CHAPTER 06

Center of Gravity and Moment of Inertia

6.1 Center of Gravity

A point at which whole of the body assumed to be concentrated is known as center of gravity of a body. In case of uniform gravity it is the same as the centre of mass and have only one centre of gravity irrespective of its all orientation.

6.2 Centroid

The plane areas such as a triangle, quadrilateral, circle etc. have only areas, but no mass. The entire area of such plane figures may be assumed to be concentrated at a point, which is known as centroid of the area.

Centroid is used for geometrical figure like line, areas and volumes and depend only geometry of the body. While center of gravity is used for physical bodies like wires, plates and solids and depends upon the physical properties of the body.

However for plane areas, the centroid and centre of gravity are the same and can be used synonymously.

6.3 Centroid of Given Lamina

Consider the of definite area as shown in figure 6.1. The body may be considered to be composed of number of small body elements.

Let the area of the small element be a_1, a_2.... a_n and the locations of the elements in $X-Y$ plane are (x_1, y_1), (x_2, y_2),.... (x_n, y_n). The area of the whole body is A and the centroid of body is G whose location in X - Y plane is $(\bar{X}, \bar{Y})$. Now we have to find out the values $\bar{X}$ and $\bar{Y}$ for locating G.

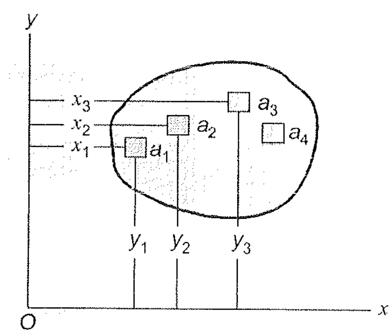


Fig.: 6.1

If G is centroid of the total area A, whose distance from the axis OY is $\bar{X}$, then moment of total area about OY axis is $A\bar{X}$. The moments of all the elemental area about the OX axis is

$$= a_1 x_1 + a_2 x_2 + a_3 x_3 + ...$$

The moment of all elemental area about the axis OY must be equal to the moment of total area about the same axis. Therefore

$$A\bar{X} = a_1 x_1 + a_2 x_2 + a_3 x_3 + ...$$

or

$$\bar{X} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3 + ...}{A}$$

Similarly $$\bar{Y} = \frac{a_1y_1 + a_2y_2 + a_3y_3 + \ldots\ldots}{A}$$

Above two equation gives the location of centroid of body. Following point should be kept in mind while determining the location of centroid of a body:

- Values of $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4 \ldots$ and $\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4 \ldots$ should be measured on same side of axis of reference. If, however the figure is on both side of the axis of reference, then the distances in one direction are taken as positive and those in the opposite direction must be taken as negative.
- If given section is symmetrical about x axis or y axis, then we only have to calculate either $\bar{X}$ or $\bar{Y}$ because centroid or C.G. of the section lies on the axis of symmetry.

6.4 C.G. of a Uniform Rectangular Lamina

Figure 6.2 shows a rectangle *ABCD* of width *b* and height *h*. Consider an element *dy* at a distance *y* from x axis, then the area of this element is

$$da = bdy$$

The moment about the x axis is

$$= y \cdot bdy$$

The sum of moments of all such elementary area is

$$= \int_0^h y \cdot bdy = b\int_0^h ydy$$

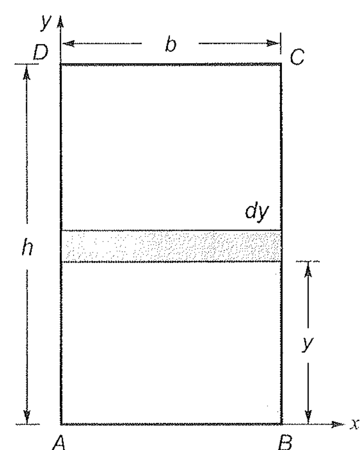


Fig.: 6.2

If $\bar{Y}$ is distance of centroid from x axis then from principal of moment we can write

$$bh\bar{Y} = b\int_0^h ydy$$

or $$h\bar{Y} = \left[\frac{y^2}{2}\right]_0^h = \frac{h^2}{2}$$

or $$\bar{Y} = \frac{h}{2}$$

Similarly $$\bar{X} = \frac{b}{2}$$

6.5 Centroid of Triangular Lamina

Consider a triangle *ABC* of base *b* and height h shown in figure 6.3. Consider an elemental width of *dy* at a distance *y* as shown in figure whose width is b_1.

The area of elemental strip is given by

$$da = b_1 \cdot dy$$

Considering the triangles *ABC* and *ADE* which are equiangular, we can write

$$\frac{b_1}{b} = \frac{h-y}{h}$$

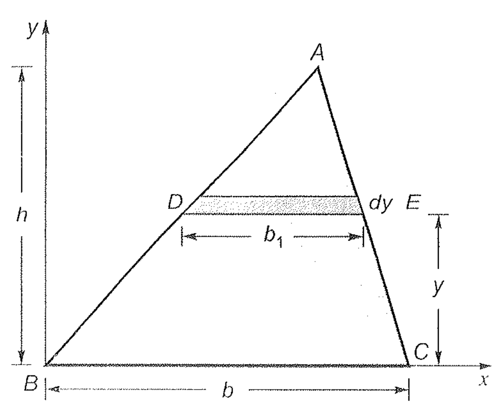


Fig.: 6.3

or
$$b_1 = \frac{b(h-y)}{h}$$

Substituting this we get
$$da = \frac{b}{h}(h-y)dy$$

Moment of this elemental strip about x axis or base BC is
$$= \frac{b}{h}(h-y)dy \cdot y$$

The sum of the moments of all such elementary strip of the triangular lamina about base BC
$$= \int_0^h y \cdot \frac{b}{h}(h-y)dy$$

If $\bar{Y}$ is distance of centroid from x axis then from principal of moment we can write
$$A\bar{Y} = \int_0^h y \cdot \frac{b}{h}(h-y)dy$$

or
$$\frac{1}{2}bh\bar{Y} = \int_0^h y \cdot \frac{b}{h}(h-y)dy$$

or
$$\frac{1}{2}h^2\bar{Y} = \int_0^h (hy - y^2)dy = \left| h\frac{y^2}{2} - \frac{y^3}{3} \right|_0^h$$

or
$$\frac{1}{2}h^2\bar{Y} = \frac{h^3}{6}$$

or
$$\bar{Y} = \frac{h}{3}$$

Thus height of centroid of any triangle from its base is $h/3$. If triangle is right angle then its $\bar{X}$ can be determine the inspection only. Consider the right angle triangle ΔABC shown in figure 6.4. As discussed above, its centroid $\bar{Y}$ must be at $h/3$ above from its base. If we consider AB its base and BC height then its centroid must be $b/3$ from BC.

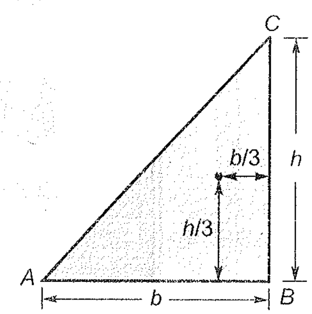


Fig.: 6.4

Now draw a perpendicular AN from point A on BC in triangle as shown in figure 6.5. Let $AN = h$, $NB = a$ and $NC = h$ where $a + c = b$. Now we have two right angle triangle ΔANC and ΔANB whose centroid are known and shown in figure.

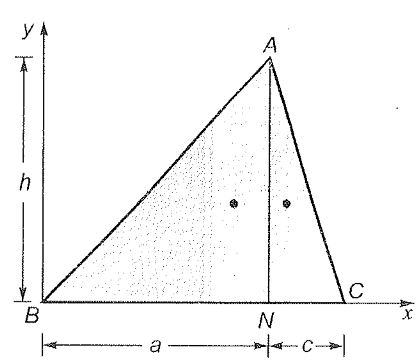


Fig.: 6.5

Assume $C.G.$ of the triangle from point B along x-axis is $\bar{X}$. Sum of moments of areas ΔANC and ΔANB about y-axis is equal to moment of area ΔABC. Therefore
$$\frac{1}{2}bh\bar{X} = \frac{1}{2}ah \times \frac{2}{3}a + \frac{1}{2}ch\left(a + \frac{c}{3}\right)$$

or
$$b\bar{X} = a \times \frac{2}{3}a + c\left(a + \frac{c}{3}\right)$$

or $$\bar{X} = \frac{2a^2+3ac+c^2}{3b} = \frac{2a^2+2ac+ac+c^2}{3b}$$

or $$\bar{X} = \frac{(2a+c)(a+c)}{3(a+c)} = \frac{(2a+c)}{3}$$

or $$\bar{X} = \frac{(2a+c)}{3} = \frac{(a+a+c)}{3} = \frac{(a+b)}{3}$$

Similarly the C.G. from the point C is $\frac{(c+b)}{3}$

6.6 Centroid of a Semicircular Lamina

Consider the semicircle shown in figure 6.6. This semicircle is symmetrical about its y axis. So centroid lie on this axis. Here $\bar{X}=0$ and we have to determined $\bar{Y}$. Consider the elemental radial area OPQ at an angle θ as shown in figure 6.6 whose include angle is $d\theta$. This can be considered as a triangle with base $PQ = r \cdot d\theta$ and height r.

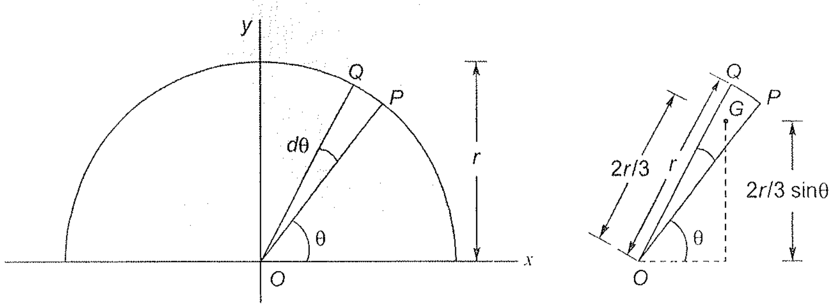


Fig. 6.6

Area of OPQ is $$da = \frac{1}{2}(rd\theta)\cdot r = \frac{1}{2}r^2 \cdot d\theta$$

The distance C.G. of this area from the point O is $\frac{2}{3}r$. The height of centroid from x-axis

$$= \frac{2}{3}r\sin\theta$$

Moment of area OPQ about x-axis is area multiply by height. Therefore

$$= \frac{1}{2}r^2 d\theta \times \frac{2}{3}r\sin\theta = \frac{r^3}{3}\sin\theta d\theta$$

Moment of whole half circle area $$= \int_0^{\pi}\frac{r^3}{3}\sin\theta d\theta = \frac{r^3}{3}\int_0^{\pi}\sin\theta d\theta = \frac{2}{3}r^3\int_0^{\pi/2}\sin\theta d\theta = \frac{2}{3}r^3$$

If $\bar{Y}$ is distance if centroid from x axis then from principal of moment we can write

$$\frac{\pi}{2}r^2\bar{Y} = \frac{2}{3}r^3 \quad \text{or} \quad \bar{Y} = \frac{4r}{3\pi}$$

6.7 C.G. of a Right Circular Cone

Consider a cone of base radius r and height h as shown in figure 6.7. The density of material is ρ. The cone is symmetrical about its y axis. So centroid lie on this axis. Thus we have to find out the position of C.G. from the base above the x axis along y axis.

Consider an element volume of thickness dy at a distance y from base as shown in figure. The weight of this elemental volume is

$$dW = \pi x^2 \cdot dy \cdot \rho$$

Considering equiangular triangles OAB and BCD, we can write

$$\frac{CD}{BC} = \frac{OA}{OB}$$

or
$$x = \frac{OA}{OB}BC = \frac{r}{h}(h-y)$$

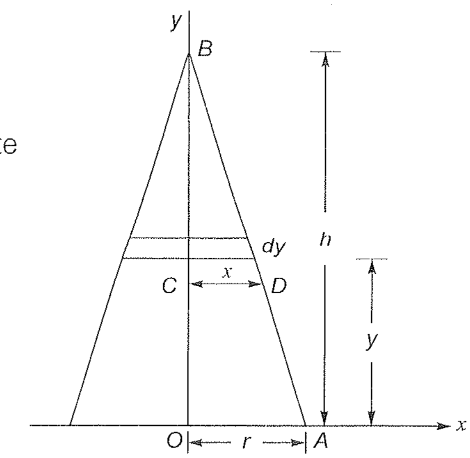


Fig: 6.7

Substituting this value we get weight of elemental volume considered

$$dW = \pi \cdot \frac{r^2}{h^2}(h-y)^2\rho \cdot dy$$

Moment of this elemental weight about x axis

$$= \pi\rho\left(\frac{r}{h}\right)^2 (h-y)^2 y dy$$

Total moment of the given cone is

$$= \int_0^h \pi\rho\left(\frac{r}{h}\right)^2 (h-y)^2 y dy = \pi\rho\left(\frac{r}{h}\right)^2 \int_0^h [h^2 - 2hy + y^2] y dy$$

$$= \pi\rho\left(\frac{r^2}{h^2}\right)\int_0^h [h^2 y - 2hy^2 + y^3] dy$$

$$= \pi\rho\frac{r^2}{h^2}\left| h^2 \cdot \frac{y^2}{2} - 2h\frac{y^3}{3} + \frac{y^4}{4} \right|_0^h = \frac{\pi\rho r^2 h^2}{12}$$

If $\bar{Y}$ is distance of C.G. from x axis then from principal of moment we can write

$$\frac{\pi}{3}r^2 h \cdot \rho \cdot \bar{Y} = \frac{\pi\rho}{12}r^2 h^2$$

or
$$\bar{Y} = \frac{h}{4}$$

6.8 C.G. of a Hemispherical Solid

Consider a solid hemisphere of radius r and where density of material is ρ. Consider a differential disk elementary strip of thickness dy at a distance y from x axis as shown in figure 6.8.

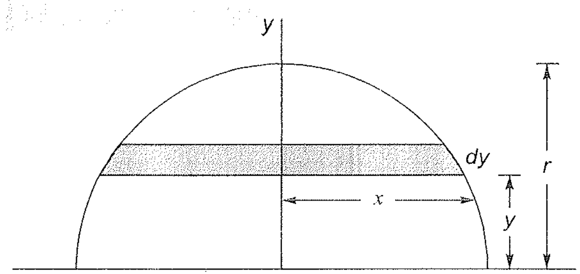


Fig.: 6.8

The weight of this elementary strip

$$dW = \rho\pi x^2 \cdot dy$$

From the geometry of figure 6.8

$$x^2 = (r^2 - y^2) \qquad (\because \text{ equation of circle is: } x^2 + y^2 = r^2)$$

Thus
$$dW = \rho\pi\,(r^2 - y^2) \cdot dy$$

The moment of this weight of elementary strip about x axis

$$= \int_0^r \rho\pi(r^2 - y^2) y dy = \rho\pi\int_0^r (r^2 - y^2) y dy$$

If $\bar{Y}$ is distance C.G. from x axis then from principal of moment we can write

$$\left[\frac{1}{2}\left(\frac{4}{3}\pi r^3\right)\rho\right]\cdot\bar{Y} = \rho\pi\int_0^r (r^2 - y^2)y\,dy$$

or

$$\frac{2}{3}r^3\cdot\bar{Y} = \int_0^r (r^2 y - y^3)dy$$

or

$$\frac{2}{3}r^3\cdot\bar{Y} = \left[r^2\frac{y^2}{2} - \frac{y^4}{4}\right]_0^r = \frac{r^4}{4}$$

or

$$\bar{Y} = \frac{3}{8}r$$

6.9 Moment of Inertia

Moment of inertia is a mathematical expression which is related to a body's tendency to resist rotation. The moment of force about any point is product of the force and the perpendicular distance between them, it is also called first moment of forces. If this first moment is again multiplied by the perpendicular distance between them, then the product so obtained is called second moment of force or moment of moment of the force. If instead of force we consider the area of the lamina or mass of the body, it is called the second moment of area or second moment of mass. They are also termed broadly as moment of inertia.

6.10 Moment of Inertia of an Area

Consider an area as shown in figure 6.9. Consider an elemental area dA which is located at y from x axis, then the moment of inertia of dA about x axis is given by

$$dI_x = dA\cdot y^2$$

Total moment of inertia of given area is

$$I_x = \int dA\cdot y^2$$

Similarly

$$I_y = \int dA\cdot x^2$$

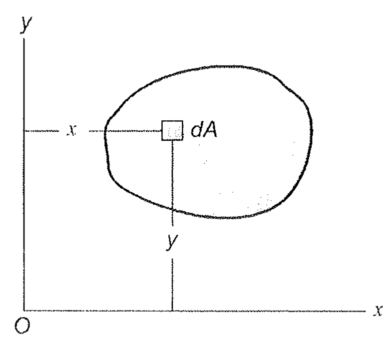


Fig.: 6.9

Thus, the moment of inertia of an area about the given axis is the sum of moments of inertia of an elementary area constituting the given area about the given axis.

6.11 Theorem of Parallel Axis

This theorem states that,

The moment of inertia of lamina about any axis in the plane of the lamina equals the sum of moment of inertia of the area about a parallel centroidal axis in the plane of lamina and the product of the area of the lamina and square of the distance between two axis.

As per above statement for the given area as shown in figure 6.10. we can write

$$I_{x'} = I_G + d^2 A$$

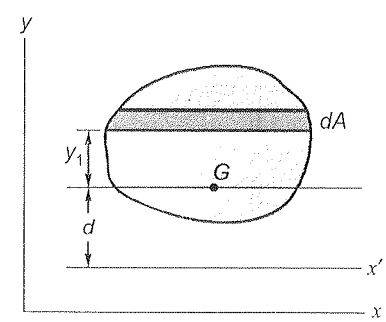


Fig.: 6.10

Proof:

Figure 6.10 shows a plane lamina of area A. We have to determine moment of inertia about x' axis. Consider an elemental area dA located as shown in figure, then we can write

$$dI_G = dA \cdot y_1^2$$

or $$I_G = \int dA \cdot y_1^2$$

The moment of inertia of dA about x' axis is

$$I_{x'} = \int (y_1 + d)^2 dA = \int dA \cdot y_1^2 + \int dA \cdot d^2 + \int 2dAy_1 d$$

or $$= I_G + d^2 \int dA + 2d \int dA \cdot y_1$$

Here $\int dA \cdot y_1$ is the moment of are about it centroid, that is always zero. Therefore

$$I_{x'} = I_G + d^2 A$$

6.12 Theorem of Perpendicular Axis

This theorem states that,

If I_x and I_y be the moment of inertia about two mutually perpendicular x axis and y axis in the plane of the lamina and I_z be the moment of inertia of the lamina about an axis normal to the plane of lamina and passing through the point of intersection of the x axis and y axis.

As per statement of theorem

$$I_x = I_x + I_y$$

where z axis is perpendicular to x and y axes.

Consider an area dA from total area A located in the three dimensions and whose coordinates are x and y in the $x - y$ plane and its distance from z axis is r as shown in figure 6.11.

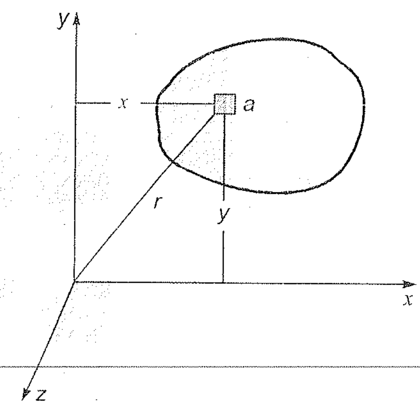


Fig.: 6.11

$$r^2 = x^2 + y^2$$

$$dI_z = dA \cdot r^2 = dA\,(x^2 + y^2)$$

$$= dA \cdot x^2 + dA \cdot y^2$$

Thus $$I_z = \int_A dA \cdot x^2 + \int_A dA \cdot y^2$$

or $$I_z = I_x + I_y$$

6.13 Radius of Gyration

The distance from the reference axis to a point where whole area of given lamina can considered to be concentrated and produce the same moment of inertia with respect to given reference axis is known as radius of gyration.

Let I is the moment of inertia about given axis and K is the distance from axis of rotation at which whole magnitude of the given area is assumed to be concentrated to have the same moment of inertia as that of the area about that axis, then,

$$I = A \cdot K^2$$

or $$K = \sqrt{\frac{I}{A}}$$

If the body is considered solid, then the above equations are written as

$$K = \sqrt{\frac{I_{xx}}{m}}$$

where m is the mass and I_{xx} is the mass moment of inertia of solid body considered

6.14 Moment of Inertia of a Rectangular Lamina

Consider rectangular of $b \times d$ as shown in figure 6.12, where x and y axes are located passing through the C.G. of the rectangular lamina. Consider an elemental strip of thickness dy at a distance y from x axis, then as per definition, the moment of inertia of this elemental area about x-axis is given by

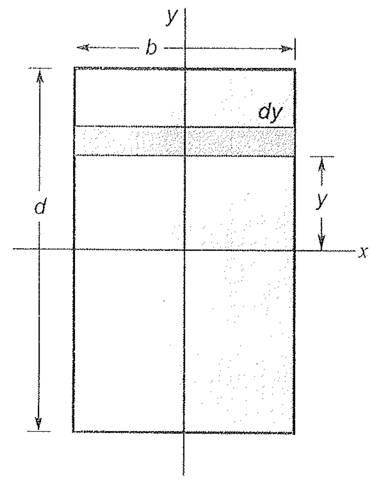


Fig.: 6.12

$$dI = dA \cdot y^2 = (bdy) \cdot y^2$$

or

$$I_x = \int_0^{d/2} (b \cdot dy) y^2 = 2b \int_0^{d/2} y^2 dy$$

$$= 2b\left[\frac{y^2}{3}\right]_0^{d/2} = \frac{2b}{3}\left(\frac{d}{2}\right)^3 = \frac{1}{12} bd^3$$

Similarly, we can also write $I_y = \frac{1}{12} db^3$

As per the theorem of perpendicular axis,

$$I_z = I_x + I_y$$

$$\Rightarrow \quad I_z = \frac{1}{12} bd(b^2 + d^2)$$

6.15 Moment of Inertia of a Circular Lamina

Consider a circular lamina of diameter d as shown in figure 6.13 where x and y axes are located passing through the C.G. of the circular lamina. Consider an z axis perpendicular to x and y axes and passing through O and then consider an elemental area dA at a distance r of thickness dr, then the moment of area of dA about the axis z is given by

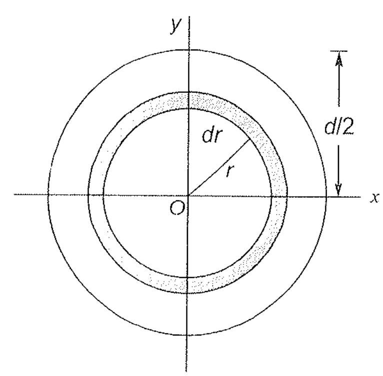


Fig.: 6.13

$$dI_s = dA \cdot r^2 = (2\pi r \cdot dr) \cdot r^2 = 2\pi r^3\, dr$$

$$I_z = \int_0^{d/2} 2\pi r^3 \cdot dr = 2\pi\left[\frac{r^4}{4}\right]_0^{d/2} = \frac{\pi d^4}{32}$$

But $\quad I_x + I_y = I_z$

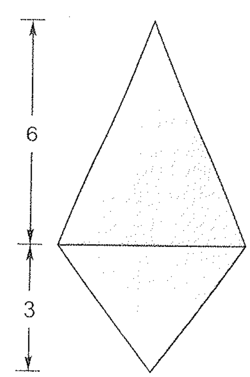
or $\quad 2I_x = 2I_y = I_z \quad$ as $I_x = I_y$

$$I_x = I_y = \frac{1}{2}\left(\frac{\pi d^4}{32}\right) = \frac{\pi d^4}{64}$$

If lamina is hollow with outer diameter d_2 and inner diameter d_1, then the moment of inertia is given by

$$I_x = I_y = \frac{\pi}{64}(d_2^4 - d_1^4)$$

Example 6.1 **Two isosceles triangles having the same base b and having heights of 6 cm and 3 cm are attached to each other at the base as shown in figure. Determine the centroid of combined body.**



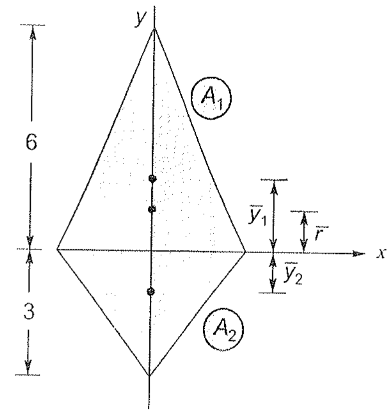
Solution:

The figure redrawn as shown in figure. Reference axis are chosen as shown in figure. As the section is symmetrical about y axis, therefore its center of gravity will lie on this axies.

The distance of centroid from x axis is

$$\bar{Y} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2}{A_1 + A_2}$$

where A_1 and A_2 are the area of respective triangle $\bar{y}_1$ and $\bar{y}_2$ are distance of respective center of gravity from axis x.



Thus $$A_1 = \frac{1}{2}\times b\times 6 = 3b\text{ cm}^2$$

$$A_2 = \frac{1}{2}\times b\times 3 = 1.5b\text{ cm}^2$$

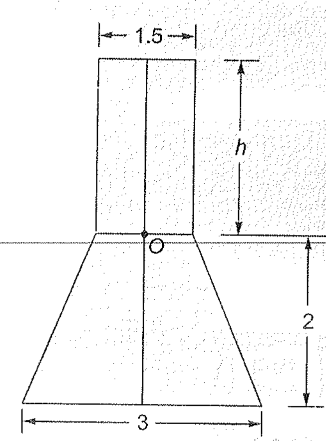
$$\bar{y}_1 = \frac{1}{3}\times 6 = 2\text{ cm}$$

$$\bar{y}_2 = -\frac{1}{3}\times 3 = -1\text{ cm}$$

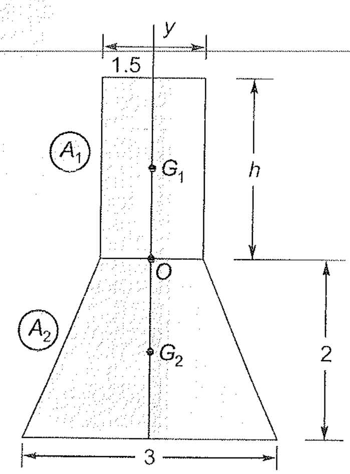
Thus $$\bar{Y} = \frac{3b\times 2 + 1.5b\times(-1)}{3b + 1.5b} = 1\text{ cm}$$

Thus centroid is at 1 cm above x axis.

Example 6.2 The C.G. of lamina shown in figure lie at the point O. Determine the height of the rectangle.



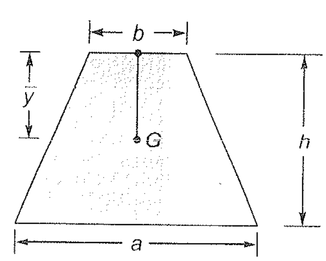
Solution:



The figure has been redrawn as shown in figure. Here G_1 and G_2 are the C.G. of the rectangle A_1 and Trapezium A_2. Here O is the center of gravity G also.

Thus $$OG_1 = GG_1 = \frac{h}{2}$$

C.G. of Trapezium, $$\bar{y} = \frac{h}{3}\left(\frac{2a+b}{a+b}\right)$$



$$OG_2 = GG_2 = \frac{h}{3}\left(\frac{2a+b}{a+b}\right) = \frac{2}{3}\left(\frac{2\times 3+1.5}{1.5+3}\right) = \frac{10}{9}\text{ cm}$$

$$A_1 = h\times 1.5 = 1.5h$$

$$A_2 = \left(\frac{a+b}{2}\right)h = \left(\frac{3+2}{2}\right)\times 2 = 4.5\text{ cm}$$

Taking moment of area about point G or O we can write

$$A_1\,GG_1 = A_2\,GG_2$$

$$1.5 \times \frac{h}{2} = \frac{10}{9} \times 4.5 = 5$$

$$h^2 = \frac{10}{1.5} \text{ or } h = 2.59 \text{ cm}$$

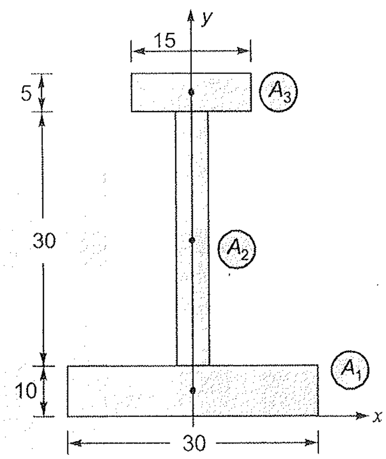
Example 6.3 An I-section has the following dimension:

Bottom flange 30 cm × 10 cm

Top flange 15 cm × 5 cm

Web 30 cm × 5 cm

Determine the position of center of gravity of the section.



Solution:

As per problem statement configuration is shown in figure. Reference axis are chosen as shown in figure. As the section is symmetrical about y axis, bisecting the web, therefore its centre of gravity will lie on this axis. Given section may be dividing into three parts.

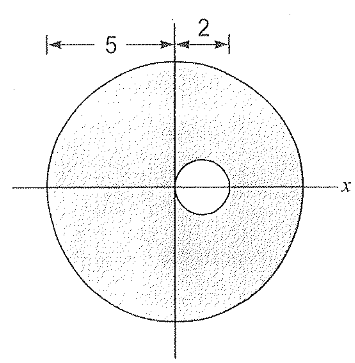
1. A_1 – Bottom flange (30 × 10)
2. A_2 – Web (30 × 5)
3. A_3 – Top flange (15 × 5)

To determine the location of the centroid of the plane figure we have the following table:

A	Area (cm^2) (A)	$\bar{y}$ from x	$A\bar{y}$ (cm^3)
A_1	30 × 10 = 300	5	1500
A_2	30 × 5 = 300	10 + 30/2 = 25	3750
A_3	15 × 5 = 300	10 + 30 + 5/2 = 425	3187.5
	ΣA = 525		$\Sigma A\bar{y}$ = 8437.5

Thus $$\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{8437.5}{525} = 16.07 \text{ cm}$$

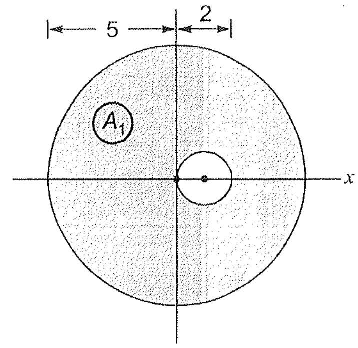
Example 6.4 Determine the C.G. of the remaining portion of a circular sheet metal of radius 5 cm when a hole of 1 cm radius is cut out from the circular disc along its horizontal diameter as shown in figure.



Solution:

As the section is symmetrical about x axis, therefore its centre of gravity will lie on this axis. We have to find out the $\bar{X}$ on x axis from axis y. Here the distance along x axis from y axis towards right are taken as positive and toward, left are taken negative.

Here A_1 is the area of the whole circle of radius 5 cm and A_2 is the area of the cut part of radius 1 cm. Area of cut part is taken as negative.



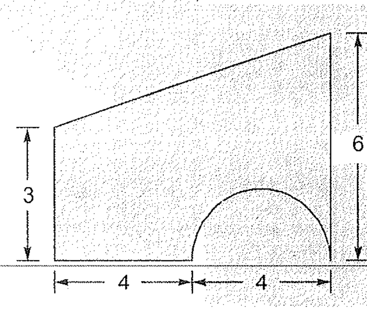
The area of both component, the centroidal distance from the y axis and the moment of both individual component about these axis are tabulated below.

A	Area (cm^2)	$\bar{x}$ from y	$A\bar{x}$ (cm^3)
A_1	$\pi 5^2$	0	0
A_2	$-\pi 1^2$	1	$-\pi$
	$\Sigma A = \pi 24$		$\Sigma A\bar{x} = -\pi$

Thus $$\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A} = \frac{-\pi}{\pi 24} = \frac{-1}{24} = -0.042 \text{ cm}$$

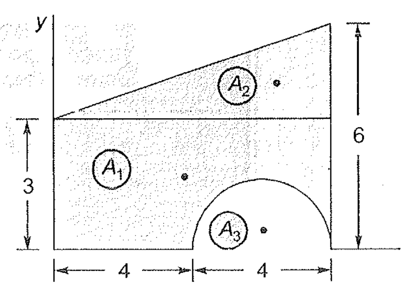
Thus centroid of remainder lamina is at 0.042 cm left to y on x.

Example 6.5 A semicircular area is remove from a trapezium as shown in figure. Determine the centroid of the remaining area.



Solution:

Reference axis are chosen as shown in figure. As the section is not symmetrical about any axis, therefore we have to find out the values of $\bar{X}$ and $\bar{Y}$ for the area. The lamina is divided into the three part as marked A_1, A_2 and A_3. Here A_1 is the area of the whole rectangle of 3 × 8, A_2 is the area of the triangle and A_3 is the area of cut part semicircle of diameter of 4 which is taken as negative.



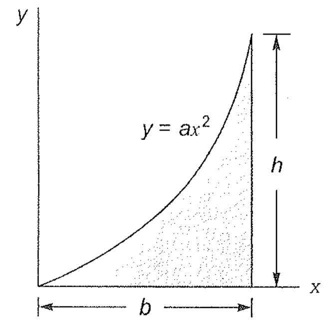
The area of these component, their centroidal distance from the y-axis and x-axis, the moment of both individual component about these axis are tabulated below.

A	Area (cm^2)	$\bar{x}$	$\bar{y}$	$A\bar{x}$	$A\bar{y}$
A_1	8 × 3 = 24	4	1.5	96	36
A_2	½ × 8 × 3 = 12	$\frac{8\times 2}{3} = 5.33$	$3+\frac{3}{3} = 4$	63.96	48
A_3	$-\frac{1}{2}\pi 2^2 = -6.283$	6	$\frac{4\times 2}{3\pi} = 0.849$	–37.7	–5.334
	$\Sigma A = 29.717$			$\Sigma A\bar{x}$ = 122.26	$\Sigma A\bar{y}$ = 78.666

Thus $$\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A} = \frac{122.26}{29.717} = 4.114 \text{ cm}$$

$$\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{78.666}{29.717} = 2.647 \text{ cm}$$

Example 6.6 Determine the centroid of shaded enclosed by $y = ax^2$ and x axis as shown in figure.



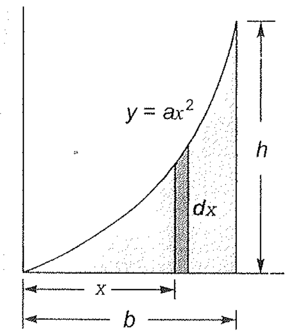
Solution:

Consider a element dx as shown in figure. Its centroidal distance are

$$\bar{x} = x$$

$$\bar{y} = \frac{y}{2} = \frac{ax^2}{2}$$

and its area is



$$da = ydx = ax^2\,dx$$

$$\int_A da = \int_0^b ax^2 dx = a\left[\frac{x^3}{3}\right]_0^b = \frac{ab^3}{3}$$

$$\bar{x}da = x \cdot ax^2\,dx$$

$$\int_A \bar{x}da = \int_0^b x \cdot ax^2 dx = a\left[\frac{x^4}{4}\right]_0^4 = \frac{ab^4}{4}$$

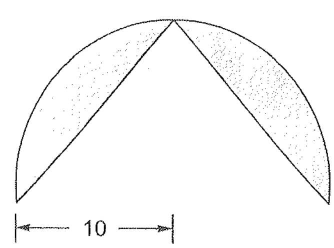
$$\bar{y}da = \frac{ax^2}{2} \cdot ax^2 dx = \frac{a^2x^4}{2}$$

$$\int_A \bar{y}da = \int_0^b \frac{a^2x^4}{2} dx = \frac{a^2}{2}\left[\frac{x^5}{5}\right]_0^b = \frac{a^2b^5}{10}$$

Therefore $$\bar{X} = \frac{\int_A \bar{x}da}{\int_A da} = \frac{\frac{1}{4}ab^4}{\frac{1}{3}ab^3} = \frac{3b}{4}$$

$$\bar{Y} = \frac{\int_A \bar{y}da}{\int_A da} = \frac{(1/10)a^2b^5}{(1/3)ab^3} = \frac{3ab^2}{10} = \frac{3h}{10} \quad \text{since } ab^2 = h$$

Example 6.7 The figure shows the sectional elevation of a hemispherical body with a central portion removed in the form of right circular cone. Find out the C.G. of the remaining part of the body.



Solution:

As the body is symmetrical about y axis, therefore its center of gravity will lie on this axis.

We know that $$\bar{Y} = \frac{V_1\bar{y}_1 + V_2\bar{y}_2}{V_1 + V_2}$$

where V_1 and V_2 are the volume of hemisphere and cone and y_1 and y_2 are distance of respective center of gravity from axis x or bottom

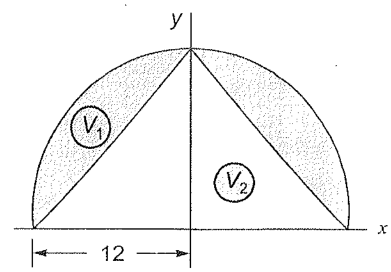
Thus

$$V_1 = \frac{2}{3}\pi 10^3 = \frac{2000}{3}\pi \text{ cm}^3$$

$$V_2 = -\frac{1}{3}\pi 10^2 \times 10 = -\frac{1000}{3}\pi \text{ cm}^3$$

$$\bar{y}_1 = \frac{3}{8}\times 10 = 3.75 \text{ cm}$$

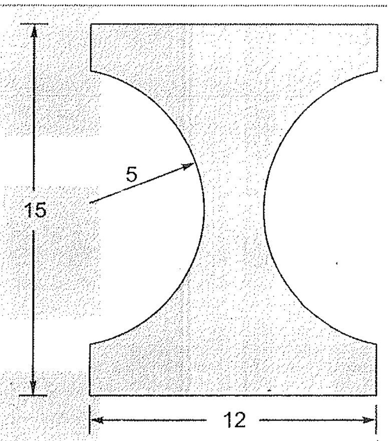
$$\bar{y}_2 = \frac{1}{4}\times 10 = 2.5 \text{ cm}$$



Thus

$$\bar{Y} = \frac{\frac{2000}{3}\pi\times 3.75 - \frac{1000}{3}\pi \times 2.5}{\frac{2000}{3}\pi - \frac{1000}{3}\pi} = 5 \text{ cm}$$

Example 6.8 The cross-section of a cast iron beam, is shown in figure. Determine the moments of inertia of the section about horizontal and vertical axis passing through the centroid of the section.



Solution:

As the section is symmetrical about its horizontal and vertical axes, therefore enter of gravity of the section will lie at the center of the rectangle. It may be easily shown that if two semicircles are placed together, it will form a circular hole with 5 mm radius.

The moment of inertia of the rectangular section about its horizontal axis passing through its centre of gravity,

$$I_{x1} = \frac{bd^3}{12} = \frac{12\times(15)^3}{12} = 3375 \text{ mm}^4$$

and moment of inertia of the circular section about a horizontal axis passing through its center of gravity.

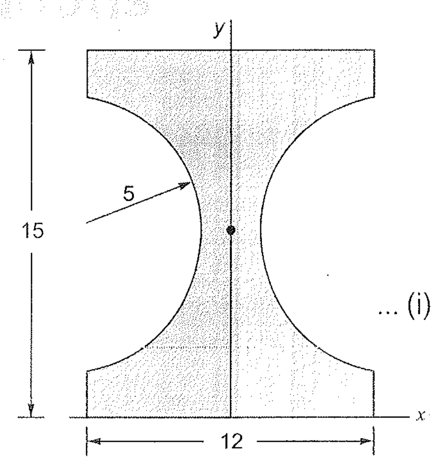
$$I_{x2} = \frac{\pi}{4}(r)^4 = \frac{\pi}{4}(5)^4 = 490.87 \text{ mm}^4$$

Moment of inertia of the whole section about horizontal axis passing through the centroid of the section,

$$I_x = I_{x1} - I_{x2}$$
$$= 3375 - 490.87 = 2884.13 \text{ cm}^4$$

Moment of inertia of the rectangular section about the vertical axis passing through its center of gravity,

$$I_{y1} = \frac{db^3}{12} = \frac{15\times(2)^3}{12} = 2160 \text{ mm}^4 \quad \text{... (i)}$$



In this problem reference axis x and y′ are also centroidal axis x′ and y′

and are of one semicircular section with 5 cm radius,

$$A = \frac{\pi r^2}{2} = \frac{\pi(5)^2}{2} = 39.27 \text{ mm}^2$$

The moment of inertia of a semicircular section about a vertical axis passing through its center of gravity,

$$I_{G2} = 0.11\,r^4 = 0.11 \times (5)^4 = 68.75 \text{ mm}^4$$

and distance between centre of gravity of the semicircular section and its base

$$\frac{4r}{3\pi} = \frac{4\times 5}{3\pi} = 2.12 \text{ mm}$$

Distance between center of gravity of the semicircular section and center of gravity of the whole section,

$$d_2 = 6 - 2.12 = 3.88 \text{ mm}$$

and moment of inertia of one semicircular section about center of gravity of the whole section,

$$I_G + Ad^2 = 68.75 + 39.27 \times (3.88)^2 = 660 \text{ mm}^4$$

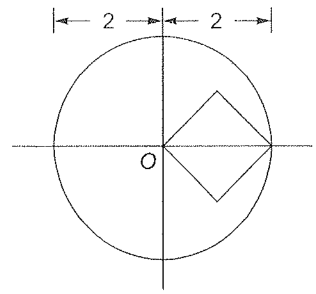
Moment of inertia of both the semicircular section about center of gravity of the whole section,

$$I_{y2} = 2 \times 660 = 1320 \text{ mm}^4 \quad \text{... (ii)}$$

and moment of inertia of the whole section about a vertical axis passing through the centroid of the section,

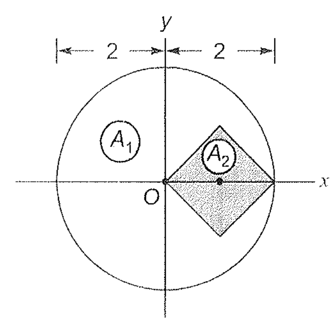
$$I_y = I_{y1} - I_{y2} = 2160 - 1320 = 840 \text{ mm}^4$$

Example 6.9 Find the moment of inertia of a section shown in figure along its centroidal x' axis.



Solution:

Reference axis are chosen as shown in figure. As the section is symmetrical about x-axis, therefore its center of gravity will lie on this axis. There is no need to determine the centroid because we have to find out the moment of inertia about centroidal axis x or x'. We can write easily the moment of inertia of circular section and rectangular section about x.



Moment of inertia of circular section about x axis

$$I_{x1} = \frac{\pi}{4} \times 4^4 = 12.566$$

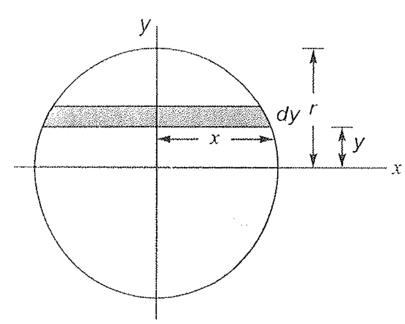
Moment of inertia of square section about x-axis

$$I_{x2} = \frac{a^4}{12} = \frac{1}{12}\left(\frac{2}{\sqrt{2}}\right)^4 = 0.33$$

$$I_x = I_{x1} - I_{x2} = 12.566 - 0.33 = 12.236$$

Example 6.10 Determine the moment of inertia I_x of the sphere and express the result in terms of the total mass m of the sphere. The sphere has a constant density ρ.

Solution:



Consider a differential disk element as shown in figure.

The mass of the differential disk element is

$$dm = \rho dV = \rho\pi x^2\, dy = \rho\pi\,(r^2 - y^2)\, dy$$
$$(\because x^2 + y^2 = r^2)$$

The mass moment of inertial of the element is

$$dI_y = \frac{1}{2}x^2 dm = \frac{1}{2}(r^2 - y^2)\rho\pi(r^2 - y^2)dy$$

$$= \frac{1}{2}\rho\pi(r^2 - y^2)^2 dy$$

Mass Moment of Inertia of disk is

$$I_y = \int_{-r}^{r} \frac{1}{2}\rho\pi(r^2 - y^2)^2 dy = \frac{8}{15}\pi\rho r^5$$

The total mass of sphere is

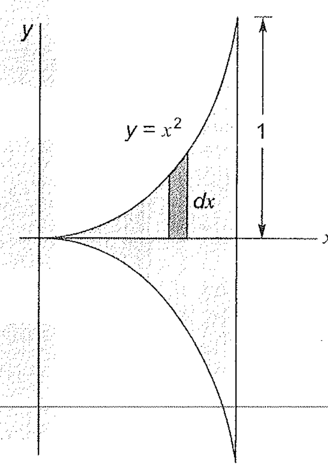
$$m = \int_{-r}^{r} \rho\pi(r^2 - x^2)^2 dx = \frac{4}{3}\rho\pi r^3$$

Thus $$I_y = \frac{2}{5}mr^2$$

Example 6.11 The curve $y = x^2$ is revolved about the x-axis from $x = 0$ to $x = 1$ cm to form a homogeneous solid of revolution of mass m. Determine the radius of gyration k_y.

Solution:

Consider a differential disk element as shown in figure.



The mass of the differential disc element is

$$dm = \rho dV = \rho \times (\pi y^2)\, dx = \pi\rho x^4 dx$$

d

The mass of moment of inertial of the element is,

$$dI_y = \frac{1}{2}x^2 dm = \frac{1}{2}x^2\rho\pi x^4 dx = \frac{1}{2}\rho\pi x^6 dx$$

Mass Moment of Inertia of disk is,

$$I_y = \int_0^1 \frac{1}{2}\rho\pi x^6 dx = \frac{1}{2}\rho\pi\int_0^1 y^6 dx = \frac{1}{2}\rho\pi\left[\frac{x^7}{7}\right]_0^1 = \frac{1}{14}\rho\pi$$

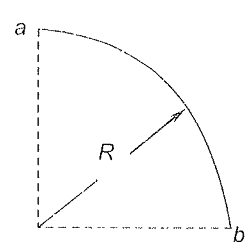
The total mass of sphere is

$$m = \int_0^1 \rho\pi x^4 dx = \rho\pi\left[\frac{x^5}{5}\right]_0^1 = \frac{\rho\pi}{5}$$

Thus $$k_y = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{\frac{1}{14}\rho\pi}{\frac{\rho\pi}{5}}} = \sqrt{\frac{5}{14}} = 0.597 \text{ cm}$$

Objective Brain Teasers

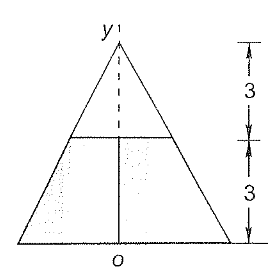
Q.1 What is $\bar{x}$ of quarter of a circle of radius R as shown in



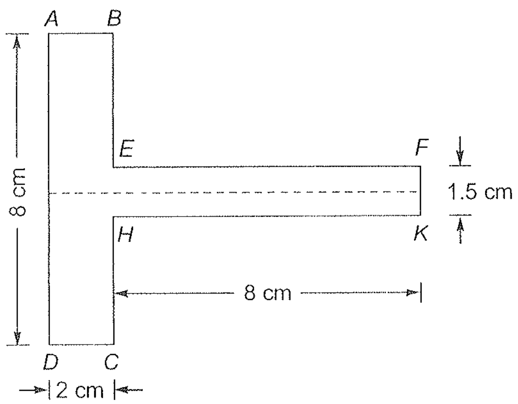
(a) 0.8R
(b) 0.75R
(c) 0.636R
(d) 0.577R

Q.2 What is $\bar{y}$ of shaded area of a triangle?

(a) 2.4 cm
(b) 1.5 cm
(c) 1.33 cm
(d) None of these



Q.3 In a thin uniform lamina having symmetrical central axis as shown in figure. The distance of centre of gravity from AD is



(a) 3 cm (b) $\frac{22}{7}$ cm

(c) $\frac{23}{7}$ cm (d) $\frac{24}{7}$ cm

ANSWERS

1. (c) 2. (c) 3. (b)

Hints & Explanation

1. (c)

$$\bar{x} = 0.636R.$$

2. (c)

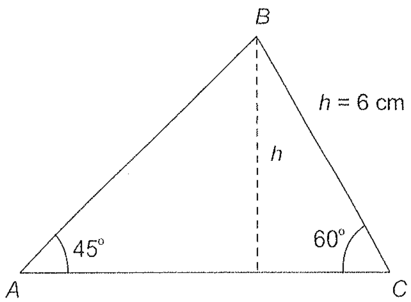
$$\bar{y} = \frac{A.2 - \frac{A}{4} \times 4}{\frac{3A}{4}} = \frac{4A}{3A} = \frac{4}{3}, \quad A = \text{area of triangle}$$

3. (b)

$$\bar{x} \text{ from } AD = \frac{16 \times 1 + 12 \times 6}{28} = \frac{88}{28} = \frac{22}{7}$$

Student's Assignments

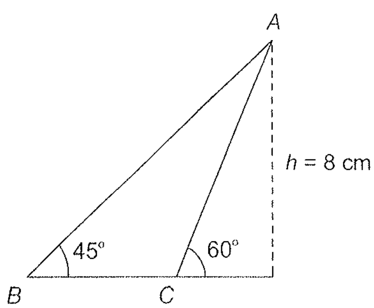
Q.1 A triangle ABC, has attitude h = 6 cm. What moment is of inertia about centroidal axis which is parallel to base AC?



(a) 56.77 cm^4 (b) 113.54 cm^4

(c) 170.35 cm^4 (d) None of these

Q.2 What is moment of inertia of triangle ABC as shown in figure about base BC. ?



(a) 48.13 cm^4 (b) 144.26 cm^4

(d) 341.3 cm^4 (d) None of these.

Q.3 What is $\bar{x}$ of circular arc A, subtending an angle of 60° at O?

(a) 0.985R (b) 0.955R

(c) 0.866R (d) 0.75R