Fill Ups of Permutations and Combinations

Q.1. In a certain test, a_i students gave wrong answers to atleast i questions, where i = 1, 2, …, k. No student gave more than k wrong answers. The total number of wrong answers given is .... (1982 - 2 Marks)

Ans. Sol. Number of students who gave wrong answers to exactly one question = $a_1 - a_2$, Two questions = $a_2 - a_3$

Three questions = $a_3 - a_4$, k–1 question = $a_{k-1} - a_k$, k question = a_k

∴ Total number of wrong answers

$= 1 \ (a_1 - a_2) + 2\ (a_2 - a_3) + 3\ (a_3 - a_4) + \ldots\ (k-1)\ (a_{k-1} - a_k) + k\, a_k$

$= a_1 + a_2 + a_3 + \ldots a_k$

Q.2. The side AB, BC and CA of a triangle ABC have 3, 4 and 5 interior points respectively on them. The number of triangles that can be constructed using these interior points as vertices is .... (1984 - 2 Marks)

Ans. 205

Sol. We have total 3 + 4 + 5 = 12 points out of which 3 fall on one line, 4 on other line and 5 on still other line. So number of D‘s that can be formed using 12 such points are

$= {}^{12}C_3 - {}^{3}C_3 - {}^{4}C_3 - {}^{5}C_3$

$= \frac{12\times11\times10}{6} - 1 - 4 - \frac{5\times4}{2\times1} = 220 - 15 = 205$

Q.3. Total number of ways in which six ‘+’ and four ‘–’ signs can be arranged in a line such that no two '–' signs occur together is .... (1988 - 2 Marks)

Ans. 35

Sol. ‘+’ signs can be put in a row in 1 way, creating 7 ticked places to keep ‘–’ sign so that no two ‘–’ signs occur together

√ +√ + √ +√ +√ +√ +√

Out of these 7 places 4 can be chosen in ${}^7C_4$ ways.

∴ Required no. of arrangements are

$$= {}^7C_4 = {}^7C_3 = \frac{7.6.5}{3.2.1} = 35$$

Q.4. There are four balls of different colours and four boxes of colours, same as those of the balls. The number of ways in which the balls, one each in a box, could be placed such that a ball does not go to a box of its own colour is __________.

Ans. 9

Sol. KEY CONCEPT : We know that number of dearrangements of n objects

$$= n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-.....\frac{1}{n!}\right]$$

∴ No. of ways of putting all the 4 balls into boxes of different colour

$$= 4!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right] = 4!\left(\frac{1}{2}-\frac{1}{6}+\frac{1}{24}\right)$$

$$= 24\left(\frac{12-4+1}{24}\right) = 9$$

True Fales of Permutations and Combinations

Q.1. The product of any r consecutive natural numbers is always divisible by r!. (1985 - 1 Mark)

Ans. T

Sol. Consider

$$\frac{(n+1)(n+2)...(n+r)}{r!}$$

$$= \frac{1.2.3...(n-1)n\ (n+1)(n+2)...(n+r)}{1.2.3...n.r!}$$

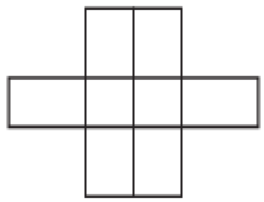
$$= \frac{(n+r)!}{n!r!} = {}^{n+r}C_r$$ = some integral value

$\Rightarrow$ (n + 1) (n + 2) ...(n + r) is divisible by r!

Thus given statement is true.

Subjective questions of Permutations and Combinations

1. Six X's have to be placed in the squares of figure below in such a way that each row contains at least one X. In how many different ways can this be done. (1978)



Ans. Sol. As all the X's are identical, the question is of selection of 6 squares from 8 squares, so that no row remains empty. Here R_1 has 2 squares, R_2 has 4 squares and R_3 has 2 squares.

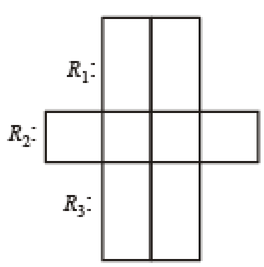
The selection scheme is as follows :

	R_1	R_2	R_3
	1	4	1
or	1	3	2
or	2	3	1
or	2	2	2

∴ Number of selections are

$${}^2C_1 \times {}^4C_4 \times {}^2C_1 + {}^2C_1 \times {}^4C_3 \times {}^2C_2$$

$$+ {}^2C_2 \times {}^4C_3 \times {}^2C_1 + {}^2C_2 \times {}^4C_2 \times {}^2C_2$$



Q.2. Five balls of different colours are to be placed in there boxes of different size. Each box can hold all five. In how many different ways can we place the balls so that no box remains empty ?

Ans.

Sol. The various possibilities to put 5 different balls in 3 different size boxes, when no box remains empty : The balls can be 1, 1 and 3 in different boxes or 2, 2, 1.

Case I : To put 1, 1 and 3 balls in different boxes. Selection of 1, 1 and 3 balls out of 5 balls can be done in ${}^5C_1 \times {}^4C_1 \times {}^3C_3$ ways and then 1, 1, 3 can permute (as defferent size boxes) in 3! ways.

$\therefore$ No. of ways = ${}^5C_1 \times {}^4C_1 \times {}^3C_3 \times 3! = 5 \times 4 \times 1 \times 6 = 120$

Case II : To put 2, 2 and 1 ball in different boxes. Selection of 2, 2 and 1 balls out of 5 balls can be done in ${}^5C_2 \times {}^3C_2 \times {}^1C_1$ ways and then 2, 2, 1 can permute (different boxes) in 3! ways

$\therefore$ No. of ways = ${}^5C_1 \times {}^3C_2 \times {}^1C_1 \times 3! = 10 \times 3 \times 1 \times 6 = 180$

Combining case I and II, total number of required ways are = 120 + 180 = 300.

Q.3. m men and n women are to be seated in a row so that no two women sit together. If m > n, then show that the number of ways in which they can be seated is
$\frac{m!(m+1)!}{(m-n+1)!}$ **(1983 - 2 Marks)**

Ans.

Sol. m men can be seated in m! ways creating (m +1) places for ladies to sit. n ladies out of (m + 1) places (as n < m) can be seated in ${}^{m+1}P_n$ ways
$\therefore$ Total ways = $m! \times {}^{m+1}P_n$

$$= m! \times \frac{(m+1)!}{(m+1-n)!} = \frac{(m+1)!m!}{(m-n+1)!}$$

Q.4. 7 relatives of a man comprises 4 ladies and 3 gentlemen; his wife has also 7 relatives; 3 of them are ladies and 4 gentlemen. In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man’s relatives and 3 of the wife’s relatives? (1985 - 5 Marks)

Ans.

Sol. There are four possibilities:

(i) 3 ladies from husband's side and 3 gentlemen from wife's side.

No. of ways in this case = ${}^4C_3 \times {}^4C_3 = 4 \times 4 = 16$

(ii) 3 gentlemen from husband's side and 3 ladies from wife's side.

No. of ways in this case= ${}^3C_3 \times {}^3C_3 = 1 \times 1 = 1$

(iii) 2 ladies and one gentlemen from husband's side and one lady and 2 gentlemen from wife's side.

No. of ways in this case = $({}^4C_2 \times {}^3C_1) \times ({}^3C_1 \times {}^4C_2) = 6 \times 3 \times 3 \times 6 = 324$

(iv) One lady and 2 gentlemen from husband's side and 2 ladies and one gentlemen from wife's side.

No. of ways in this case = $({}^4C_1 \times {}^3C_2) \times ({}^3C_2 \times {}^4C_1) = 4 \times 3 \times 3 \times 4 = 144$

Hence the total no. of ways are = 16 + 1 + 324 + 144 = 485

Q.5. A box contains two white balls, three black balls and four red balls. In how many ways can three balls be drawn from the box if at least one black ball is to be included in the draw? (1986 - 2½ Marks)

Ans.

Sol. Number of ways of drawing at least one black ball

= 1 black and 2 other or 2 black and 1 other or 3 black

$= {}^3C_1 \times {}^6C_2 + {}^3C_2 \times {}^6C_1 + {}^3C_3$

$= 3 \times 15 + 3 \times 6 + 1$

$= 45 + 18 + 1 = 64$

Q.6. Eighteen guests have to be seated, half on each side of a long table. Four particular guests desire to sit on one particular side and three others on the other side. Determine the number of ways in which the sitting arrangements can be made. (1991 - 4 Marks)

Ans.

Sol. Out of 18 guests half i.e. 9 to be seated on side A and rest 9 on side B. Now out of 18 guests, 4 particular guests desire to sit on one particular side say side A and other 3 on other side B. Out of rest 18 – 4 – 3 = 11 guests we can select 5 more for side A and rest 6 can be seated on side B. Selection of 5 out of 11 can be done in ${}^{11}C_5$ ways and 9 guests on each sides of table can be seated in 9! × 9! ways. Thus there are total ${}^{11}C_5 \times 9! \times 9!$ arrangements.

Q.7. A committee of 12 is to be formed from 9 women and 8 men. In how many ways this can be done if at least five women have to be included in a committee? In how many of these committees (1994 - 4 Marks) (a) The women are in majority? (b) The men are in majority?

Ans.

Sol. Given that there are 9 women and 8 men. A committee of 12 is to be formed including at least 5 women.

This can be doen in the following ways.

=5W and 7M or 6W and 6M

or 7W and 5M

or 8W and 4M

or 9W and 3M

No. of ways of forming committee is

$$= {}^9C_5 \times {}^8C_7 + {}^9C_6 \times {}^8C_6 + {}^9C_7 \times {}^8C_5 + {}^9C_8 \times {}^8C_4 + {}^9C_9 \times {}^8C_3$$

$$= \frac{9.8.7.6}{4.3.2.1} \times 8 + \frac{9.8.7}{3.2.1} \times \frac{8.7}{2.1} + \frac{9.8}{2.1} \times \frac{8.7.6}{3.2.1}$$

$$+ 9 \times \frac{8.7.6.5}{4.3.2.1} + 1 \times \frac{8.7.6}{3.2.1}$$

= 126 × 8 + 84 × 28 + 36 × 56 + 9× 70 + 56 = 6062 ways.

(a) The women are in majority in 2016 + 630 + 56 = 2702 ways.

(b) The men are in majority in 1008 ways.

Q.8. Prove by permutation or oth erwise $\frac{(n^2)!}{(n!)^n}$ is an integer ($n \in I^+$). (2004 - 2 Marks)

Ans.

Sol. Let there be n sets of different objects each set containing n identical objects [eg (1, 1, 1 ... 1 (n times)), (2, 2, 2 ..., 2

(n times) ... (n, n, n ... n (n times))]

Then the no. of ways in which these $n \times n = n^2$ objects can be arranged in a row

$$= \frac{(n^2)!}{n!n!...n!} = \frac{(n^2)!}{(n!)^n}$$

But these number of ways should be a natural number.

Hence $\frac{(n^2)!}{(n!)^n}$ is an integer. ($n \in I^+$)

Q.9. If total number of runs scored in n matches is $\left(\frac{n+1}{4}\right)(2^{n+1}-n-2)$ where n > 1, and the runs scored in the k^{th} match are given by $k.2^{n+1-k}$, where $1 \le k \le n$. Find n. (2005 - 2 Marks)

Ans. Sol. Given that runs scored in k^{th} match $k.2^{n+1-k}$, $1 \le k \le n$

and runs scored in n matches $= \frac{n+1}{4}(2^{n+1}-n-2)$

$$\therefore \sum_{k=1}^{n} k.2^{n+1-k} = \frac{n+1}{4}(2^{n+1}-n-2)$$

$$\Rightarrow 2^{n+1}\left[\sum_{k=1}^{n} \frac{k}{2^k}\right] = \frac{n+1}{4}(2^{n+1}-n-2)$$

$$\Rightarrow \quad 2^{n+1}\left[\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{n}{2^n}\right]$$

$$=\frac{n+1}{4}(2^{n+1}-n-2) \qquad \ldots.(\text{i})$$

Let $\quad S=\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots.+\frac{n}{2^n}$

$$\frac{1}{2}S=\frac{1}{2^2}+\frac{2}{2^3}+\ldots.+\frac{n-1}{2^n}+\frac{n}{2^{n+1}}$$

Subtracting the above two, we get

$$\frac{1}{2}S=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}.+\ldots+\frac{1}{2^n}-\frac{n}{2^{n+1}}$$

$$\frac{1}{2}S=\frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}-\frac{n}{2^{n+1}} \Rightarrow \quad S=2\left[1-\frac{1}{2^n}-\frac{n}{2^{n+1}}\right]$$

$\therefore$ Equation (i) becomes

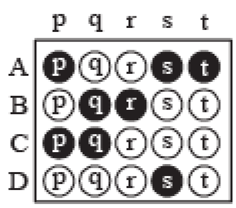
$$2.2^{n+1}\left[1-\frac{1}{2^n}-\frac{n}{2^{n+1}}\right]=\frac{n+1}{4}\left[2^{n+1}-n-2\right]$$

$$\Rightarrow 2.\left[2^{n+1}-2-n\right] \; =\frac{n+1}{4}\left[2^{n+1}-2-n\right]$$

$$\Rightarrow \frac{n+1}{4}=2 \Rightarrow n=7$$

Match the following of Permutations and Combinations

Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in ColumnII. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :



If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.

Q.1. Consider all possible permutations of the letters of the word ENDEANOEL. Match the Statements / Expressions in Column I with the Statements / Expressions in Column II and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in the ORS. (2008)

Column I	**Column II**
(A) The number of permutations containing the word ENDEA is	**(p) 5!**
(B) The number of permutations in which the letter E occurs in the first and the last positions is	**(q) 2 × 5!**
(C) The number of permutations in which none of the letters D, L, N occurs in the last five positions is	**(r) 7 × 5!**
(D) The number of permutations in which the letters A, E, O occur only in odd positions is	**(s) 21 × 5!**

Ans. (A)-p; (B)-s; (C)-q; (D)-q (A)

Sol.

(A) For the permutations containing the word ENDEA we consider 'ENDEA' as single letter. Then we have total ENDEA, N, O, E, L i.e. 5 letters which can be arranged in 5! ways. ∴ (A) → (p)

(B) If E occupies the first and last position, the middle 7 positions can be filled by N, D, E, A, N, O, L.

These can be arranged in $\frac{7!}{2!}$ = 21 x 5! ways.

∴ (B) → (s)

(C) If none of the letters D, L, N occur in the last five positions then we should arrange D, D, L, N at first four positions and rest five i.e. E, E, E, A,O at last five positions. This can be done in

$\frac{4!}{2!} \times \frac{5!}{3!}$ ways . (C) → (q)

(D) As per question A, E, E, E, O can be arranged at 1st, 3rd, 5th, 7th and 9th positions and rest N, D, N, L at rest 4 positions. This can be done in

$\frac{5!}{3!} \times \frac{4!}{2!}$ ways = 2 x 5! ways (D) → (q)

Integar Type ques of Permutations and Combinations

Q.1. Consider the set of eight vectors $V=\{a\hat{i}+b\hat{j}+c\hat{k}:a,b,c\in\{-1,1\}\}$. Three non- coplanar vectors can be chosen from V in 2p ways. Then p is (JEE Adv. 2013)

Ans. (5)

Sol. Given 8 vectors are (1, 1, 1), (–1, –1, –1); (–1, 1, 1), (1, –1, –1); (1, –1, 1), (–1, 1, –1); (1, 1, –1), (–1, –1, 1)

These are 4 diagonals of a cube and their opposites.

For 3 non coplanar vectors first we select 3 groups of diagonals and its opposite in ${}^4C_3$ ways.

Then one vector from each group can be selected in 2 × 2 × 2 ways.

∴ Total ways = ${}^4C_3 \times 2 \times 2\times 2 = 32 = 2^5$

∴ p = 5

Q.2. Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positive integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$. Then the number of such distinct arrangements $(n_1, n_2, n_3, n_4, n_5)$ is (JEE Adv. 2014)

Ans. (7)

Sol. ∵ n_1, n_2, n_3, n_4 and n_5 are positive integers such that $n_1 < n_2 < n_3 < n_4 < n_5$

Then for $n_1 + n_2 + n_3 + n_4 + n_5 = 20$ If n_1, n_2, n_3, n_4

take minimum values 1, 2, 3, 4 respectively then n_5 will be maximum 10.

∴ Corresponding to $n_5 = 10$,

there is only one solution $n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 4$.

Corresponding to $n_5 = 9$, we can have, only solution $n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 5$ i.e., one solution

Corresponding to $n_5 = 8$, we can have, only solution $n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 6$ or $n_1 =$

1, $n_2 = 2$, $n_3 = 4$, $n_4 = 4$ i.e., 2 solution

For $n_5 = 7$, we can have $n_1 = 1$, $n_2 = 1$, $n_3 = 4$, $n_4 = 6$ or $n_1 = 1$, $n_2 = 3$, $n_3 = 4$, $n_4 = 5$ i.e. 2 solutions

For $n_5 = 6$, we can have $n_1 = 2$, $n_2 = 3$, $n_3 = 4$, $n_4 = 5$ i.e., one solution

Thus there can be 7 solutions.

Q.3. Let n ≥ 2 be an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, then the value of n is (JEE Adv. 2014)

Ans. (5)

Sol. Number of adjacent lines = n Number of non adjacent lines = $^nC_2 - n$

$$\therefore {}^nC_2 - n = n \Rightarrow \frac{n(n-1)}{2} - 2n = 0$$

$$\Rightarrow n^2 - 5n = 0 \Rightarrow n = 0 \text{ or } 5$$

But $n \geq 2 \Rightarrow n = 5$

Q.4. Let n be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that all the girls stand consecutively in the queue. Let m be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that exactly four girls stand consecutively in the queue.

Then the value of $\frac{m}{n}$ is (JEE Adv. 2015)

Ans. (5)

Sol. $n = 5! \times 6!$

For second arrangement, 5 boys can be made to stand in a row in 5! ways, creating 6 alternate space for girls. A group of 4 girls can be selected in $^5C_4$ ways. A group of 4 and single girl can be arranged at 2 places out of 6 in $^6P_2$ ways. Also 4 girls can arrange themselves in 4! ways.

$\therefore m = 5! \times {}^6P_2 \times {}^5C_4 \times 4!$

$$\frac{m}{n} = \frac{5! \times 6 \times 5 \times 5 \times 4!}{5! \times 6!} = 5$$