Liquids Capillary Effects (Part - 1)

Q. 160. Find the capillary pressure
(a) in mercury droplets of diameter d = 1.5 pm;
(b) inside a soap bubble of diameter d = 3.0 mm if the surface tension of the soap water solution is α = 45 mN/m.

Solution. 160.

(a) $$\Delta p = \alpha\left(\frac{1}{d/2} + \frac{1}{d/2}\right) = \frac{4\alpha}{d}$$

$$= \frac{4 \times 490 \times 10^{-3}}{1{\cdot}5 \times 10^{-6}}\frac{\text{N}}{\text{m}^2} = 1{\cdot}307 \times 10^{6}\frac{\text{N}}{\text{m}^2} = 13 \text{ atmosphere}$$

(b) The soap bubble has two surfaces

so $$\Delta p = 2\alpha\left(\frac{1}{d/2} + \frac{1}{d/2}\right) = \frac{8\alpha}{d}$$

$$= \frac{8 \times 45}{3 \times 10^{-3}} \times 10^{-3} = 1{\cdot}2 \times 10^{-3} \text{ atomsphere.}$$

Q. 161. In the bottom of a vessel with mercury there is a round hole of diameter d = 70 µm. At what maximum thickness of the mercury layer will the liquid still not flow out through this hole?

Solution. 161. The pressure just inside the hole will be less than the outside pressure by 4 α/d. This can support a height h of Hg where

$$\rho g h = \frac{4\alpha}{d} \quad \text{or} \quad h = \frac{4\alpha}{\rho g d}$$

$$= \frac{4 \times 490 \times 10^{-3}}{13{\cdot}6 \times 10^{3} \times 9{\cdot}8 \times 70 \times 10^{-6}} = \frac{200}{13{\cdot}6 \times 70} \approx {\cdot}21 \text{ m of Hg}$$

Q. 162. A vessel filled with air under pressure P_0 contains a soap bubble of diameter d. The air pressure having been reduced isothermally n-fold, the bubble diameter increased η-fold. Find the surface tension of the soap water solution.

Solution. 162. By Boyle's law

$$\left(p_0 + \frac{8\alpha}{d}\right)\frac{4\pi}{3}\left(\frac{d}{2}\right)^3 = \left(\frac{p_0}{n} + \frac{8\alpha}{\eta d}\right)\frac{4\pi}{3}\left(\frac{\eta d}{2}\right)^3$$

or $$p_0\left(1 - \frac{\eta^3}{n}\right) = \frac{8\alpha}{d}(\eta^2 - 1)$$

Thus $$\alpha = \frac{1}{8}p_0 d\left(1 - \frac{\eta^3}{n}\right)(\eta^2 - 1)$$

Q. 163. Find the pressure in an air bubble of diameter d = 4.0 µm, located in water at a depth h = 5.0 m. The atmospheric pressure has the standard value P_0.

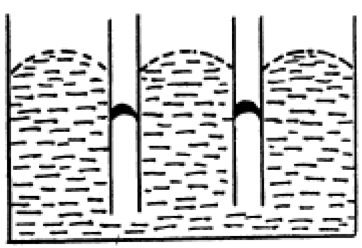
Solution. 163. The pressure has terms due to hydrostatic pressure and capillarity and they add

$$p = p_0 + \rho g h + \frac{4\alpha}{d}$$

$$= \left(1 + \frac{5 \times 9{\cdot}8 \times 10^3}{10^5} + \frac{4 \times {\cdot}73 \times 10^{-3}}{4 \times 10^{-6}} \times 10^{-5}\right) \text{atoms} = 2{\cdot}22 \text{ atom.}$$

Q. 164. The diameter of a gas bubble formed at the bottom of a pond is d = 4.0 µm. When the bubble rises to the surface its diameter increases n = 1.1 times. Find how deep is the pond at that spot. The atmospheric pressure is standard, the gas expansion is assumed to be isothermal.

Solution. 164. By Boyle's law



$$\left(p_0 + h g \rho + \frac{4\alpha}{d}\right)\frac{\pi}{6}d^3 = \left(p_0 + \frac{4\alpha}{nd}\right)\frac{\pi}{6}n^3 d^3$$

or $$\left[hg\rho - p_0(n^3 - 1)\right] = \frac{4\alpha}{d}(n^2 - 1)$$

or $$h = \left[p_0(n^3-1) + \frac{4\alpha}{d}(n^2-1)\right] \Big/ g\rho = 4.98$$ meter of water

Q. 165. Find the difference in height of mercury columns in two communicating vertical capillaries whose diameters are d_1 = 0.50 mm and d_2 = 1.00 mm, if the contact angle θ = 138°.

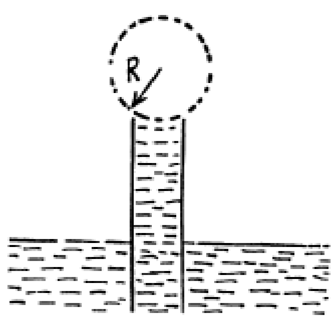
Solution. 165. Clearly

$$\Delta h\, \rho\, g = 4\,\alpha\,|\cos\theta|\left(\frac{1}{d_1} - \frac{1}{d_2}\right)$$

$$\Delta h = \frac{4\,\alpha\,|\cos\theta|\,(d_2 - d_1)}{d_1\, d_2\, \rho g} = 11 \text{ mm}$$

Q. 166. A vertical capillary with inside diameter 0.50 mm is submerged into water so that the length of its part protruding over the water surface is equal to h = 25 mm. Find the curvature radius of the meniscus.

Solution. 166. In a capillary with diameter d = 0.5 mm water will rise to a height



$$\frac{2\alpha}{\rho g r} = \frac{4\alpha}{\rho g d}$$

$$= \frac{4 \times 73 \times 10^{-3}}{10^3 \times 9.8 \times 0.5 \times 10^{-3}} = 59.6 \text{ mm}$$

Since this is greater than the height (= 25 mm) of the tube, a meniscus of radius R will be formed at the top of the tube, where

$$R = \frac{2\alpha}{\rho g h} = \frac{2 \times 73 \times 10^{-3}}{10^3 \times 9.8 \times 25 \times 10^{-3}} = 0.6 \text{ mm}$$

Q. 167. A glass capillary of length l = 110 mm and inside diameter d = 20 µm is

submerged vertically into water. The upper end of the capillary is sealed. The outside pressure is standard. To what length x has the capillary to be submerged to make the water levels inside and outside the capillary coincide?

Solution. 167. Initially the pressure of air in the capillary is p_0 and it's length is *Z*.

When submerged under water, the pressure of air in the portion above water must

be $p_0 + 4\frac{\alpha}{d}$, since the level of water inside the capillary is the same as the level outside.

Thus by Boyle's law

$$\left(p_0 + \frac{4\alpha}{d}\right)(l - x) = p_0 l$$

or $$\frac{4\alpha}{d}(l - x) = p_0 x \quad \text{or} \quad x = \frac{l}{1 + \frac{p_0 d}{4\alpha}}$$

Q. 168. When a vertical capillary of length l with the sealed upper end was brought in contact with the surface of a liquid, the level of this liquid rose to the height h. The liquid density is p, the inside diameter of the capillary is d, the contact angle is θ, the atmospheric pressure is P_0. Find the surface tension of the liquid.

Solution. 168. We have by Boyle's law

$$\left(p_0 - \rho g h + \frac{4\,\alpha \cos\theta}{d}\right)(l - h) = p_0 l$$

or, $$\frac{4\,\alpha\cos\theta}{d} = \rho g h + \frac{p_0 h}{l - h}$$

Hence, $$\alpha = \left(\rho g h + \frac{p_0 h}{l - h}\right)\frac{d}{4\cos\theta}$$

Q. 169. A glass rod of diameter d_1 = 1.5 mm is inserted symmetrically into a glass capillary with inside diameter d_2 = 2.0 mm. Then the whole arrangement is vertically oriented and brought in contact with the surface of water. To what height will the water rise in the capillary?

Solution. 169. Suppose the liquid rises to a height h. Then the total energy of the liquid in the capillary is

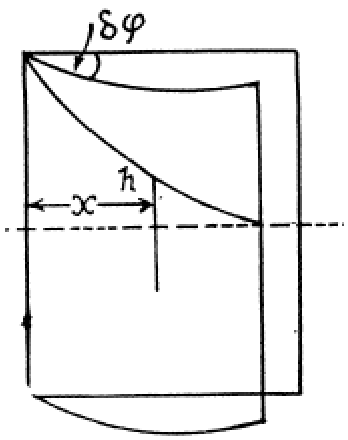
$$E(h) = \frac{\pi}{4}(d_2^2 - d_1^2)\, h \times \rho g \times \frac{h}{2} - \pi(d_2 - d_1)\,\alpha h$$

Minimising E we get

$$h = \frac{4\alpha}{\rho g (d_2 - d_1)} = 6 \text{ cm.}$$

Q. 170. Two vertical plates submerged partially in a wetting liquid form a wedge with a very small angle δφ. The edge of this wedge is vertical. The density of the liquid is p, its surface tension is α, the contact angle is θ. Find the height h, to which the liquid rises, as a function of the distance x from the edge.

Solution. 170. Let h be the height of the water level at a distance x from the edge. Then the total eneigy of water in the wedge above the level outside is.



$$E = \int x\,\delta\varphi \cdot dx \cdot h \cdot \rho g \frac{h}{2} - 2\int dx \cdot h \cdot \alpha \cos 0$$

$$= \int dx \frac{1}{2} x \rho g \delta\varphi \left(h^2 - 2\frac{2\alpha\cos\theta}{x\rho g\delta\varphi} h\right)$$

$$= \int dx \frac{1}{2} x \rho g \delta\varphi \left[\left(h - \frac{2\alpha\cos 0}{x\rho g\delta\varphi}\right)^2 - \frac{4\alpha^2\cos^2 0}{x^2\rho^2 g^2\delta\varphi^2}\right]$$

This is minimum when $h = \frac{2\alpha\cos 0}{x\rho g\delta\varphi}$

Q. 171. A vertical water jet flows out of a round hole. One of the horizontal sections of the jet has the diameter d = 2.0 mm while the other section located l =

20 mm lower has the diameter which is n = 1.5 times less. Find the volume of the water flowing from the hole each second.

Solution. 171. From the equation of continuity

$$\frac{\pi}{4}d^2 \cdot v = \frac{\pi}{4}\left(\frac{d}{n}\right)^2 \cdot V \quad \text{or} \quad V = n^2 v.$$

We then apply Bernoulli's theorem

$$\frac{p}{\rho} + \frac{1}{2}v^2 + \Phi = \text{constant}$$

The pressure p differs from the atmospheric pressure by capillary effects. At the upper section

$$p = p_0 + \frac{2\alpha}{d}$$

Neglecting the curvature in the vertical plane. Thus,

$$\frac{p_0 + \frac{2\alpha}{d}}{\rho} + \frac{1}{2}v^2 + gl = \frac{p_0 + \frac{2n\alpha}{d}}{\rho} + \frac{1}{2}n^4 v^2$$

or $$v = \sqrt{\frac{2gl - \frac{4\alpha}{\rho d}(n-1)}{n^4 - 1}}$$

Finally, the liquid coming out per second is,

$$V = \frac{1}{4}\pi d^2 \sqrt{\frac{2gl - \frac{4\alpha}{\rho d}(n-1)}{n^4 - 1}}$$

Q. 172. A water drop falls in air with a uniform velocity. Find the difference between the curvature radii of the drop's surface at the upper and lower points of the drop separated by the distance h = 2.3 mm.

Solution. 172. The radius of curvature of the drop is R_1 at the upper end of the drop and R_2 at the lower end. Then the pressure inside the drop is $p_0 + \frac{2\alpha}{R_1}$ at the top end and $p_0 + \frac{2\alpha}{R_2}$ at the bottom end. Hence

$$p_0 + \frac{2\alpha}{R_1} = p_0 + \frac{2\alpha}{R_2} + \rho g h \quad \text{or} \quad \frac{2\alpha (R_2 - R_1)}{R_1 R_2} = \rho g h$$

To a first approximation $R_1 \approx R_2 \approx \frac{h}{2}$ so $R_2 - R_1 \approx \frac{1}{8} \rho g h^3 / \alpha. \approx 0{\cdot}20\ \text{mm}$

If $h = 2{\cdot}3\ \text{mm},\ \alpha = 73\ \text{mN/m}$

Liquids Capillary Effects (Part - 2)

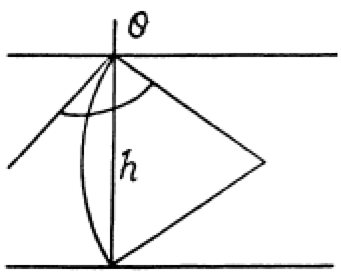
Q. 173. A mercury drop shaped as a round tablet of radius R and thickness h is located between two horizontal glass plates. Assuming that h ≪ R , find the mass m of a weight which has to be placed on the upper plate to diminish the distance between the plates n-times. The contact angle equals θ. Calculate m if R = 2.0 cm, h = 0.38 mm, n = 2.0, and θ = 135°.

Solution. 173. We must first calculate the pressure difference inside the film from that outside. This is

$$p = \alpha\left(\frac{1}{r_1}+\frac{1}{r_2}\right).$$

Here $2 r_1 |\cos\theta| = h$ and $r_2 \sim -R$ the radius of the tablet and can be neglected. Thus the total force exerted by mercury drop on the upper glass plate i

$$\frac{2\pi R^2 \alpha |\cos\theta|}{h} \text{ typically}$$



We should put h / n for h because the tablet is com pressed n times. Then since Hg is nearly, in com pressible, $\pi R^2 h$ = constants so $R \to R\sqrt{n}$. Thus,

$$\text{total force} = \frac{2\pi R^2 \alpha |\cos\theta|}{h} n^2$$

Part of the force is needed to keep the Hg in the shape of a table rather than in the shape of infinitely thin sheet. This part can be calculated being putting n = 1 above. Thus

$$mg + \frac{2\pi R^2 \alpha |\cos\theta|}{h} = \frac{2\pi R^2 \alpha |\cos\theta|}{h} n^2$$

or $$m = \frac{2\pi R^2 \alpha |\cos\theta|}{hg}(n^2 - 1) = 0{\cdot}7 \text{ kg}$$

Q. 174. Find the attraction force between two parallel glass plates, separated by a distance h = 0.10 mm, after a water drop of mass m = 70 mg was introduced between them. The wetting is assumed to be complete.

Solution. 174. The pressure inside the film is less than that outside by an amount

a $\alpha\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$ where r1and r_2 are the principal radii of curvature of the meniscus. One of

these is small being given by $h = 2r_1 \cos\theta$ while the other is large and will be ignored.

Then $F \approx \frac{2A\cos\theta}{h}\alpha$ where A = area of the water film between the plates.

Now $A = \frac{m}{\rho h}$ so $F = \frac{2m\alpha}{\rho h^2}$ when θ (The angle of contact) = 0

Q. 175. Two glass discs of radius R = 5.0 cm were wetted with water and put together so that the thickness of the water layer between them was h = 1.9 p.m. Assuming the wetting to he complete, find the force that has to be applied at right angles to the plates in order to pull them apart.

Solution. 175. This is analogous to the previous problem except that : A = π R^2

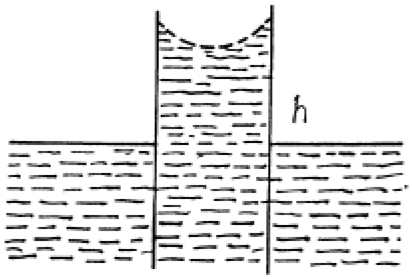
So $$F = \frac{2\pi R^2 \alpha}{h} = 0{\cdot}6 \text{ kN}$$

Q. 176. Two vertical parallel glass plates are partially submerged in water. The distance between the plates is d = 0.10 mm, and their width is l = 12 cm. Assuming that the water between the plates does not reach the upper edges of the plates and that the wetting is complete, find the force of their mutual attraction.

Solution. 176. The energy of the liquid between the plates is

$$E = l d h \rho g \frac{h}{2} - 2\alpha l h = \frac{1}{2}\rho g l d h^2 - 2\alpha l h$$

$$= \frac{1}{2}\rho g l d\left(h - \frac{2\alpha}{\rho g d}\right)^2 - \frac{2\alpha^2 l}{\rho g d}$$



This energy is minimum when, $h = \frac{2\alpha}{\rho g d}$ and the minimum potential energy is

the $E_{min} = -\frac{2\alpha^2 l}{\rho g d}$

The force of attraction between the plates can be obtained from this as

$F = \frac{-\partial E_{min}}{\partial d} = -\frac{2\alpha^2 l}{\rho g d^2}$ (Minus sign means the force is attractive.)

Thus $F = -\frac{\alpha l h}{d} = 13\text{ N}$

Q. 177. Find the lifetime of a soap bubble of radius R connected with the atmosphere through a capillary of length l and inside radius r. The surface tension is α, the viscosity coefficient of the gas is η.

Solution. 177. Suppose the radius of the bubble is x at some instant. Then the pressure

inside is $p_0 + \frac{4\alpha}{x}$. The flow through the capillary is by Poiseuille's equation,

$$Q = \frac{\pi r^4}{8\eta l}\frac{4\alpha}{x} = -4\pi^2\frac{dx}{dt}$$

Integrating $\frac{\pi r^4 \alpha}{2\eta l}t = \pi(R^4 - x^4)$ where we have used the fact that t = 0 where x = R.

This gives $t = \frac{2\eta l R^4}{\alpha r^4}$ as the life time of the bubble corresponding to x = 0

Q. 178. A vertical capillary is brought in contact with the water surface. What

amount of heat is liberated while the water rises along the capillary? The wetting is assumed to be complete, the surface tension equals α.

Solution. 178. If the liquid rises to a height h', the energy of the liquid column becomes

$$E = \rho g \pi r^2 h \cdot \frac{h}{2} - 2\pi r h \alpha = \frac{1}{2}\rho g \pi\left(rh - 2\frac{\alpha}{\rho g}\right)^2 - \frac{2\pi\alpha^2}{\rho g}$$

This is minimum when $rh = \frac{2\alpha}{\rho g}$ and that is relevant height to which water must rise.

At this point, $E_{min} = -\frac{2\pi\alpha^2}{\rho g}$

Since E = 0 in the absence o f surface tension a heat $Q = \frac{2\pi\alpha^2}{\rho g}$ must have been liberated.

Q. 179. Find the free energy of the surface layer of
(a) a mercury droplet of diameter d = 1.4 mm;
(b) a soap bubble of diameter d = 6.0 mm if the surface tension of the soap water solution is equal to α = 45 mN/m.

Solution. 179. (a) The free energy per unit area being α,

$$F = \pi \alpha d^2 = 3\ \mu J$$

(b) $F = 2\pi\alpha d^2$ Because the soap bubble has two surfaces. Substitution gives $F = 10\ \mu J$

Q. 180. Find the increment of the free energy of the surface layer when two identical mercury droplets, each of diameter d = 1.5 mm, merge isothermally.

Solution. 180. When two mercury drops each of diameter d merge, the resulting drop has diameter d_1

Where $\frac{\pi}{6}d_1^3 = \frac{\pi}{6}d^3 \times 2 \quad \text{or,} \quad d_1 = 2^{1/3} d$

The increase in free energy is

$$\Delta F = \pi 2^{2/3} d^2 \alpha - 2\pi d^2 \alpha = 2\pi d^2 \alpha (2^{-1/3} - 1) = -1{\cdot}43\ \mu J$$

Q. 181. Find the work to be performed in order to blow a soap bubble of radius R if the outside air pressure is equal to p_0 and the surface tension of the soap water

solution is equal to α.

Solution. 181. Work must be done to stretch the soap film and compress the air inside. The former is simply $2\alpha \times 4\pi R^2 = 8\pi R^2\alpha,$ here being two sides o f the film. To get the latter we note that the compression is isothermal and work done is

$$-\int_{V_i = V_0}^{V_f = V} p dV \quad \text{where} \quad V_0 p_0 = \left(p_0 + \frac{4\alpha}{R}\right) \cdot V, \; V = \frac{4\pi}{3} R^3$$

or $V_0 = \frac{pV}{p_0}, \; p = p_0 + \frac{4\alpha}{R}$

and minus sign is needed because we are calculating work done on the system. Thus

since pV remains constants, the work done is

$$pV \ln \frac{V_0}{V} = p V \ln \frac{p}{p_0}$$

So $A' = 8\pi R^2 \alpha + p V \ln \frac{p}{p_0}$

Q. 182. A soap bubble of radius r is inflated with an ideal gas. The atmospheric pressure is p_0, the surface tension of the soap water solution is α. Find the difference between the molar heat capacity of the gas during its heating inside the bubble and the molar heat capacity of the gas under constant pressure, C — C_p.

Solution. 182. When heat is given to a soap bubble the temperature of the air inside rises and the bubble expands but unless the bubble bursts, the amount of air inside does not change. Further we shall neglect the variation of the surface tension with temperature. Then from the gas equations

$$\left(p_0 + \frac{4\alpha}{r}\right) \frac{4\pi}{3} r^3 = \nu R T, \; \nu = \text{Constant}$$

Differentiating

$$\left(p_0 + \frac{8\alpha}{3r}\right) 4\pi r^2 dr = \nu R \, d T$$

or

$$dV = 4\pi r^2\, dr = \frac{\nu R\, dT}{p_0 + \frac{8\alpha}{3r}}$$

Now from the first law

$$đQ = \nu C dT = \nu C_V dT + \frac{\nu R\, dT}{p_0 + \frac{8\alpha}{3r}} \cdot \left(p_0 + \frac{4\alpha}{r}\right)$$

or

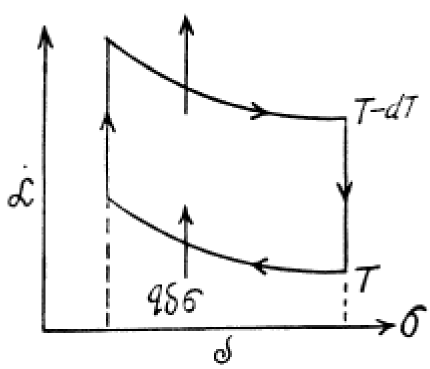
$$C = C_V + R\frac{p_0 + \frac{4\alpha}{r}}{p_0 \frac{8\alpha}{3r}}$$

using

$$C_p = C_V + R\,,\ C = C_p + \frac{\frac{1}{2}R}{1 + \frac{3 p_0 r}{8\alpha}}$$

Q. 183. Considering the Carnot cycle as applied to a liquid film, show that in an isothermal process the amount of heat required for the formation of a unit area of the surface layer is equal to q = —T•dα/dT, where dα/dT is the temperature derivative of the surface tension.

Solution. 183. Consider an infinitesimal Carnot cycle with isotherms at T - dT and T. Let A be the work done during the cycle. Then



$$A = [\alpha(T - dT) - \alpha(T)]\,\delta\sigma = -\frac{d\alpha}{dT} dT\,\delta\sigma$$

Where δa is the change in the area of film (we are considering only one surface).

Then $\eta = \frac{A}{Q_1} = \frac{dT}{T}$ by Carnot therom.

or $\frac{-\frac{d\alpha}{dT} dT \, \delta\sigma}{q \, \delta\sigma} = \frac{dT}{T}$ or $q = -T \frac{d\alpha}{dT}$

Q. 184. The surface of a soap film was increased isothermally by Δα at a temperature T. Knowing the surface tension of the soap water solution α and the temperature coefficient dα/dT, find the increment
(a) of the entropy of the film's surface layer;
(b) of the internal energy of the surface layer.

Solution. 184. As before we can calculate the heat required. It, is taking into account two sides of the soap film

$$\delta q = -T \frac{d\alpha}{dT} \delta\sigma \times 2$$

Thus $\Delta S = \frac{\delta q}{T} = -2 \frac{d\alpha}{dT} \delta\sigma$

Now $\Delta F = 2\alpha\,\delta\sigma$ so, $\Delta U = \Delta F + T\Delta S = 2\left(\alpha - T\frac{d\alpha}{dT}\right)\delta\sigma$