2 Deductive Proof

DO YOU KNOW THAT ....

- If someone offers you a ticket to Europe tour or Asia tour then Logic is on your side, if you accept the ticket for Europe but not Asia, You can prove the Conclusion by showing that its denial is impossible.
- When an idividual says '6 + 4' is same as '4 + 6' then that individual is using the rule of Logic.

2.1 Formal Proof of Validity :

There are two types of methods used by the logicians, for deciding or proving the validity of arguments.

1) Decision Procedure such as Truth Table Method, Shorter truth table method, Truth tree etc. are used to decide validity of arguments.

2) Methods that are not Decision procedure such as Deductive proof, Conditional proof, Indirect proof are used to prove validity of arguments.

Truth-table is a purely mechanical method for deciding whether an argument is valid or invalid, however it is not a convenient method when an argument contains many different truth-functional statements. In such cases there are other methods in Logic for establishing the validity of arguments and one of the method is the 'Method of Deductive Proof'.

The Deductive Proof is of three types. They are :

(1) The Direct Deductive Proof

(2) Conditional Proof

(3) Indirect Proof

In the Method of Direct Deductive Proof, the conclusion is deduced directly from the premises by a sequence of Elementary valid argument forms. The Elementary valid argument forms, used for this purpose are called the 'Rules of Inference'; we have already dealt with direct deductive proof and we know that the Direct Deductive proof is based on nine rules of inference and ten rules based on rule of replacement as follows.

Rules of Inference :

(i) Rule of Modus Ponens (M.P.) $p \supset q$ p $\therefore q$	(ii) Rule of Modus Tollens (M.T.) $p \supset q$ $\sim q$ $\therefore \sim p$
(iii) Rule of Hypothetical syllogism (H.S.) $p \supset q$ $q \supset r$ $\therefore p \supset r$	(iv) Rule of Disjunctive syllogism (D.S.) $p \vee q$ $\sim p$ $\therefore q$
(v) Rule of Constructive Dilemma (D.D.) $(p \supset q) \cdot (r \supset s)$ $p \vee r$ $\therefore q \vee s$	(vi) Rule of Destructive Dilemma (D.D.) $(p \supset q) \cdot (r \supset s)$ $\sim q \vee \sim s$ $\therefore \sim p \vee \sim r$

(vii) Rule of Conjunction (Conj.) p q $\therefore$ p · q	(viii) Rule of Simplification (Simp.) p · q $\therefore$ p
(ix) Rule of Addition (Add.) p $\therefore$ p ∨ q	

Rules based on the rule of Replacement:

(i) Rule of Double Negation (D.N.) $\sim \sim p \equiv p$	(ii) De-Morgan's Law (De. M.) $\sim (p \cdot q) \equiv (\sim p \lor \sim q)$ $\sim (p \lor q) \equiv (\sim p \cdot \sim q)$
(iii) Associative Laws (Assoc.) $[(p \cdot q) \cdot r)] \equiv [p \cdot (q \cdot r)]$ $[(p \lor q) \lor r)] \equiv [p \lor (q \lor r)]$	(iv) Distributive Laws (Dist.) $[p \cdot (q \lor r) \equiv [(p \cdot q) \lor (p \cdot r)]$ $[p \lor (q \cdot r) \equiv [(p \lor q) \cdot (p \lor r)]$
(v) Commutative Law (Comm.) $(p \cdot q) \equiv (q \cdot p)$ $(p \lor q) \equiv (q \lor p)$	(vi) Rule of Transposition (Trans.) $(p \supset q) \equiv (\sim q \supset \sim p)$
(vii) Rule of Material Implication (M. Imp.) $(p \supset q) \equiv (\sim p \lor q)$	(viii) Rule of Material Equivalence (M. Equi) $(p \equiv q) \equiv [(p \supset q) \cdot (q \supset p)]$ $(p \equiv q) \equiv [(p \cdot q) \lor (\sim p \cdot \sim q)]$
(ix) Rule of Exportation (Export.) $[(p \cdot q) \supset r] \equiv [p \supset (q \supset r)]$	(x) Rule of Tautology (Taut.) $p \equiv (p \cdot p)$ $p \equiv (p \lor p)$

2.2 Conditional Proof

The method of Conditional Proof is used to establish the validity of arguments, when the conclusion of an argument is an implicative (conditional) proposition. The method of Conditional Proof is based upon the Rule of Conditional Proof.

The Rule of Conditional Proof enables us to construct shorter proofs of validity for some arguments. Further by using it, we can prove the validity of some arguments which cannot be proved by using the above nineteen rules.

The Rule of Conditional Proof may be expressed in a simple way :

"By assuming the antecedent of the conclusion as an additional premise, when its consequent is deduced as the conclusion, the original conclusion will be taken to have been proved".

While using Conditional Proof, it should be noted that the conclusion can be any statement equivalent to a conditional statement. In such a case, first the equivalent conditional statement is derived and then the Rule of Conditional Proof is used. **However, in this chapter, we will use Conditional Proof only when the conclusion is a conditional statement.**

To illustrate let us construct a Conditional Proof of Validity for the following argument :

Example : 1

~ M ⊃ N

∴ ~ N ⊃ M

The proof may be written as follows :

1.	~ M ⊃ N	/ ∴ ~ N ⊃ M	
2.	~ N	Assumption	
3.	~ ~ M	1, 2 . M.T.	
4.	M	3 . D.N.	

Here the step 2 is the antecedent of the conclusion. It is used as an assumption. (The assumption should be indicated by bent arrow.)

From the premise 1 and the assumption, one has deduced the consequent of the conclusion by the Rule of M.T.

However the proof is not complete. One has yet to arrive at the conclusion. To do so one more step remains to be taken, i.e. to write down the conlcusion, '~ N ⊃ M'.

The proof is now written by adding step 5 thus :

1.	~ M ⊃ N	/ ∴ ~ N ⊃ M	
2.	~ N	Assumption	
3.	~ ~ M	1, 2 . M.T.	
4.	M	3. D.N.	
5.	~ N ⊃ M		2 - 4, C.P.

The conclusion step 5 has not been deduced from the assumption. So the conclusion lies outside the scope of the assumption. i.e. the scope of the assumption ends up with the last step which follows from step 4. To mark this out clearly the device of a bent arrow (↱) is used. The head of the arrow points at the assumption and its shaft runs down till it reaches the last statement which is deduced on its basis, then the arrow bends inwards and discharges (closes) the assumption. The last step i.e. step 5, where the conclusion is written, will lie outside the scope of assumption.

The proof may now be written down as :

1.	~ M ⊃ N	/ ∴ ~ N ⊃ M	
2.	~ N		
3.	~ ~ M	1, 2 . M.T.	
4.	M	5 . D.N.	
5.	~ N ⊃ M		2 - 4, C.P.

The head of the arrow indicates that step 2 is an assumption. So the word "assumption" need not be written as the justification.

If the conclusion has a compound proposition with more than one conditional statement as its components, then the antecedents of all the conditional statements can be assumed as additional premises.

Let us take an example of this type :

Example : 2

1.	(X ∨ Y) ⊃ Z	
2.	A ⊃ (B • C)	/ ∴ (X ⊃ Z) • (A ⊃ B)
3.	X	
4.	X ∨ Y	3, Add.
5.	Z	1, 4 M.P.
6.	X ⊃ Z	3 - 5, C.P.
7.	A	
8.	(B • C)	2, 7, M.P.
9.	B	8, Simp.
10.	A ⊃ B	7 - 9, C.P.
11 .	(X ⊃ Z) • (A ⊃ B)	6, 10 Conj.

Here the scope of the assumption in step 3 is independent of the scope of assumption in step 7.

Hence assumption in step 7 lies outside the scope of the assumption in step 3.

But in the next example-3 given below, the scope of one assumption lies within the scope of the other assumption.

Example : 3

1. (M • N) ⊃ O / ∴ ~ O ⊃ (M ⊃ ~ N)
2. ~ O
3. ~ (M • N) 1, 2 . M.T.
4. ~ M ∨ ~ N 3, De.M.
5. M
6. ~ ~ M 5, D.N.
7. ~ N 4, 6 . D.S.
8. M ⊃ ~ N 5-7, C.P.
9. ~ O ⊃ (M ⊃ ~ N) 2-8, C.P.

Here the assumption at step 5, lies within the scope of the assumption of step 2.

Give justifications for each step of the following formal proofs of validity by the method of conditional proof.

1. (P • Q) ⊃ S / ∴ ~ S ⊃ [P ⊃ (~ Q ∨ T)]
2. ~ S
3. ~ (P • Q)
4. ~ P ∨ ~ Q
5. P
6. ~ ~ P
7. ~ Q
8. ~ Q ∨ T
9. P ⊃ (~ Q ∨ T)
10. ~ S ⊃ [P ⊃ (~ Q ∨ T)]

2.3 Indirect Proof :

The methods of Direct Deductive Proof and Conditional Proof have one thing in common while using them we deduce the conclusion from the premises. The method of Indirect Proof is completely different from these methods.

The method of Indirect Proof is based on **the principle of reductio-ad-absurdum**. Here one assumes the opposite of what is to be proved and this leads to an absurdity. i.e. this method consists in proving the conclusion by showing that its negation leads to contradiction.

An Indirect Proof of validity for an argument is constructed by assuming the negation of the conclusion as an additional premise. From this additional premise, along with original premise/s a contradiction is derived. A contradiction is a conjunction in which one conjunct is the denial of the other conjunct. Eg. 'A • ~ A' , '(A ∨ B) • ~ (A ∨ B)', are contradictions.

By assuming the negation of the conclusion, we obtain a contradiction. This shows that the assumption is false. The assumption is the negation of the conclusion. Since the assumption is false, the original conclusion is taken to be proved.

When this method of proof is used, the validity of the original argument is said to follow by the rule of Indirect proof. **Unlike conditional proof the method of Indirect proof can be used irrespective of the nature of the conclusion.**

Let us construct an Indirect proof of validity for the following argument :

Example : 1

1. ~ M ∨ N
2. ~ N / ~ M
3. ~ ~ M I.P.
4. N 1, 3 D.S.
5. N • ~ N 4, 2 Conj.

In the above proof, the expression 'I.P' shows that the Rule of Indirect Proof is being used. In the above example, we first assume the negation of the conclusion then by using rules of inference and rules based on the rule of replacement, we arrive at a contradiction.

The last step of the proof is a contradiction, which is a demonstration of the absurdity derived by assuming ~ ~ M in the step 3. This contradiction is formally expressed in the last step exhibits the absurdity and completes the proof.

Let us construct few more Indirect Proof of validity for the following arguments :

Example : 2

1.	M ⊃ T	
2.	G ⊃ T	
3.	M	/ ∴ T
4.	~ T	1.P.
5.	~ M	1, 4. M. T.
6	M • ~ M	3, 5 Conj

Example : 3

1.	(B • D) ∨ E	
2.	C ⊃ ~ E	
3.	F ⊃ ~ E	
4.	C ∨ F / ∴ B • D	
5.	~ (B • D)	..... I.P.
6.	E	1,5 D.S.
7.	(C ⊃ ~ E) • (F ⊃ ~ E)	2, 3 Conj.
8.	~ E ∨ ~ E	7,4 C.D.
9.	~ E	8, Taut.
10.	E • ~ E	6, 9 Conj.

Example : 4

1.	(Q ∨ ~ P) ⊃ S	/ ∴ Q ⊃ S
2.	~ (Q ⊃ S)	..... I.P.
3.	~ (~ Q ∨ S)	2, m. Imp.
4.	~ ~ Q • ~ S	3, De. M
5.	~ ~ Q	4, Simp.
6.	Q	5, D.N.
7.	Q ∨ ~ P	6, Add.
8.	S	1, 7 M.P.
9.	~ S • ~ ~ Q	4, Com.
10.	~ S	9, Simp.
11.	S • ~ S	8,10 Conj.

In the fourth argument given above, the conclusion is a conditional statement. So the method of Conditional Proof could have been used. Infact the proof would have been shorter.

Give justifications for each step of the following formal proofs of validity by the method of Indirect proof :

1.	(H ∨ K) ⊃ (N • B)	
2.	B ⊃ ~ C	
3.	C	/ ∴ ~ H
4.	~ ~ H	
5.	H	
6.	H ∨ K	
7.	N • B	
8.	B • N	
9.	B	
10.	~ C	
11.	C • ~ C	

Summary

There are three types of Deductive Proofs :

(1) Direct Deductive Proof : In this method conclusion is derived directly from the premises.

(2) Conditional Proof : This method is used only when the conclusion of an argument is a conditional statement. In this method the antecedent of the conclusion is taken as an additional premise and the consequent of the conclusion is deduced with the help of the required rules of Inference and rules based on the rule of replacement.

(3) Indirect Proof : This method is preferably used when the conclusion of an argument is other than a conditional statement. In this method we assume the negation of the conclusion as an additional premise.

From this, along with the original premises, we obtain a contradiction. And this is taken to be the proof of validity of arguments.

Exercises

Q. 1. Fill in the blanks with suitable words from those given in the brackets:

(1) [(p ⊃ q) • p] ⊃ q is the rule of ……… . *(Modus Ponens / Modus Tollens)*

(2) The rule of ……… consists in interchanging the antecedent and the consequent by negating both of them. *(Commutation / Transposition)*

(3) The rule of Addition is based on the basis truth table of ……… . *(Conjunction / Disjunction)*

(4) The ……… can be applied to the part of the statement. *(rules of inference / rules based on rule of replacement)*

(5) ~ (~ p ∨ q) ≡ ………, according to De. Morgan's Law. *((p • ~ q) / (~ p • q)*

(6) (p ⊃ q) ≡ (~ p ∨ q) is the rule of ………. *(Material Implication / Material Equivalance)*

(7) The method of ……… is used only when the conclusion of an argument is an implicative statement. *(Conditional Proof / Indirect Proof)*

(8) In the method of ………, we assume the negation of the conclusion as an additional premise. *(Conditional Proof / Indirect Proof)*

(9) The rule of ……… states that if an implication is true and its consequent is false, then its antecedent must also be false. *(M.P./ M.T.)*

(10) (p • p) ≡ p is the rule of ……… . *(Simplification / Tautology)*

(11) The method of ……… is based on the principle of reductio-ad-absurdum. *(Conditional Proof / Indirect Proof)*

Q. 2. State whether the following statements are true or false.

(1) The rule of Disjunctive Syllogism can be applied to the part of the statement.

(2) ~ ~ p ≡ p is the rule of Tautology.

(3) When the denial of the conclusion leads to contradiction, the argument is proved to be valid in the method of indirect proof.

(4) Conditional Proof decides whether the argument is valid or invalid.

(5) Indirect proof is constructed for establishing the validity of arguments.

(6) Conditional proof is a mechanical procedure.

(7) (p ∨ q) ≡ (q ∨ p) is Commutative Law.

(8) The rule of inference can be applied to the whole statement only.

(9) The Elementary valid arguments forms are called the rule of Replacement.

Q. 3. Match the columns :

	A		B
(1)	Elementary valid argument forms	(a)	Antecendent of the conclusion is assumed.
(2)	Conditional Proof	(b)	Principle of reductio-ad absurdum.
(3)	Indirect Proof	(c)	Rule based on rule of replacement.
(4)	De. Morgan's Law	(d)	Rules of Inference

Q. 4. Give Logical Terms for the following :

(1) The rules that can be applied only for the whole statement.

(2) The elementary valid argument forms.

(3) The method of establishing the validity of an argument by assuming the negation of the conclusion.

(4) The deductive proof which is based on the principle of reductio-ad-absurdum.

(5) The method which is used to establish the validity of argument, only when its conclusion is an implicative statement.

Q. 5. Construct Conditional proof or Indirect proof of validity for the following arguments:

(1) ~ A / ∴ A ⊃ B

(2) 1. (L ∨ M) ⊃ (P • Q)

2. ~ P / ∴ ~ L

(3) 1. (S • A) ⊃ R

2. ~ R

3. A / ∴ ~ S

(4) 1. Q ∨ (P ∨ R) / ∴ ~ Q ⊃ [~ R ⊃ (P ∨ S)]

(5) 1. A ∨ (B ⊃ D)

2. A ⊃ C

3. B / ∴ ~ C ⊃ D

(6) 1. D ⊃ E

2. D ∨ G / ∴ E ∨ G

(7) 1. W ⊃ L

2. T ⊃ (~ P • L)

3. W ∨ T / ∴ L

(8) 1. T ∨ B

2. (T ∨ N) ⊃ (L • S)

3. ~ S / ∴ B

(9) 1. R ⊃ (Q ⊃ P)

2. S ⊃ R

3. T ⊃ Q

4. ~ P / ∴ S ⊃ ~ T

(10) 1. (A ∨ B)

2. (C ∨ D)⊃ E

/ ∴ [~A ⊃ (B ∨ F)] • (D ⊃ E)

(11) 1. (G ⊃ H) ⊃ J

2. ~ J / ∴ G

(12) 1. L ⊃ (M ∨ N)

2. T ∨ L / ∴ ~ M ⊃ (~ T ⊃ N)

(13) 1. A ⊃ B

2. C ⊃ D / ∴ (A • C) ⊃ (B • D)

(14) 1. K ∨ (T • ~ W)

2. W ∨ S / ∴ K ∨ S

(15) 1. A ∨ (B ⊃ C)

2. C ⊃ D

3. ~ D

4. B ∨ E / ∴ ~ A ⊃ E

(16) 1. P ⊃ (Q ⊃ R)

2. (Q • S) ∨ W / ∴ ~ R ⊃ (P ⊃W)

(17) 1. (A • B) ∨ C

2. (C ∨ D) ⊃ E / ∴ ~ A ⊃ E

(18) 1. ~ K ∨ G

2. G ⊃ I

3. ~ I / ∴ ~ K

(19) 1. D ⊃ E / ∴ D ⊃ (D • E)

(20) 1. F ⊃ (G ⊃ H)

2. G ⊃ (H ⊃ J) / ∴ F ⊃ (G ⊃ J)

(21) 1. R ⊃ (S • T)

2. (S ∨ U) ⊃ W

3. U ∨ R / ∴ W

(22) 1. (P ∨ Q) ⊃ [(R ∨ S) ⊃ T]

/ ∴ P ⊃ [(R • U) ⊃ T]

(23) 1. (A ⊃ B) • (C ⊃ D)

2. ~ B / ∴ (A ∨ C) ⊃ D

(24) 1. (K ∨ G) ⊃ (H • I)

2. (I ∨ M) ⊃ O / ∴ K ⊃ O

(25) 1. (R • R) ⊃ Q

2. Q ⊃ ~ R / ∴ ~ R

(26) 1. ~ P ⊃ S

2. ~ Q ⊃ P

3. ~ Q ∨ ~ S / ∴ P

(27) 1. (~ P ∨ Q) ⊃ S / ∴ ~ S ⊃ ~ Q

(28) 1. ~ F ⊃ (G ⊃ ~ H)

2. L ∨ ~ F

3. H ∨ ~ M / ∴ ~ L ⊃ (G ⊃ ~ M)

(29) 1. B ⊃ C

2. D ⊃ E

3. (C • E) ⊃ G / ∴ (B • D) ⊃ G

(30) 1. U ⊃ (W ∨ X)

2. ~ ~ U • ~ X

3. (Y ∨ W) ⊃ Z / ∴ Z

(31) 1. D ⊃ G

2. D ∨ H / ∴ G ∨ H

(32) 1. ~ (P ⊃ Q) ⊃ ~ R

2. S ∨ R / ∴ ~ S ⊃ (~ P ∨ Q)

(33) 1. J ⊃ K

2. ~ (K • L)

3. L / ∴ ~ J

(34) (P ∨ Q) ⊃ R

2. ~ R ∨ S

3. ~ P ⊃ T

4. ~ S / ∴ T

(35) 1. C ∨ (W • S)

2. C ⊃ S / ∴ ~ W ⊃ S

(36) 1. (A ∨ B) ⊃ C

2. (B ∨ C) ⊃ (A ⊃ E)

3. D ⊃ A / ∴ D ⊃ E

(37) 1. R ⊃ (~ P ∨ ~ Q)

2. S ⊃ T

3. T ⊃ Q

4. P / ∴ S ⊃ ~ R

(38) 1. A ⊃ (B ⊃ C)

2. B

3. (E ⊃ T) ⊃ K

/ ∴ (A ⊃ C) • (T ⊃ K)

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