1.2 Indices and Surds

1.2.1 Definition of Indices

If a is any non zero real or imaginary number and m is the positive integer, then $a^m = a.a.a.a.........a$ (m times). Here a is called the base and m the index, power or exponent.

1.2.2 Laws of Indices

(1) $a^0 = 1$, $(a \neq 0)$

(2) $a^{-m} = \frac{1}{a^m}$, $(a \neq 0)$

(3) $a^{m+n} = a^m.a^n$, where m and n are rational numbers

(4) $a^{m-n} = \frac{a^m}{a^n}$, where m and n are rational numbers, $a \neq 0$

(5) $(a^m)^n = a^{mn}$ (6) $a^{p/q} = \sqrt[q]{a^p}$

(7) If $x = y$, then $a^x = a^y$, but the converse may not be true.

For example: $(1)^6 = (1)^8$, but $6 \neq 8$

(i) If $a \neq \pm 1$, or 0, then $x = y$ (ii) If $a = 1$, then x, y may be any real number

(iii) If $a = -1$, then x, y may be both even or both odd (iv) If $a = 0$, then x, y may be any non-zero real number

But if we have to solve the equations like $[f(x)]^{\phi(x)} = [f(x)]^{\Psi(x)}$ then we have to solve :

(a) $f(x) = 1$ (b) $f(x) = -1$ (c) $f(x) = 0$ (d) $\phi(x) = \Psi(x)$

Verification should be done in (b) and (c) cases

(8) $a^m.b^m = (ab)^m$ is not always true

In real domain, $\sqrt{a}\sqrt{b} = \sqrt{(ab)}$, only when $a \geq 0, b \geq 0$

In complex domain, $\sqrt{a}.\sqrt{b} = \sqrt{(ab)}$, if at least one of a and b is positive.

(9) If $a^x = b^x$ then consider the following cases :

(i) If $a \neq \pm b$, then $x = 0$ (ii) If $a = b \neq 0$, then x may have any real value (iii) If $a = -b$, then x is even.

If we have to solve the equation of the form $[f(x)]^{\phi(x)} = [g(x)]^{\phi(x)}$ *i.e.*, same index, different bases, then we have to solve

(a) $f(x) = g(x)$, (b) $f(x) = -g(x)$, (c) $\phi(x) = 0$

Verification should be done in (b) and (c) cases.

Example: 1 For $x \neq 0, \left(\frac{x^l}{x^m}\right)^{(l^2+lm+m^2)} \left(\frac{x^m}{x^n}\right)^{(m^2+nm+n^2)} \left(\frac{x^n}{x^l}\right)^{(n^2+nl+l^2)} =$

(a) 1 (b) x (c) Does not exist (d) None of these

Solution: (a) $\left(\frac{x^l}{x^m}\right)^{l^2+lm+m^2} \left(\frac{x^m}{x^n}\right)^{m^2+nm+n^2} \left(\frac{x^n}{x^l}\right)^{n^2+nl+l^2}$

$= (x^{l-m})^{(l^2+lm+m^2)} (x^{m-n})^{m^2+nm+n^2} (x^{n-l})^{n^2+nl+l^2} = x^{l^3-m^3}.x^{m^3-n^3}.x^{n^3-l^3} = x^{l^3-m^3+m^3-n^3+n^3-l^3} = x^0 = 1$

Example: 2 If $2^x = 4^y = 8^z$ and $xyz = 288$, then $\frac{1}{2x} + \frac{1}{4y} + \frac{1}{8z} =$

(a) 11/48 (b) 11/24 (c) 11/8 (d) 11/96

Solution: (d) $2^x = 2^{2y} = 2^{3z}$ *i.e.*, $x = 2y = 3z = k$ (say). Then $xyz = \frac{k^3}{6} = 288$, So $k = 12$

$\therefore x = 12, y = 6, z = 4$. Therefore, $\frac{1}{2x} + \frac{1}{4y} + \frac{1}{8z} = \frac{11}{96}$

Example: 3 $\frac{2.3^{n+1} + 7.3^{n-1}}{3^{n+2} - 2(1/3)^{1-n}} =$

(a) 1 (b) 3 (c) –1 (d) 0

Solution: (a) $\frac{2.3^{n+1} + 7.3^{n-1}}{3^{n+2} - 2\left(\frac{1}{3}\right)^{1-n}} = \frac{2.3^{n-1}.3^2 + 7.3^{n-1}}{3^{n-1}.3^3 - 2.3^{n-1}} = \frac{3^{n-1}[18+7]}{3^{n-1}[27-2]} = 1$

Example: 4 If $\left(\frac{2}{3}\right)^{x+2} = \left(\frac{3}{2}\right)^{2-2x}$, then $x =$ **[UPSEAT 1999]**

(a) 1 (b) 3 (c) 4 (d) 0

Solution: (c) $\left(\frac{2}{3}\right)^{x+2} = \left(\frac{3}{2}\right)^{2-2x} \Rightarrow \left(\frac{2}{3}\right)^{x+2} = \left(\frac{2}{3}\right)^{2x-2}$. Clearly $x + 2 = 2x - 2 \Rightarrow x = 4$

Example: 5 The equation $4^{(x^2+2)} - 9.2^{(x^2+2)} + 8 = 0$ has the solution

(a) $x = 1$ (b) $x = -1$ (c) $x = \sqrt{2}$ (d) $x = -\sqrt{2}$

Solution: (a, b) $4^{(x^2+2)} - 9.2^{(x^2+2)} + 8 = 0 \Rightarrow \left(2^{(x^2+2)}\right)^2 - 9.2^{(x^2+2)} + 8 = 0$

Put $2^{(x^2+2)^2} = y$. Then $y^2 - 9y + 8 = 0$, which gives $y = 8, y = 1$

When $y = 8 \Rightarrow 2^{x^2+2} = 8 \Rightarrow 2^{x^2+2} = 2^3 \Rightarrow x^2 + 2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = 1, -1$

When $y = 1 \Rightarrow 2^{x^2+2} = 1 \Rightarrow 2^{x^2+2} = 2^o \Rightarrow x^2 + 2 = 0 \Rightarrow x^2 = -2$, which is not possible.

1.2.3 Definition of Surds

Any root of a number which can not be exactly found is called a surd.

Let *a* be a rational number and *n* is a positive integer. If the n^{th} root of *x* *i.e.*, $x^{1/n}$ is irrational, then it is called surd of order *n*.

Order of a surd is indicated by the number denoting the root.

For example $\sqrt{7}, \sqrt[3]{9}, (11)^{3/5}, \sqrt[n]{3}$ are surds of second, third, fifth and n^{th} order respectively.

A second order surd is often called a quadratic surd, a surd of third order is called a cubic surd.

Note : ❑ If a is not rational, $\sqrt[n]{a}$ is not a surd.

For example, $\sqrt{(5+\sqrt{7})}$ is not a surd as $5+\sqrt{7}$ is not a rational number.

1.2.4 Types of Surds

(1) **Simple surd :** A surd consisting of a single term. For example $2\sqrt{3}, 6\sqrt{5}, \sqrt{5}$ etc.

(2) **Pure and mixed surds :** A surd consisting of wholly of an irrational number is called pure surd.

Example : $\sqrt{5}, \sqrt[3]{7}$

A surd consisting of the product of a rational number and an irrational number is called a mixed surd.

Example : $5\sqrt{3}$.

(3) **Compound surds :** An expression consisting of the sum or difference of two or more surds.

Example : $\sqrt{5}+\sqrt{2},\ 2-\sqrt{3}+3\sqrt{5}$ etc.

(4) **Similar surds :** If the surds are different multiples of the same surd, they are called similar surds.

Example : $\sqrt{45}, \sqrt{80}$ are similar surds because they are equal to $3\sqrt{5}$ and $4\sqrt{5}$ respectively.

(5) **Binomial surds :** A compound surd consisting of two surds is called a binomial surd.

Example : $\sqrt{5} - \sqrt{2}, 3+\sqrt[3]{2}$ etc.

(6) **Binomial quadratic surds:** Binomial surds consisting of pure (or simple) surds of order two *i.e.,* the surds of the form $a\sqrt{b} \pm c\sqrt{d}$ or $a \pm b\sqrt{c}$ are called binomial quadratic surds.

Two binomial quadratic surds which differ only in the sign which connects their terms are said to be conjugate or complementary to each other. The product of a binomial quadratic surd and its conjugate is always rational.

For example: The conjugate of the surd $2\sqrt{7}+5\sqrt{3}$ is the surd $2\sqrt{7}-5\sqrt{3}$.

1.2.5 Properties of Quadratic Surds

(1) The square root of a rational number cannot be expressed as the sum or difference of a rational number and a quadratic surd.

(2) If two quadratic surds cannot be reduced to others, which have not the same irrational part, their product is irrational.

(3) One quadratic surd cannot be equal to the sum or difference of two others, not having the same irrational part.

(4) If $a+\sqrt{b} = c+\sqrt{d}$, where a and c are rational, and $\sqrt{b}, \sqrt{d}$ are irrational, then $a=c$ and $b = d$.

Example: 6 The greatest number among $\sqrt[3]{9}, \sqrt[4]{11}, \sqrt[6]{17}$ is

(a) $\sqrt[3]{9}$ (b) $\sqrt[4]{11}$ (c) $\sqrt[6]{17}$ (d) Can not be determined

Solution: (a) $\sqrt[3]{9}, \sqrt[4]{11}, \sqrt[6]{17}$

$\because L.C.M$ of 3, 4, 6 is 12

$\therefore \sqrt[3]{9} = 9^{1/3} = (9^4)^{1/12} = (6561)^{1/12}$, $\sqrt[4]{11} = (11)^{1/4}(11^3)^{1/12} = (1331)^{1/12}$, $\sqrt[6]{17} = (17)^{1/6} = (17^2)^{1/2} = (289)^{1/12}$

Hence $\sqrt[3]{9}$ is the greatest number.

Example: 7 The value of $\dfrac{15}{\sqrt{10}+\sqrt{20}+\sqrt{40}-\sqrt{5}-\sqrt{80}}$ is

(a) $\sqrt{5}(5+\sqrt{2})$ (b) $\sqrt{5}(2+\sqrt{2})$ (c) $\sqrt{5}(1+\sqrt{2})$ (d) $\sqrt{5}(3+\sqrt{2})$

Solution: (c) Given fraction $= \dfrac{15}{\sqrt{10}+\sqrt{20}+\sqrt{40}-\sqrt{5}-\sqrt{80}} = \dfrac{15}{\sqrt{10}+2\sqrt{5}+2\sqrt{10}-\sqrt{5}-4\sqrt{5}}$

$$= \frac{15}{3\sqrt{10}-3\sqrt{5}} = \frac{5}{\sqrt{10}-\sqrt{5}}.\frac{\sqrt{10}+\sqrt{5}}{\sqrt{10}+\sqrt{5}} = \sqrt{10}+\sqrt{5} = \sqrt{5}(\sqrt{2}+1)$$

Example: 8 If $x = \sqrt[3]{(\sqrt{2}+1)} - \sqrt[3]{(\sqrt{2}-1)}$; then $x^3+3x =$

(a) 2 (b) 6 (c) $6x$ (d) None of these

Solution: (a) $x = (\sqrt{2}+1)^{1/3} - (\sqrt{2}-1)^{1/3}$

$x^3 = (\sqrt{2}+1)-(\sqrt{2}-1)-3(\sqrt{2}+1)^{1/3}(\sqrt{2}-1)^{1/3}\left[\sqrt[3]{(\sqrt{2}+1)}-\sqrt[3]{\sqrt{2}-1}\right]$

$x^3 = 2-3(2-1)^{1/3}x \Rightarrow x^3+3x = 2$.

1.2.6 Rationalisation Factors

If two surds be such that their product is rational, then each one of them is called rationalising factor of the other.

Thus each of $2\sqrt{3}$ and $\sqrt{3}$ is a rationalising factor of each other. Similarly $\sqrt{3}+\sqrt{2}$ and $\sqrt{3}-\sqrt{2}$ are rationalising factors of each other, as $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}) = 1$, which is rational.

To find the factor which will rationalize any given binomial surd :

Case I: Suppose the given surd is $\sqrt[p]{a} - \sqrt[q]{b}$

suppose $a^{1/P} = x, b^{1/q} = y$ and let n be the L.C.M. of p and q. Then x^n and y^n are both rational.

Now $x^n - y^n$ is divisible by $x-y$ for all values of n, and $x^n - y^n = (x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+.....+y^{n-1})$.

Thus the rationalizing factor is $x^{n-1}+x^{n-2}y+x^{n-3}y^2+.....+y^{n-1}$ and the rational product is $x^n - y^n$.

Case II: Let the given surd be $\sqrt[p]{a}+\sqrt[q]{b}$.

Let x, y, n have the same meaning as in Case I.

(1) If n is even, then $x^n - y^n$ is divisible by $x+y$ and $x^n - y^n = (x+y)(x^{n-1}-x^{n-2}y+x^{n-3}y^3-.....-y^{n-1})$

Thus the rationalizing factor is $x^{n-1} - x^{n-2}y + x^{n-3}y^2 - ..... - y^{n-1}$ and the rational product is $x^n - y^n$.

(2) If *n* is odd, $x^n + y^n$ is divisible by $x + y$, and $x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + ... + y^{n-1})$

Thus the rationalizing factor is $x^{n-1} - x^{n-2}y + ... - xy^{n-2} + y^{n-1}$ and the rational product is $x^n + y^n$.

Example: 9 The rationalising factor of $a^{1/3} + a^{-1/3}$ is

(a) $a^{1/3} - a^{-1/3}$ (b) $a^{2/3} + a^{-2/3}$ (c) $a^{2/3} - a^{-2/3}$ (d) $a^{2/3} + a^{-2/3} - 1$

Solution: (d) Let $x = a^{1/3}, y = a^{-1/3}$ then $a = x^3, a^{-1} = y^3$

$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

So rationlising factor is $(x^2 - xy + y^2)$. Put the value of *x* and *y*

Thus the required rationlising factor is $a^{2/3} + a^{-2/3} - 1$.

1.2.7 Square Roots of *a* +√*b* and *a* + √*b* + √*c* + √*d* Where √*b* , √*c* , √*d* are Surds

Let $\sqrt{(a+\sqrt{b})} = \sqrt{x} + \sqrt{y}$, where $x, y > 0$ are rational numbers.

Then squaring both sides we have, $a + \sqrt{b} = x + y + 2\sqrt{x}\sqrt{y}$

$\Rightarrow a = x + y, \sqrt{b} = 2\sqrt{xy} \quad \Rightarrow b = 4xy$

So, $(x-y)^2 = (x+y)^2 - 4xy = a^2 - b$

After solving we can find *x* and *y*.

Similarly square root of $a - \sqrt{b}$ can be found by taking $\sqrt{(a-\sqrt{b})} = \sqrt{x} - \sqrt{y}, \ x > y$

To find square root of a + √b + √c + √d : Let $\sqrt{(a+\sqrt{b}+\sqrt{c}+\sqrt{d})} = \sqrt{x} + \sqrt{y} + \sqrt{z}, (x, y, z > 0)$ and take $\sqrt{(a+\sqrt{b}-\sqrt{c}-\sqrt{d})} = \sqrt{x} + \sqrt{y} - \sqrt{z}$. Then by squaring and equating, we get equations in *x, y, z*. On solving these equations, we can find the required square roots.

Note :

- ❑ If $a^2 - b$ is not a perfect square, the square root of $a + \sqrt{b}$ is complicated *i.e.*, we can't find the value of $\sqrt{(a+\sqrt{b})}$ in the form of a compound surd.
- ❑ If $\sqrt{(a+\sqrt{b})} = \sqrt{x} + \sqrt{y}, x > y$ then $\sqrt{(a-\sqrt{b})} = \sqrt{x} - \sqrt{y}$
- ❑ $\sqrt{a+\sqrt{(b)}} = \sqrt{\left(\frac{a+\sqrt{a^2-b}}{2}\right)} + \sqrt{\left(\frac{a-\sqrt{a^2-b}}{2}\right)}$
- ❑ $\sqrt{a-\sqrt{(b)}} = \sqrt{\left(\frac{a+\sqrt{a^2-b}}{2}\right)} - \sqrt{\left(\frac{a-\sqrt{a^2-b}}{2}\right)}$
- ❑ If *a* is a rational number, $\sqrt{b}, \sqrt{c}, \sqrt{d}$, are surds then

(i) $\sqrt{a+\sqrt{b}+\sqrt{c}+\sqrt{d}} = \sqrt{\frac{bd}{4c}} + \sqrt{\frac{bc}{4d}} + \sqrt{\frac{cd}{4b}}$ (ii) $\sqrt{a-\sqrt{b}-\sqrt{c}+\sqrt{d}} = \sqrt{\frac{bd}{4c}} + \sqrt{\frac{cd}{4b}} + \sqrt{\frac{bc}{4d}}$,

(iii) $\sqrt{a-\sqrt{b}-\sqrt{c}+\sqrt{d}} = \sqrt{\dfrac{bc}{4d}} - \sqrt{\dfrac{bd}{4c}} - \sqrt{\dfrac{cd}{4b}}$

Example: 10 $\sqrt{(3+\sqrt{5})}$ is equal to

(a) $\sqrt{5}+1$ (b) $\sqrt{3}+\sqrt{2}$ (c) $(\sqrt{5}+1)/\sqrt{2}$ (d) $\dfrac{1}{2}(\sqrt{5}+1)$

Solution: (c) Let $\sqrt{3+\sqrt{5}} = \sqrt{x}+\sqrt{y}$

$3+\sqrt{5} = x+y+2\sqrt{xy}$. Obviously $x+y=3$ and $4xy=5$. So $(x-y)^2 = 9-5=4$ or $(x-y)=2$

After solving $x=\dfrac{5}{2}, y=\dfrac{1}{2}$. Hence $\sqrt{3+\sqrt{5}} = \sqrt{\dfrac{5}{2}}+\sqrt{\dfrac{1}{2}} = \dfrac{\sqrt{5}+1}{\sqrt{2}}$

Example: 11 $\sqrt{[10-\sqrt{(24)}-\sqrt{(40)}+\sqrt{(60)}]} =$

(a) $\sqrt{5}+\sqrt{3}+\sqrt{2}$ (b) $\sqrt{5}+\sqrt{3}-\sqrt{2}$ (c) $\sqrt{5}-\sqrt{3}+\sqrt{2}$ (d) $\sqrt{2}+\sqrt{3}-\sqrt{5}$

Solution: (b) Let $10-\sqrt{24}-\sqrt{40}+\sqrt{60} = (\sqrt{a}-\sqrt{b}+\sqrt{c})^2$

$10-\sqrt{24}-\sqrt{40}+\sqrt{60} = a+b+c-2\sqrt{ab}-2\sqrt{bc}+2\sqrt{ca},\ a,b,c>0$. Then $a+b+c=10,\ ab=6$, $bc=10,\ ca=15$

$a^2b^2c^2 = 900 \Rightarrow abc = 30\ (\neq \pm 30)$. So $a=3,\ b=2,\ c=5$

Therefore, $\sqrt{(10-\sqrt{24}-\sqrt{40}+\sqrt{60})} = \pm(\sqrt{3}+\sqrt{5}-\sqrt{2})$

Example: 12 $\sqrt[4]{(17+12\sqrt{2})} =$

(a) $\sqrt{2}+1$ (b) $2^{1/4}(\sqrt{2}+1)$ (c) $2\sqrt{2}+1$ (d) None of these

Solution: (a) $\sqrt{(17+12\sqrt{2})} = \sqrt{[3^2+(2\sqrt{2})^2+2.3.2\sqrt{2}]} = 3+2\sqrt{2}$

$\therefore \sqrt[4]{(17+12\sqrt{2})} = \sqrt{(3+2\sqrt{2})} = \sqrt{2}+1$.

1.2.8 Cube Root of a Binomial Quadratic Surd

If $(a+\sqrt{b})^{1/3} = x+\sqrt{y}$ then $(a-\sqrt{b})^{2/3} = x-\sqrt{y}$, where a is a rational number and b is a surd.

Procedure of finding $(a+\sqrt{b})^{1/3}$ is illustrated with the help of an example :

Taking $(37-30\sqrt{3})^{1/3} = x+\sqrt{y}$ we get on cubing both sides, $37-30\sqrt{3} = x^3+3xy-(3x^2+y)\sqrt{y}$

$\therefore x^3+3xy = 37$

$(3x^2+y)\sqrt{y} = 30\sqrt{3} = 15\sqrt{12}$

As $\sqrt{3}$ can not be reduced, let us assume $y=3$ we get $3x^2+y = 3x^2+3 = 30 \therefore x=3$

Which doesn't satisfy $x^3+3xy = 37$

Again taking $y=12,$ we get

$3x^2+12 = 15$, $\therefore x=1$

$x=1, y=12$ satisfy $x^3+3xy = 37$

$\therefore \sqrt[3]{37-30\sqrt{3}} = 1-\sqrt{12} = 1-2\sqrt{3}$

Example: 13 $\sqrt[3]{(61-46\sqrt{5})} =$

(a) $1-2\sqrt{5}$ (b) $1-\sqrt{5}$ (c) $2-\sqrt{5}$ (d) None of these

Solution: (a) $\sqrt[3]{61-46\sqrt{5}}=a-\sqrt{b} \Rightarrow 61-46\sqrt{5}=(a-\sqrt{b})^3=a^3+3ab-(3a^2+b)\sqrt{b}$

$\Rightarrow 61=a^3+3ab, 46\sqrt{5}=(3a^2+b)\sqrt{b} \Rightarrow 61=(a^2+3b)a$, $23\sqrt{20}=(3a^2+b)\sqrt{b}$

So $a=1$, $b=20$. Therefore $\sqrt[3]{61-46\sqrt{5}}=1-\sqrt{20}=1-2\sqrt{5}$.

1.2.9 Equations Involving Surds

While solving equations involving surds, usually we have to square, on squaring the domain of the equation extends and we may get some extraneous solutions, and so we must verify the solutions and neglect those which do not satisfy the equation.

Note that from $ax=bx$, to conclude $a=b$ is not correct. The correct procedure is $x(a-b)=0$ *i.e.* $x=0$ or $a=b$. Here, necessity of verification is required.

Example: 14 The equation $\sqrt{(x+1)}-\sqrt{(x-1)}=\sqrt{(4x-1)}$, $x\in R$ has

(a) One solution (b) Two solution (c) Four solution (d) No solution

Solution: (d) Given $\sqrt{(x+1)}-\sqrt{(x-1)}=\sqrt{(4x-1)}$ .....(i)

Squaring both sides, we get, $-2\sqrt{(x^2-1)}=2x-1$

Squaring again, we get, $x=\frac{5}{4}$, which does not satisfy equation (i)

Hence, there is no solution of the given equation.

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