Arithmetic Progressions

The Concept Of Arithmetic Progression

Any given series of numbers may exhibit some special properties. A sequence of numbers in arithmetic progression is one of these special series.

Let us look at the following example.

Example 1: Check whether the following sequences form an A.P. or not.

(i) $5\sqrt{2}-7\sqrt{5},\ 6\sqrt{2}-6\sqrt{5},\ 7\sqrt{2}-5\sqrt{5},\ 8\sqrt{2}-4\sqrt{5}$ ...

(ii) $\dfrac{1}{3},\ \dfrac{5}{6},\ \dfrac{4}{3},\ \dfrac{11}{6}$ ...

(iii) $\dfrac{1}{1\times 2},\ \dfrac{1}{2\times 3},\ \dfrac{1}{3\times 4},\ \dfrac{1}{4\times 5}$ ...

(iv) A list of prime numbers greater than 2

(v) A list of the squares of natural numbers

Solution:

(i) The given sequence is $5\sqrt{2}-7\sqrt{5},\ 6\sqrt{2}-6\sqrt{5},\ 7\sqrt{2}-5\sqrt{5},\ 8\sqrt{2}-4\sqrt{5}$ ...

Difference between the second and the first term $=\left(6\sqrt{2}-6\sqrt{5}\right)-\left(5\sqrt{2}-7\sqrt{5}\right)$

$=\sqrt{2}+\sqrt{5}$

Difference between the third and the second term $=\left(7\sqrt{2}-5\sqrt{5}\right)-\left(6\sqrt{2}-6\sqrt{5}\right)$

$=\sqrt{2}+\sqrt{5}$

Difference between the fourth and the third term $=\left(8\sqrt{2}-4\sqrt{5}\right)-\left(7\sqrt{2}-5\sqrt{5}\right)$

$=\sqrt{2}+\sqrt{5}$

Since the difference between any two consecutive terms is a constant, the sequence is an arithmetic progression.

(ii) The given sequence is $\dfrac{1}{3}, \dfrac{5}{6}, \dfrac{4}{3}, \dfrac{11}{6} \ldots$

Difference between the second and the first term $= \dfrac{5}{6} - \dfrac{1}{3} = \dfrac{1}{2}$

Difference between the third and the second term $= \dfrac{4}{3} - \dfrac{5}{6} = \dfrac{1}{2}$

Difference between the fourth and the third term $= \dfrac{11}{6} - \dfrac{4}{3} = \dfrac{1}{2}$

Since the difference between any two consecutive terms is a constant, the sequence is an arithmetic progression.

(iii) The given sequence is $\dfrac{1}{1\times2}, \dfrac{1}{2\times3}, \dfrac{1}{3\times4}, \dfrac{1}{4\times5} \ldots$

This sequence can be rewritten as $\dfrac{1}{2}, \dfrac{1}{6}, \dfrac{1}{12}, \dfrac{1}{20} \ldots$

Difference between the second and the first term $= \dfrac{1}{6} - \dfrac{1}{2} = \dfrac{-1}{3}$

Difference between the third and the second term $= \dfrac{1}{12} - \dfrac{1}{6} = \dfrac{-1}{12}$

Difference between the fourth and the third term $= \dfrac{1}{20} - \dfrac{1}{12} = \dfrac{-1}{30}$

Since the difference between the consecutive terms is not a constant, the sequence is **not** an arithmetic progression.

(iv) The list of the prime numbers greater than 2 is: 3, 5, 7, 11 …

Difference between the second and the first term = 5 – 3 = 2

Difference between the third and the second term = 7 – 5 = 2

Difference between the fourth and the third term = 11 – 7 = 4

Since the difference between the consecutive terms is not a constant, the sequence is **not** an arithmetic progression.

(v) The list of the squares of natural numbers is: $1^2, 2^2, 3^2, 4^2$ ... = 1, 4, 9, 16 ...

Difference between the second and the first term = 4 – 1 = 3

Difference between the third and the second term = 9 – 4 = 5

Difference between the fourth and the third term = 16 – 9 = 7

Since the difference between the consecutive terms is not a constant, the sequence is **not** an arithmetic progression.

The Terminologies Related To Arithmetic Progressions

An arithmetic progression (A.P.) is a sequence in which the difference between any two consecutive terms is constant.

Now there are many common terms associated with an arithmetic progression, which one has to understand to solve any A.P. related problem.

Let us solve some more examples to understand this concept better.

Example 1: Find the common difference of the A.P.
$2+\sqrt{2},\ 2+2\sqrt{2},\ 2+3\sqrt{2},\ 2+4\sqrt{2}$ ... **Also, state whether it is finite or infinite and write down its first term.**

Solution:

The given A.P. is $2+\sqrt{2},\ 2+2\sqrt{2},\ 2+3\sqrt{2},\ 2+4\sqrt{2}$ ...

$a_1 = 2+\sqrt{2}$

$a_2 = 2+2\sqrt{2}$

$a_3 = 2+3\sqrt{2}$

$a_4 = 2+4\sqrt{2}$

$$a_2 - a_1 = \left(2 + 2\sqrt{2}\right) - \left(2 + \sqrt{2}\right) = \sqrt{2}$$
$$a_3 - a_2 = \left(2 + 3\sqrt{2}\right) - \left(2 + 2\sqrt{2}\right) = \sqrt{2}$$

∴Common difference of the A.P. $= d = \sqrt{2}$

First term of the A.P. $= a_1 = 2 + \sqrt{2}$

The given A.P. is infinite because its last term cannot be calculated.

Example 2: State whether the following statements are true or false.

(i) The sequence $\sqrt{5} - \sqrt{6}, -2\sqrt{6}, -\sqrt{5} - 3\sqrt{6}, -2\sqrt{5} - 4\sqrt{6}$ ... **forms an A.P. with the first term as** $-\sqrt{5} - \sqrt{6}$ **and the common difference as** $\sqrt{5} - \sqrt{6}$.

(ii) The common difference of the A.P. $\dfrac{1}{3}, \dfrac{5}{6}, \dfrac{4}{3}, \dfrac{11}{6}$ ... **is** $\dfrac{1}{2}$.

(iii) The nth term of an A.P. is given by $3n - 4$. Its common difference is 4.

Solution:

(i) The given sequence is $\sqrt{5} - \sqrt{6}, \ -2\sqrt{6}, \ -\sqrt{5} - 3\sqrt{6}, \ -2\sqrt{5} - 4\sqrt{6}$ ...

Difference between the first and the second term $= -2\sqrt{6} - \left(\sqrt{5} - \sqrt{6}\right) = -\sqrt{5} - \sqrt{6}$

Difference between the second and the third term $= -\sqrt{5} - 3\sqrt{6} - \left(-2\sqrt{6}\right) = -\sqrt{5} - \sqrt{6}$

Difference between the third and the fourth term $= -2\sqrt{5} - 4\sqrt{6} - \left(-\sqrt{5} - 3\sqrt{6}\right) = -\sqrt{5} - \sqrt{6}$

Since the difference between the consecutive terms of the sequence is constant, the given sequence is an A.P. with the first term as $\sqrt{5} - \sqrt{6}$ and the common difference as $-\sqrt{5} - \sqrt{6}$.

Thus, the statement is **false**.

(ii) The given A.P. is $\dfrac{1}{3}, \dfrac{5}{6}, \dfrac{4}{3}, \dfrac{11}{6}$ ...

Common difference of the A.P. = $\dfrac{5}{6} - \dfrac{1}{3} = \dfrac{4}{3} - \dfrac{5}{6} = \dfrac{11}{6} - \dfrac{4}{3} = \dfrac{1}{2}$

Thus, the statement is **true**.

(iii) It is given that the nth term of the A.P. is given by $3n - 4$.

Thus, first term of the A.P. = $a = 3 \times 1 - 4 = 3 - 4 = -1$

Second term of the A.P. = $a_2 = 3 \times 2 - 4 = 6 - 4 = 2$

Third term of the A.P. = $a_3 = 3 \times 3 - 4 = 9 - 4 = 5$

Fourth term of the A.P. = $a_4 = 3 \times 4 - 4 = 12 - 4 = 8$

Thus, common difference of the A.P. = $8 - 5 = 5 - 2 = 2 - (-1) = 3$

Thus, the statement is **false**.

Missing Terms In Arithmetic Progressions

An arithmetic progression (A.P.) is a sequence in which the difference between any two consecutive terms is constant.

Now if we are given an A.P. 3, 6, 9, ?, 15 and asked to find the missing term, then how will we go about it?

Let us now solve some problems based on this concept.

Example 1: Find the missing terms in the following arithmetic progressions.

(i) $\dfrac{1}{a}, \dfrac{3}{a}, \dfrac{\square}{\square}, \dfrac{7}{a}, \ldots$

(ii) $-\dfrac{11}{6}, \dfrac{\square}{\square}, \dfrac{13}{6}, \dfrac{25}{6}$

Solution:

(i) In the given A.P.,

$$a_1 = \frac{1}{a}, a_2 = \frac{3}{a}$$

$$\therefore d = a_2 - a_1 = \frac{3}{a} - \frac{1}{a} = \frac{2}{a}$$

$$\therefore a_3 = a_2 + d$$

$$= \frac{3}{a} + \frac{2}{a}$$

$$= \frac{5}{a}$$

Thus, missing term, $a_3 = \dfrac{5}{a}$

(ii) In the given A.P.,

$$a_1 = -\frac{11}{6}, \ a_3 = \frac{13}{6}, \ a_4 = \frac{25}{6}$$

$$\therefore d = a_4 - a_3$$

$$= \frac{25}{6} - \frac{13}{6}$$

$$= \frac{12}{6}$$

$$= 2$$

$$\therefore a_2 = a_1 + d$$

$$= -\frac{11}{6} + 2$$

$$= \frac{1}{6}$$

Thus, missing term, $a_2 = \dfrac{1}{6}$

Example 2: Find the first two terms of an A.P. in which the third term is 10 and the common difference is 7.

Solution:

Here, $d = 7$ and $a_3 = 10$

Hence, the value of a_2 can be found by subtracting 7 from a_3.

$\therefore a_2 = 10 - 7 = 3$

Similarly, the value of a_1 can be found by subtracting 7 from a_2.

$\therefore a_1 = 3 - 7 = -4$

Thus, the first two terms of the A.P. are −4 and 3.

nth Term Of An Arithmetic Progression

We know what an arithmetic progression (A.P.) is. Also, we have learnt that there is a common difference between any two consecutive terms of an A.P..

Now, can we find the required term of a given A.P. with this information?

Let us consider the A.P. 3, 7, 11, 15,...

Here, first term (a) = 3 and common difference (d) = 4

Now, if we want to find the 5th term of this A.P., then we will simply add the common difference to 4th term. Thus, 5th term of this A.P. will be 15 + 4 = 19.

What would we do if we are asked to find the 20th term or 100th term or n^{th} term?

Obviously, the process of adding common difference will be very time consuming.

For such problems, we must have a short cut or a formula to find the general term of an A.P.

Let us derive the same.

Consider the A.P. $a, a + d, a + 2d, a + 3d, ...$

For this A.P., we have

$a_1 = a$

$a_2 - a_1 = d$

$a_3 - a_2 = d$

.   .   .

.   .   .

.   .   .

$$a_{n-1} - a_{n-2} = d$$

$$a_n - a_{n-1} = d$$

Adding all these equations, we get

$$a_1 + (a_2 - a_1) + (a_3 - a_2) + ... + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}) = a + \{d + d + ... + d \ (n-1 \ \text{times})\}$$

$$\Rightarrow (a_1 - a_1) + (a_2 - a_2) + (a_3 - a_3) + ... + (a_{n-1} - a_{n-1}) + a_n = a + (n-1)d$$

$$\Rightarrow a_n = a + (n-1)d$$

Hence, the general term or n^{th} term i.e., a_n of an A.P. whose first term is a and common difference is d can be found by the following formula:

$$\boldsymbol{a_n = a + (n - 1)d}$$

Sometimes, we need to find three, four or five consecutive terms of an A.P. then it is convenient to take them as follows:

- Three consecutive terms can be taken as $a - d, a, a + d$
- Four consecutive terms can be taken as $a - 3d, a - d, a + d, a + 3d$. Here, common difference is $2d$.
- Five consecutive terms can be taken as $a - 2d, a - d, a, a + d, a + 2d$

Result:

In an A.P., common difference, $d = \dfrac{T_p - T_q}{p - q}$, where T_p and T_q are p^{th} and q^{th} term respectively.

In particular, $d = \dfrac{T_n - a}{n - 1}$.

Proof:

$$T_p = a + (p - 1)d, \ T_q = a + (q - 1)d$$

$$\Rightarrow T_p - T_q = a + (p - 1)d - \{a + (q - 1)d\} = (p - q)d$$

$$\Rightarrow d = \dfrac{T_p - T_q}{p - q}$$

In particular, take $T_1 = a$.

Then, $d = \dfrac{T_n - a}{n - 1}$

Now, let us solve some examples to understand the concept better.

Example 1: Find the 20th term of the following arithmetic progression.

0.4, 1.5, 2.6, 3.7, 4.8 …

Solution:

Here, $a = 0.4$ and $d = 1.5 - 0.4 = 1.1$

Thus, the 20th term is given by,

$a_{20} = a + (20 - 1)\, d$

$\qquad = 0.4 + (20 - 1)\, 1.1$

$\qquad = 0.4 + 19 \times 1.1$

$\qquad = 0.4 + 20.9$

$\qquad = 21.3$

Thus, the 20th term of the given A.P. is 21.3.

Example 2: If the 7th term of an A.P. is – 21 and 15th term is – 53, then find the first term and common difference.

Solution:

Let the first term and common difference of the A.P. be a and d respectively.

It is given that $a_7 = -21$ and $a_{15} = -53$

Using the formula for nth term, we obtain

$a_7 = a + (7 - 1)\, d$

$-21 = a + 6d \,…\, (1)$

and, $a_{15} = a + (15 - 1)\, d$

– 53 = a + 14d ... (2)

Subtracting equation (1) from (2), we obtain

–32 = 8d

$\Rightarrow d = -4$

Substituting the value of d in equation (1), we obtain

–21 = a + 6 (–4)

–21 = a – 24

$\Rightarrow a = 3$

Thus, the first term is 3 and the common difference is –4.

Example 3: Is 102 a term of the A.P., 5, 11, 17, 23 ...?

Solution:

Let 102 be the nth term of the given sequence.

$\therefore a_n = 102$

Using the formula for n^{th} term, we obtain

$a_n = a + (n - 1) d$

$\therefore 102 = a + (n - 1) d$

For the given A.P., $a = 5$ and $d = 11 - 5 = 6$

$\therefore 102 = 5 + (n - 1) 6$

$102 - 5 = (n - 1) 6$

$97 = (n - 1) 6$

$n - 1 = \dfrac{97}{6}$

$$n = \frac{97}{6} + 1$$

$$n = \frac{103}{6}$$

However, n should be a positive integer. Therefore, 102 is not a term of the given A.P.

Example 4: Find the number of three-digit numbers that are divisible by 5.

Solution:

The first three-digit number which is divisible by 5 is 100, second is 105, third is 110, and so on. The last three-digit number which is divisible by 5 is 995.

Thus, we obtain the following A.P.

100, 105 … 995

Here, we have to find the number of terms, n.

∴Last term of A.P. = 995

The number of terms in the A.P. is n, so the last term is the nth term.

$a + (n - 1)d = 995$

Here, $a = 100$ and $d = 5$

$100 + (n - 1)5 = 995$

$(n - 1)5 = 995 - 100$

$5n - 5 = 895$

$5n = 895 + 5$

$5n = 900$

$$n = \frac{900}{5}$$

$n = 180$

Thus, there are 180 three-digit numbers, which are divisible by 5.

Example 5: The fare of a bus is Rs 10 for the first kilometre and Rs 5 for each additional kilometre. Find the fair after 12 kilometres.

Solution:

The fare after each kilometre forms an A.P. as follows.

Fare after one kilometre = Rs 10

Fare after two kilometres = 10 + 5 = Rs 15

Fare after three kilometres = 15 + 5 = Rs 20

Now the arithmetic progression is 10, 15, 20 …

Here, first term, a = 10 and common difference, d = 5

Now the fare after 12 kilometres is the 12th term of the A.P.

$\therefore a_{12} = a + (12 - 1)\, d$

$a_{12} = 10 + 11 \times 5$

$= 10 + 55$

$= 65$

Thus, the fare after 12 kilometres is Rs 65.

Example 6: Mohit borrowed a sum of money at a simple interest rate of 2% per annum. He has to pay an amount of Rs 1120 after 6 years. How much money did he borrow?

Solution:

Let the amount of money Mohit borrowed be Rs x. We know that the amount after T years is

$$A = P + \frac{P \times R \times T}{100}$$

Where, P and R denotes the principal and rate respectively

The amount after every year forms an A.P.

Amount after first year $= x + \dfrac{x \times 2 \times 1}{100}$

$= x + \dfrac{2x}{100}$

Amount after second year $= x + \dfrac{x \times 2 \times 2}{100}$

$= x + \dfrac{4x}{100}$

Thus, the A.P. is as follows.

$x + \dfrac{2x}{100}, x + \dfrac{4x}{100} \ldots$

Here, the first term is $x + \dfrac{2x}{100}$ and common difference is $\dfrac{2x}{100}$.

Now, it is given that the amount after 6 years is Rs 1120 i.e., 6th term of the A.P. is 1120.

Now using the formula, $a_n = a + (n - 1)d$, we obtain

$1120 = x + \dfrac{2x}{100} + (6 - 1)\dfrac{2x}{100}$

$1120 = x + \dfrac{2x}{100} + 5 \times \dfrac{2x}{100}$

$1120 = x + \dfrac{2x}{100} + \dfrac{10x}{100}$

$1120 = \dfrac{100x + 2x + 10x}{100}$

$1120 \times 100 = 112x$

$x = 1000$

Thus, Mohit borrowed Rs 1000.

Example 7: Find four consecutive terms of an A.P. such that the difference of the middle terms is 8 and the product of the extreme terms is 217.

Solution:

Let four consecutive terms of required A.P. be $a - 3d, a - d, a + d, a + 3d$.

According to the question, we have

$a + d - (a - d) = 8$

$\Rightarrow a + d - a + d = 8$

$\Rightarrow 2d = 8$

$\Rightarrow d = 4$

Also,

$(a - 3d)(a + 3d) = 217$

$\Rightarrow a^2 - (3d)^2 = 217$

$\Rightarrow a^2 - 9d^2 = 217$

$\Rightarrow a^2 - 9(4^2) = 217$

$\Rightarrow a^2 - 144 = 217$

$\Rightarrow a^2 = 361$

$\Rightarrow a = \pm 19$

When $d = 4$ and $a = 19$, then four consecutive terms are:

$19 - 3(4), (19 - 4), (19 + 4), 19 + 3(4)$

i.e., 7, 15, 23, 31

When $d = 4$ and $a = -19$, then four consecutive terms are:

$-19 - 3(4), (-19 - 4), (-19 + 4), -19 + 3(4)$

i.e., −31, −23, −15, −7

Example 8: Find five consecutive terms of an A.P. such that the product of the extreme terms is –63 and product of second and fourth terms is –15.

Solution:

Let five consecutive terms of required A.P. be $a - 2d, a - d, a, a + d, a + 2d$.

According to the question, we have

$(a - 2d)(a + 2d) = -63$

$\Rightarrow a^2 - (2d)^2 = -63$

$\Rightarrow a^2 - 4d^2 = -63$ ...(1)

Also,

$(a - d)(a + d) = -15$

$\Rightarrow a^2 - d^2 = -15$ ...(2)

On subtracting (1) from (2), we get

$3d^2 = 48$

$\Rightarrow d^2 = 16$

$\Rightarrow d = \pm 4$

On substituting $d^2 = 16$ in (2), we get

$a^2 - 16 = -15$

$\Rightarrow a^2 = 1$

$\Rightarrow a = \pm 1$

When $a = 1$ and $d = 4$, A.P. will be –7, –3, 1, 5, 9

When $a = 1$ and $d = -4$, A.P. will be 9, 5, 1, –3, –7

When $a = -1$ and $d = 4$, A.P. will be –9, –5, –1, 3, 7

When $a = -1$ and $d = -4$, A.P. will be 7, 3, –1, –5, –9

Example 9: If the 25th and 35th terms of an arithmetic progression are 121 and 171 respectively, then find the common difference of the A.P.

Answer:

$T_{25} = 121$, $T_{35} = 171$

It is known that common difference, $d = \dfrac{T_p - T_q}{p - q}$.

$\therefore d = \dfrac{T_{35} - T_{25}}{35 - 25} = \dfrac{171 - 121}{10} = 5$

Example 10: In an A.P, show that $d = \dfrac{T_{200} - T_{100}}{100}$.

Answer:

$T_{100} = a + (100 - 1)d = a + 99d$

$T_{200} = a + (200 - 1)d = a + 199d$

$\quad = a + 99d + 100d$

$\quad = T_{100} + 100d$

Therefore, $d = \dfrac{T_{200} - T_{100}}{100}$

Sum of n Terms of an Arithmetic Progression

We know what an arithmetic progression (A.P.) is. Sometimes, we may come across the situations when we have to find the sum of all terms involved in a series and if the series is an AP, then there is a formula which can make the process very simple.

Let us consider a similar situation.

Harry saved Rs 2000 from his salary in the first month. He increases his savings by Rs 50 every month.

Can we calculate his total savings for the first four months?

Let us try to find it.

To find the total savings for the first 4 months, we have to take the sum of the savings for the first four months.

It is given that, savings of Harry for the first month = Rs 2000

Every month, he increases his savings by Rs 50.

Thus, savings for second month = Rs (2000 + 50) = Rs 2050

Similarly, savings for the third month = Rs (2050 + 50) = Rs 2100

nd, savings for the fourth month = Rs (2100 + 50) = Rs 2150

Thus, the total savings of Harry for the first four months = Rs (2000 + 2050 + 2100 + 2150)

= Rs 8300

Now, can we calculate the total savings of Harry for 2 years?

Yes, we can find it as above but it is a very lengthy as well as time consuming process as we have to find the savings for 24 months.

We can also find the total savings of Harry for first two years using a formula. Now, let us see how we can find it.

The savings of Harry for each month forms an A.P., which is as follows.

2000, 2050, 2100, 2150 ….

The sum of savings of Harry = Rs (2000 + 2050 + 2100 + 2150 ….)

Here, we can observe that the total savings of Harry for the first month is the first term of the A.P., i.e. Rs 2000. The total savings for the first two months is the sum of first two terms of the A.P., i.e. Rs (2000 + 2050). In the same way, the total savings of Harry for first 2 years, i.e. 24 months, is the sum of first 24 terms of the A.P. We can find it by using the formula for finding the sum of n terms of an A.P.

Now, let us find the sum of first 24 terms of the above discussed A.P. which is as follows:

2000, 2050, 2100, 2150 ….

Here, first term, a = 2000

Common difference, d = 2050 − 2000 = 50

The sum of first 24 terms of the A.P. is

$$S_{24} = \frac{24}{2}\left[2 \times 2000 + (24-1)50\right]$$
$$= 12(4000 + 23 \times 50)$$
$$= 12(4000 + 1150)$$
$$= 12 \times 5150$$
$$= 61800$$

Therefore, the total savings of Harry for the first two years is Rs 61800.

This formula is used when we are given the first term and the common difference of the arithmetic progression.

We can also find the sum of n terms of an A.P., if we know the first and the last term.

The sum of n terms of an A.P. whose first term is a and last term is l is given by the formula:
$$S_n = \frac{n}{2}\{a + l\}$$

For example, consider an A.P. whose first term is 2 and 30th term is 263. Then, what will be the sum of 30 terms?

Here, $a = 2$, $l = 263$ and $n = 30$

Therefore, sum of 30 terms $= \frac{n}{2}[a + l]$

$$= \frac{30}{2}[2 + 263]$$

$= 15 \times 265$

$= 3975$

Thus, the sum of 30 terms is 3975.

Result: The sum of the first n natural numbers is given by $\frac{n(n+1)}{2}$.

Proof:

This can be proved by two methods.

1st method (Using concept of A.P.):

The first n natural numbers can be listed as 1, 2, 3, ..., n.

Here, $a = 1$, $d = 1$.

$$\therefore S_n = \frac{n}{2}\left[2a+(n-1)d\right] = \frac{n}{2}\left[2\times1+(n-1)1\right] = \frac{n(n+1)}{2}$$

2nd method (Without using concept of A.P.):

$S_n = 1 + 2 + 3 + ... + n$ ... (1)

$S_n = n + (n - 1) + (n - 2) + ... + 3 + 2 + 1$ ... (2)

Adding (1) and (2):

$2S_n = (n + 1) + (n + 1) + ... + (n + 1)$

Here, there are n terms in the RHS.

$$\therefore S_n = \frac{n(n+1)}{2}$$

Now, let us discuss some more examples based on sum of n terms of an A.P.

Example 1: Find the sum of first 25 terms of the following A.P.

2, 7, 12 ...

Solution:

Here, $a = 2$ and $d = 7 - 2 = 5$.

Sum of the first 25 terms is given by

$$S_{25} = \frac{25}{2}\left[2a + (25-1)d\right]$$

$$= \frac{25}{2}\left[2 \times 2 + (25-1)5\right]$$

$$= \frac{25}{2}\left[4 + 24 \times 5\right]$$

$$= \frac{25}{2}\left[4 + 120\right]$$

$$= \frac{25}{2}\left[124\right]$$

= 25 × 62

= 1550

Thus, the sum of first 25 terms of the given A.P. is 1550.

Example 2: Find the sum of first 8 terms of the A.P whose nth term is given by $6n - 5$.

Solution:

The nth term is given by

$a_n = 6n - 5$

On replacing n by 1, 2, 3 ... respectively, we get the first, second, third ... terms of the A.P.

∴ $a_1 = 6(1) - 5 = 1$

$a_2 = 6(2) - 5 = 7$

$a_3 = 6(3) - 5 = 13$ ... and so on.

The A.P. so obtained is as follows.

1, 7, 13 ...

Here, the first term, $a = 1$

and the common difference $d = a_2 - a_1 = 7 - 1 = 6$

Using the formula, $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$, the sum of first 8 terms is given by

$$S_8 = \frac{8}{2}\left[2 \times 1 + (8-1)6\right]$$
$$= 4\left[2 + 7 \times 6\right]$$
$$= 4\left[2 + 42\right]$$
$$= 4\left[44\right]$$

= 176

Thus, the sum of first 8 terms is 176.

Example 3: How many terms of the A.P. −28, −24, −20 ... should be taken so that the sum will be zero?

Solution:

Let the sum of n terms be zero.

Here, $a = -28$ and $d = -24 - (-28) = 4$

Sum of n terms of an A.P. is given by

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

But, it is given that the sum of n terms of the given A.P. is zero.

$$S_n = 0$$
$$\frac{n}{2}\left[2a + (n-1)d\right] = 0$$
$$\frac{n}{2}\left[2 \times (-28) + (n-1) \times 4\right] = 0$$
$$-56 + 4n - 4 = 0$$
$$4n - 60 = 0$$
$$4n = 60$$
$$n = \frac{60}{4}$$
$$n = 15$$

Thus, the sum of 15 terms of the A.P is zero.

Example 4: Sapna's father planted 4 trees in his garden, when he was 22 years old. After that, every year he planted one more tree than the number of trees he planted in the previous year. How many trees will be there in his garden when he will become 40 years old?

Solution:

We can write the given information in the form of an A.P. as follows

Number of trees he planted in the first year = 4

Number of trees he planted in the second year = 4 + 1 = 5

Number of trees he planted in the third year = 5 + 1 = 6

And so on.

Now, the A.P. is 4, 5, 6 …

He planted trees from the age of 22 years to 40 years, i.e. for 19 years.

Thus, we have to find the sum of 19 terms of this A.P.

Here, $a = 4$, $d = 1$

And, $n = 19$

Using the formula, $S_n = \dfrac{n}{2}\left[2a+(n-1)d\right]$, the sum of 19 terms of this A.P. is given by

$$S_{19} = \frac{19}{2}\left[2 \times 4 + (19 - 1)\,1\right]$$
$$= \frac{19}{2} \times \left[8 + 18\right]$$
$$= \frac{19}{2} \times 26 = 19 \times 13$$
$$= 247$$

Thus, there will be 247 trees in his garden when he will become 40 years old.

Example 5: The pth term of an A.P. is q and the qth term of the A.P. is p. What is the sum of $(p + q)$ terms of the A.P.?

Solution:

Let the first term and the common difference of the A.P. be *a* and *d* respectively.

It is given that the pth term is q.

$\therefore a_p = q$

$\Rightarrow a + (p - 1) d = q$ ... (1)

Similarly, the qth term is *p*, therefore we obtain

$a + (q - 1) d = p$ ... (2)

On subtracting equation (1) from (2), we obtain

$p - q = (q - 1) d - (p - 1) d$

$p - q = d [q - 1 - p + 1]$

$d = \dfrac{p-q}{q-p}$

$d = -1$

By putting the value of *d* in equation (1), we obtain

$q = a + (p - 1) (-1)$

$q = a + 1 - p$

$a = p + q - 1$

But we know that the sum of *n* terms of an A.P. is

$S_n = \dfrac{n}{2}\left[2a + (n-1)d\right]$

Thus, the sum of (*p* + *q*) terms is

$$S_{p+q} = \frac{p+q}{2}\left[2(p+q-1)+(p+q-1)(-1)\right]$$

$$S_{p+q} = \frac{p+q}{2}\left[2p+2q-2-p-q+1\right]$$

$$S_{p+q} = \frac{(p+q)}{2}\left[p+q-1\right]$$

$$S_{p+q} = \frac{(p+q)(p+q-1)}{2}$$

Thus, the sum of $(p + q)$ terms is $\dfrac{(p+q)(p+q-1)}{2}$.

Example 6: Find the nth term of the A.P., the sum of whose n terms is $n^2 + 2n$.

Solution:

Let S_n be the sum of n terms.

It is given that the sum of n terms of the A.P. is $n^2 + 2n$.

$\therefore S_n = n^2 + 2n$ ... (1)

On replacing n by $(n - 1)$ in the equation, we obtain

$$S_{n-1} = (n-1)^2 + 2(n-1)$$

Let a_n be the nth term of the A.P.

Therefore, we can write

$S_n = S_{n-1} + a_n$

Thus, $a_n = S_n - S_{n-1}$

$= n^2 + 2n - [(n - 1)^2 + 2(n - 1)]$

$= n^2 + 2n - [n^2 + 1 - 2n + 2n - 2]$

$= n^2 + 2n - n^2 + 1$

$= 2n + 1$

Thus, the nth term of the A.P is $(2n + 1)$.

Example 7: Find the sum of first 1000 natural numbers.
Solution:

The sum of first n natural number is given by $S_n = \dfrac{n(n+1)}{2}$.

$$\therefore S_{1000} = \dfrac{1000 \times (1000+1)}{2} = 500 \times 1001 = 500500$$

Hence, the sum of first 1000 natural numbers is 500500.

Example 8: If sum of the first n natural numbers is 5050, find the value of n.
Solution:

The sum of first n natural number is given by $S_n = \dfrac{n(n+1)}{2}$.

Now,

$$5050 = \dfrac{n \times (n+1)}{2}$$
$$n^2 + n - 2 \times 5050 = 0$$
$$n^2 - 100n + 101n - 2 \times 5050 = 0$$
$$(n-100)(n+101)$$
$$n = 100$$

Hence, $n = 100$

Properties of Arithmetic Progressions and the Concept of Arithmetic Mean

Arithmetic progression is a sequence of numbers such that the difference between the consecutive terms is a constant. It exhibits some properties which are used in solving various problems.

Properties of an Arithmetic Progression

- If a constant is added to each term of an A.P., the resulting sequence will also be an A.P.
- If a constant is subtracted from each term of an A.P., the resulting sequence will also be an A.P.
- If a constant is multiplied to each term of an A.P., the resulting sequence will also be an A.P.
- If each term of an A.P. is divided by a non-zero constant, the resulting sequence is also an A.P.

Arithmetic Mean

- If we are given two numbers *a* and *b*, then we can insert a number *A* between these two numbers so that the sequence *a*, *A*, *b* becomes an A.P. Such a number i.e., *A* is called an **arithmetic mean (A.M.)** of the numbers *a* and *b*.

- If *A* is the A.M. of the numbers *a* and *b*, then *A* is given by $A = \dfrac{a+b}{2}$.

  For example, the A.M. of the two numbers 18 and 16 is $\dfrac{18+16}{2} = 17$.

- For any two given numbers *a* and *b*, we can insert as many numbers between them as we want such that the resulting sequence becomes an A.P.

  Example 1: Between −12 and 40, *p* numbers have been inserted in such a way that the resulting sequence is an A.P. Find the value of *p* if the ratio of the 4th and the (*p* − 3)th number is 1 : 6.

  Solution:

  Let $A_1, A_2, \ldots A_p$ be *p* numbers such that −12, $A_1, A_2, \ldots A_p$, 40 is an A.P.

  Here, *a* = −12, *b* = 40, *n* = *p* + 2

  $\therefore 40 = -12 + (p + 2 - 1)\,(d)$

  $\Rightarrow 52 = (p + 1)\,d$

  $\Rightarrow d = \dfrac{52}{p+1}$ \qquad ...(1)

  $A_1 = a + d$

  $A_2 = a + 2d$

  $A_3 = a + 3d \ldots$

  $\therefore A_4 = a + 4d$

  $A_{p-3} = a + (p - 3)\,d$

  According to the given information,

$$\frac{a+4d}{a+(p-3)d}=\frac{1}{6}$$

$$\Rightarrow \frac{-12+4\left(\dfrac{52}{p+1}\right)}{-12+(p-3)\left(\dfrac{52}{p+1}\right)}=\frac{1}{6} \qquad \text{[From (1)]}$$

$$\Rightarrow \frac{-12(p+1)+4(52)}{-12(p+1)+52(p-3)}=\frac{1}{6}$$

$$\Rightarrow \frac{-12p-12+208}{-12p-12+52p-156}=\frac{1}{6}$$

$$\Rightarrow \frac{-12p+196}{40p-168}=\frac{1}{6}$$

$$\Rightarrow 6(-12p+196)=40p-168$$

$$\Rightarrow -72p+1176=40p-168$$

$$\Rightarrow 40p-168+72p-1176=0$$

$$\Rightarrow 112p=1344$$

$$\Rightarrow p=\frac{1344}{112}$$

$$\Rightarrow p=12$$

Thus, the value of p is 12.

Example 2: Between 10 and 30, m numbers are inserted such that the resulting sequence is an A.P. If the sum of all the terms of the A.P. is 140, then find the value of m.

Solution:

It is given that between 10 and 30, m numbers are inserted such that the resulting sequence is an A.P.

It is also given that the sum of all the terms in the A.P. is 140.

We know that the sum of n terms of an A.P. is given by

$$S_n=\frac{n}{2}[a+l]$$
, where a is the first term and l is the last term

Here, $a = 10, l = 30$

Therefore,

$$140 = \frac{n}{2}[10+30]$$

$$\Rightarrow 140 = \frac{n}{2}[40]$$

$$\Rightarrow 140 = 20n$$

$$\Rightarrow n = 7$$

Thus, the total number of terms in the A.P. is 7.

Thus, the value of m is 7 − 2 = 5.