said that ${}^3C_2 = 3$. In general, the total number of combinations of selecting $r$ ($\leq n$) things out of $n$ different things will be ${}^nC_r$. It is also denoted by ${}_nC_r$, $C(n, r)$, $C^n_r$ . We will use the notation ${}^nC_r$.

Thus, total combinations of selecting $r$ things out of $n$ different things is ${}^nC_r$.

Now, let us see how the value of ${}^nC_r$ is obtained. Total combinations of selecting $r$ things out of $n$ distinct things is ${}^nC_r$. In each type of selection, $r$ things are involved. $r$ things among themselves can be arranged in ${}^rP_r = r!$ ways. Thus, we get $r!$ permutations corresponding to each combination. Hence, ${}^nC_r$ combinations give ${}^nC_r \times r!$ permutations. But as we have seen earlier, total number of permutations of $r$ things out of $n$ distinct things is ${}^nP_r$.

Hence, $${}^nP_r = {}^nC_r \times r!$$

$$\therefore \quad {}^nC_r = \frac{{}^nP_r}{r!}$$

Substituting the formula of ${}^nP_r$,

$$\therefore \quad {}^nC_r = \frac{n!}{(n-r)!\,r!}$$

$$= \frac{n!}{r!\,(n-r)!}$$

$\therefore\ {}^nC_r = \dfrac{n!}{r!(n-r)!}$ where, $n > 0$, $r \geq 0$, $n \geq r$, $n$ is positive integer and $r$ is non-negative integer.

Using this formula, following results can be derived :

| Some important results | | | | |
|---|---|---|---|---|
| $^n\mathbf{C}_0 = 1$ | $^n\mathbf{C}_n = 1$ | $^n\mathbf{C}_1 = n$ | $^n\mathbf{C}_{n-1} = n$ | $^n\mathbf{C}_r = {}^n\mathbf{C}_{n-r}$ |
| $^nC_r = \frac{n!}{r!(n-r)!}$ <br> $\therefore\ {}^nC_0 = \frac{n!}{0!(n-0)!}$ <br> $= \frac{n!}{n!}$ <br> $= 1$ | $^nC_r = \frac{n!}{r!(n-r)!}$ <br> $\therefore\ {}^nC_n = \frac{n!}{n!(n-n)!}$ <br> $= \frac{n!}{n!}$ <br> $= 1$ | $^nC_r = \frac{n!}{r!(n-r)!}$ <br> $\therefore\ {}^nC_1 = \frac{n!}{1!(n-1)!}$ <br> $= \frac{n(n-1)!}{1\times(n-1)!}$ <br> $= n$ | $^nC_r = \frac{n!}{r!(n-r)!}$ <br> $\therefore\ {}^nC_{n-1} = \frac{n!}{(n-1)![n-(n-1)]!}$ <br> $= \frac{n!}{(n-1)!1!}$ <br> $= n$ | $^nC_r = \frac{n!}{r!(n-r)!}$ <br> $\therefore\ {}^nC_{n-r} = \frac{n!}{(n-r)![n-(n-r)]!}$ <br> $= \frac{n!}{(n-r)!\,r!}$ <br> $= {}^nC_r$ |
| e.g. $^5C_0 = 1$ <br> $^{10}C_0 = 1$ | e.g. $^5C_5 = 1$ <br> $^{10}C_{10} = 1$ | e.g. $^5C_1 = 5$ <br> $^{10}C_1 = 10$ | e.g. $^5C_4 = 5$ <br> $^{10}C_9 = 10$ | e.g. $^5C_4 = {}^5C_1$ <br> $^{10}C_7 = {}^{10}C_3$ |
| If $^nC_x = {}^nC_y$ then, $x + y = n$ or $x = y$ | | | | |

**Illustration 16 : Find the values of the following :**

**(1)** $^8\mathbf{C}_3$ **(2)** $^{20}\mathbf{C}_3$ **(3)** $^5\mathbf{C}_4$ **(4)** $^6\mathbf{C}_6$

**(1)** $^nC_r = \frac{n!}{r!\,(n-r)!}$

$\therefore\quad {}^8C_3 = \frac{8!}{3!\,(8-3)!}$

$= \frac{8!}{3! \times 5!}$

$= \frac{8\times7\times6\times5!}{3\times2\times1\times5!}$

$= 56$

**Alternative method :**

According to definition of $^nC_r$

$^nC_r = \frac{^nP_r}{r!}$

$\therefore\quad {}^8C_3 = \frac{^8P_3}{3!}$

$= \frac{8\times7\times6}{3\times2\times1}$

$= 56$

**(2)** According to the definition of $^nC_r$,

$^nC_r = \frac{^nP_r}{r!}$

$\therefore\quad {}^{20}C_3 = \frac{^{20}P_3}{3!}$

$= \frac{20 \times 19 \times 18}{3 \times 2 \times 1}$

$= 1140$

**(3)** According to the definition of ${}^nC_r$,

$${}^nC_r = \frac{{}^nP_r}{r!}$$

$$\therefore \quad {}^5C_4 = \frac{{}^5P_4}{4!}$$

$$= \frac{5\times4\times3\times2}{4\times3\times2\times1}$$

$$= 5$$

**(4)** According to the definition of ${}^nC_r$,

$${}^nC_r = \frac{{}^nP_r}{r!}$$

$$\therefore \quad {}^6C_6 = \frac{{}^6P_6}{6!}$$

$$= \frac{6\times5\times4\times3\times2\times1}{6\times5\times4\times3\times2\times1}$$

$$= 1$$

**Illustration 17 : If ${}^nC_2 = 45$, find the value of $n$.**

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

$$\therefore \quad {}^nC_2 = \frac{n!}{2!(n-2)!}$$

$$\therefore \quad 45 = \frac{n\,(n-1)\,(n-2)!}{2\times1\times(n-2)!}$$

$\therefore \quad 90 = n\,(n-1)$

$\therefore \quad n\,(n-1) = 10\,(10-1)$

$\therefore \quad n = 10$

**Alternative method :**

${}^nC_2 = 45$

$$\therefore \quad \frac{n\,(n-1)}{2\times1} = 45$$

$\therefore \quad n\,(n-1) = 90$

$\therefore \quad n\,(n-1) = 10\,(10-1)$

$\therefore \quad n = 10$

**Illustration 18 : If $3\cdot{}^{2n}C_3 = 44\cdot{}^nC_2$ then find the value of $n$.**

$3\cdot{}^{2n}C_3 = 44\cdot{}^nC_2$

$$\therefore \quad \frac{3\times2n\,(2n-1)\,(2n-2)}{3\times2\times1} = \frac{44\times n\,(n-1)}{2\times1}$$

$$\therefore \quad \frac{2n\,(2n-1)\,2\,(n-1)}{2} = \frac{44n\,(n-1)}{2}$$

$\therefore \quad 4\,(2n-1) = 44$

$\therefore \quad 2n - 1 = 11$

$\therefore \quad 2n = 12$

$\therefore \quad n = 6$

**Illustration 19 : If $^nC_{n-3} = 56$, find the value of *n*.**

$^nC_{n-3} = {^nC_3}$ $\quad [\because \ {^nC_r} = {^nC_{n-r}}]$

$\therefore \quad {^nC_3} = 56$

$\therefore \quad \frac{n\,(n-1)\,(n-2)}{3 \times 2 \times 1} = 56$

$\therefore \quad n\,(n-1)\,(n-2) = 336$

$\therefore \quad n\,(n-1)\,(n-2) = 8\,(8-1)\,(8-2)$

$\therefore \quad n = 8$

**Illustration 20 : If $^nC_4 = {^nC_6}$ then find the value of *n*.**

we know that if $^nC_x = {^nC_y}$ then,

**Option 1 :**

$x + y = n$

$\therefore \quad 4 + 6 = n$

$\therefore \quad n = 10$

**Option 2 :**

$x = y$

$\therefore \ 4 = 6$

Which is impossible.

Thus, the value of *n* is 10.

**Illustration 21 : If $^{50}C_{r+2} = {^{50}C_{2r-3}}$ then find the value of *r*.**

We know that if $^nC_x = {^nC_y}$ then,

**Option 1 :**

$x + y = n$

$\therefore \quad (r + 2) + (2r - 3) = 50$

$\therefore \quad r + 2 + 2r - 3 = 50$

$\therefore \quad 3r = 51$

$\therefore \quad r = 17$

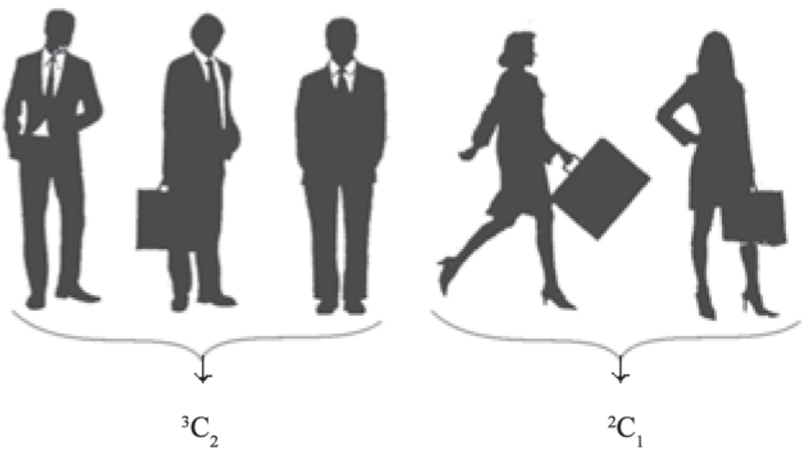
**Option 2 :**

$x = y$

$\therefore \quad r + 2 = 2r - 3$

$\therefore \quad r = 5$

**Illustration 22 : In a company 2 male managers and 1 female manager are to be selected from 3 male managers and 2 female managers for training. How many ways this selection can be done ?**




2 male managers out of 3 male managers can be selected in $^3C_2$ ways and 1 female manager out of 2 female managers can be selected in $^2C_1$ ways.

$\therefore$ **Total Combinations** $= {^3C_2} \times {^2C_1}$

$= 3 \times 2$

$= 6$

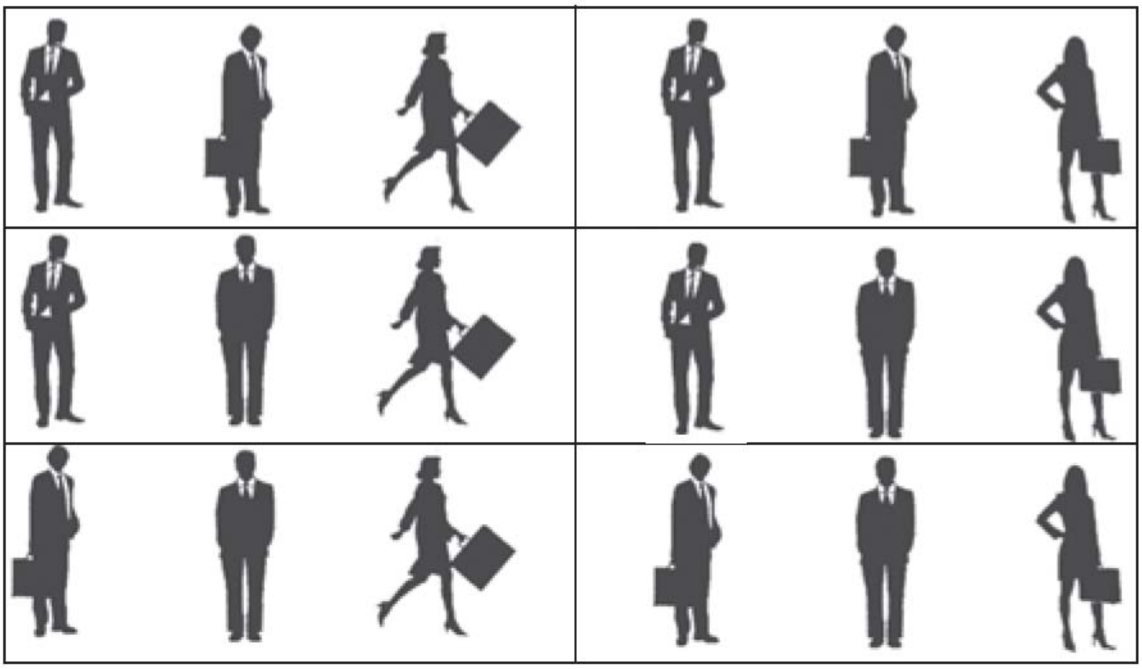


**Illustration 23 : There are 5 yellow, 4 white and 3 pink flowers in a basket. In how many ways 3 yellow, 2 white and 1 pink flower can be selected from it ?**

3 yellow flowers from 5 yellow flowers can be selected in ${}^5C_3$ ways, 2 white flower from 4 white flowers can be selected in ${}^4C_2$ ways and 1 pink flower from 3 pink flowers can be selected in ${}^3C_1$ ways.

$\therefore$ **Total combinations** $= {}^5C_3 \times {}^4C_2 \times {}^3C_1$

$= 10 \times 6 \times 3$

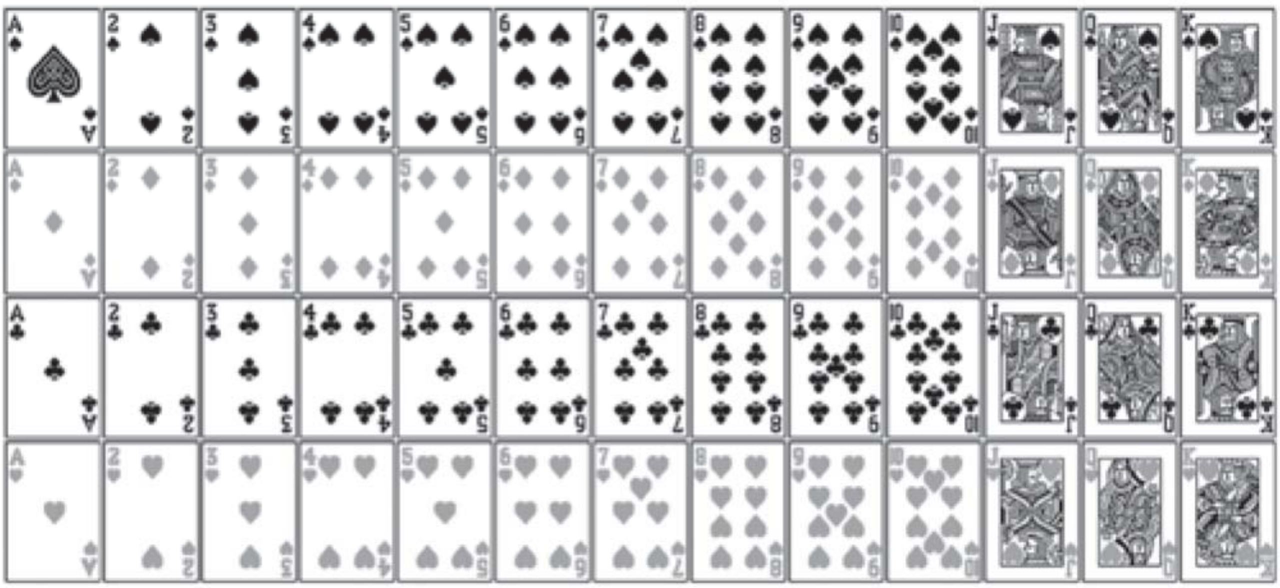
$= 180$

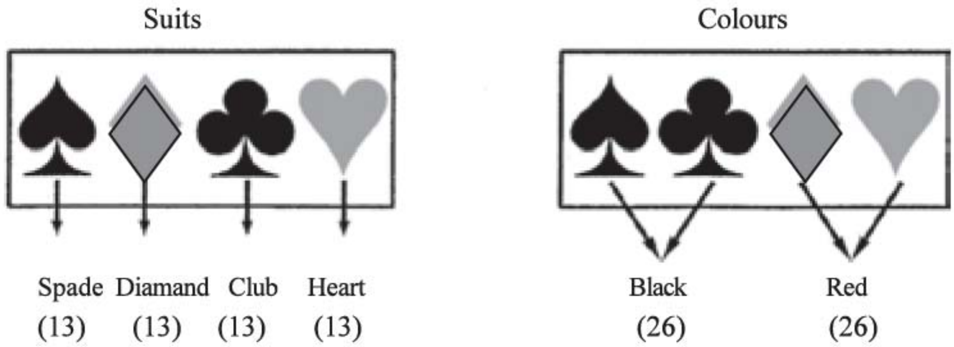
**Illustration 24 : 2 cards are selected from a pack of 52 cards. In how many ways, this selection can be done such that**

**(1) One is a face card and the other is a number card ?**

**(2) both are of different colours ?**

**(3) both are of same suit ?**






(1) There are 12 face cards (king, Queen, Jack) and 40 number cards in a pack of 52 cards. One face card can be selected in ${}^{12}C_1$ ways and one number card can be selected in ${}^{40}C_1$ ways.

$\therefore$ **Total Combinations** $= {}^{12}C_1 \times {}^{40}C_1$

$= 12 \times 40$

$= 480$

(2) There are 26 black and 26 red cards in a pack of 52 cards. One black card can be selected in ${}^{26}C_1$ ways and one red card can be selected in ${}^{26}C_1$ ways.

$\therefore$ **Total Combinations** $= {}^{26}C_1 \times {}^{26}C_1$

$= 26 \times 26$

$= 676$

(3) There are 4 suits, spade, diamond, club and heart, in a pack of 52 cards and each suit has 13 cards. 2 cards are of same suit i.e. 2 of spade or 2 of diamond or 2 of club or 2 of heart.

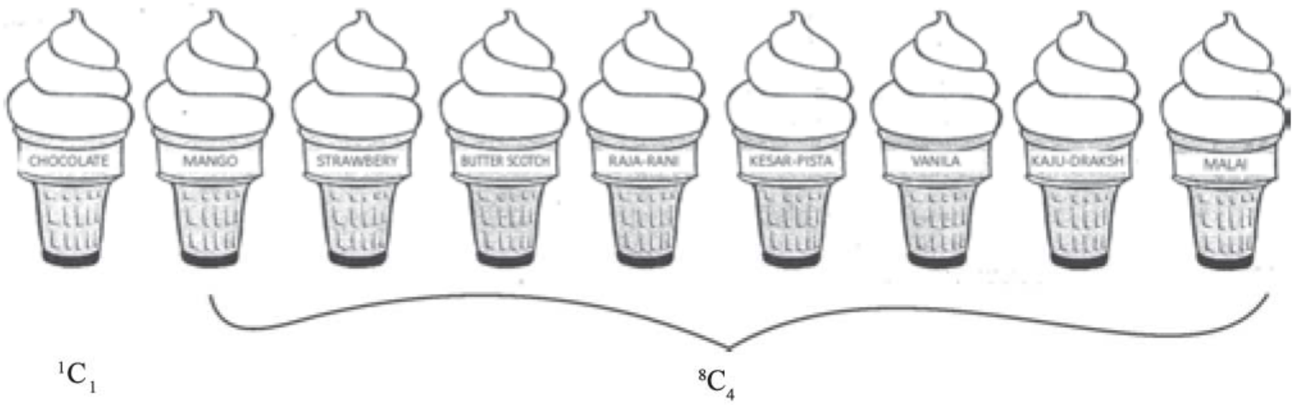
$\therefore$ **Total Combinations** $= {}^{13}C_2 + {}^{13}C_2 + {}^{13}C_2 + {}^{13}C_2$

$= 78 + 78 + 78 + 78$

$= 312$

**Illustration 25 : A boy named Kathan wants to select 5 different flavours of ice-cream cones out of 9 different flavours of ice-cream cones. If he wants one of the selected ice-cream cone to be of chocolate flavour then find total number of selections ?**

Kathan wants to select 5 different flavours of ice-cream cones out of 9 different flavours of ice-cream cones. Chocolate ice-cream cone has to be selected which can be selected in ${}^{1}C_1$ ways. Now out of remaining 8 ice-cream cones, Kathan can select remaining 4 ice-cream cones in ${}^{8}C_4$ ways.



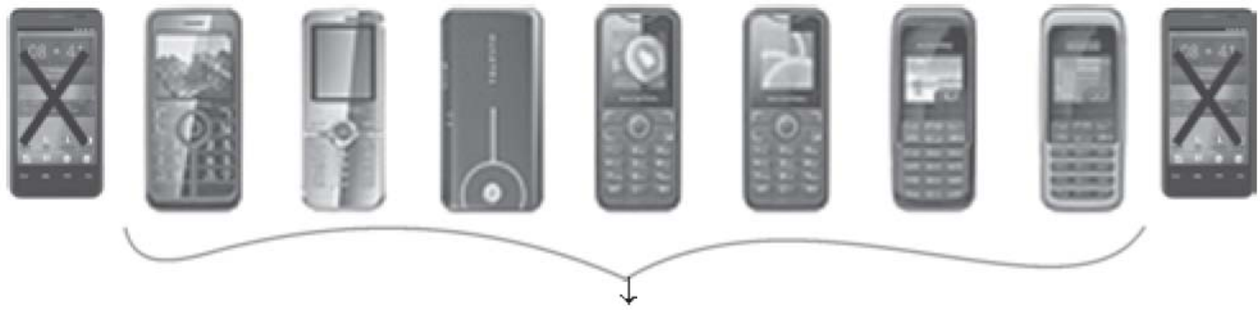

$\therefore$ **Total Combinations** $= {}^{1}C_1 \times {}^{8}C_4$

$$= 1 \times \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1}$$

$= 70$

**Illustration 26 : A person wishes to purchase 3 mobile handsets from 10 different handsets of a company. But 2 mobile handsets do not fit into his budget. In how many ways can a person puchase 3 different handsets ?**

Here there are 10 different handsets. But 2 mobile hand sets do not fit into his budget. So, selection of 3 mobile handsets out of remaining 8 mobile handsets can be done in $^8C_3$ ways.



$^8C_3$

$\therefore$ **Total Combinations** $= {}^8C_3$

$= \frac{8 \times 7 \times 6}{3 \times 2 \times 1}$

$= 56$

**Illustration 27 : There are 6 engineers and 4 managers in a company. In how many ways can a committee of 5 members be made such that**

**(1) there are at least 2 managers ?**

**(2) there are at the most 2 engineers ?**

**(3) engineers are in majority ?**

(1) In a committee of 5 members, selection of at least 2 managers can be done in following ways :

| Manager (4) | | Engineer (6) |
|---|---|---|
| 2 | and | 3 |
| | **OR** | |
| 3 | and | 2 |
| | **OR** | |
| 4 | and | 1 |

$\therefore$ **Total Combinations** $= ({}^4C_2 \times {}^6C_3) + ({}^4C_3 \times {}^6C_2) + ({}^4C_4 \times {}^6C_1)$

$= (6 \times 20) + (4 \times 15) + (1 \times 6)$

$= 120 + 60 + 6$

$= 186$

(2) In a committee of 5 persons, selection of at the most 2 engineers can be done in following ways:

| **Engineer (6)** | | **Manager (4)** |
|---|---|---|
| 2 | and | 3 |
| | **OR** | |
| 1 | and | 4 |

$\therefore$ **Total Combinations** $= ({}^6C_2 \times {}^4C_3) + ({}^6C_1 \times {}^4C_4)$

$= (15 \times 4) + (6 \times 1)$

$= 60 + 6$

$= 66$

(3) The selection of committe of 5 members such that engineers are in majority can be done in following ways :

| **Engineer (6)** | | **Manager (4)** |
|---|---|---|
| 5 | and | 0 |
| | **OR** | |
| 4 | and | 1 |
| | **OR** | |
| 3 | and | 2 |

$\therefore$ **Total Combinations** $= ({}^6C_5 \times {}^4C_0) + ({}^6C_4 \times {}^4C_1) + ({}^6C_3 \times {}^4C_2)$

$= (6 \times 1) + (15 \times 4) + (20 \times 6)$

$= 6 + 60 + 120$

$= 186$

**Illustration 28 : In an interview, 6 questions are asked to a person. In how many ways can a person give (1) at least 4 correct answers ? (2) at the most 3 correct answers ?**

(1) If a person has given at least 4 correct answers of 6 questions asked then it means he has given 4 or 5 or 6 correct answers.

$\therefore$ **Total Combinations** $= {}^6C_4 + {}^6C_5 + {}^6C_6$

$= 15 + 6 + 1$

$= 22$

(2) If a person has given at the most 3 correct answers of 6 questions asked then it means he has given 3 or 2 or 1 or 0 correct answers.

$\therefore$ **Total Combinations** $= {}^6C_3 + {}^6C_2 + {}^6C_1 + {}^6C_0$

$= 20 + 15 + 6 + 1$

$= 42$

**Activity**

Make a group of 10 friends including yourself. Each one shakes hand with remaining in the group. Count total number of handshakes. Now obtain the solution of this problem with the formula of combination and compare both the answers.

## EXERCISE 6.2

**1.** Obtain the values of the following :

(1) $^{11}C_4$ (2) $^{9}C_0$ (3) $^{25}C_{23}$ (4) $^{8}C_8$

**2.** Find the unknown value :

(1) $^{n}C_2 = 28$ (2) $^{27}C_{r+4} = {}^{27}C_{2r-1}$ (3) $^{n}C_{n-2} = 15$ (4) $4 \cdot {}^{n}C_4 = 7 \cdot {}^{n}C_3$

**3.** 8 candidates applied for 2 posts of peon in a school. In how many ways can 2 peons be selected from 8 candidates ?

**4.** 5 countries participate in a cricket tournament. In the first round, every country plays a match with the other country. How many matches will be played in this round ?

**5.** There are 200 items in a box and 5% of them are defective. In how many ways can 3 items can be selected from the box so that all the items selected are defective ?

**6.** In how many ways can 3 clerks and 1 peon be selected from 14 clerks and 6 peons working in a bank ?

**7.** There are 3 white and 5 pink flowers in a box. In how many ways can

(1) three flowers of same colour be selected ?

(2) 2 flowers of different colours be selected ?

**8.** Two cards are randomly selected from a pack of 52 cards. In how many ways can 2 cards be selected such that,

(1) both are of heart ?

(2) one is a king and the other is a queen ?

**9.** There are 9 employees in a bank of which 6 are clerks, 2 are peons and 1 is a manager. In how many ways can a committee of 4 members be formed such that

(1) the manager must be selected ?

(2) two peons are not to be selected and the manager is to be selected ?

**10.** In an office, there are 8 employees of which 3 are females and remaining are males. 3 employees are to be selected from the office for training. In how many ways can the selection be done so that at least one male is selected ?

**11.** A person has 6 friends. In how many ways can he invite atleast one friend to his house ?

**12.** In how many ways can 5 books be selected from 8 different books so that,

(1) a particular book is always selected ?

(2) a particluar book is never selected ?

**13.** A student in 12th standard commerce stream has to appear for exam in 7 subjects. It is necessary to pass in all the subjects to pass an exam. Certain minimum marks must be obtained to pass in a subject. In how many ways can a student appearing for the exam fail ?

**14.** In how many ways can a hotel owner subscribe 3 newspapers and 2 magazines from 8 different newspapers and 5 different magazines available in the city ? If a particular newspaper is to be selected and a particular magazine is not to be selected then in how many ways can this selection be done ?

**15.** If $^{n}P_2 + {}^{n}C_2 = 84$ then find the value of $n$.

**16.** If $^{n}P_r \div {}^{n}C_r = 24$ then find the value of $r$.

*

## 6.3 Meaning of Binomial Expansion

Any expression consisting of two terms connected by + or – sign is called a binomial expression. For example, $a + b$, $x - y$, $4a + 3b$, $x + 2a$ etc. are called binomial expressions. We have already studied the expansions of binomial expressions up to third power, which are as follows :

- $(x + a)^1 = x + a$
- $(x + a)^2 = x^2 + 2xa + a^2$
- $(x - a)^2 = x^2 - 2xa + a^2$
- $(x + a)^3 = x^3 + 3x^2a + 3xa^2 + a^3$
- $(x - a)^3 = x^3 - 3x^2a + 3xa^2 - a^3$

Now, let us think about an easy way to obtain the expansion of binomial expression with positive integer power greater than 3.

This expansion can be obtained using binomial theorem. We can write the coefficients of different terms of the above binomial expansions using combinations as follows :

- $(x + a)^1 = x + a$

  $= {}^1C_0\, x + {}^1C_1\, a$
- $(x + a)^2 = x^2 + 2x\,a + a^2$

  $= {}^2C_0\, x^2 + {}^2C_1\, xa + {}^2C_2\, a^2$
- $(x + a)^3 = x^3 + 3x^2\, a + 3xa^2 + a^3$

  $= {}^3C_0\, x^3 + {}^3C_1 x^2a + {}^3C_2\, xa^2 + {}^3C_3\, a^3$

Similarly, expansion of $(x + a)^4$ and $(x + a)^5$ can be written as follows :

$$(x + a)^4 = {}^4C_0x^4 + {}^4C_1\, x^3a + {}^4C_2\, x^2a^2 + {}^4C_3\, xa^3 + {}^4C_4\, a^4$$
$$= x^4 + 4x^3a + 6x^2\, a^2 + 4xa^3 + a^4$$
$$(x + a)^5 = {}^5C_0x^5 + {}^5C_1\, x^4a + {}^5C_2\, x^3a^2 + {}^5C_3\, x^2a^3 + {}^5C_4\, xa^4 + {}^5C_5\, a^5$$
$$= x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5xa^4 + a^5$$

If we observe the above expansion in detail, we come to know that the power of $x$ decreases by one in each successive term and power of '$a$' increases by one in each successive terms. From this, the expansion of binominal expression with positive integer power $n$ can be written as follows :

$$(x + a)^n = {}^nC_o x^n a^0 + {}^nC_1 x^{n-1} a^1 + {}^nC_2 x^{n-2} a^2 + ... + {}^nC_n\, x^0\, a^n$$

Expansion of $(x + a)^n$ is called binomial expansion. The general term of this expansion is ${}^nC_r\, x^{n-r}\, a^r$. If we put $r = 0, 1, 2, ..., n$ in this term we get all the terms of binomial expansion.

| $r$ | **Order in the expansion** | ${}^nC_r\, x^{n-r}\, a^r$ |
|---|---|---|
| 0 | First term | ${}^nC_0\, x^{n-0}\, a^0 = x^n$ |
| 1 | Second term | ${}^nC_1\, x^{n-1}\, a^1$ |
| 2 | Third term | ${}^nC_2\, x^{n-2}\, a^2$ |
| . | . | . |
| . | . | . |
| . | . | . |
| $n$ | $(n + 1)$th term | ${}^nC_n\, x^{n-n}\, a^n = a^n$ |

Thus, ${}^nC_r\, x^{n-r} a^r$ is the $(r + 1)$th term of the binomial expansion of $(x+a)^n$ which is called as general term of $(x + a)^n$.

i.e. $T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

We can see the following charasteristics in the above expansion :

(1) Total number of terms in the expansion of $(x + a)^n$ is $n + 1$ i.e. the number of terms is 1 more than the power of the binomial expression $(x + a)$.

(2) The coefficients of the terms of expansion are ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$ respectively.

(3) The first term of the expansion is $x^n$. In the successive terms of the expansion, the power of '$x$' goes on decreasing by 1 and the power of '$a$' goes on increasing by 1. The last term of the expansion is $a^n$.

(4) The sum of powers of $x$ and $a$ is equal to $n$ in any term of the expansion.

(5) The coefficients of terms equidistant from the middle term or terms are equal.

The coefficients of binomial expansion are given below in a triangular form. This triangular arrangement was constructed by the French mathematician **Blaise Pascal.**

**Pascal's triangle**

| power $n$ | Cofficients | Sum of coefficients $= 2^n$ |
|---|---|---|
| 1 | 1 1 | $2^1 = 2$ |
| 2 | 1 2 1 | $2^2 = 4$ |
| 3 | 1 3 3 1 | $2^3 = 8$ |
| 4 | 1 4 6 4 1 | $2^4 = 16$ |
| 5 | 1 5 10 10 5 1 | $2^5 = 32$ |
| 6 | 1 6 15 20 15 6 1 | $2^6 = 64$ |
| 7 | 1 7 21 35 35 21 7 1 | $2^7 = 128$ |
| 8 | 1 8 28 56 70 56 28 8 1 | $2^8 = 156$ |
| 9 | 1 9 36 84 126 126 84 36 9 1 | $2^9 = 512$ |

**Illustration 29 : Expand $(x + y)^6$ .**

$(x + y)^6$

$= {}^6C_0 (x)^6 (y)^0 + {}^6C_1 (x)^5 (y)^1 + {}^6C_2 (x)^4 (y)^2 + {}^6C_3 (x)^3 (y)^3$

$+ {}^6C_4 (x)^2 (y)^4 + {}^6C_5 (x)^1 (y)^5 + {}^6C_6 (x)^0 (y)^6$

$= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

**Illustration 30 : Expand $(1 + x)^4$.**

$(1 + x)^4$

$= {}^4C_0 (1)^4 (x)^0 + {}^4C_1 (1)^3 (x)^1 + {}^4C_2 (1)^2 (x)^2 + {}^4C_3 (1)^1 (x)^3 + {}^4C_4 (1)^0 (x)^4$

$= 1 + 4x + 6x^2 + 4x^3 + x^4$

**Illustration 31 : Expand $(3a + 2y)^3$.**

$(3a + 2y)^3$

$= {}^3C_0 \, (3a)^3 \, (2y)^0 + {}^3C_1 \, (3a)^2 \, (2y)^1 + {}^3C_2 \, (3a)^1 \, (2y)^2 + {}^3C_3 \, (3a)^0 \, (2y)^3$

$= 27a^3 + 3 \, (9a^2) \, (2y) + 3 \, (3a) \, (4y^2) + 8y^3$

$= 27a^3 + 54a^2y + 36ay^2 + 8y^3$

**Illustration 32 : Expand $(3x - y)^4$ .**

$(3x - y)^4 = [3x + (-y)]^4$

$= {}^4C_0 \, (3x)^4 \, (-y)^0 + {}^4C_1 \, (3x)^3 \, (-y)^1 + {}^4C_2 \, (3x)^2 \, (-y)^2 + {}^4C_3 \, (3x)^1 \, (-y)^3 + {}^4C_4 \, (3x)^0 \, (-y)^4$

$= 81x^4 + 4 \, (27x^3) \, (-y) + 6 \, (9x^2) \, (y^2) + 4 \, (3x) \, (-y^3) + y^4$

$= 81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4$

**Note :** If there is a negative sign between two terms of binomial expression, then its binomial expansion has the sign of the first term +, sign of the second term –, sign of the third term + and so on.

**Illustration 33 : Expand $(2x - a)^5$.**

$(2x - a)^5$

$= {}^5C_0 \, (2x)^5 \, (a)^0 - {}^5C_1 \, (2x)^4 \, (a)^1 + {}^5C_2 \, (2x)^3 \, (a)^2 - {}^5C_3 \, (2x)^2 \, (a)^3$

$+ {}^5C_4 \, (2x)^1 \, (a)^4 - {}^5C_5 \, (2x)^0 \, (a)^5$

$= 32x^5 - 5(16x^4) \, (a) + 10 \, (8x^3) \, (a^2) - 10 \, (4x^2) \, (a^3) + 5(2x) \, (a^4) - a^5$

$= 32x^5 - 80x^4a + 80x^3a^2 - 40x^2a^3 + 10xa^4 - a^5$

**Illustration 34 : Expand $\left(x - \frac{2}{x}\right)^5$.**

$\left(x - \frac{2}{x}\right)^5$

$= {}^5C_0 \, (x)^5 \left(\frac{2}{x}\right)^0 - {}^5C_1 \, (x)^4 \left(\frac{2}{x}\right)^1 + {}^5C_2 \, (x)^3 \left(\frac{2}{x}\right)^2 - {}^5C_3 \, (x)^2 \left(\frac{2}{x}\right)^3 + {}^5C_4 \, (x)^1 \left(\frac{2}{x}\right)^4 - {}^5C_5 \, (x)^0 \left(\frac{2}{x}\right)^5$

$= x^5 - 5 \, (x)^4 \left(\frac{2}{x}\right) + 10 \, (x)^3 \left(\frac{4}{x^2}\right) - 10 \, (x^2) \left(\frac{8}{x^3}\right) + 5 \, (x) \left(\frac{16}{x^4}\right) - \frac{32}{x^5}$

$= x^5 - 10x^3 + 40x - \frac{80}{x} + \frac{80}{x^3} - \frac{32}{x^5}$

**Activity**

To obtain the value of $(11)^4$ using binomial expansion, first of all convert 11 to a binomial expression $(10 + 1)$. Now obtain the value of $(11)^4 = (10 + 1)^4$ using binomial expansion and check whether the value obtained is correct or not using calculator.

**Illustration 35 : Expand $\left(\frac{a}{b} - \frac{b}{a}\right)^4$.**

$$\left(\frac{a}{b} - \frac{b}{a}\right)^4$$

$$= {}^4C_0 \left(\frac{a}{b}\right)^4\left(\frac{b}{a}\right)^0 - {}^4C_1 \left(\frac{a}{b}\right)^3\left(\frac{b}{a}\right)^1 + {}^4C_2 \left(\frac{a}{b}\right)^2\left(\frac{b}{a}\right)^2 - {}^4C_3 \left(\frac{a}{b}\right)^1\left(\frac{b}{a}\right)^3 + {}^4C_4 \left(\frac{a}{b}\right)^0\left(\frac{b}{a}\right)^4$$

$$= \frac{a^4}{b^4} - 4\left(\frac{a^3}{b^3}\right)\left(\frac{b}{a}\right) + 6\left(\frac{a^2}{b^2}\right)\left(\frac{b^2}{a^2}\right) - 4\left(\frac{a}{b}\right)\left(\frac{b^3}{a^3}\right) + \frac{b^4}{a^4}$$

$$= \frac{a^4}{b^4} - \frac{4a^2}{b^2} + 6 - \frac{4b^2}{a^2} + \frac{b^4}{a^4}$$

**Illustration 36 : Expand $\left(\sqrt{a} + \frac{1}{\sqrt{a}}\right)^4$.**

$$\left(\sqrt{a} + \frac{1}{\sqrt{a}}\right)^4$$

$$= {}^4C_0 \left(\sqrt{a}\right)^4\left(\frac{1}{\sqrt{a}}\right)^0 + {}^4C_1 \left(\sqrt{a}\right)^3\left(\frac{1}{\sqrt{a}}\right)^1 + {}^4C_2 \left(\sqrt{a}\right)^2\left(\frac{1}{\sqrt{a}}\right)^2$$

$$+ {}^4C_3 \left(\sqrt{a}\right)^1\left(\frac{1}{\sqrt{a}}\right)^3 + {}^4C_4 \left(\sqrt{a}\right)^0\left(\frac{1}{\sqrt{a}}\right)^4$$

$$= a^2 + 4\left(\sqrt{a}\right)^2 + 6 + 4\frac{1}{\left(\sqrt{a}\right)^2} + \frac{1}{a^2}$$

$$= a^2 + 4a + 6 + \frac{4}{a} + \frac{1}{a^2}$$

**Illustration 37 : Expand $(1 + x)^6$ and verify it by putting $x = 1$ on both the sides.**

$(1 + x)^6$

$= {}^6C_0 (1)^6 (x)^0 + {}^6C_1 (1)^5 (x)^1 + {}^6C_2 (1)^4 (x)^2 + {}^6C_3 (1)^3 (x)^3 + {}^6C_4 (1)^2 (x)^4 + {}^6C_5 (1)^1 (x)^5 + {}^6C_6 (1)^0 (x)^6$

$= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

Thus,

$(1 + x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

Putting $x = 1$,

L.H.S. $= (1 + x)^6 = (1 + 1)^6 = (2)^6 = 64$

R.H.S. $= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

$= 1 + 6(1) + 15(1)^2 + 20(1)^3 + 15(1)^4 + 6(1)^5 + (1)^6$

$= 1 + 6 + 15 + 20 + 15 + 6 + 1$

$= 64$

Thus, L.H.S. = R.H.S.

**Illustration 38 : Obtain the value of $(\sqrt{3}+2)^5 - (\sqrt{3}-2)^5$ using binomial expansion method.**

$$(\sqrt{3}+2)^5 - (\sqrt{3}-2)^5$$

$$= \begin{bmatrix} {}^5C_0(\sqrt{3})^5(2)^0 \\ + {}^5C_1(\sqrt{3})^4(2)^1 \\ + {}^5C_2(\sqrt{3})^3(2)^2 \\ + {}^5C_3(\sqrt{3})^2(2)^3 \\ + {}^5C_4(\sqrt{3})^1(2)^4 \\ + {}^5C_5(\sqrt{3})^0(2)^5 \end{bmatrix} - \begin{bmatrix} {}^5C_0(\sqrt{3})^5(2)^0 \\ - {}^5C_1(\sqrt{3})^4(2)^1 \\ + {}^5C_2(\sqrt{3})^3(2)^2 \\ - {}^5C_3(\sqrt{3})^2(2)^3 \\ + {}^5C_4(\sqrt{3})^1(2)^4 \\ - {}^5C_5(\sqrt{3})^0(2)^5 \end{bmatrix}$$

(As there is a negative sign between two binomial expressions, the signs of terms of the second expression will change. As a result, the first, third and the fifth terms in the expansion of both expressions will get cancelled.)

$= 2\ [{}^5C_1\ (\sqrt{3})^4\ (2)^1 + {}^5C_3\ (\sqrt{3})^2\ (2)^3 + {}^5C_5\ (\sqrt{3})^0\ (2)^5]$

$= 2\ [5\ (9)\ (2) + 10\ (3)\ (8) + 32]$

$= 2\ [90 + 240 + 32]$

$= 2\ [362]$

$= 724$

**Illustration 39 : Obtain the value of $(\sqrt{3}+\sqrt{2})^4 + (\sqrt{3}-\sqrt{2})^4$ using binomial expansion method.**

$$(\sqrt{3} + \sqrt{2})^4 + (\sqrt{3} - \sqrt{2})^4$$

$$= \begin{bmatrix} {}^4C_0(\sqrt{3})^4(\sqrt{2})^0 \\ + {}^4C_1(\sqrt{3})^3(\sqrt{2})^1 \\ + {}^4C_2(\sqrt{3})^2(\sqrt{2})^2 \\ + {}^4C_3(\sqrt{3})^1(\sqrt{2})^3 \\ + {}^4C_4(\sqrt{3})^0(\sqrt{2})^4 \end{bmatrix} + \begin{bmatrix} {}^4C_0(\sqrt{3})^4(\sqrt{2})^0 \\ - {}^4C_1(\sqrt{3})^3(\sqrt{2})^1 \\ + {}^4C_2(\sqrt{3})^2(\sqrt{2})^2 \\ - {}^4C_3(\sqrt{3})^1(\sqrt{2})^3 \\ + {}^4C_4(\sqrt{3})^0(\sqrt{2})^4 \end{bmatrix}$$

(As there is a positive sign between two binomial expressions, the signs of terms of the second expression will not change. As a result, the second, fourth terms in the expansion of both the expressions will be cancelled.)

$= 2\ [4_{C_0}\ (\sqrt{3})^4\ (2)^0 + 4_{C_2}\ (\sqrt{3})^2\ (\sqrt{2})^2 + 4_{C_4}\ (\sqrt{3})^0\ (\sqrt{2})^4]$

$= 2\ [\ 9 + 6\ (3)\ (2) + 4]$

$= 2\ [9 + 36 + 4]$

$= 2\ [49]$

$= 98$

## EXERCISE 6.3

**1.** Obtain the expansion of following binomial expressions :

(1) $(3a + 4b)^3$ (2) $(1 + x)^7$ (3) $\left(\frac{3}{x} - \frac{4x}{3}\right)^4$ (4) $\left(\frac{\sqrt{x}}{3} + \frac{3}{\sqrt{x}}\right)^6$ (5) $\left(\frac{a}{2} - \frac{b}{3}\right)^5$

**2.** Obtain the values using binomial expansion :

(1) $(\sqrt{5} + 1)^5 - (\sqrt{5} - 1)^5$

(2) $(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$

(3) $(\sqrt{5}+\sqrt{3})^4 + (\sqrt{5}-\sqrt{3})^4$

**3.** Expand $(1 + x)^5$ and verify by putting $x = 1$ on both sides.

**4.** Expand $(1 + a)^6$ and verify by putting $a = 2$ on both sides.

**Summary**

- If there are $m$ distinct things in Group 1 and $n$ distinct things in Group 2 then selection of one thing from combined groups can be done in $m + n$ ways.
- If the first action can be done in $m$ ways and second action can be done in $n$ ways then two actions together can be done in $m \times n$ ways.
- $n! = n(n-1)(n-2) \times ... \times 3 \times 2 \times 1$
- Total permutations of $n$ different things in $r$ places will be ${}^nP_r$.
- The fundamental difference between permutation and combination is that the order is important in permutation and the order is not important in combination. i.e. in permutation *ab* and *ba* are different while in combination *ab* and *ba* are same.
- The expansion of $(x + a)^n$ has $(n + 1)$ terms with coefficients ${}^nC_0, {}^nC_1, {}^nC_2, ... , {}^nC_n$ respectively.

**List of formulae**

- ${}^nP_r = \frac{n!}{(n-r)!}$
- ${}^nP_0 = 1,\ {}^nP_n = n!,\ {}^np_1 = n,\ {}^np_{n-1} = n!$
- Out of N total things, A things are identical of the first type, B things are identical of the second type, C things are identical of the third type and remaining things are different then total number of permutations of N things is $\frac{N!}{A!\,B!\,C!}$.
- ${}^nC_r = \frac{n!}{r!\,(n-r)!}$
- The mathematical relation between ${}^nP_r$ and ${}^nC_r$ with reference to permutations and combinations :

  ${}^nC_r = \frac{{}^nP_r}{r!}$
- Expansion of expression $(x + a)^n$ :

  ${}^nC_0\,(x)^n\,(a)^0 + {}^nC_1\,(x)^{n-1}\,(a)^1 + {}^nC_2\,(x)^{n-2}\,(a)^2 + ... + {}^nC_n\,(x)^0\,(a)^n$

## EXERCISE 6

### Section A

**For the following multiple choice questions select the correct option :**

1. In one group there are $m$ distinct things and in the other group there are $n$ distinct things. In how many ways can one thing be selected from both the groups ?

   (a) $mn$ (b) $\frac{m}{n}$ (c) $m - n$ (d) $m + n$

2. If the first action can be done in $m$ ways and second action can be done in $n$ ways then in how many ways can both the actions be done together ?

   (a) $mn$ (b) $\frac{m}{n}$ (c) $m - n$ (d) $m + n$

3. What is $n!$ ?

   (a) Addition of 1 to $n$ natural numbers

   (b) Multiplication of 1 to $n$ natural numbers

   (c) Multiplication of 1 to $n-r$ natural numbers

   (d) Multiplication of 0 to $n$ natural numbers

4. In usual notations, which of the following relation between permutation and combination is true ?

   (a) ${}^nC_r = {}^nP_r \times r!$ (b) ${}^nP_r = {}^nC_r + r!$ (c) ${}^nP_r = \frac{{}^nC_r}{r!}$ (d) ${}^nC_r = \frac{{}^nP_r}{r!}$

5. Which of the following is equivalent to ${}^nC_r$ ?

   (a) $\frac{n!}{(n-r)!}$ (b) ${}^nC_{n-r}$ (c) ${}^nC_{r-1}$ (d) $\frac{{}^nC_{r+1}}{r}$

6. Find the value of ${}^nC_0 + {}^nC_n$.

   (a) 0 (b) 1 (c) 2 (d) $2n$

7. If $(n + 1)! = 120$ then find the value of $n$.

   (a) 3 (b) 4 (c) 5 (d) 6

8. How many terms are there in the expansion of $(x + a)^{n-1}$ ?

   (a) $n$ (b) $n - 2$ (c) $n + 1$ (d) $n + 2$

9. If $10 \times n ! = 240$ then find value of $n$.

   (a) 6 (b) 3 (c) 5 (d) 4

10. State the last term in the expansion of $(x + a)^n$.

    (a) $a^n$ (b) $a^{n-1}$ (c) $x^0$ (d) $x^{n-1}$

11. A ride in a fun fair has 8 seats. In how many ways can 3 persons be arranged in this ride ?

    (a) ${}^8C_3$ (b) ${}^3P_8$ (c) ${}^3C_8$ (d) ${}^8P_3$

### Section B

**Answer the following questions in one sentence :**

1. What is the main difference between permutation and combination ?
2. Write the fundamental principle of counting for addition.
3. Write the fundamental priciple of counting for multiplication.

4. Write the mathematical relationship between permutation and combination in usual notations.

5. Write the coefficients of the terms in the expansion of $(x + a)^n$ for $n = 6$.

6. Write the general term of the expansion of $(x + a)^n$.

7. There are 5 empty seats in the coach of a train. In how many ways will 3 persons be seated ?

8. If ${}^nC_2 = 15$ then find the value of $n$.

9. If ${}^nP_3 = 210$, find the value of $n$.

10. How many new arrangements can be made using all the letters of the word TUESDAY ?

11. How many arrangements can be made using all the letters of the word VIAAN ?

12. In how many ways can 5 different letters be placed in 5 covers ?

13. What is ${}^nP_r$ ?

14. Write the coefficients of $(n + 1)$ terms in the expansion of $(x + a)^n$.

15. If ${}^nC_x = {}^nC_y$ then write the two possible relationships between $x$ and $y$.

## Section C

**Answer the following questions :**

1. Write the characteristics of binomial expansion.
2. 10 schools participate in a science fair. In how many ways can the first, second and the third prizes be distributed among these schools ?
3. In how many ways can 4 boys and 3 girls be arranged in a row such that no two boys and no two girls are together ?
4. There are 6 different books of Statistics, 5 different books of Accounts and and 4 different books of English on a table. In how many ways can these books be arranged in a row such that the books of same subject remain together ?
5. How many 5 digit numbers can be formed using all the digits 3, 8, 0, 7, 6 ?
6. In how many ways can all the letters of the word TANI be arranged so that vowels remain together ?
7. In how many ways can all the letters of the word MANGO be arranged so that vowels are not together ?
8. How many numbers can be formed using all the digits of the number 1234321 such that odd digits occupy odd places only ?
9. What will be the ratio of number of arrangements obtained using all the letters of the word ROLLS and DOLLS ?
10. There are 2 defective screws in a box having 6 screws. In how many ways can 2 non-defective screws be selected from the box ?
11. In how many ways can 2 cards of queen or king can be selected from a well shuffled pack of 52 cards ?
12. In how many ways can 3 cards of same colour be selected from a well shuffled pack of 52 cards ?
13. Expand $(2x + 3y)^3$.
14. Expand $\left(x - \frac{1}{x}\right)^3$.
15. Expand $(y + k)^5$.

## Section D

**Answer the following :**

**1.** Using all the first five natural numbers

(1) how many numbers can be formed ?

(2) how many numbers greater than 30,000 can be formed ?

(3) how many numbers divisible by 5 can be formed ?

**2.** In how many ways can 4 boys and 4 girls be arranged in a row such that no two boys and no two girls appear together ?

**3.** In how many ways can 3 boys and 2 girls be arranged in a row such that

(1) both the girls remain together ?

(2) boys and girls are alternately arranged ?

(3) all the three boys remain together ?

**4.** If all the arrangements formed using all the letters of the word WAKEFUL are arranged in the order of dictionary then what will be the rank of the word WAKEFUL ?

**5.** 4 couples (husband-wife) attend a party. In how many ways can 2 persons be selected from these 8 persons such that

(1) two persons selected are husband and wife ?

(2) one is a male and the other is a female ?

(3) one is a male and the other is a female but they are not husband and wife ?

**6.** There are 4 different books of Statistics and 3 different books of Economics on a table. In how many ways can 2 books be selected such that

(1) both the books are of the same subject ?

(2) both the books are of different subject ?

(3) no book of Economics is selected ?

**7.** 3 dolls, 4 kitchen sets and 3 cars are displayed in a toy shop. In how many ways can 3 toys be seleted such that,

(1) all are dolls ?

(2) all are different toys ?

(3) two are dolls and one is a kitchen set ?

**8.** A committee of 3 members is to be formed from 4 chartered accountants and 5 doctors associated with a social organization. In how many ways can the committee be formed such that,

(1) the chartered accountants are in majority ?

(2) the doctors are in majority ?

**9.** Obtain the value of $(\sqrt{7}+1)^3-(\sqrt{7}-1)^3$ using binomial expansion method.

**10.** Obtain the value of $(\sqrt{3}+1)^6+(\sqrt{3}-1)^6$ using binomial expansion method.

**Blaise Pascal**
**(1623 - 1662)**

French mathematician – Pascal's inventions and discoveries have been instrumental to development in the fields of geometry, physics and computer science. His exploration of binomial coefficients infulenced Sir Isaac Newton, leading him to uncover his "general binomial theorem for fractional and negative powers."

In the 1970s, the Pascal (Pa) unit was named after Blaise Pascal the in honour of his contributions. Pascal is also credited with building the foundation of probability theory.