Exercise 14.3

Answer 1E.

$T = f(x, y, t)$

(A)

Then $\frac{\partial T}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y, t) - f(x, y, t)}{h}$

It denotes the rate of change of temperature as longitude varies with latitude and time fixed

$$\frac{\partial T}{\partial y} = \lim_{h \to 0} \frac{f(x, y+y, t) - f(x, y, t)}{h}$$

It denotes the rate of change of temperature as latitude varies with longitude and time fixed

And $\frac{\partial T}{\partial t} = \lim_{h \to 0} \frac{f(x, y, t th) - f(x, y, t)}{h}$

It denotes the rate of change of temperature as time varies with latitude and longitude fixed.

(B)

For the given place $T = f(158, 21, 9)$

Then $f_x(158, 21, 9) = \frac{\partial T}{\partial x} = \lim_{h \to 0} \frac{f(158+h, 21, 9) - f(158, 21, 9)}{h}$

It denotes the rate of change of temperature as longitude varies with latitude and time fixed since the air to the west and south is warm and the air to the north and east is cooler, then as the longitude varies the temperature increases with increase in longitude and then f_x will be positive

Now $f_y(158, 21, 9) = \frac{\partial T}{\partial y} = \lim_{h \to 0} \frac{f(158, 21+h, 9) - f(158, 21, 9)}{h}$

It denotes the rate of change of temperature as latitude varies with longitude and time fixed. Since the air to the west and south is warm and the air to the north and east is cooler, then as the latitude varies with longitude and time fixed, the temperature decreases with the increase of latitude and hence f_y will be negative

And $f_t(158, 21, 9) = \frac{\partial T}{\partial t} = \lim_{h \to 0} \frac{f(158, 21, 9+h) - f(158, 21, 9)}{h}$

It denotes the rate of change of temperature as time varies with longitude and latitude fixed. As the time increase from 9AM then the temperature goes on increasing and thus $\frac{\partial f}{\partial t}$ that is $f_t = (158, 21, 9)$ will be positive

Answer 2E.

$$I = f(T, H)$$

$$f_T = (92, 60) = \lim_{h \to 0} \frac{f(92+h, 60) - f(92, 60)}{h}$$

Using values in table 1 and by taking h = 2 and h = - 2

$$f_T(92, 60) \approx \frac{f(94, 60) - f(92, 60)}{2} = \frac{111 - 105}{2} = 3$$

And $$f_T = (92, 60) \approx \frac{f(90, 60) - f(92, 60)}{-2} = \frac{100 - 105}{-2} = 2.5$$

On averaging these values

$$f_T(92, 60) \approx 2.75$$

Which interprets that when the temperature is 92°F and relative humidity is 60%, the heat index rises about 2.75°F for every degree that the actual temperature rises.

And $$f_H(92, 60) = \lim_{h \to 0} \frac{f(92, 60+h) - f(92, 60)}{h}$$

Using the values in table 1 and by taking $h = 5, -5$

$$f_H(92, 60) \approx \frac{f(92, 65) - f(92, 60)}{5} = \frac{108 - 105}{5} = 0.6$$

And $$f_H = (92, 60) \approx \frac{f(92, 55) - f(92, 60)}{-5} = \frac{103 - 105}{-5} = 0.4$$

On averaging these values:

$$f_H(92, 60) \approx 0.5$$

Which says that when the temperature is 92°F and relative humidity is 60%, the heat index rises about 0.5°F for every percent that the relative humidity rises

Hence we see that the heat index increase rapidly with the increase in temperature than the increase in humidity

Answer 3E.

The wind-chill index W is the perceived temperature when the actual temperature is T and the wind speed is v.

$$w = f(T, v)$$

Where the index W is a subjective temperature that depends on the actual temperature T and the wind speed v

Consider the following table

Table 1

Actual temperature (°C) T \ v (Wind speed (km/h))	20	30	40	50	60	70
−10	−18	−20	−21	−22	−23	−23
−15	−24	−26	−27	−29	−30	−30
−20	−30	−33	−34	−35	−36	−37
−25	−37	−39	−41	−42	−43	−44

(a) If we concentrated on the highlighted column of the Table 1, which corresponds to a wind speed of $v = 30\,\text{km/h}$. We are considering wind-chill index as a function of the single variable T for a fixed value of v.

The derivative of f when $T = -15^\circ C$ is the rate of change of w with respect to T when $T = -15^\circ C$

$$f_T(-15, 30) = \lim_{h\to 0} \frac{f(-15+h, 30) - f(-15, 30)}{h}$$

We can approximate $f_T(-15, 30)$ using the values in Table 1 by taking $h = 5$ and $h = -5$

$$f_T = (-15, 30) \approx \frac{f(-10, 30) - f(-15, 30)}{5} = \frac{-20-(-26)}{5} = 1.2$$

$$f_T(-15, 30) \approx \frac{f(-20, 30) - f(-15, 30)}{-5} = \frac{-33-(-26)}{-5} = 1.4$$

On averaging these values:

$$f_T = (-15, 30) \approx 1.3$$

This means when the actual temperature is $-15^\circ C$ and the wind speed is 30 km/h the wind-chill index rises about 1.3°C for every degree that the actual temperature rises

Now let's look at the highlighted row in Table 1, which corresponds a fixed temperature of $T = -15^\circ C$.

The derivative of f when $v = 30\,\text{km/h}$ is the rate of change of w with respect to v

When $v = 30\,\text{km/h}$

$$f_v(-15, 30) = \lim_{h\to 0} \frac{f(-15, 30+h) - f(-15, 30)}{h}$$

We can approximate $f_v(-15,30)$ using the values in Table 1 by taking $h=10$ and $h=-10$

$$f_v(-15, 30) \approx \frac{f(-15, 40) - f(-15, 30)}{10} = \frac{-27-(-26)}{10} = -0.1$$

$$f_v(-15, 30) \approx \frac{f(-15, 20) - f(-15,30)}{10} = \frac{-24-(-26)}{-10} = -0.2$$

On averaging these values:

$$f_v(-15, 30) \approx -0.15$$

This means when the actual temperature is – 15°C and the wind speed is 30 km/h, the wind-chill index decreases by – 0.15°C for each km/h the wind speed increases.

(b)

$\frac{\partial w}{\partial T}$, denotes the rate of change of wind – chill index as temperature varies with wind – speed fixed. From the table we see that $\frac{\partial w}{\partial T}$ is positive

And $\frac{\partial w}{\partial v}$ denotes the rate of change of wind – chill index as temperature remains fixed and wind – speed varies. From the table we see that $\frac{\partial w}{\partial v}$ is negative

(c)

Since $\frac{\partial w}{\partial v}$ denotes the rate of change of wind – chill index us temperature remains fixed and wind speed varies. We see that $\frac{\partial w}{\partial v}$ decrease with the increase in the value of v therefore

$$\lim_{v\to\infty} \frac{\partial w}{\partial v} = 0$$

Answer 4E.

a)

The partial derivative $\frac{\partial h}{\partial v}$ represents the rate of change of wave height as the time remains fixed and speed varies.

The partial derivative $\frac{\partial h}{\partial t}$ denotes the rate of change of wave heights as the speed of the wind remains fixed and time varies.

(b)

The objective is to estimate $f_v(40,15)$ and $f_t(40,15)$.

$$f_v(40,15) = \lim_{h\to 0}\frac{f(40+h,15)-f(40,15)}{h}$$

This can be approximated by taking the average of the following two values:

$$\frac{f(50,15)-f(40,15)}{50-40} \text{ and } \frac{f(30,15)-f(40,15)}{30-40}$$

From the table,

$$f(50,15)=36, f(40,15)=25 \text{ and } f(30,15)=16$$

$$\begin{aligned}\frac{f(50,15)-f(40,15)}{50-40} &= \frac{36-25}{10}\\ &= \frac{11}{10}\\ &= 1.1\end{aligned}$$

$$\begin{aligned}\frac{f(30,15)-f(40,15)}{30-40} &= \frac{16-25}{-10}\\ &= \frac{-9}{-10}\\ &= 0.9\end{aligned}$$

The value of $f_v(40,15)$ estimated as follows:

$$\begin{aligned}f_v(40,15) &\approx \frac{1.1+0.9}{2}\\ &= 1.0\end{aligned}$$

Therefore, when 40-knot wind has been blowing for 15 hours, wave heights should increase by about 1 foot for every knot as the wind speed increases.

Compute $f_t(40,15)$ as follows:

$$f_t(40,15) = \lim_{h \to 0} \frac{f(40,15+h) - f(40,15)}{h}$$

This can be approximated by taking the average of the following two values:

$$\frac{f(40,20) - f(40,15)}{20-15} \text{ and } \frac{f(40,10) - f(40,15)}{10-15}$$

From the table,

$$f(40,20) = 28, f(40,15) = 25 \text{ and } f(40,10) = 21$$

$$\begin{aligned}\frac{f(40,20) - f(40,15)}{20-15} &= \frac{28-25}{5} \\ &= \frac{3}{5} \\ &= 0.6\end{aligned}$$

$$\begin{aligned}\frac{f(40,10) - f(40,15)}{10-15} &= \frac{21-25}{-5} \\ &= \frac{-4}{-5} \\ &= 0.8\end{aligned}$$

The value of $f_t(40,15)$ is estimated as follows:

$$\begin{aligned}f_t(40,15) &\approx \frac{0.6+0.8}{2} \\ &= 0.7\end{aligned}$$

Therefore, when 40-knot wind has been blowing for 15 hours, wave height should increase by about 0.7 foot for every knot as the time increases.

(C)

The partial derivate $\frac{\partial h}{\partial t}$ denotes the rate of change of wave heights as the speed of

The wind remains fixed and time varies. For fixed values of v the function $f(v,t)$ appear

to increase in smaller interments and becoming nearly constant as t increases.

Thus, the corresponding rate of change is nearly 0 as t increase and hence $\lim_{t \to \infty} \frac{\partial h}{\partial t} = 0$.

Answer 5E.

(A)

Since the graph of f increase, if we start at (1, 2) and move in the positive x – direction, this reflects the positive values of f_x

Then $f_x(1,2)$ is positive

(B)

Since the graph of f decrease if we start at (1, 2) and move in the positive y – direction, this reflects the negative values of f_y

Then $f_y(1,2)$ is negative

Answer 6E.

A) Since the graph of f decreases if we start at (-1, 2) and move in the positive x direction initially, this reflects the negative values of fx

Then fx (-1, 2) is negative

B) Since the graph of f decreases if we start at (-1, 2) and move in the positive y direction, this reflects the negative values of fy

Then fy (-1, 2) is negative

Answer 7E.

The second derivative determines concavity. As we move along the x axis at f(-1,2) the graph concaves up so (a) is positive

as we move along the y axis at f(-1,2) the graph concaves down, so (b) is negative

Answer 8E.

(a)

Around the point (1,2) fx is positive since the value of the function increases as x increases. Around the point, the slope in the x direction becomes less steep as y increases. So, fxy is negative.

(b)

Near the point (-1,2) fx is negative since the function decreases as x increases. The slope in the positive x direction becomes more steep (takes a greater negative number) as y increases so fxy is negative.

Answer 9E.

The partial derivative of the function f of two variables is defined as,

$$f_x(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

And,

$$f_y(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

Consider the graph of the function given in the problem with surface labeled as, a,b and c. Observe that both b and c decreases while a increases as the point $(3,-3)$ moves along the positive y-direction, also at the point $(3,-1.5)$ a is zero. This show a is definitely a graph of f_y.

To find the graphs of the surfaces b and c, start at the point $(-3,-1.5)$ and move in the positive x-direction. Note that b traces out a line with negative slope while c traces out a parabola opening downward. This tells us that the surface b is the x-derivative of c. So c is the graph of f, b is the graph of f_x, and a is the graph of f_y.

Hence, the surface a is the graph of f_y; the surface b is the graph of f_x and c is the graph of f.

Answer 10E.

Consider the contour map for a function f shown in figure 1:

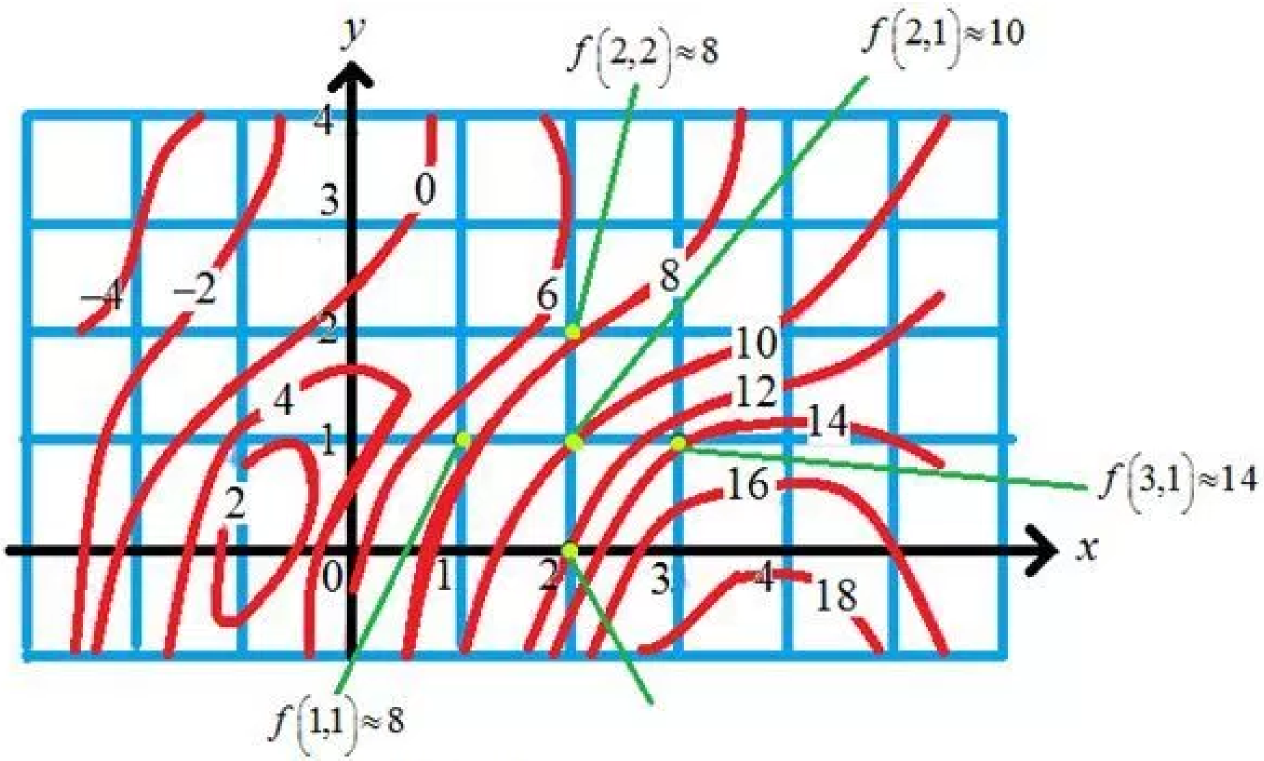


Figure 1

Estimate $f_x(2,1)$ and $f_y(2,1)$:

List some function values at some points (x,y):

$$f(3,1) \approx 14$$
$$f(2,1) \approx 10$$
$$f(1,1) \approx 8$$
$$f(2,0) \approx 12$$
$$f(2,2) \approx 8$$

The partial derivative of f with respect to x:

$$f_x(2,1) = \lim_{h \to 0} \frac{f(2+h,1) - f(2,1)}{h}.$$

Take $h = \Delta x = 1.$

$$\begin{aligned} f_x(2,1) &\approx \frac{f(2+1,1) - f(2,1)}{1} \\ &\approx f(3,1) - f(2,1) \\ &\approx 14 - 10 \\ &\approx 4 \end{aligned}$$

Take $h = \Delta x = -1.$

$$\begin{aligned} f_x(2,1) &\approx \frac{f(2-1,1) - f(2,1)}{-1} \\ &\approx \frac{f(1,1) - f(2,1)}{-1} \\ &\approx \frac{8-10}{-1} \\ &\approx 2 \end{aligned}$$

Take the average value of these two values.

The partial derivative of f with respect to x at the point $(2,1)$ is calculated as follows:

$$\begin{aligned} f_x(2,1) &\approx \frac{4+2}{2} \\ &\approx 3 \end{aligned}$$

Therefore, $f_x(2,1) \approx \boxed{3}$.

The partial derivative of f with respect to y:

$$f_y(2,1) = \lim_{h \to 0} \frac{f(2,1+h) - f(2,1)}{h}.$$

Take $h = \Delta y = -1$.

$$\begin{aligned} f_y(2,1) &\approx \frac{f(2,0) - f(2,1)}{-1} \\ &\approx \frac{12-10}{-1} \\ &\approx -2 \end{aligned}$$

Take $h = \Delta y = 1$.

$$\begin{aligned} f_y(2,1) &\approx \frac{f(2,2) - f(2,1)}{1} \\ &\approx \frac{8-10}{1} \\ &\approx -2 \end{aligned}$$

Take the average value of these two values.

The partial derivative of f with respect to y at the point $(2,1)$ is,

$$\begin{aligned} f_y(2,1) &\approx \frac{-2-2}{2} \\ &\approx -2 \end{aligned}$$

Therefore, $f_y(2,1) \approx \boxed{-2}$.

Answer 11E.

Consider the function $f(x,y)=16-4x^2-y^2$.

Need to find $f_x(1,2)$ and $f_y(1,2)$.

To find $f_x(x,y)$ first holding y constant and differentiating with respect to x, get

$$f_x(x,y)=-4(2x)$$
$$=-8x$$

At $(x,y)=(1,2)$.

$$f_x(1,2)=-8(1)$$
$$=-8$$

To find $f_y(x,y)$ first holding x constant and differentiating with respect to y, get

$$f_y(x,y)=-2y$$

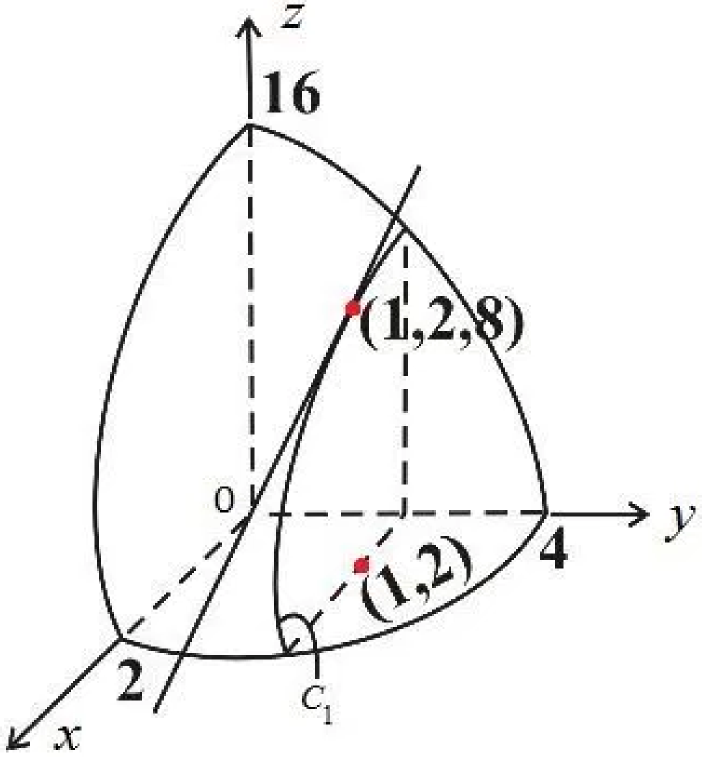
At $(x,y)=(1,2)$.

$$f_y(1,2)=-2(2)$$
$$=-4$$

The graph of the function f is the paraboloid $z=16-4x^2-y^2$ and the vertical plane $y=2$ intersects it in the parabola $z=12-4x^2$, $y=2$ let this curve be $C1$.

The slope of the tangent line to this parabola at the point $(1,2,8)$ is $\boxed{f_x(1,2)=-8}$.

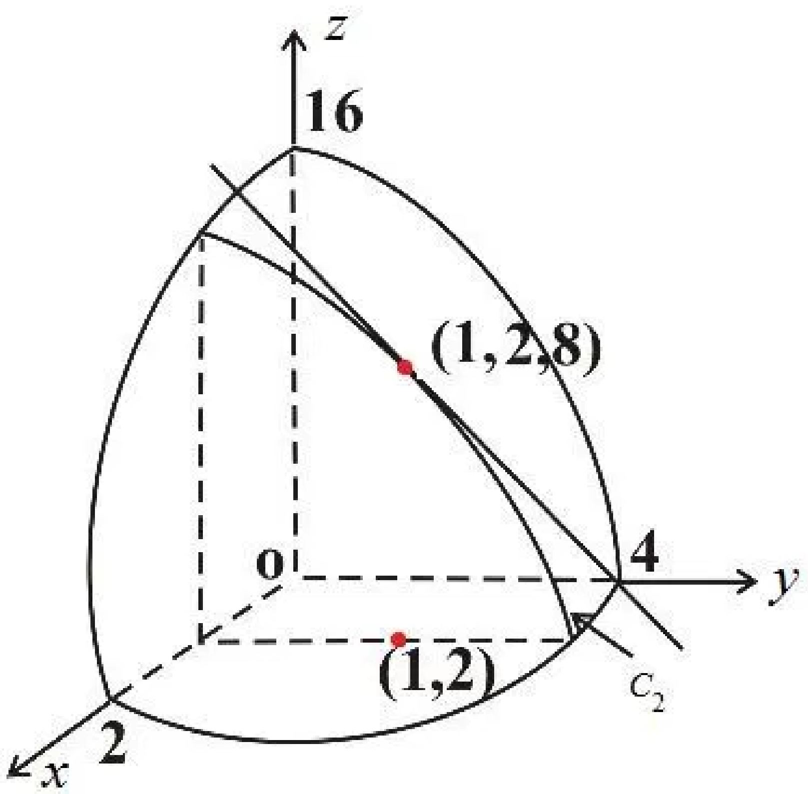
See the below figure:



Similarly the vertical plane $x=1$ intersects the paraboloid $z=16-4x^2-y^2$ in the parabola $z=12-y^2$, $x = 1$ let this curve be $C2$.

The slope of the tangent line to this parabola at the point $(1,2,8)$ is $\boxed{f_y(1,2)=-4}$.

See the below figure:



Answer 12E.

$$f(x,y)=\sqrt{4-x^2-4y^2}$$

Differentiating partially with respect to x,

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\sqrt{4-x^2-4y^2}$$

$$=\frac{-2x}{2\sqrt{4-x^2-4y^2}}$$

$$=-\frac{x}{\sqrt{4-x^2-4y^2}}$$

Therefore, $\quad f_x(1,0)=\left(\frac{\partial f}{\partial x}\right)_{(1,0)}$

$$=\frac{-1}{\sqrt{4-(1)^2-4(0)^2}}$$

$$=-\frac{1}{\sqrt{3}}$$

Differentiating f partially with respect to y,

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\sqrt{4-x^2-4y^2}$$

$$= \frac{1}{2\sqrt{4-x^2-4y^2}} \cdot \frac{\partial}{\partial y}\left(4-x^2-4y^2\right)$$

$$= \frac{1}{2\sqrt{4-x^2-4y^2}} \cdot (-8y)$$

$$= \frac{-4y}{\sqrt{4-x^2-4y^2}}$$

Therefore, $$f_y(1,0) = \left(\frac{\partial f}{\partial y}\right)_{(1,0)}$$

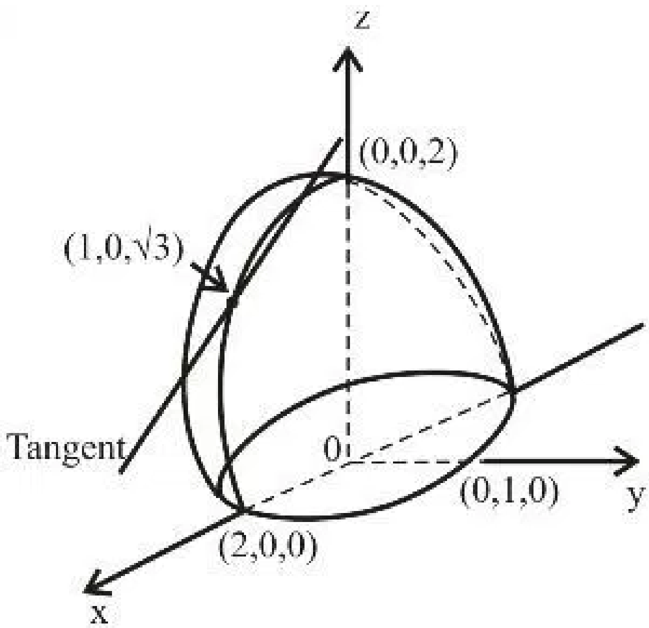
$$= \frac{-4\times 0}{\sqrt{4-(-1)^2-4(0)^2}}$$

$$= 0.$$

The graph of f(x,y) is the upper half of ellipsoid $x^2+4y^2+z^2=4$, $z \ge 0$ and the vertical plane y = 0 cuts it in the semi-circle

$x^2+z^2=4$; $y=0$, $z \ge 0$

$f_x(1,0)$ Represents the slope of the tangent line to this semi-circle at the point $\left(1,0,\sqrt{3}\right)$



Plane x = 1, cuts the graph of f(x,y) i.e., upper half of ellipsoid

$x^2+4y^2+z^2=4 \quad ; \quad z \geq 0$

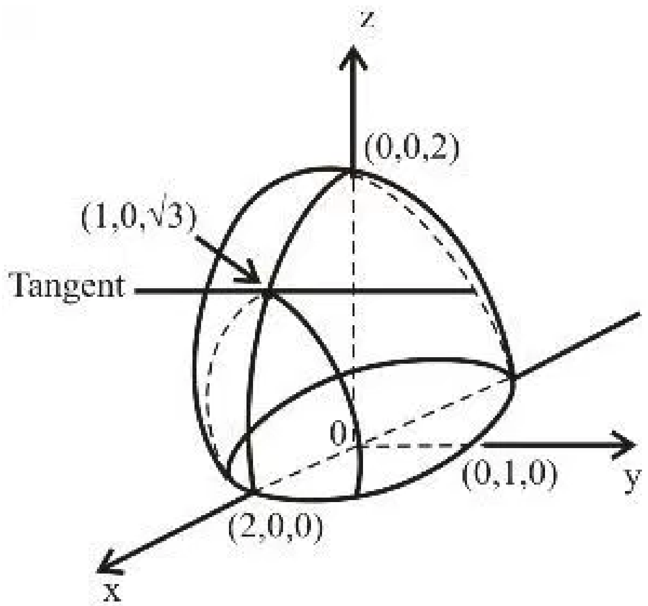
In the semi ellipse $\quad 1+4y^2+z^2=4 \; ; \quad x=1$

i.e. $\; 4y^2+z^2=3 \quad ; \quad x=1$

And $f_y(1,0)$ represents the slope of tangent line to this semi ellipse at the point $(1,0,\sqrt{3})$

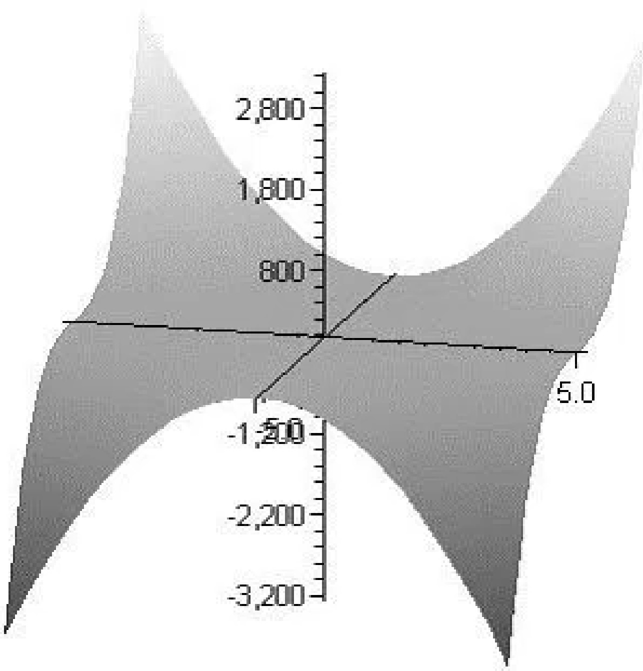
Hence,

$$\boxed{f_x(1,0)=\frac{-1}{\sqrt{3}} \; , \quad f_y(1,0)=0}$$



Answer 13E.

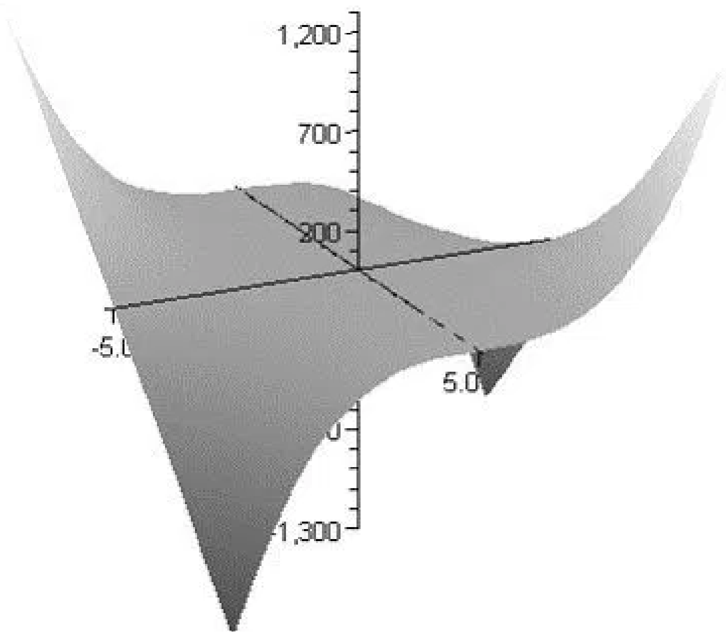
Let us sketch the graph for $f(x,y)=x^2y^3$.



Find $f_x(x, y)$ by differentiating the given function with respect to x keeping y as constant.

$$f_x(x, y) = \frac{\partial}{\partial x}(x^2y^3)$$
$$= 2xy^3$$

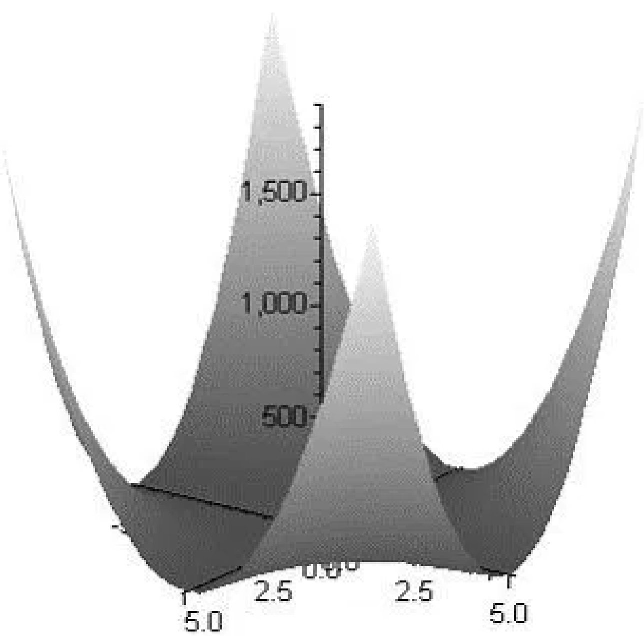
Now, sketch the graph for $f_x(x, y) = 2xy^3$.



Now differentiate the given function with respect to y keeping x as constant to find $f_y(x, y)$.

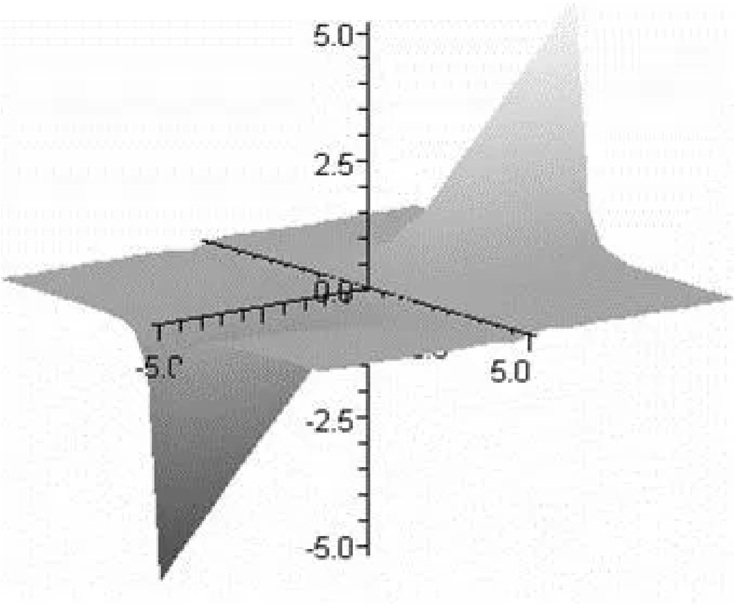
$$f_y(x, y) = \frac{\partial}{\partial y}(x^2y^3)$$
$$= 3x^2y^2$$

Sketch $f_y(x, y) = 3x^2y^2$.



Answer 14E.

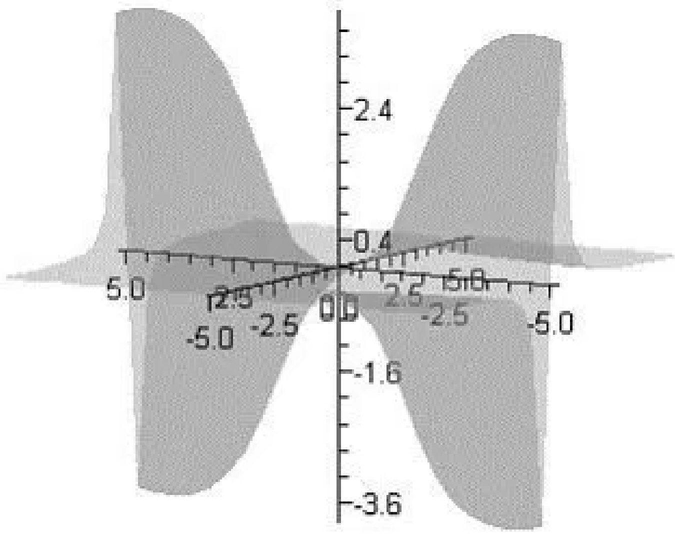
Let us sketch the graph for $f(x, y) = \dfrac{y}{1 + x^2 y^2}$.



Find $f_x(x, y)$ by differentiating the given function with respect to x keeping y as constant.

$$f_x(x, y) = \frac{\partial}{\partial x}\left(\frac{y}{1 + x^2 y^2}\right)$$
$$= -\frac{2xy^3}{\left(1 + x^2 y^2\right)^2}$$

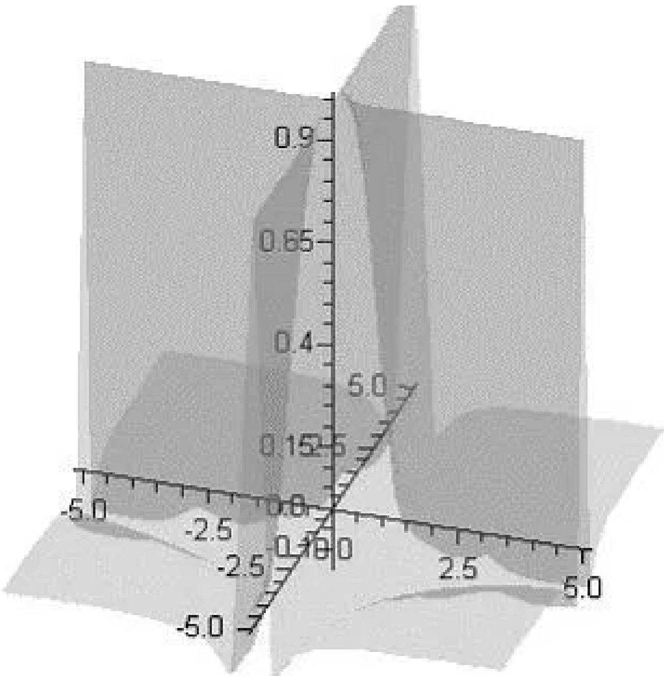
Now, sketch the graph for $f_x(x, y) = -\dfrac{2xy^3}{\left(1 + x^2 y^2\right)^2}$.



Now differentiate the given function with respect to y keeping x as constant to find $f_y(x, y)$.

$$f_y(x, y) = \frac{\partial}{\partial y}\left(\frac{y}{1+x^2y^2}\right)$$
$$= \frac{1}{1+x^2y^2} - \frac{2x^2y^2}{\left(1+x^2y^2\right)^2}$$
$$= -\frac{x^2y^2-1}{\left(1+x^2y^2\right)^2}$$

Sketch $f_y(x, y) = -\dfrac{x^2y^2-1}{\left(1+x^2y^2\right)^2}$.



Answer 15E.

Given function is $f(x, y) = y^5 - 3xy$

$$f_x(x, y) = -3y$$

$$f_y(x, y) = 5y^4 - 3x$$

Answer 16E.

Given function is $f(x, y) = x^4 y^3 + 8x^2 y$

$$f_x(x, y) = 4x^3 y^3 + 16xy$$

$$f_y(x, y) = 3x^4 y^2 + 8x^2$$

Answer 17E.

Given that $f(x, t) = e^{-t} \cos \pi x$

$$f_x(x, t) = -\pi e^{-t} \sin \pi x$$

$$f_t(x, t) = -e^{-t} \cos \pi x$$

Answer 18E.

Given function is $f(x, t) = \sqrt{x} \ln t$

$$f_x(x, t) = \frac{\ln t}{2\sqrt{x}}$$

$$f_t(x, t) = \sqrt{x} / t$$

Answer 19E.

Given function is $z = (2x + 3y)^{10}$

Differentiate the given function with respect to x, we have

$$\partial z / \partial x = 20(2x + 3y)^9$$

Differentiate the given function with respect to y, we have

$$\partial z / \partial y = 30(2x + 3y)^9$$

Answer 20E.

Given function is $z = \tan xy$

$$\frac{\partial z}{\partial x} = y\sec^2 xy$$

$$\frac{\partial z}{\partial y} = x\sec^2 xy$$

Answer 21E.

Let us start by finding f_x.

We know that $f_x = \frac{\partial}{\partial x} f(x, y)$.

$$f_x = \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$= \frac{1}{y}$$

We get $\boxed{f_x = \frac{1}{y}}$.

Now, we have $f_y = \frac{\partial}{\partial y} f(x, y)$.

$$f_y = \frac{\partial}{\partial y}\left(\frac{x}{y}\right)$$

$$= -\frac{x}{y^2}$$

Thus, we get $\boxed{f_y = -\frac{x}{y^2}}$.

Answer 22E.

To find the first partial derivatives of the function,

$$f(x,y)=\frac{x}{(x+y)^2}$$

Quotient rule for differentiation:

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right)=\frac{v(x)u'(x)-u(x)v'(x)}{(v(x))^2}$$

And,

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right)=-\frac{1}{(f(x))^2}\frac{d}{dx}f(x)$$

Differentiate with respect x (keeping y as constant) on both sides of the function f,

$$f_x=\frac{\partial}{\partial x}f(x,y)$$

$$=\frac{\partial}{\partial x}\left[\frac{x}{(x+y)^2}\right]$$

$$=\frac{(x+y)^2\frac{\partial}{\partial x}(x)-x\frac{\partial}{\partial x}(x+y)^2}{\left((x+y)^2\right)^2}$$

$$=\frac{(x+y)^2-2x(x+y)\frac{\partial}{\partial x}(x+y)}{(x+y)^4}$$

$$=\frac{(x+y)^2-2x(x+y)(1+0)}{(x+y)^4}$$

$$=\frac{(x+y)-2x}{(x+y)^3}$$

$$=\frac{y-x}{(x+y)^3}$$

$$=-\frac{x-y}{(x+y)^3}$$

Therefore, the first partial derivative of f with respect to x is, $f_x=\boxed{-\frac{(x-y)}{(x+y)^3}}$

Differentiate with respect y (keeping x as constant) on both sides of the function f,

$$f_y = \frac{\partial}{\partial y} f(x, y)$$

$$= \frac{\partial}{\partial y}\left[\frac{x}{(x+y)^2}\right]$$

$$= x\frac{\partial}{\partial y}\left[\frac{1}{(x+y)^2}\right] \quad (\text{since } x \text{ is constant})$$

$$= x\left(-\frac{1}{\left((x+y)^2\right)^2}\right)\frac{\partial}{\partial y}(x+y)^2$$

$$= -\frac{x}{(x+y)^4}\left[2(x+y)\frac{\partial}{\partial y}(x+y)\right]$$

$$= -\frac{x}{(x+y)^4}\left[2(x+y)(0+1)\right]$$

$$= -\frac{2x}{(x+y)^3}$$

Therefore, the first partial derivative of f with respect to y is,

$$f_y = \boxed{-\frac{2x}{(x+y)^3}}$$

Answer 23E.

Consider the following function:

$$f(x,y)=\frac{ax+by}{cx+dy}.$$

The objective is to find the first partial derivatives of the function with respect to x and y.

Now partially differentiate the function with respect to x and treat the variable y as a constant.

$$f_x(x,y)=\frac{\partial}{dx}\left(\frac{ax+by}{cx+dy}\right)$$

$$=\frac{(cx+dy)\frac{\partial}{dx}(ax+by)-(ax+by)\frac{\partial}{dx}(cx+dy)}{(cx+dy)^2}$$ Since $\left(\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{vdu-udv}{v^2}\right)$

$$=\frac{(cx+dy)(a)-(ax+by)(c)}{(cx+dy)^2}$$

$$=\frac{(acx+ady)-(acx+bcy)}{(cx+dy)^2}$$

$$=\frac{ady-bcy}{(cx+dy)^2}$$

$$=\frac{y(ad-bc)}{(cx+dy)^2}$$

The partial derivative with respect to x is $f_x(x,y)=\frac{y(ad-bc)}{(cx+dy)^2}$.

Again partially differentiate the function with respect to y, treat the variable x as a constant.

$$f_y(x,y) = \frac{\partial}{dy}\left(\frac{ax+by}{cx+dy}\right)$$

$$= \frac{(cx+dy)\frac{\partial}{dy}(ax+by)-(ax+by)\frac{\partial}{dy}(cx+dy)}{(cx+dy)^2} \quad \text{Since } \left(\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{vdu-udv}{v^2}\right)$$

$$= \frac{(cx+dy)(b)-(ax+by)(d)}{(cx+dy)^2}$$

$$= \frac{(bcx+bdy)-(adx+bdy)}{(cx+dy)^2}$$

$$= \frac{bcx-adx}{(cx+dy)^2}$$

$$= \frac{(bc-ad)x}{(cx+dy)^2}$$

The partial derivative with respect to y is $f_y(x,y)=\frac{(bc-ad)x}{(cx+dy)^2}$.

Therefore, the first partial derivatives of the functions are $\boxed{f_x(x,y)=\frac{y(ad-bc)}{(cx+dy)^2}}$ and

$\boxed{f_y(x,y)=\frac{(bc-ad)x}{(cx+dy)^2}}$.

Answer 24E.

Consider the function,

$$w = \frac{e^v}{u+v^2}.$$

The objective is to find the first partial derivatives of the function.

Differentiate *w* partially with respect to u, treating v as a constant:

$$\begin{aligned}
w_u &= \frac{\partial}{\partial u}\left(\frac{e^v}{u+v^2}\right) \\
&= e^v \cdot \frac{-1}{\left(u+v^2\right)^2} \cdot \frac{\partial}{\partial u}\left(u+v^2\right) \qquad \text{(Chain Rule)} \\
&= e^v \cdot \frac{-1}{\left(u+v^2\right)^2} \cdot (1) \\
&= \boxed{\frac{-e^v}{\left(u+v^2\right)^2}}
\end{aligned}$$

Differentiate *w* partially with respect to v, treating u as a constant:

$$\begin{aligned}
w_v &= \frac{\partial}{\partial v}\left(\frac{e^v}{u+v^2}\right) \\
&= \frac{\frac{\partial}{dv}\left(e^v\right)\cdot\left(u+v^2\right)-\left(e^v\right)\cdot\frac{\partial}{dv}\left(u+v^2\right)}{\left(u+v^2\right)^2} \qquad \text{(Quotient Rule)} \\
&= \frac{\left(e^v\right)\cdot\left(u+v^2\right)-\left(e^v\right)\cdot(2v)}{\left(u+v^2\right)^2} \\
&= \boxed{\frac{e^v}{\left(u+v^2\right)} - \frac{2ve^v}{\left(u+v^2\right)^2}}
\end{aligned}$$

Therefore, the first partial derivatives of the function $w = \frac{e^v}{u+v^2}$ are,

$w_u = \frac{-e^v}{\left(u+v^2\right)^2}$, and $w_v = \frac{e^v}{\left(u+v^2\right)} - \frac{2ve^v}{\left(u+v^2\right)^2}$.

Answer 25E.

Let us start by finding g_u.

We know that $g_u = \dfrac{\partial}{\partial x} g(u, v)$.

$$g_u = \frac{\partial}{\partial u}\left(u^2 v - v^3\right)^5$$
$$= 10uv\left(u^2 v - v^3\right)^4$$

We get $\boxed{g_u = 10uv\left(u^2 v - v^3\right)^4}$.

Now, we have $g_v = \dfrac{\partial}{\partial y} g(u, v)$.

$$g_v = \frac{\partial}{\partial v}\left(u^2 v - v^3\right)^5$$
$$= 5\left(u^2 - 3v^2\right)\left(u^2 v - v^3\right)^4$$

Thus, we get $\boxed{g_v = 5\left(u^2 - 3v^2\right)\left(u^2 v - v^3\right)^4}$.

Answer 26E.

Let us start by finding u_r.

We know that $u_r = \dfrac{\partial}{\partial x} u(r, \theta)$.

$$u_r = \frac{\partial}{\partial r}\sin(r\cos\theta)$$
$$= \left[\cos(r\cos\theta)\right]\cos\theta$$

We get $\boxed{u_r = \cos(\theta)\left[\cos(r\cos\theta)\right]}$.

Now, we have $u_\theta = \dfrac{\partial}{\partial \theta} u(r, \theta)$.

$$u_\theta = \frac{\partial}{\partial r}\sin(r\cos\theta)$$
$$= -\left[\cos(r\cos\theta)\right]r\sin\theta$$

Thus, we get $\boxed{u_\theta = -r\sin(\theta)\left[\cos(r\cos\theta)\right]}$.

Answer 27E.

Let us start by finding R_p.

We know that $f_x = \frac{\partial}{\partial x} f(x, y)$.

$$R_p = \frac{\partial}{\partial p}\left[\tan^{-1}\left(pq^2\right)\right]$$

$$= \frac{q^2}{1 + p^2 q^4}$$

We get $\boxed{R_p = \frac{q^2}{1 + p^2 q^4}}$.

Now, we have $R_q = \frac{\partial}{\partial q} R(p, q)$.

$$R_q = \frac{\partial}{\partial q}\left[\tan^{-1}\left(pq^2\right)\right]$$

$$= \frac{2pq}{1 + p^2 q^4}$$

Thus, we get $\boxed{R_q = \frac{2pq}{1 + p^2 q^4}}$.

Answer 28E.

The given function is

$f(x,y) = x^y$

Taking logarithms on both sides

$\ln f = y \ln x$ ------------- (1)

Partially differentiating both sides with respect to x

$$\frac{1}{f}\frac{\partial f}{\partial x} = y \times \frac{1}{x}$$

i.e. $$\frac{\partial f}{\partial x} = f\frac{y}{x}$$

i.e. $$\frac{\partial f}{\partial x} = \frac{x^y y}{x}$$

i.e. $$\boxed{f_x(x,y) = y\,x^{y-1}}$$

Now partially differentiating both sides of (1) with respect to y

$$\frac{1}{f}\frac{\partial f}{\partial y} = \ln x$$

i.e. $$\frac{\partial f}{\partial y} = f \ln x$$

i.e. $$\boxed{f_y(x,y) = x^y \ln x}$$

Answer 29E.

Let us start by evaluating the integral.

$$\begin{aligned} F(x,y) &= \int_y^x \cos\left(e^t\right) dt \\ &= \left[\mathrm{Ci}\left(e^t\right)\right]_y^x \\ &= \mathrm{Ci}\left(e^x\right) - \mathrm{Ci}\left(e^y\right) \end{aligned}$$

We get $F(x,y) = \mathrm{Ci}\left(e^x\right) - \mathrm{Ci}\left(e^y\right)$, where Ci is the cosine integral.

Now, find F_x. We know that $f_x = \frac{\partial}{\partial x} f(x,y)$.

$$\begin{aligned} F_x &= \frac{\partial}{\partial x}\left[\mathrm{Ci}\left(e^x\right) - \mathrm{Ci}\left(e^y\right)\right] \\ &= \cos\left(e^x\right) \end{aligned}$$

We get $\boxed{F_x = \cos\left(e^x\right)}$.

Now, we have $F_y = \frac{\partial}{\partial y} F(x,y)$.

$$\begin{aligned} F_y &= \frac{\partial}{\partial y}\left[\mathrm{Ci}\left(e^x\right) - \mathrm{Ci}\left(e^y\right)\right] \\ &= -\cos\left(e^y\right) \end{aligned}$$

Thus, we get $\boxed{F_y = -\cos\left(e^y\right)}$.

Answer 30E.

Consider the function $F(\alpha,\beta)=\int_{\alpha}^{\beta}\sqrt{t^3+1}dt$

The objective is to find the first partial derivatives of the function.

In order to find the partial derivatives, need to apply the Fundamental Theorem of Calculus.

According to Fundamental Theorem of Calculus we have,

If $f(x)$ is continuous on $[a,b]$ and let $F(x)$ be an antiderivative of $f(x)$ on $[a,b]$ then $\int_{a}^{b} f(x)dx = F(b)-F(a)$.

To find the first partial derivative in terms of α, find the derivative of the function in terms of α treating β as a constant.

$$F_{\alpha}(\alpha,\beta)=\frac{\partial}{\partial\alpha}\int_{\alpha}^{\beta}\sqrt{t^3+1}dt$$

$=\sqrt{\alpha^3+1}\frac{\partial}{\partial\alpha}(\alpha)-\sqrt{\beta^3+1}\frac{\partial}{\partial\alpha}(\beta)$ Use Fundamental Theorem of Calculus

$=\sqrt{\alpha^3+1}\cdot(1)-0$ Use $\frac{\partial}{\partial x}(x)=1,\frac{\partial}{\partial y}(c)=0$

$$=\boxed{\sqrt{\alpha^3+1}}$$

To find the first partial derivative in terms of β, find the derivative of the function in terms of β treating α as a constant.

$$F_{\beta}(\alpha,\beta)=\frac{\partial}{\partial\beta}\int_{\alpha}^{\beta}\sqrt{t^3+1}dt$$

$=\sqrt{\alpha^3+1}\frac{\partial}{\partial\beta}(\alpha)-\sqrt{\beta^3+1}\frac{\partial}{\partial\beta}(\beta)$ Use Fundamental Theorem of Calculus

$=0-\sqrt{\beta^3+1}(1)$ Use $\frac{\partial}{\partial x}(x)=1,\frac{\partial}{\partial y}(c)=0$

$$=\boxed{-\sqrt{\beta^3+1}}$$

Answer 31E.

Consider the function $f(x,y,z) = xz - 5x^2y^3z^4$.

Need to find the first partial derivatives of the function.

To find the first partial derivative in terms of *x*, find the derivative of the function in terms of *x* treating *y* and *z* as a constants.

The function is differentiating by using the power rule.

The power rule is, $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$

$$f_x(x,y,z) = \frac{\partial}{\partial x}(xz - 5x^2y^3z^4)$$

$$= \frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial x}(5x^2y^3z^4)$$

$$= z\frac{\partial}{\partial x}(x) - 5y^3z^4\frac{\partial}{\partial x}(x^2)$$

$= z - 5y^3z^4(2x)$ Use power rule, $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$

$$= \boxed{z - 10xy^3z^4}$$

To find the first partial derivative in terms of *y*, find the derivative of the function in terms of *y* treating *x* and *z* as constants.

$$f_y(x,y,z) = \frac{\partial}{\partial y}(xz - 5x^2y^3z^4)$$

$$= \frac{\partial}{\partial y}(xz) - \frac{\partial}{\partial y}(5x^2y^3z^4)$$

$$= 0 - 5x^2z^4\frac{\partial}{\partial y}(y^3)$$

$= -5x^2(3y^2)z^4$ Use power rule, $\frac{d}{dx}(x^n) = n \cdot x^{n-1}, \frac{\partial}{\partial x}(c) = 0$

$$= \boxed{-15x^2y^2z^4}$$

To find the first partial derivative in terms of z, find the derivative of the function in terms of z treating x and y as constants.

$$f_z(x,y,z) = \frac{\partial}{\partial z}\left(xz - 5x^2y^3z^4\right)$$

$$= \frac{\partial}{\partial z}(xz) - \frac{\partial}{\partial z}\left(5x^2y^3z^4\right)$$

$$= x\frac{\partial}{\partial z}(z) - 5y^3x^2\frac{\partial}{\partial z}\left(z^4\right)$$

$$= x(1) - 5x^2y^3\left(4z^3\right)$$ Use power rule, $\frac{d}{dx}\left(x^n\right) = n\cdot x^{n-1}$

$$= x - 20x^2y^3z^3$$

$$= \boxed{x - 20xy^3z^3}$$

Answer 32E.

Given function is $f(x, y, z) = x\sin(y - z)$

Differentiate the given function with respect to x, we have

$$f_x(x, y, z) = \sin(y - z)$$

Differentiate the given function with respect to y, we have

$$f_y(x, y, z) = x\cos(y - z)$$

Differentiate the given function with respect to z, we have

$$f_z(x, y, z) = -x\cos(y - z)$$

Answer 33E.

$$w = \ln\left(x + 2y + 3z\right)$$

Differentiating partially with respect to x,

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\ln\left(x + 2y + 3z\right)$$

$$= \frac{1}{x + 2y + 3z}.(1)$$

$$= \frac{1}{x + 2y + 3z}$$

Differentiating w partially with respect to y,

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\ln(x+2y+3z)$$
$$= \frac{1}{x+2y+3z}(2)$$
$$= \frac{2}{x+2y+3z}$$

Differentiating w partially with respect to z,

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}\ln(x+2y=3z)$$
$$= \frac{1}{x+2y+3z}(3)$$
$$= \frac{3}{x+2y+3z}$$

Hence $\boxed{\frac{\partial w}{\partial x} = \frac{1}{x+2y+3z}; \frac{\partial w}{\partial y} = \frac{2}{x+2y+3z}; \frac{\partial w}{\partial z} = \frac{3}{x+2y+3z}}$

Answer 34E.

Given function is $w = ze^{xyz}$

$$\partial w / \partial x = yz^2 e^{xyz}$$

$$\partial w / \partial y = xz^2 e^{xyz}$$

$$\partial w / \partial z = e^{xyz} + xyze^{xyz}$$

Answer 35E.

Consider the function,

$$u = xy\sin^{-1}(yz).$$

Find the first partial derivatives of the given function.

Find the partial derivative with respect to x, treat y and z as constants.

$$u_x = \frac{\partial}{\partial x}\left(xy\sin^{-1}(yz)\right)$$
$$= \sin^{-1}(yz)\cdot\frac{\partial}{\partial x}(xy)$$
$$= \sin^{-1}(yz)\cdot y$$
$$= y\sin^{-1}(yz)$$

Find the partial derivative with respect to y, treat x and z as a constants.

$$u_y = \frac{\partial}{\partial y}\left(xy\sin^{-1}(yz)\right)$$

$$= xy\cdot\frac{\partial}{\partial y}\left(\sin^{-1}(yz)\right) + \sin^{-1}(yz)\cdot\frac{\partial}{\partial y}(xy)$$

$$= xy\cdot\frac{1}{\sqrt{1-(yz)^2}}\cdot\frac{\partial}{\partial y}(yz) + \sin^{-1}(yz)\cdot x$$

$$= xy\cdot\frac{1}{\sqrt{1-y^2z^2}}\cdot z + \sin^{-1}(yz)\cdot x$$

$$= \frac{xyz}{\sqrt{1-y^2z^2}} + x\sin^{-1}(yz)$$

Find the partial derivative with respect to z, treat x and y as constants.

$$u_z = \frac{\partial}{\partial z}\left(xy\sin^{-1}(yz)\right)$$

$$= xy\cdot\frac{\partial}{\partial z}\left(\sin^{-1}(yz)\right)$$

$$= xy\cdot\frac{1}{\sqrt{1-(yz)^2}}\cdot\frac{\partial}{\partial z}(yz)$$

$$= xy\cdot\frac{1}{\sqrt{1-y^2z^2}}\cdot y$$

$$= \frac{xy^2}{\sqrt{1-y^2z^2}}$$

Therefore, the first partial derivatives of the given function are,

$$\boxed{u_x = y\sin^{-1}(yz), u_y = \frac{xyz}{\sqrt{1-y^2z^2}} + x\sin^{-1}(yz), u_z = \frac{xy^2}{\sqrt{1-y^2z^2}}}.$$

Answer 36E.

$$u = x^{y/z}$$

Taking logarithms on both sides

$$\ln u = \frac{y}{z}\ln x \quad \text{--------------- (1)}$$

Differentiating (1) partially with respect to x,

$$\frac{1}{u}\frac{\partial u}{\partial x}=\frac{y}{z}\times\frac{1}{x}$$

$$\frac{\partial u}{\partial x}=\frac{uy}{zx}$$

i.e. $$\boxed{u_x=\frac{x^{\frac{y}{z}}y}{xz}}$$

Differentiating (1) partially with respect to y

$$\frac{1}{u}\frac{\partial u}{\partial y}=\frac{1}{z}\ln x$$

$$\frac{\partial u}{\partial y}=\frac{u}{z}\ln x$$

i.e. $$\boxed{u_y=\frac{x^{y/z}}{z}\ln x}$$

Differentiating (1) partially with respect to z

$$\frac{1}{u}\frac{\partial u}{\partial z}=\frac{-y}{z^2}\ln x$$

i.e. $$\frac{\partial u}{\partial z}=-\frac{uy}{z^2}\ln x$$

i.e. $$\boxed{u_z=\frac{-x^{y/z}y\ln x}{z^2}}$$

Answer 37E.

Consider t, y, and z to be a constant and differentiate the given function with respect to x to find $h_x(x, y, z, t)$.

$$\begin{aligned}h_x(x, y, z, t) &= \frac{\partial}{\partial x}\left(x^2y\cos(z/t)\right)\\ &= 2xy\cos(z/t)\end{aligned}$$

Now, find $h_y(x, y, z, t)$.

$$\begin{aligned}h_y(x, y, z, t) &= \frac{\partial}{\partial y}\left(x^2y\cos(z/t)\right)\\ &= x^2\cos(z/t)\end{aligned}$$

Evaluate $h_z(x, y, z, t)$.

$$\begin{aligned}h_z(x, y, z, t) &= \frac{\partial}{\partial z}\left(x^2y\cos(z/t)\right)\\ &= -\frac{x^2y\sin(z/t)}{t}\end{aligned}$$

Find $h_t(x, y, z, t)$.

$$h_t(x, y, z, t) = \frac{\partial}{\partial t}\left(x^2 y\cos(z/t)\right)$$
$$= \frac{x^2 yz\sin(z/t)}{t^2}$$

Therefore, we get $h_x(x, y, z, t) = 2xy\cos(z/t)$, $h_y(x, y, z, t) = x^2\cos(z/t)$,

$h_z(x, y, z, t) = -\frac{x^2 y\sin(z/t)}{t}$, and $h_t(x, y, z, t) = \frac{x^2 yz\sin(z/t)}{t^2}$.

Answer 38E.

Let us start by finding ϕ_x.

We know that $f_x = \frac{\partial}{\partial x} f(x, y)$.

$$\phi_x = \frac{\partial}{\partial x}\left[\frac{\alpha x + \beta y^2}{\gamma z + \delta t^2}\right]$$
$$= \frac{\alpha}{\gamma z + \delta t^2}$$

Therefore, we get $\phi_x = \frac{\alpha}{\gamma z + \delta t^2}$.

Now, we have $\phi_y = \frac{\partial}{\partial y}\phi(x, y, z, t)$.

$$\phi_y = \frac{\partial}{\partial y}\left[\frac{\alpha x + \beta y^2}{\gamma z + \delta t^2}\right]$$
$$= \frac{2\beta y}{\gamma z + \delta t^2}$$

Thus, ϕ_y is obtained as $\frac{2\beta y}{\gamma z + \delta t^2}$

Answer 39E.

$$u = \sqrt{x_1^2 + x_2^2 + \ldots\ldots + x_n^2}$$

i.e. $u = \left(x_1^2 + x_2^2 + \ldots\ldots + x_n^2\right)^{1/2}$ ---------------------- (1)

On differentiating partially both sides with respect to x_1

$$\frac{\partial u}{\partial x_1} = \frac{1}{2}\left(x_1^2 + x_2^2 + \dots\dots\dots + x_n^2\right)^{-1/2}(2x_1)$$

i.e. $$\frac{\partial u}{\partial x_1} = \frac{x_1}{\sqrt{x_1^2 + x_2^2 + \dots\dots\dots + x_n^2}}$$

On differentiating partially both sides of (1) with respect to x_2

$$\frac{\partial u}{\partial x_2} = \frac{1}{2}\left(x_1^2 + x_2^2 + \dots\dots\dots + x_n^2\right)^{-1/2}(2x_2)$$

i.e. $$\frac{\partial u}{\partial x_2} = \frac{x_2}{\sqrt{x_1^2 + x_2^2 + \dots\dots + x_n^2}}$$

Proceeding in the same manner we find in general

$$\frac{\partial u}{\partial x_i} = \frac{x_i}{\sqrt{x_1^2 + x_2^2 + \dots\dots\dots + x_n^2}}$$

$i = 1, 2, \dots\dots\dots\dots, n$

Answer 40E.

$u = \sin\left(x_1 + 2x_2 + \dots\dots + n\,x_n\right)$ -------------------- (1)

Differentiating (1) partially with respect to x

$$\frac{\partial u}{\partial x_1} = \cos\left(x_1 + 2\,x_2 + \dots\dots\dots\dots + n\,x_n\right).(1)$$

Differentiating (1) partially with respect to x_2:

$$\frac{\partial u}{\partial x_2} = \cos\left(x_1 + 2x_2 + \dots\dots\dots + n\,x_n\right).(2)$$

⋮

Differentiating (1) partially with respect to x_n:

$$\frac{\partial u}{\partial x_n} = \cos\left(x_1 + 2x_2 + \dots\dots\dots + nx_n\right).n$$

Hence in general

$$\boxed{\frac{\partial u}{\partial x_i} = i\cos\left(x_1 + 2x_2 + \dots\dots\dots + n\,x_n\right)}$$

Where $\boxed{i = 1, 2, \dots\dots\dots, n}$

Answer 41E.

Given function is $f\left(x, y\right) = \ln\left(x + \sqrt{x^2 + y^2}\right)$

$$f_x\left(x, y\right) = 1/\left(x + \sqrt{x^2 + y^2}\right)\cdot\left[1 + 1/2\left(x^2 + y^2\right)^{-1/2}\left(2x\right)\right]$$

$$f_x(3, 4) = 1/(3 + \sqrt{25})\cdot[1 + (1/2 \times 1/5)\cdot 6]$$

$$f_x(3, 4) = (1/8)\cdot(1 + 3/5)$$

$$f_x\left(3, 4\right) = \left(1/8\right) \times \left(8/5\right) = 8/40 = \frac{1}{5}$$

Answer 42E.

Given function is $f(x, y) = \arctan(y/x)$

$$f_x\left(x, y\right) = \left[1/1 + (y/x)^2\right]\cdot\left(-y/x^2\right)$$

$$f_x\left(2, 3\right) = \left[1/1 + (3/2)^2\right]\cdot\left(-3/4\right)$$

$$f_x(2, 3) = 4/13(-3/4)$$

$$f_x(2, 3) = -12/52 = -3/13$$

Answer 43E.

Consider the following function:

$$f(x, y, z) = \frac{y}{x + y + z}$$

Find the partial derivative with respect to ‘ y ’.

Use quotient rule of derivative as shown below:

According to the quotient rule of derivative, “If $u(x)$ and $v(x)$ are two differentiable functions, then

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{v(x)\frac{d}{dx}(u(x)) - u(x)\frac{d}{dx}(v(x))}{[v(x)]^2}$$ is called the quotient rule”.

Consider the following function:

$$f(x,y,z)=\frac{y}{x+y+z}$$

Differentiate the function partially with respect to y.

$$f_y(x,y,z)=\frac{(x+y+z)\frac{d}{dy}(y)-y\frac{d}{dy}(x+y+z)}{(x+y+z)^2};$$

$$=\frac{(x+y+z)(1)-y(0+1+0)}{(x+y+z)^2};$$

$$=\frac{x+y+z-y}{(x+y+z)^2};$$

$$=\frac{x+z}{(x+y+z)^2}.$$

Therefore, the partial derivative with respect to ' y ' is given as follows:

$$f_y(x,y,z)=\boxed{\frac{x+z}{(x+y+z)^2}}.$$

Consider the point $(2,1,-1)$.

The partial derivative at the point $(2,1,-1)$ is computed as under:

$$f_y(x,y,z)=\frac{x+z}{(x+y+z)^2}$$

$$f_y(2,1,-1)=\frac{2-1}{(2+1-1)^2};$$

$$=\frac{1}{4}.$$

Therefore, $f_y(2,1,-1)=\boxed{\frac{1}{4}}.$

Answer 44E.

Consider the following function:

$$f(x,y,z)=\sqrt{\sin^2 x+\sin^2 y+\sin^2 z}$$

Find the partial derivative with respect to ' z ':

$$\frac{\partial f}{\partial z}=f_z(x,y,z)$$
$$=\frac{1}{2\sqrt{\sin^2 x+\sin^2 y+\sin^2 z}}\frac{\partial}{\partial z}\left(\sin^2 x+\sin^2 y+\sin^2 z\right)$$
$$=\frac{1}{2\sqrt{\sin^2 x+\sin^2 y+\sin^2 z}}\left(0+0+2\sin z\cos z\right)$$
$$=\frac{\sin z\cos z}{\sqrt{\sin^2 x+\sin^2 y+\sin^2 z}}$$

Therefore, partial derivative with respect to ' z ' is as follows:

$$f_z(x,y,z)=\frac{\sin z\cos z}{\sqrt{\sin^2 x+\sin^2 y+\sin^2 z}}.$$

Consider the point $\left(0,0,\frac{\pi}{4}\right)$.

Partial derivative at $\left(0,0,\frac{\pi}{4}\right)$ is calculated as follows:

$$f_z(x,y,z)=\frac{\sin z\cos z}{\sqrt{\sin^2 x+\sin^2 y+\sin^2 z}};$$
$$f_z\left(0,0,\frac{\pi}{4}\right)=\frac{\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)}{\sqrt{\sin^2(0)+\sin^2(0)+\sin^2\left(\frac{\pi}{4}\right)}};$$
$$=\frac{\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)}{\sqrt{0+0+\frac{1}{2}}};$$
$$=\frac{\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)}{\left(\frac{1}{\sqrt{2}}\right)};$$
$$=\frac{1}{\sqrt{2}};$$

Therefore, the value of the partial derivative is $f_z\left(0,0,\frac{\pi}{4}\right)=\boxed{\frac{1}{\sqrt{2}}}$.

Answer 45E.

Consider the following function:

$$f(x,y)=xy^2-x^3y$$

It is required to find $f_x(x,y)$ and $f_y(x,y)$ using limits.

To find the first partial derivative with respect to *x*, use the following formula:

$$f_x(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

Evaluate the limit by simplifying the expression.

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}&=\lim_{h\to 0}\frac{(x+h)y^2-(x+h)^3y-(xy^2-x^3y)}{h}\\&=\lim_{h\to 0}\frac{(xy^2+hy^2)-(x^3+3x^2h+3xh^2+h^3)y-(xy^2-x^3y)}{h}\end{aligned}$$

$$\begin{aligned}&=\lim_{h\to 0}\frac{xy^2+hy^2-x^3y-3x^2hy-3xh^2y-h^3y-xy^2+x^3y}{h}\\&=\lim_{h\to 0}\frac{hy^2-3x^2hy-3xh^2y-h^3y}{h}\end{aligned}$$

Calculate the limit.

$$\begin{aligned}\lim_{h\to 0}\frac{hy^2-3x^2hy-3xh^2y-h^3y}{h}&=\lim_{h\to 0}\frac{h(y^2-3x^2y-3xhy-h^2y)}{h}\\&=\lim_{h\to 0}(y^2-3x^2y-3xhy-h^2y)\\&=y^2-3x^2y-3x(0)y-(0)^2y\end{aligned}$$

Therefore, the first partial derivative with respect to *x* is the following:

$$f_x(x,y)=\boxed{y^2-3x^2y}$$

To find the first partial derivative with respect to *y*, use the following formula:

$$f_y(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

Evaluate the limit by simplifying the expression.

$$\begin{aligned}\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}&=\lim_{h\to 0}\frac{\left(x(y+h)^2-x^3(y+h)\right)-(xy^2-x^3y)}{h}\\&=\lim_{h\to 0}\frac{\left(x(y^2+2yh+h^2)-x^3y-x^3h\right)-(xy^2-x^3y)}{h}\end{aligned}$$

$$\begin{aligned}&=\lim_{h\to 0}\frac{xy^2+2xyh+xh^2-x^3y-x^3h-xy^2+x^3y}{h}\\&=\lim_{h\to 0}\frac{2xyh+xh^2-x^3h}{h}\end{aligned}$$

Calculate the limit.

$$\lim_{h\to 0}\frac{2xyh+xh^2-x^3h}{h}=\lim_{h\to 0}\frac{h\left(2xy+xh-x^3\right)}{h}$$
$$=\lim_{h\to 0}\left(2xy+xh-x^3\right)$$
$$=2xy+x(0)-x^3$$

Therefore, the first partial derivative with respect to *y* is the following:

$$f_y(x,y)=\boxed{2xy-x^3}.$$

Answer 46E.

Consider the function

$$f(x,y)=\frac{x}{x+y^2}$$

To find the first partial derivative in terms of *x*, find the following limit.

$$f_x(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

Evaluate the limit to find the partial derivative.

Start by simplifying the expression.

$$\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}=\lim_{h\to 0}\frac{\frac{(x+h)}{(x+h)+y^2}-\frac{x}{x+y^2}}{h}$$
$$=\lim_{h\to 0}\left(\frac{(x+h)}{\left((x+h)+y^2\right)h}-\frac{x}{\left(x+y^2\right)h}\right)$$
$$=\lim_{h\to 0}\left(\frac{(x+h)}{\left((x+h)+y^2\right)h}\left(\frac{x+y^2}{x+y^2}\right)-\frac{x}{\left(x+y^2\right)h}\left(\frac{(x+h)+y^2}{(x+h)+y^2}\right)\right)$$
$$=\lim_{h\to 0}\frac{(x+h)\left(x+y^2\right)-x\left(x+h+y^2\right)}{h\left(x+h+y^2\right)\left(x+y^2\right)}$$

Then calculate the limit.

$$\lim_{h\to 0}\frac{(x+h)\left(x+y^2\right)-x\left(x+h+y^2\right)}{h\left(x+h+y^2\right)\left(x+y^2\right)}=\lim_{h\to 0}\frac{x^2+xh+xy^2+hy^2-x^2-xh-xy^2}{h\left(x+h+y^2\right)\left(x+y^2\right)}$$
$$=\lim_{h\to 0}\frac{hy^2}{h\left(x+h+y^2\right)\left(x+y^2\right)}$$
$$=\lim_{h\to 0}\frac{y^2}{\left(x+(0)+y^2\right)\left(x+y^2\right)}$$
$$f_x(x,y)=\boxed{\frac{y^2}{\left(x+y^2\right)^2}}$$

To find the first partial derivative in terms of y, find the following limit.

$$f_y(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

Evaluate the limit to find the partial derivative. Start by simplifying the expression.

$$\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{x}{x+(y+h)^2}-\frac{x}{x+y^2}}{h}$$

$$=\lim_{h\to 0}\frac{x}{h(x+y^2+2yh+h^2)}-\frac{x}{h(x+y^2)}$$

$$=\lim_{h\to 0}\frac{x}{h(x+y^2+2yh+h^2)}\left(\frac{(x+y^2)}{(x+y^2)}\right)-\frac{x}{h(x+y^2)}\left(\frac{(x+y^2+2yh+h^2)}{(x+y^2+2yh+h^2)}\right)$$

$$=\lim_{h\to 0}\frac{x((x+y^2))-x(x+y^2+2yh+h^2)}{h(x+y^2+2yh+h^2)(x+y^2)}$$

Then calculate the limit.

$$\lim_{h\to 0}\frac{x((x+y^2))-x(x+y^2+2yh+h^2)}{h(x+y^2+2yh+h^2)(x+y^2)}=\lim_{h\to 0}\frac{x^2+xy^2-x^2-xy^2-2xyh-xh^2}{h(x+y^2+2yh+h^2)(x+y^2)}$$

$$=\lim_{h\to 0}\frac{-2xyh-xh^2}{h(x+y^2+2yh+h^2)(x+y^2)}$$

$$=\frac{-2xy-x(0)}{(x+y^2+2y(0)+(0)^2)(x+y^2)}$$

$$f_y(x,y)=\boxed{\frac{-2xy}{(x+y^2)^2}}$$

Answer 47E.

Consider the equation $x^2+2y^2+3z^2=1$.

The objective is to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ using implicit differentiation.

Here z is defined implicitly in terms of x, and y.

Differentiate with respect to *x* on both sides of the equation $x^2+2y^2+3z^2=1$ to get,

$$2x+0+3(2z)\frac{\partial z}{\partial x}=0 \quad (\text{keeping } y \text{ as constant})$$

$$6z\frac{\partial z}{\partial x}=-2x$$

$$\frac{\partial z}{\partial x}=-\frac{2x}{6z}$$

$$=-\frac{x}{3z}$$

Thus, $\frac{\partial z}{\partial x}=\boxed{-\frac{x}{3z}}$.

Differentiate with respect to *y* on both sides of the equation $x^2+2y^2+3z^2=1$ to get,

$$0+2(2y)+3(2z)\frac{\partial z}{\partial y}=0 \quad (\text{keeping } x \text{ as constant})$$

$$6z\frac{\partial z}{\partial x}=-4y$$

$$\frac{\partial z}{\partial x}=-\frac{4y}{6z}$$

$$=-\frac{2y}{3z}$$

Thus, $\frac{\partial z}{\partial x}=\boxed{-\frac{2y}{3z}}$.

Answer 48E.

We can start by rewriting the given equation.

$x^2 - y^2 + z^2 - 2z - 4 = 0$

Let $F(x, y, z) = x^2 - y^2 + z^2 - 2z - 4$.

Evaluate $F_x(x, y, z)$, $F_y(x, y, z)$, and $F_z(x, y, z)$.

$$\begin{aligned} F_x(x, y, z) &= \frac{\partial}{\partial x}\left(x^2 - y^2 + z^2 - 2z - 4\right) \\ &= 2x \end{aligned}$$

$$\begin{aligned} F_y(x, y, z) &= \frac{\partial}{\partial y}\left(x^2 - y^2 + z^2 - 2z - 4\right) \\ &= -2y \end{aligned}$$

$$\begin{aligned} F_z(x, y, z) &= \frac{\partial}{\partial z}\left(x^2 - y^2 + z^2 - 2z - 4\right) \\ &= 2z - 2 \end{aligned}$$

Now, we know that $\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)}$ and $\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}$.

$$\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{2x}{2z - 2} \\ &= -\frac{x}{z - 1} \end{aligned} \qquad \begin{aligned} \frac{\partial z}{\partial y} &= -\frac{-2y}{2z - 2} \\ &= \frac{y}{z - 1} \end{aligned}$$

Thus, we get the partial derivatives of z as $\boxed{\frac{\partial z}{\partial x} = -\frac{x}{z - 1} \text{ and } \frac{\partial z}{\partial y} = \frac{y}{z - 1}}$.

Answer 49E.

We can start by rewriting the given equation.
$e^z - xyz = 0$

Let $F(x, y, z) = e^z - xyz$.

Evaluate $F_x(x, y, z)$, $F_y(x, y, z)$, and $F_z(x, y, z)$.

$$\begin{aligned} F_x(x, y, z) &= \frac{\partial}{\partial x}\left(e^z - xyz\right) \\ &= -yz \end{aligned}$$

$$\begin{aligned} F_y(x, y, z) &= \frac{\partial}{\partial y}\left(e^z - xyz\right) \\ &= -xz \end{aligned}$$

$$\begin{aligned} F_z(x, y, z) &= \frac{\partial}{\partial z}\left(e^z - xyz\right) \\ &= e^z - xy \end{aligned}$$

Now, we know that $\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)}$ and $\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}$.

$$\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{-yz}{e^z - xy} \\ &= \frac{yz}{e^z - xy} \end{aligned} \qquad \begin{aligned} \frac{\partial z}{\partial y} &= -\frac{-xz}{e^z - xy} \\ &= \frac{xz}{e^z - xy} \end{aligned}$$

Thus, we get the partial derivatives of z as $\boxed{\frac{\partial z}{\partial x} = \frac{yz}{e^z - xy} \text{ and } \frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}}$.

Answer 50E.

Consider,

$$yz + x\ln y = z^2$$

Here z is defined implicitly in terms of x, and y.

Use implicit differentiation, to find $\frac{\partial z}{\partial x}$, and $\frac{\partial z}{\partial y}$

Differentiate with respect to x (keeping y as constant) on both sides of the equation,

$$yz + x\ln y = z^2$$

$$\frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial x}(x\ln y) = \frac{\partial}{\partial x}(z^2)$$

$$y\frac{\partial z}{\partial x} + \ln y\frac{\partial x}{\partial x} = 2z\frac{\partial z}{\partial x}$$

$$(y-2z)\frac{\partial z}{\partial x} = -\ln y$$

$$\frac{\partial z}{\partial x} = \boxed{-\frac{\ln y}{y-2z}}$$

Differentiate with respect to y (keeping x as constant) on both sides of the equation,

$$yz + x\ln y = z^2$$

$$\frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial y}(x\ln y) = \frac{\partial}{\partial y}(z^2)$$

$$y\frac{\partial z}{\partial y} + z\frac{\partial y}{\partial y} + x\frac{\partial}{\partial y}(\ln y) = 2z\frac{\partial z}{\partial y}$$

$$y\frac{\partial z}{\partial y} + z + x\left(\frac{1}{y}\right) = 2z\frac{\partial z}{\partial y}$$

$$(y-2z)\frac{\partial z}{\partial y} = -\left(z + \frac{x}{y}\right)$$

$$\frac{\partial z}{\partial y} = \boxed{-\left(\frac{x+yz}{y(y-2z)}\right)}$$

Answer 51E.

(A)

$$z = f(x) + g(y) \qquad \text{----------------- (1)}$$

Differentiating (1) partially with respect to x:

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}$$

But g is a function of y only, then $\frac{\partial g}{\partial x} = 0$

And f is a function of x only, then $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}$

$$= f'(x)$$

Hence $\frac{\partial z}{\partial x} = f'(x)$

Now differentiating (1) partially with respect to y:

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}$$

But $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial g}{\partial y} = \frac{\partial g}{\partial y}$

$$= g'(y)$$

Hence $\frac{\partial z}{\partial y} = g'(y)$

(B)

$$z = f(x+y) \quad \text{----------------- (2)}$$

Differentiating (2) partially with respect to x:

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}$$

Since $f = f(x+y)$

If we take $(x+y) = u$

Then $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}$

$$= \frac{\partial f}{\partial u}(1+0)$$

$$= \frac{\partial f}{\partial u} = f'(u)$$

Hence $\frac{\partial z}{\partial x} = f'(x+y)$

Now differentiating (2) partially with respect to y:

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial y}$$

Since $f=f(x+y)$, if we take $x+y=u$

Then $\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}.\frac{\partial}{\partial y}(x+y)$

$$=\frac{\partial f}{\partial u}$$

$$=f'(u)$$

$$=f'(x+y)$$

Hence $\frac{\partial z}{\partial y}=f'(x+y)$

Answer 52E.

(A)

$$z=f(x)g(y) \qquad \text{------------------- (1)}$$

Differentiating (1) partially with respect to x

$$\frac{\partial z}{\partial x}=\frac{\partial f(x)}{\partial x}g(y)+f(x)\frac{\partial}{\partial x}g(y)$$

$=g(y)\frac{\partial f(x)}{\partial x}+f(x)(0)$ {Since g(y) is a function of y only then $\frac{\partial g}{\partial x}=0$ }

As f is a function of x only then

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial x}=f'(x)$$

Therefore $\frac{\partial z}{\partial x}=g(y)f'(x)$

Now differentiating (1) partially with respect to y

$$\frac{\partial z}{\partial y}=\frac{\partial f(x)}{\partial y}.g(y)+f(x)\frac{\partial}{\partial y}g(y)$$

But $\frac{\partial f(x)}{\partial y}=0$ (As f is a function of x only)

And $\frac{\partial g(y)}{\partial y}=\frac{\partial g}{\partial y}=g'(y)$

Hence $\frac{\partial z}{\partial y}=f(x)g'(y)$

(B)

$$z = f(xy) \qquad \text{-------------------} \ (2)$$

Differentiating (2) partially with respect to x

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}$$

Now $f = f(xy)$, if we take $xy = u$

Then $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial u} \cdot \dfrac{\partial u}{\partial x}$

$$= \frac{\partial f}{\partial u} \cdot \frac{\partial}{\partial u}(xy)$$

$$= \frac{\partial f}{\partial u} \cdot yx$$

$$= \frac{2}{s}$$

$$= y\, f'(xy)$$

Hence $\boxed{\dfrac{\partial z}{\partial x} = y\, f'(xy)}$

Differentiating (2) partially with respect to:

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}$$

$$= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x}$$

$$= \frac{\partial f}{\partial u} \cdot \frac{\partial}{\partial x}(xy)$$

$$= f'(u)(x)$$

$$= x\, f'(xy)$$

Hence $\boxed{\dfrac{\partial z}{\partial y} = x\, f'(xy)}$

(C)

$$z = f\left(x/y\right)$$

Take $x/y = u$

Then $\dfrac{\partial u}{\partial x} = \dfrac{1}{y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{x}{y^2}$

Now $z = f(u) \qquad \text{------------} \ (3)$

Differentiating (3) partially with respect to x:

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x}$$
$$= \frac{\partial f}{\partial u} \cdot \frac{1}{y}$$
$$= \frac{f'(u)}{y}$$

Hence $\boxed{\frac{\partial z}{\partial x} = \frac{f'\left(x/y\right)}{y}}$

Differentiating (3) partially with respect to y:

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y}$$
$$= f'(u)\left(-\frac{x}{y^2}\right)$$

Hence $\boxed{\frac{\partial z}{\partial y} = -\frac{x}{y^2} f'\left(x/y\right)}$

Answer 53E.

Given function is $f(x, y) = x^3 y^5 + 2x^4 y$

$$f_x(x, y) = 3x^2 y^5 + 8x^3 y$$

$$f_{xx}(x, y) = 6xy^5 + 24x^2 y$$

$$f_{xy}(x, y) = 15x^2 y^4 + 8x^3$$

$$f_y(x, y) = 5x^3 y^4 + 2x^4$$

$$f_{yy}(x, y) = 20x^3 y^3$$

$$f_{yx}(x, y) = 15x^2 y^4 + 8x^3$$

$f_{xy}(x, y) = f_{yx}(x, y)$ by Clairaut's Theorem

Answer 54E.

Consider the function,

$$f(x,y)=\sin^2(mx+ny).$$

The objective is to find all the second partial derivatives of the given function.

To find f_x, take y as a constant and differentiate $f(x,y)$ with respect to x

So,

$$f_x=\frac{\partial}{\partial x}f(x,y)$$

$$=\frac{\partial}{\partial x}\left[\sin^2(mx+ny)\right]$$

$$=2\sin(mx+ny)\frac{\partial}{\partial x}\left[\sin(mx+ny)\right] \qquad \text{By chain rule}$$

$$=2\sin(mx+ny)\cdot\cos(mx+ny)\cdot\frac{\partial}{\partial x}(mx+ny)$$

$$=2\sin(mx+ny)\cos(mx+ny)\cdot\left[m\frac{\partial}{\partial x}(x)+n\frac{\partial}{\partial x}(y)\right]$$

$$=2\sin(mx+ny)\cos(mx+ny)\cdot[m\cdot 1+n\cdot 0]$$

$$=\sin 2(mx+ny)[m\cdot 1+n\cdot 0]$$

$$=m\sin 2(mx+ny)$$

To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

So, $f_y=\frac{\partial}{\partial y}f(x,y)$

$$=\frac{\partial}{\partial y}\left[\sin^2(mx+ny)\right]$$

$$=2\sin(mx+ny)\frac{\partial}{\partial y}\left[\sin(mx+ny)\right]$$

$$=2\sin(mx+ny)\cdot\cos(mx+ny)\cdot\frac{\partial}{\partial y}(mx+ny)$$

$$=2\sin(mx+ny)\cos(mx+ny)\cdot\left[m\frac{\partial}{\partial y}(x)+n\frac{\partial}{\partial y}(y)\right]$$

$$=2\sin(mx+ny)\cos(mx+ny)\cdot[m\cdot 0+n\cdot 1]$$

$$=\sin 2(mx+ny)[m\cdot 0+n\cdot 1]$$

$$=n\sin 2(mx+ny)$$

Now find the second order partial derivatives.

For f_{xx}, differentiate $f_x = m\sin 2(mx+ny)$ partially, with respect to x.

$$f_{xx} = \frac{\partial}{\partial x} f_x$$
$$= \frac{\partial}{\partial x}\left[m\sin 2(mx+ny)\right]$$
$$= m\frac{\partial}{\partial x}\left[\sin 2(mx+ny)\right]$$
$$= m\cdot\cos 2(mx+ny)\frac{\partial}{\partial x}\left[2(mx+ny)\right]$$
$$= m\cdot\cos 2(mx+ny)\cdot 2\cdot\frac{\partial}{\partial x}(mx+ny)$$
$$= m\cdot\cos 2(mx+ny)\cdot 2\cdot\left[m\frac{\partial}{\partial x}(x)+n\frac{\partial}{\partial x}(y)\right]$$
$$= m\cdot\cos 2(mx+ny)\cdot 2\cdot\left[m\cdot 1+n\cdot 0\right]$$
$$= 2m^2\cos 2(mx+ny)$$

Thus, $\boxed{f_{xx} = 2m^2\cos 2(mx+ny)}$

For f_{xy}, differentiate $f_x = m\sin 2(mx+ny)$ partially, with respect to y.

$$f_{xy} = \frac{\partial}{\partial y} f_x$$
$$= \frac{\partial}{\partial y}\left[m\sin 2(mx+ny)\right]$$
$$= m\frac{\partial}{\partial y}\left[\sin 2(mx+ny)\right]$$
$$= m\cdot\cos 2(mx+ny)\frac{\partial}{\partial y}\left[2(mx+ny)\right]$$
$$= m\cdot\cos 2(mx+ny)\cdot 2\cdot\frac{\partial}{\partial y}(mx+ny)$$
$$= m\cdot\cos 2(mx+ny)\cdot 2\cdot\left[m\frac{\partial}{\partial y}(x)+n\frac{\partial}{\partial y}(y)\right]$$
$$= m\cdot\cos 2(mx+ny)\cdot 2\cdot\left[m\cdot 0+n\cdot 1\right]$$
$$= 2mn\cos 2(mx+ny)$$

Thus, $\boxed{f_{xy} = 2mn\cos 2(mx+ny)}$

For f_{yy}, differentiate $f_y = n\sin 2(mx+ny)$ partially, with respect to y.

$$f_{yy} = \frac{\partial}{\partial y} f_y$$
$$= \frac{\partial}{\partial y}\left[n\sin 2(mx+ny)\right]$$
$$= n\frac{\partial}{\partial y}\left[\sin 2(mx+ny)\right]$$
$$= n\cdot\cos 2(mx+ny)\frac{\partial}{\partial y}\left[2(mx+ny)\right]$$
$$= n\cdot\cos 2(mx+ny)\cdot 2\cdot\frac{\partial}{\partial y}(mx+ny)$$
$$= n\cdot\cos 2(mx+ny)\cdot 2\cdot\left[m\frac{\partial}{\partial y}(x)+n\frac{\partial}{\partial y}(y)\right]$$
$$= n\cdot\cos 2(mx+ny)\cdot 2\cdot\left[m\cdot 0+n\cdot 1\right]$$
$$= 2n^2\cos 2(mx+ny)$$

Thus, $\boxed{f_{yy} = 2n^2\cos 2(mx+ny)}$

Answer 55E.

Consider the function.

$$w=\sqrt{u^2+v^2}$$

First, find w_u.

The partial derivative is calculated by treating v as a constant and using the derivative rules for a power function and the chain rule.

The derivative that remains in the second line is the result of the chain rule.

$$w_u = \frac{\partial}{\partial u}\left(\sqrt{u^2+v^2}\right)$$
$$= \frac{1}{2\sqrt{u^2+v^2}}\cdot\frac{\partial}{\partial u}\left(u^2+v^2\right)$$
$$= \frac{1}{2\sqrt{u^2+v^2}}\cdot\left((2u)+0\right)$$
$$= \frac{u}{\sqrt{u^2+v^2}}$$

Find w_{uu} by finding the partial derivative of w_u in terms of *u*.

The partial derivative is calculated by using the quotient rule.

Note that the first partial derivative that results from the chain rule is calculated using the rules for calculating a polynomial and the second is the same partial derivative from the calculation of w_u.

$$
\begin{aligned}
w_{uu} &= \frac{\partial}{\partial u}\left(\frac{u}{\sqrt{u^2+v^2}}\right) \\
&= \frac{\frac{\partial}{\partial u}(u)\cdot\left(\sqrt{u^2+v^2}\right)-(u)\cdot\frac{\partial}{\partial u}\left(\sqrt{u^2+v^2}\right)}{\left(\sqrt{u^2+v^2}\right)^2} \\
&= \frac{(1)\sqrt{u^2+v^2}-u\left(\frac{u}{\sqrt{u^2+v^2}}\right)}{u^2+v^2} \\
&= \frac{\sqrt{u^2+v^2}}{u^2+v^2}-\frac{u^2}{\left(u^2+v^2\right)\sqrt{u^2+v^2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{u^2+v^2-u^2}{\left(u^2+v^2\right)\sqrt{u^2+v^2}} \\
&= \boxed{\frac{v^2}{\left(u^2+v^2\right)\sqrt{u^2+v^2}}}.
\end{aligned}
$$

Find w_{uv} by finding the partial derivative of w_u in terms of *v*.

The partial derivative is calculated by treating *u* as a constant and using the derivative rules for a power function and the chain rule.

The derivative that remains in the second line is the result of the chain rule.

$$
\begin{aligned}
w_{uv} &= \frac{\partial}{\partial v}\left(\frac{u}{\sqrt{u^2+v^2}}\right) \\
&= \frac{-u}{2\left(u^2+v^2\right)^{3/2}}\cdot\frac{\partial}{\partial v}\left(u^2+v^2\right) \\
&= \frac{-u}{2\left(u^2+v^2\right)^{3/2}}\cdot\left(0+(2v)\right) \\
&= \boxed{\frac{-uv}{\left(u^2+v^2\right)^{3/2}}}
\end{aligned}
$$

To find w_{vv}, first calculate find w_v.

The partial derivative is calculated by treating *u* as a constant and using the derivative rules for a power function and the chain rule.

The derivative that remains in the second line is the result of the chain rule.

$$w_v = \frac{\partial}{\partial v}\left(\sqrt{u^2+v^2}\right)$$
$$= \frac{1}{2\sqrt{u^2+v^2}} \cdot \frac{\partial}{\partial v}\left(u^2+v^2\right) \text{ Since } \frac{\partial}{\partial x}\left(\sqrt{x} = \frac{1}{2\sqrt{x}}\right)$$
$$= \frac{1}{2\sqrt{u^2+v^2}} \cdot \left(0+(2v)\right)$$
$$= \frac{v}{\sqrt{u^2+v^2}}$$

Find w_{vv} by finding the partial derivative of w_v in terms of *u*.

The partial derivative is calculated by using the quotient rule. Note that the first partial derivative that results from the chain rule is calculated using the rules for calculating a polynomial and the second is the same partial derivative from the calculation of w_v.

$$w_{vv} = \frac{\partial}{\partial v}\left(\frac{v}{\sqrt{u^2+v^2}}\right)$$
$$= \frac{\frac{\partial}{\partial v}(v)\cdot\left(\sqrt{u^2+v^2}\right) - (v)\cdot\frac{\partial}{\partial v}\left(\sqrt{u^2+v^2}\right)}{\left(\sqrt{u^2+v^2}\right)^2}$$
$$= \frac{(1)\sqrt{u^2+v^2} - v\left(\frac{v}{\sqrt{u^2+v^2}}\right)}{u^2+v^2}$$
$$= \frac{\sqrt{u^2+v^2}}{u^2+v^2} - \frac{v^2}{\left(u^2+v^2\right)\sqrt{u^2+v^2}}$$
$$= \frac{u^2+v^2-v^2}{u^2+v^2\sqrt{u^2+v^2}}$$
$$= \boxed{\frac{u^2}{\left(u^2+v^2\right)\sqrt{u^2+v^2}}}$$

Answer 56E.

Consider the following function

$$v = \frac{xy}{x-y}$$

Its need to find all the second partial derivatives of v

First, find v_x.

The partial derivative v_x is calculated by treating y as a constant.

$$\begin{aligned}
v_x &= \frac{\partial}{\partial x}\left(\frac{xy}{x-y}\right) \\
&= y \cdot \frac{\partial}{\partial x}\left(\frac{x}{x-y}\right) \\
&= y \cdot \frac{(x-y)\frac{\partial}{\partial x}(x) - x\frac{\partial}{\partial x}(x-y)}{(x-y)^2} \\
&= y \cdot \frac{(x-y)\cdot 1 - x\cdot(1-0)}{(x-y)^2} \\
&= y \cdot \frac{x-y-x}{(x-y)^2} \\
&= -\frac{y}{(x-y)^2}
\end{aligned}$$

Thus, $v_x = -\frac{y}{(x-y)^2}$

Find v_{xx} by find the partial derivative of v_x with respect to x

$$v_{xx} = (v_x)_x$$

$$= \frac{\partial}{\partial x}\left(\frac{-y^2}{(x-y)^2}\right)$$

$$= -y^2 \cdot \frac{\partial}{\partial x}\left(\frac{1}{(x-y)^2}\right)$$

$$= -y^2 \cdot \frac{\partial}{\partial x}\left((x-y)^{-2}\right)$$

$$= (-y^2)\cdot\left(-2(x-y)^{-3}\right)\frac{\partial}{\partial x}(x-y)$$

$$= (-y^2)\cdot\left(-2(x-y)^{-3}\right)(1-0)$$

$$= 2y^2(x-y)^{-3}$$

$$= \frac{2y^2}{(x-y)^3}$$

Thus, $\boxed{v_{xx} = \frac{2y^2}{(x-y)^3}}$

Find v_{xy} by finding the partial derivative of v_x with respect to y

$$v_{xy} = (v_x)_y$$

$$= \frac{\partial}{\partial y}\left(\frac{-y^2}{(x-y)^2}\right)$$

$$= \frac{-y^2 \cdot \frac{\partial}{\partial y}\left((x-y)^2\right) - (x-y)^2 \cdot \frac{\partial}{\partial y}\left(-y^2\right)}{\left((x-y)^2\right)^2}$$

$$= \frac{-y^2 \cdot 2(x-y)\frac{\partial}{\partial y}(x-y) - (x-y)^2 \cdot (-2y)}{(x-y)^4}$$

$$= \frac{-y^2 \cdot 2(x-y)(0-1) - (x-y)^2 \cdot (-2y)}{(x-y)^4}$$

$$= \frac{2y^2(x-y) + 2y(x-y)^2}{(x-y)^4}$$

$$= \frac{\left(2y^2 + 2y(x-y)\right)(x-y)}{(x-y)^4}$$

$$= \frac{2y^2 + 2y(x-y)}{(x-y)^3}$$

$$= \frac{2xy}{(x-y)^3}$$

Thus, $\boxed{v_{xy} = \frac{2xy}{(x-y)^3}}$

The partial derivative v_y is calculated by treating x as a constant.

$$v_y = \frac{\partial}{\partial y}\left(\frac{xy}{x-y}\right)$$
$$= x \cdot \frac{\partial}{\partial y}\left(\frac{y}{x-y}\right)$$
$$= x \cdot \frac{(x-y)\frac{\partial}{\partial y}(y) - (y)\frac{\partial}{\partial y}(x-y)}{(x-y)^2}$$
$$= x \cdot \frac{(x-y)\cdot 1 - (y)(0-1)}{(x-y)^2}$$
$$= \frac{x^2}{(x-y)^2}$$

Thus, $v_y = \dfrac{x^2}{(x-y)^2}$

Find v_{yy} by finding the partial derivative of v_y with respect to y

$$v_{yy} = (v_y)_y$$
$$= \frac{\partial}{\partial y}\left(\frac{x^2}{(x-y)^2}\right)$$
$$= x^2 \cdot \frac{\partial}{\partial y}\left(\frac{1}{(x-y)^2}\right)$$
$$= x^2 \cdot \frac{\partial}{\partial y}\left((x-y)^{-2}\right)$$
$$= x^2 \cdot \left(-2(x-y)^{-3}\right)\frac{\partial}{\partial y}(x-y)$$
$$= x^2 \cdot \left(-2(x-y)^{-3}\right)(0-1)$$
$$= 2x^2(x-y)^{-3}$$
$$= \frac{2x^2}{(x-y)^3}$$

Thus, $\boxed{v_{yy} = \dfrac{2x^2}{(x-y)^3}}$

Find v_{yx} by finding the partial derivative of v_y with respect to x

$$v_{yx} = (v_y)_x$$

$$= \frac{\partial}{\partial x}\left(\frac{x^2}{(x-y)^2}\right)$$

$$= \frac{x^2 \cdot \frac{\partial}{\partial x}\left((x-y)^2\right) - (x-y)^2 \cdot \frac{\partial}{\partial x}(x^2)}{\left((x-y)^2\right)^2}$$

$$= \frac{x^2 \cdot 2(x-y)\frac{\partial}{\partial x}(x-y) - (x-y)^2 \cdot \frac{\partial}{\partial x}(x^2)}{(x-y)^4}$$

$$= \frac{x^2 \cdot 2(x-y)(1-0) - (x-y)^2 \cdot (2x)}{(x-y)^4}$$

$$= \frac{2x^2(x-y) - 2x(x-y)^2}{(x-y)^4}$$

$$= \frac{\left(2x^2 - 2x(x-y)\right)(x-y)}{(x-y)^4}$$

$$= \frac{2x^2 - 2x(x-y)}{(x-y)^3}$$

$$= \frac{2x^2 - 2x^2 + 2xy}{(x-y)^3}$$

$$= \frac{2xy}{(x-y)^3}$$

Thus, $\boxed{v_{yx} = \frac{2xy}{(x-y)^3}}$

Answer 57E.

You are asked to find all of the second partial derivatives of the following function.

$$z = \arctan\frac{x+y}{1-xy}$$

First, find z_x. The partial derivative is calculated by treating y as a constant and using the derivative rules for a inverse trigonometric function and the chain rule. The derivative that remains in the second line is the result of the chain rule. The third and fourth lines are simplifications.

$$\begin{aligned}
z_x &= \frac{\partial}{\partial x}\left(\arctan\frac{x+y}{1-xy}\right) \\
&= \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2}\cdot\frac{\partial}{\partial x}\left(\frac{x+y}{1-xy}\right) \\
&= \frac{1}{\left(\frac{(1-xy)^2+(x+y)^2}{(1-xy)^2}\right)}\cdot\frac{\partial}{\partial x}\left(\frac{x+y}{1-xy}\right) \\
&= \frac{(1-xy)^2}{(1-xy)^2+(x+y)^2}\cdot\frac{\partial}{\partial x}\left(\frac{x+y}{1-xy}\right)
\end{aligned}$$

To calculate the derivative in the second line, use the quotient rule.

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{x+y}{1-xy}\right) &= \frac{\frac{\partial}{\partial x}(x+y)\cdot(1-xy)-(x+y)\cdot\frac{\partial}{\partial x}(1-xy)}{(1-xy)^2} \\
&= \frac{(1+0)\cdot(1-xy)-(x+y)\cdot(0-y)}{(1-xy)^2} \\
&= \frac{1-xy+xy+y^2}{(1-xy)^2} \\
&= \frac{1+y^2}{(1-xy)^2}
\end{aligned}$$

Plug in the second partial derivative calculation into the first to get z_x.

$$\frac{(1-xy)^2}{(1-xy)^2+(x+y)^2}\cdot\frac{\partial}{\partial x}\left(\frac{x+y}{1-xy}\right)=\frac{(1-xy)^2}{(1-xy)^2+(x+y)^2}\cdot\frac{1+y^2}{(1-xy)^2}$$

$$=\frac{1+y^2}{(1-xy)^2+(x+y)^2}$$

$$=\frac{1+y^2}{(1-2xy+x^2y^2)+(x^2+y^2+2xy)}$$

$$=\frac{1+y^2}{(1+y^2)(1+x^2)}$$

$$=\frac{1}{1+x^2}$$

To find z_{xy}, find the partial derivative of z_x in terms of *y*.

$$z_{xy}=\frac{\partial}{\partial y}\left(\frac{1}{x^2+1}\right)$$

$$=\boxed{0}$$

To find z_{xx}, find the partial derivative of z_x in terms of *x*. The derivative is calculated using the rule for differentiating a power function, along with the chain rule. The partial derivative in the second line is the result of the chain rule.

$$z_{xx}=\frac{\partial}{\partial x}\left(\frac{1}{x^2+1}\right)$$

$$=\frac{-1}{(x^2+1)^2}\cdot\frac{\partial}{\partial x}(x^2+1)$$

$$=\frac{-1}{(x^2+1)^2}\cdot(2x+0)$$

$$=\boxed{\frac{-2x}{(x^2+1)^2}}$$

To find z_{yy}, first find z_y. The partial derivative is calculated by treating y as a constant and using the derivative rules for a inverse trigonometric function and the chain rule. The derivative that remains in the second line is the result of the chain rule. The third and fourth lines are simplifications.

$$\begin{aligned} z_y &= \frac{\partial}{\partial y}\left(\arctan\frac{x+y}{1-xy}\right) \\ &= \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2}\cdot\frac{\partial}{\partial y}\left(\frac{x+y}{1-xy}\right) \\ &= \frac{1}{\left(\frac{(1-xy)^2+(x+y)}{(1-xy)^2}\right)}\cdot\frac{\partial}{\partial y}\left(\frac{x+y}{1-xy}\right) \\ &= \frac{(1-xy)^2}{(1-xy)^2+(x+y)}\cdot\frac{\partial}{\partial y}\left(\frac{x+y}{1-xy}\right) \end{aligned}$$

To calculate the derivative in the second line, use the quotient rule.

$$\begin{aligned} \frac{\partial}{\partial y}\left(\frac{x+y}{1-xy}\right) &= \frac{\frac{\partial}{\partial y}(x+y)\cdot(1-xy)-(x+y)\cdot\frac{\partial}{\partial y}(1-xy)}{(1-xy)^2} \\ &= \frac{(0+1)\cdot(1-xy)-(x+y)\cdot(0-x)}{(1-xy)^2} \\ &= \frac{1-xy+xy+x^2}{(1-xy)^2} \\ &= \frac{1+x^2}{(1-xy)^2} \end{aligned}$$

Plug in the second partial derivative calculation into the first to get z_y.

$$\frac{(1-xy)^2}{(1-xy)^2+(x+y)}\cdot\frac{\partial}{\partial y}\left(\frac{x+y}{1-xy}\right)=\frac{(1-xy)^2}{(1-xy)^2+(x+y)}\cdot\frac{1+x^2}{(1-xy)^2}$$
$$=\frac{1+x^2}{(1-xy)^2+(x+y)}$$
$$=\frac{1+x^2}{(1-2xy+x^2y^2)+(x+y)}$$

This expression can be simplified.

$$\frac{1+x^2}{(1-2xy+x^2y^2)+(x+y)}=\frac{1}{y^2+1}$$

To find z_{yy}, find the partial derivative of z_y in terms of *y*. The derivative is calculated using the rule for differentiating a power function, along with the chain rule. The partial derivative in the second line is the result of the chain rule.

$$z_{yy}=\frac{\partial}{\partial y}\left(\frac{1}{y^2+1}\right)$$
$$=\frac{-1}{(y^2+1)^2}\cdot\frac{\partial}{\partial y}(y^2+1)$$
$$=\frac{-1}{(y^2+1)^2}\cdot(2y+0)$$
$$=\boxed{\frac{-2y}{(y^2+1)^2}}$$

Answer 58E.

Consider the following function:

$v=e^{xe^y}$.

The objective is to find all the second derivatives of the function.

The second derivatives of the function are $v_{xx}, v_{xy}, v_{yx}, v_{yy}$.

First, find the first partial derivatives v_x and v_y.

To find v_x, differentiate with respect to x treating y as a constant.

$$v_x = \frac{\partial}{\partial x}\left(e^{xe^y}\right)$$

$$= e^{xe^y}\frac{\partial}{\partial x}\left(xe^y\right) \qquad \text{Use } \frac{\partial}{\partial x}\left(e^{ax}\right) = ae^{ax}$$

$$= e^{xe^y}e^y\frac{\partial}{\partial x}(x)$$

$$= e^{xe^y}e^y$$

$$= e^{xe^y+y} \qquad \text{Use law of exponents}$$

To find v_y, differentiate v with respect to y, treating x as a constant.

$$v_y = \frac{\partial}{\partial y}\left(e^{xe^y}\right)$$

$$= e^{xe^y}\frac{\partial}{\partial y}\left(xe^y\right) \qquad \text{Use } \frac{\partial}{\partial x}\left(e^{ax}\right) = ae^{ax} \text{ and chain rule}$$

$$= e^{xe^y}x\frac{\partial}{\partial y}e^y$$

$$= xe^{xe^y}e^y$$

$$= xe^{y+xe^y} \qquad \text{Use law of exponents}$$

To find v_{xx}, differentiate v_x with respect to x, treating y as a constant.

$$v_{xx} = \frac{\partial}{\partial x}\left(e^{xe^y+y}\right)$$

$$= e^{xe^y+y}\frac{\partial}{\partial x}\left(xe^y+y\right)$$

$$= e^{xe^y+y}\left(\frac{\partial}{\partial x}xe^y + \frac{\partial}{\partial x}y\right)$$

$$= e^{xe^y+y}\left(e^y\frac{\partial}{\partial x}x + 0\right)$$

$$= e^{xe^y+y}e^y$$

$$= e^{y+xe^y+y}$$

$$= e^{2y+xe^y}$$

Hence, $\boxed{v_{xx} = e^{2y+xe^y}}$

To find v_{xy}, differentiate v_x with respect to *y*, treating *x* as a constant

$$v_{xy} = \frac{\partial}{\partial y}\left(e^{xe^y+y}\right)$$

$$= e^{xe^y+y}\frac{\partial}{\partial y}\left(xe^y+y\right) \qquad \text{Use } \frac{\partial}{\partial x}\left(e^{ax}\right) = ae^{ax} \text{and chain rule}$$

$$= e^{xe^y+y}\left(\frac{\partial}{\partial y}xe^y + \frac{\partial}{\partial y}y\right)$$

$$= e^{xe^y+y}\left(xe^y+1\right)$$

Hence, $\boxed{v_{xy} = e^{xe^y+y}\left(xe^y+1\right)}$

By Clairaut's Theorem,

$$\boxed{v_{yx} = e^{xe^y+y}\left(xe^y+1\right)} \quad \left(f_{xy}(a,b) = f_{yx}(a,b)\right)$$

To compute v_{yy}, differentiate v_y with respect to *y*, treating *x* as a constant

$$v_{yy} = \frac{\partial}{\partial y}\left(xe^{y+xe^y}\right)$$

$$= x\frac{\partial}{\partial y}\left(e^{y+xe^y}\right) + e^{y+xe^y}\left(\frac{\partial}{\partial y}x\right)$$

$$= xe^{y+xe^y}\frac{\partial}{\partial y}\left(y+xe^y\right)+0$$

$$= xe^{y+xe^y}\left(\frac{\partial}{\partial y}y + \frac{\partial}{\partial y}xe^y\right)$$

$$= xe^{y+xe^y}\left(1+x\frac{\partial}{\partial y}e^y\right)$$

$$= xe^{y+xe^y}\left(1+xe^y\right) \qquad \text{Use } \frac{\partial}{\partial x}\left(e^{ax}\right) = ae^{ax} \text{ and chain rule.}$$

Therefore, $\boxed{v_{yy} = xe^{y+xe^y}\left(1+xe^y\right)}$.

Answer 59E.

Clairaut's Theorem states that suppose f is defined on a disk D that contains the point (a, b) and if the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

Let us start by finding u_x by differentiating $u(x, y)$ with respect to x keeping y constant.

We know that $f_x = \frac{\partial}{\partial x} f(x, y)$.

$$u_x = \frac{\partial}{\partial x}(x^4y^3 - y^4)$$
$$= 4x^3y^3$$

Now, differentiate u_x with respect to y keeping x constant.

$$u_{xy} = \frac{\partial}{\partial x}(4x^3y^3)$$
$$= 12x^3y^2$$

We get $u_{xy} = 12x^3y^2$.

Now, find u_y.

$$u_y = \frac{\partial}{\partial y}(x^4y^3 - y^4)$$
$$= 3x^4y^2 - 4y^3$$

Differentiate u_y with respect to x.

$$u_{yx} = \frac{\partial}{\partial x}(3x^4y^2 - 4y^3)$$
$$= 12x^3y^2$$

We note that $u_{xy} = u_{yx}$. Thus, the Clairaut's Theorem is verified.

Answer 60E.

The Clairaut's Theorem states that suppose f is defined on a disk D that contains the point (a, b) and if the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

Let us start by finding u_x by differentiating $u(x, y)$ with respect to x keeping y constant.

We know that $f_x = \frac{\partial}{\partial x} f(x, y)$.

$$u_x = \frac{\partial}{\partial x}\left(e^{xy} \sin y\right)$$
$$= ye^{xy} \sin y$$

Now, differentiate u_x with respect to y keeping x constant.

$$u_{xy} = \frac{\partial}{\partial x}\left(ye^{xy} \sin y\right)$$
$$= e^{xy} \sin y + xye^{xy} \sin y + ye^{xy} \cos y$$

We get $u_{xy} = e^{xy} \sin y + xye^{xy} \sin y + ye^{xy} \cos y$.

Now, find u_y.

$$u_y = \frac{\partial}{\partial y}\left(e^{xy} \sin y\right)$$
$$= xe^{xy} \sin y + e^{xy} \cos y$$

Differentiate u_y with respect to x.

$$u_{yx} = \frac{\partial}{\partial x}\left(xe^{xy} \sin y + e^{xy} \cos y\right)$$
$$= e^{xy} \sin y + xye^{xy} \sin y + ye^{xy} \cos y$$

We note that $u_{xy} = u_{yx}$. Thus, the Clairaut's Theorem is verified.

Answer 61E.

The Clairaut's Theorem states that suppose f is defined on a disk D that contains the point (a, b) and if the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

Let us start by finding u_x by differentiating $u(x, y)$ with respect to x keeping y constant.

We know that $f_x = \frac{\partial}{\partial x} f(x, y)$.

$$u_x = \frac{\partial}{\partial x}\left[\cos\left(x^2 y\right)\right]$$
$$= -2xy\sin\left(x^2 y\right)$$

Now, differentiate u_x with respect to y keeping x constant.

$$u_{xy} = \frac{\partial}{\partial x}\left(-2xy\sin\left(x^2 y\right)\right)$$
$$= -2x^3 y\cos\left(x^2 y\right) - 2x\sin\left(x^2 y\right)$$

We get $u_{xy} = -2x^3 y\cos\left(x^2 y\right) - 2x\sin\left(x^2 y\right)$.

Now, find u_y.

$$u_y = \frac{\partial}{\partial y}\left[\cos\left(x^2 y\right)\right]$$
$$= -x^2\sin\left(x^2 y\right)$$

Differentiate u_y with respect to x.

$$u_{yx} = \frac{\partial}{\partial x}\left(-x^2\sin\left(x^2 y\right)\right)$$
$$= -2x^3 y\cos\left(x^2 y\right) - 2x\sin\left(x^2 y\right)$$

We note that $u_{xy} = u_{yx}$. Thus, the Clairaut's Theorem is verified.

Answer 62E.

The Clairaut's Theorem states that suppose f is defined on a disk D that contains the point (a, b) and if the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

Let us start by finding u_x by differentiating $u(x, y)$ with respect to x keeping y constant.

We know that $f_x = \frac{\partial}{\partial x} f(x, y)$.

$$u_x = \frac{\partial}{\partial x}\left[\ln(x + 2y)\right]$$

$$= \frac{1}{x + 2y}$$

Now, differentiate u_x with respect to y keeping x constant.

$$u_{xy} = \frac{\partial}{\partial x}\left(\frac{1}{x + 2y}\right)$$

$$= -\frac{2}{(x + 2y)^2}$$

We get $u_{xy} = -\frac{2}{(x + 2y)^2}$.

Now, find u_y.

$$u_y = \frac{\partial}{\partial y}\left[\ln(x + 2y)\right]$$

$$= \frac{2}{x + 2y}$$

Differentiate u_y with respect to x.

$$u_{yx} = \frac{\partial}{\partial x}\left(\frac{2}{x + 2y}\right)$$

$$= -\frac{2}{(x + 2y)^2}$$

We note that $u_{xy} = u_{yx}$. Thus, the Clairaut's Theorem is verified.

Answer 63E.

Let us start by finding $f_x(x, y)$ by differentiating $f(x, y)$ with respect to x keeping y constant.

We know that $f_x = \frac{\partial}{\partial x} f(x, y)$.

$$f_x = \frac{\partial}{\partial x}\left[x^4y^2 - x^3y\right]$$
$$= 4x^3y^2 - 3x^2y$$

Now, differentiate f_x with respect to x keeping y constant.

$$f_{xx} = \frac{\partial}{\partial x}\left(4x^3y^2 - 3x^2y\right)$$
$$= 12x^2y^2 - 6xy$$

Find $\frac{\partial}{\partial x}(f_{xx})$.

$$f_{xxx} = \frac{\partial}{\partial x}\left(12x^2y^2 - 6xy\right)$$
$$= 24xy^2 - 6y$$

Thus, we get $\boxed{f_{xxx} = 24xy^2 - 6y}$.

Similarly, find $f_{xyx}(x, y)$.

$$f_{xyx} = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(x^4y^2 - x^3y\right)\right]\right\}$$
$$= \frac{\partial}{\partial x}\left(8x^3y - 3x^2\right)$$
$$= 24x^2y - 6x$$

Thus, we get $\boxed{f_{xyx} = 24x^2y - 6x}$.

Answer 64E.

Let us start by finding $f_y(x, y)$ by differentiating $f(x, y)$ with respect to y keeping x constant.

We know that $f_y = \frac{\partial}{\partial y} f(x, y)$.

$$f_y = \frac{\partial}{\partial y}\left[\sin(2x + 5y)\right]$$
$$= 5\cos(2x + 5y)$$

Now, differentiate f_y with respect to x keeping y constant.

$$f_{yx} = \frac{\partial}{\partial x}\left(5\cos(2x + 5y)\right)$$
$$= -10\sin(2x + 5y)$$

Find $\frac{\partial}{\partial y}(f_{yx})$.

$$f_{yxy} = \frac{\partial}{\partial y}\left(-10\sin(2x + 5y)\right)$$
$$= -50\cos(2x + 5y)$$

Thus, we get $\boxed{f_{yxy} = -50\cos(2x + 5y)}$.

Answer 65E.

Let us start by finding $f_x(x, y, z)$ by differentiating $f(x, y, z)$ with respect to x keeping y and z constant.

We know that $f_x = \frac{\partial}{\partial x} f(x, y, z)$.

$$f_x = \frac{\partial}{\partial x}\left[e^{xyz^2}\right]$$
$$= yz^2 e^{xyz^2}$$

Now, differentiate f_x with respect to y keeping x and z constant.

$$f_{xy} = \frac{\partial}{\partial y}\left(yz^2 e^{xyz^2}\right)$$
$$= z^2 e^{xyz^2} + xyz^4 e^{xyz^2}$$

Find $\frac{\partial}{\partial z}(f_{xy})$.

$$f_{xyz} = \frac{\partial}{\partial z}\left(z^2 e^{xyz^2} + xyz^4 e^{xyz^2}\right)$$
$$= 2ze^{xyz^2} + 6xyz^3 e^{xyz^2} + 2x^2y^2z^5 e^{xyz^2}$$
$$= 2e^{xyz^2}\left(z + 3xyz^3 + x^2y^2z^5\right)$$

Thus, we get $\boxed{f_{xyz} = 2e^{xyz^2}\left(z + 3xyz^3 + x^2y^2z^5\right)}$.

Answer 66E.

Let us start by finding $g_r(r, s, t)$ by differentiating $g(r, s, t)$ with respect to r keeping s and t constant.

We know that $f_x = \frac{\partial}{\partial x} f(x, y, z)$.

$$g_r = \frac{\partial}{\partial r}\left[e^r \sin(st)\right]$$
$$= e^r \sin(st)$$

Now, differentiate g_r with respect to s keeping r and t constant.

$$g_{rs} = \frac{\partial}{\partial s}\left(e^r \sin(st)\right)$$
$$= e^r t \cos(st)$$

Find $\frac{\partial}{\partial t}(g_{rs})$.

$$g_{rst} = \frac{\partial}{\partial t}\left(e^r t \cos(st)\right)$$
$$= -e^r st \sin(st) + e^r \cos(st)$$

Thus, we get $\boxed{g_{rst} = -e^r st \sin(st) + e^r \cos(st)}$.

Answer 67E.

Consider the function $u = e^{r\theta}\sin\theta.$

The objective is to find $\frac{\partial^3 u}{\partial r^2 \partial\theta}$ for the above function.

First differentiate the function with respect to r by keeping θ as constant.

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial}{\partial r}\left(e^{r\theta}\sin\theta\right)\\ &= \sin\theta\cdot\frac{\partial}{\partial r}\left(e^{r\theta}\right)\\ &= \sin\theta\cdot\theta e^{r\theta}\\ &= \theta\sin\theta e^{r\theta}\end{aligned}$$

Again differentiate partially with respect to r by keeping θ as constant.

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)\\ &= \frac{\partial}{\partial r}\left(\theta\sin\theta e^{r\theta}\right)\\ &= \theta\sin\theta\cdot\frac{\partial}{\partial r}\left(e^{r\theta}\right)\\ &= \theta\sin\theta\cdot\theta e^{r\theta}\\ &= \theta^2\sin\theta e^{r\theta}\end{aligned}$$

Now differentiate partially with respect to θ by keeping r as constant using **Product Rule**.

$$\begin{aligned}\frac{\partial^3 u}{\partial r^2\partial\theta} &= \frac{\partial}{\partial\theta}\left(\frac{\partial^2 u}{\partial r^2}\right)\\ &= \frac{\partial}{\partial\theta}\left(\theta^2\sin\theta e^{r\theta}\right)\\ &= \frac{\partial}{\partial\theta}\left(\theta^2\right)\cdot\sin\theta e^{r\theta} + \theta^2\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\right)\cdot e^{r\theta} + \theta^2\sin\theta\cdot\frac{\partial}{\partial\theta}\left(e^{r\theta}\right)\\ &= 2\theta\cdot\sin\theta e^{r\theta} + \theta^2\cdot\cos\theta\cdot e^{r\theta} + \theta^2\sin\theta\cdot re^{r\theta}\\ &= \theta e^{r\theta}\left(2\sin\theta + \theta\cos\theta + r\theta\sin\theta\right)\end{aligned}$$

Thus, the value of $\frac{\partial^3 u}{\partial r^2\partial\theta}$ is $\frac{\partial^3 u}{\partial r^2\partial\theta} = \boxed{\theta e^{r\theta}\left(2\sin\theta + \theta\cos\theta + r\theta\sin\theta\right)}.$

Answer 68E.

$$z = u\sqrt{v-w}$$

Differentiating partially with respect to w:

$$\frac{\partial z}{\partial w} = u \times \frac{1}{2\sqrt{v-w}} \times (-1)$$

$$= \frac{-u}{2\sqrt{v-w}}$$

Then differentiating partially with respect to v

$$\frac{\partial^2 z}{\partial v\, \partial w} = \frac{-u}{2} \times -\frac{1}{2}(v-w)^{-3/2}(1)$$

$$= \frac{u}{4(v-w)^{3/2}}$$

Differentiating partially with respect to u

$$\boxed{\frac{\partial^3 z}{\partial u\, \partial v\, \partial w} = \frac{1}{4(v-w)^{3/2}}}$$

Answer 69E.

$$w = \frac{x}{y+2z} \qquad \text{---------------} \ (1)$$

Differentiating partially with respect to x

$$\frac{\partial w}{\partial x} = \frac{1}{y+2z}$$

Then differentiating partially with respect to y

$$\frac{\partial^2 w}{\partial y\, \partial x} = \frac{-1}{(y+2z)^2}$$

Differentiating partially with respect to z

$$\frac{\partial^3 w}{\partial z\, \partial y\, \partial x} = -\frac{(-2)}{(y+2z)^3}(2)$$

i.e. $$\boxed{\frac{\partial^3 w}{\partial z\, \partial y\, \partial x} = \frac{4}{(y+2z)^3}}$$

Now differentiating (1) partially with respect to y

$$\frac{\partial w}{\partial y} = -\frac{x}{(y+2z)^2}$$

Then differentiating partially with respect to x

$$\frac{\partial^2 w}{\partial x\, \partial y} = \frac{-1}{(y+2z)^2}$$

Again differentiating partially with respect to x

$$\boxed{\frac{\partial^3 w}{\partial x^2\, \partial y} = 0}$$

Answer 70E.

$u = x^a y^b z^c$

Differentiating partially thrice with respect to z

$$\frac{\partial u}{\partial z} = c\, x^a y^b z^{c-1}$$

$$\frac{\partial^2 u}{\partial z^2} = c(c-1)\, x^a y^b z^{c-2}$$

$$\frac{\partial^3 u}{\partial z^3} = c(c-1)(c-2)\, x^a y^b z^{c-3}$$

Now differentiating partially twice with respect to y

$$\frac{\partial^4 u}{\partial y\, \partial z^3} = c(c-1)(c-2)\, b\, x^a y^{b-1} z^{c-3}$$

$$\frac{\partial^5 u}{\partial y^2\, \partial z^3} = c(c-1)(c-2)\, b(b-1)\, x^a y^{b-2} z^{c-3}$$

Now differentiating partially with respect to x

$$\frac{\partial^6 u}{\partial x\, \partial y^2\, \partial z^3} = c(c-1)(c-2)\, b(b-1)\, a\, x^{a-1} y^{b-2} z^{c-3}$$

Hence $$\boxed{\frac{\partial^6 u}{\partial x\, \partial y^2\, \partial z^3} = ab(b-1)c(c-1)(c-2)\, x^{a-1} y^{b-2} z^{c-3}}$$

Answer 71E.

The Clairaut's Theorem states that suppose f is defined on a disk D that contains the point (a, b) and if the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.
Then, we can say that $f_{xzy} = f_{yxz} = f_{zyx}$.

Since the second term of the given function does not contain y term, differentiating $f(x, y, z)$ with respect to y makes it zero. So, let us start by finding f_y.

$$\begin{aligned} f_y &= \frac{\partial}{\partial y}\left[xy^2z^3 + \arcsin\left(x\sqrt{z}\right)\right] \\ &= 2xyz^3 + 0 \\ &= 2xyz^3 \end{aligned}$$

Now, let us find f_{yxz} given by $\frac{\partial^2}{\partial z \partial x}\left[f_y\right]$.

$$f_{yxz} = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial x}\left(2xyz^3\right)\right]$$
$$= \frac{\partial}{\partial z}\left(2yz^3\right)$$
$$= 6yz^2$$

Since $f_{xzy} = f_{yxz}$, we get $\boxed{f_{xzy} = 6yz^2}$.

Answer 72E.

The Clairaut's Theorem states that suppose f is defined on a disk D that contains the point (a, b) and if the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

We know that $g_{xyz} = \frac{\partial^3 g}{\partial z \partial y \partial x}$.

From the Clairaut's Theorem, we can say that $g_{xyz} = g_{zxy} = g_{yzx}$. Then,

$$g_{xyz} = \frac{\partial^3}{\partial x \partial z \partial y}\left(\sqrt{1 + xz}\right) + \frac{\partial^3}{\partial y \partial x \partial z}\left(\sqrt{1 - xy}\right).$$

$$g_{xyz} = \frac{\partial^2}{\partial x \partial z}\left[\frac{\partial}{\partial y}\left(\sqrt{1 + xz}\right)\right] + \frac{\partial^2}{\partial y \partial x}\left[\frac{\partial}{\partial z}\left(\sqrt{1 - xy}\right)\right]$$
$$= \frac{\partial^2}{\partial x \partial z}[0] + \frac{\partial^2}{\partial y \partial x}\left[\frac{\partial}{\partial z}(0)\right]$$
$$= 0$$

Thus, we get $g_{xyz} = 0$.

Answer 73E.

Consider the table values

$x \backslash y$	1.8	2.0	2.2
2.5	12.5	10.2	9.3
3.0	18.1	17.5	15.9
3.5	20.0	22.4	26.1

Use the table of values of $f(x,y)$ to estimate the values of $f_x(3,2)$, $f_x(3,2.2)$ and $f_{xy}(3,2)$.

Use the definition of partial derivatives

$$f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

Using the above formula to find the $f_x(3,2)$

$$f_x(3,2) = \lim_{h\to 0}\frac{f(3+h,2)-f(3,2)}{h}$$

Using the given table by taking $h = 0.5, -0.5$

$$\begin{aligned} f_x(3,2) &\approx \frac{f(3.5,2)-f(3,2)}{0.5} \\ &= \frac{22.4-17.5}{0.5} \\ &= \frac{4.9}{0.5} \\ &= 9.8 \end{aligned}$$

$$\begin{aligned} f_x(3,2) &\approx \frac{f(2.5,2)-f(3,2)}{-0.5} \\ &= \frac{10.2-17.5}{-0.5} \\ &= 14.6 \end{aligned}$$

On averaging these values find

$$= \frac{9.8 + 14.6}{2}$$

$$= \frac{24.4}{2}$$

$$= 12.2$$

$$\boxed{f_x(3,2) \approx 12.2}$$

Find $f_x(3,2.2)$

Using the above formula

$$\begin{aligned} f_x(3,2.2) &\approx \frac{f(3.5,\ 2.2) - f(3,2.2)}{0.5} \\ &= \frac{26.1 - 15.9}{0.5} \\ &= \frac{10.2}{0.5} \\ &= 20.4 \end{aligned}$$

And

$$\begin{aligned} f_x(3,2.2) &\approx \frac{f(2.5,2.2) - f(3,2.2)}{-0.5} \\ &= \frac{9.3 - 15.9}{-0.5} \\ &= \frac{-6.6}{-0.5} \\ &= 13.2 \end{aligned}$$

On averaging these values

$$= \frac{20.4 + 13.2}{2}$$

$$= \frac{33.6}{2}$$

$$= 16.8$$

$$\boxed{f_x(3, 2.2) \approx 16.8}$$

Find $f_{xy}(3,2)$

Using the definition of partial derivative

Now $f_{xy}(x,y) = \frac{\partial}{\partial y}\left[f_x(x,y)\right]$ and $f_x(x,y)$ it self a function of 2 variables.

$$\begin{aligned} f_{xy}(x,y) &= \frac{\partial}{\partial y}\left[f_x(x,y)\right] \\ &= \lim_{h \to 0} \frac{f_x(x, y+h) - f_x(x,y)}{h} \end{aligned}$$

$$f_{xy}(3,2) = \lim_{h \to 0} \frac{f_x(3, 2+h) - f_x(3,2)}{h}$$

To estimate $f_{xy}(3,2)$,first need to estimate $f_x(3,1.8)$

$$\begin{aligned} f_x(3,1.8) &\approx \frac{f(3.5, 1.8) - f(3,1.8)}{0.5} \\ &= \frac{20 - 18.1}{0.5} \\ &= 3.8 \end{aligned}$$

$$\begin{aligned} f_x(3,1.8) &\approx \frac{f(2.5, 1.8) - f(3,1.8)}{-0.5} \\ &= \frac{12.5 - 18.1}{-0.5} \\ &= 11.2 \end{aligned}$$

Averaging these values, get $f_x(3,1.8) \approx 7.5$

Using the above value to estimate $f_{xy}(3,2)$ with $h=0.2$ and $h=-0.2$

$$f_{xy}(3,2) \approx \frac{f(3,2.2)-f_x(3,2)}{0.2}$$
$$= \frac{16.8-12.2}{0.2}$$
$$= 23$$

$$f_{xy}(3,2) \approx \frac{f(3,1.8)-f(3,2)}{-0.2}$$
$$= \frac{7.5-12.2}{-0.2}$$
$$= 23.5$$

On averaging these values

$$= \frac{23+23.5}{2}$$
$$= \frac{46.5}{2}$$
$$= 23.25$$

Therefore, $\boxed{f_{xy}(3,2) = 23.5}$

Answer 74E.

(a) We take y as a constant, and allow x to vary through point P, then we see that the value of the function decreases as we move in positive x direction
So f_x will be negative at point P

(b) We take x as a constant, and allow y to vary through point P, then we see that the value of the function increases as we move in positive y direction
So f_y will be positive at point P

(c) f_{xx} is the rate of change of f_x with respect to x, we see that space between level curves increases as we move in positive x direction. So value of f_x increases as we move in positive x direction.
Thus f_{xx} is positive at point P

(d) f_{xy} is the rate of change of f_x with respect to y, we see that space between level curves decreases in x direction as we move in positive y direction. So value of f_x decreases as we move in positive y direction.
Thus f_{xy} is negative at point P

(e) f_{yy} is the rate of change of f_y with respect to y, we see that space between level curves increases in y direction as we move in positive y direction. So value of f_y increases as we move in positive y direction.

Thus f_{yy} is positive at point P

Answer 75E.

Consider the function:

$$u = e^{-\alpha^2 k^2 t}\sin(kx).$$

The Partial derivative with respect to ' t ' is calculated as follows:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(e^{-\alpha^2 k^2 t}\sin(kx)\right);$$

$$u_t = -\alpha^2 k^2 e^{-\alpha^2 k^2 t}\sin(kx).$$

Find the partial derivative with respect to ' x ' as follows:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(e^{-\alpha^2 k^2 t}\sin(kx)\right);$$

$$u_x = ke^{-\alpha^2 k^2 t}\cos(kx).$$

Once again partially differentiate with respect to ' x ':

$$\frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial x}\left[ke^{-\alpha^2 k^2 t}\cos(kx)\right];$$

$$u_{xx} = -k^2 e^{-\alpha^2 k^2 t}\sin(kx).$$

RHS of the heat conduction equation is as follows:

$$\alpha^2 u_{xx} = -\alpha^2 k^2 e^{-\alpha^2 k^2 t}\sin(kx);$$

$$\alpha^2 u_{xx} = u_t.$$

Hence, $\boxed{u = e^{-\alpha^2 k^2 t}\sin(kx)}$ is the solution of the heat conduction equation $\boxed{\alpha^2 u_{xx} = u_t}$.

Answer 76E.

(A)

$$u = x^2 + y^2 \quad \text{------------ (1)}$$

Differentiating partially with respect to x

$$u_x = 2x$$

Again differentiating partially with respect to x

$$u_{xx} = 2$$

Differentiating (1) partially with respect to y

$$u_y = 2y$$

Again differentiating partially with respect to y

$$u_{yy} = 2$$

Now $\quad u_{xx} + u_{yy} = 2 + 2 = 4 \neq 0$

Therefore the given function (1) is not a solution of equation $u_{xx} + u_{yy} = 0$

(B)

$$u = x^2 - y^2 \qquad \text{------------ (2)}$$

Then $\quad u_x = 2x$

$$u_{xx} = 2$$

And $\quad u_y = -2y$

$$u_{yy} = -2$$

Since $\quad u_{xx} + u_{yy} = 2 - 2 = 0$

Therefore the given function (2) is a solution of $u_{xx} + u_{yy} = 0$

(C)

$$u = x^3 + 3xy^2 \qquad \text{------------- (3)}$$

Then $\quad u_x = 3x^2 + 3y^2$

$$u_{xx} = 6x$$

And $\quad u_y = 6xy$

$$u_{yy} = 6x$$

Now $\quad u_{xx} + u_{yy} = 6x + 6x = 12x \neq 0$

Thus the given function (3) is not a solution of $u_{xx} + u_{yy} = 0$

(D)

$$u = \ln \sqrt{x^2 + y^2} \quad \text{----------------} \quad (4)$$

Then $u_x = \frac{1}{\sqrt{x^2+y^2}} \times \frac{1}{2} \frac{(2x)}{\sqrt{x^2+y^2}}$

$$= \frac{x}{x^2+y^2}$$

$$u_{xx} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$= \frac{-(x^2-y^2)}{(x^2+y^2)^2}$$

And $u_y = \frac{1}{\sqrt{x^2+y^2}} \times \frac{1}{2} \frac{(2y)}{\sqrt{x^2+y^2}}$

$$= \frac{y}{x^2+y^2}$$

$$u_{yy} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

Now $u_{xx} + u_{yy} = \frac{-(x^2-y^2)}{(x^2+y^2)^2} + \frac{(x^2-y^2)}{(x^2+y^2)^2} = 0$

Thus given function (4) is a solution of $y_{xx} + u_{yy} = 0$

(E)

$$u = \sin x \cosh y + \cos x \sinh y \qquad \text{------------------ (5)}$$

Then $u_x = \cos x \cosh y - \sin x \sinh y$

$u_{xx} = -\sin x \cosh y - \cos x \sinh y$

And $u_y = \sin x \sinh y + \cos x \cosh y$

$u_{yy} = \sin x \cosh y + \cos x \sinh y$

Now $u_{xx} + u_{yy} = -\sin x \cosh y - \cos x \sinh y + \sin x \cosh y + \cos x \sinh y$

$= 0$

Hence given function (5) is a solution of equation $u_{xx} + u_{yy} = 0$

(F)

$$u = e^{-x} \cos y - e^{-y} \cos x \qquad \text{------------------ (6)}$$

Then $u_x = -e^{-x} \cos y + e^{-y} \sin x$

$u_{xx} = e^{-x} \cos y + e^{-y} \cos x$

And $u_y = -e^{-x} \sin y + e^{-y} \cos x$

$u_{yy} = -e^{-x} \cos y - e^{-y} \cos x$

Now $u_{xx} + u_{yy} = e^{-x} \cos y + e^{-y} \cos x - e^{-x} \cos y - e^{-y} \cos x$

$= 0$

Hence the given function (6) is a solution of equation $u_{xx} + u_{yy} = 0$

Answer 77E.

$$u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

Or $u = \left(x^2 + y^2 + z^2\right)^{-1/2}$

Then $u_x = \frac{-1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}(2x)$

$= -x\left(x^2 + y^2 + z^2\right)^{-3/2}$

And $u_{xx} = -\left(x^2 + y^2 + z^2\right)^{-3/2} - x\left(\frac{-3}{2}\right)\left(x^2 + y^2 + z^2\right)^{-5/2}(2x)$

$= -\left(x^2 + y^2 + z^2\right)^{-3/2} + 3x^2\left(x^2 + y^2 + z^2\right)^{-5/2}$

$= \frac{-\left(x^2 + y^2 + z^2\right) + 3x^2}{\left(x^2 + y^2 + z^2\right)^{5/2}}$

$= \frac{2x^2 - y^2 - z^2}{\left(x^2 + y^2 + z^2\right)^{5/2}}$

And $u_y = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2y)$

$$= -y\left(x^2+y^2+z^2\right)^{-3/2}$$

$$u_{yy} = -\left(x^2+y^2+z^2\right)^{-3/2} - y\left(\frac{-3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}(2y)$$

$$= \frac{-\left(x^2+y^2+z^2\right)+3y^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$= \frac{2y^2-x^2-z^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

And $u_z = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2z)$

$$= -z\left(x^2+y^2+z^2\right)^{-3/2}$$

$$u_{zz} = \frac{-\left(x^2+y^2+z^2\right)+3z^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$= \frac{\left(2z^2-x^2-y^2\right)}{\left(x^2+y^2+z^2\right)^{5/2}}$$

Now $u_{xx}+u_{yy}+u_{zz} = \frac{2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2}{\left(x^2+y^2+z^2\right)^{5/2}}$

$$= 0$$

Hence given function is a solution of the Laplace equation $u_{xx}+u_{yy}+u_{zz}=0$

Answer 78E.

(A)

$$u = \sin(kx)\sin(akt) \quad \text{---------------- (1)}$$

Then $u_t = ak\sin(kx)\cos(akt)$

$$u_{tt} = -a^2k^2\sin(kx)\sin(akt)$$

And $u_x = k\cos(kx)\sin(akt)$

$$u_{xx} = -k^2\sin(kx)\sin(akt)$$

Now $u_{tt} = -a^2k^2\sin(kx)\sin(akt) = a^2u_{xx}$

Hence given function (1) is a solution of $u_{tt} = a^2u_{xx}$

(B)

$$u = \frac{t}{\left(a^2t^2 - x^2\right)} \qquad \text{---------------- (2)}$$

Then $u_t = \dfrac{1}{\left(a^2t^2 - x^2\right)} - t\left(a^2t^2 - x^2\right)^{-2}\left(2a^2t\right)$

$$= \frac{\left(a^2t^2 - x^2\right) - 2a^2t^2}{\left(a^2t^2 - x^2\right)^2}$$

$$= \frac{-\left(x^2 + a^2t^2\right)}{\left(a^2t^2 - x^2\right)^2}$$

$$u_{tt} = -\frac{2a^2t}{\left(a^2t^2 - x^2\right)^2} + \frac{2\left(x^2 + a^2t^2\right)\left(2a^2t\right)}{\left(a^2t^2 - x^2\right)^3}$$

$$= \frac{-2a^2t\left(a^2t^2 - x^2\right) + 2\left(x^2 + a^2t^2\right)2a^2t}{\left(a^2t^2 - x^2\right)^3}$$

$$= \frac{-2a^4t^3 + 2a^2tx^2 + 4a^2x^2t + 4a^4t^3}{\left(a^2t^2 - x^2\right)^3}$$

$$= \frac{2a^2t\left(a^2t^2 + 3x^2\right)}{\left(a^2t^2 - x^2\right)^3}$$

On differentiating (2) partially with respect to x

$$u_x = \frac{2tx}{\left(a^2t^2 - x^2\right)^2}$$

And $u_{xx} = \dfrac{2t}{\left(a^2t^2 - x^2\right)^2} - \dfrac{4tx(-2x)}{\left(a^2t^2 - x^2\right)^3}$

$$= \frac{2t\left(a^2t^2 - x^2\right) + 8t\,x^2}{\left(a^2t^2 - x^2\right)^3}$$

$$= \frac{2a^2t^3 - 2tx^2 + 8tx^2}{\left(a^2t^2 - x^2\right)^3}$$

$$= \frac{2t\left(a^2t^2 + 3x^2\right)}{\left(a^2t^2 - x^2\right)^3}$$

Now $u_{tt} = a^2\left[\dfrac{2t\left(a^2t^2 + 3x^2\right)}{\left(a^2t^2 - x^2\right)^3}\right] = a^2u_{xx}$

Hence the given function (2) is a solution of $u_{tt} = a^2u_{xx}$

(C)

$$u = (x - at)^6 + (x + at)^6 \qquad \text{----------------- (3)}$$

Then $u_t = 6(x - at)^5(-a) + 6(x + at)^5(a)$

$$= -6a(x - at)^5 + 6a(x + at)^5$$

$$u_{tt} = 30a^2(x - at)^4 + 30a^2(x + at)^4$$

And $u_x = 6(x - at)^5 + 6(x + at)^5$

$$u_{xx} = 30(x - at)^4 + 30(x + at)^4$$

Now $u_{tt} = a^2\left(30(x - at)^4 + 30(x + at)^4\right) = a^2u_{xx}$

Hence the given function (3) is a solution of $u_{tt} = a^2u_{xx}$

(D)

$$u = \sin(x - at) + \ln(x + at) \qquad \text{--------------- (4)}$$

Then $u_t = -a\cos(x - at) + \dfrac{a}{x + at}$

$$u_{tt} = -a^2\sin(x - at) - \frac{a^2}{(x + at)^2}$$

And $u_x = \cos(x - at) + \dfrac{1}{x + at}$

$$u_{xx} = -\sin(x - at) - \frac{1}{(x + at)^3}$$

Now $u_{tt} = a^2\left[-\sin(x - at) - \dfrac{1}{(x + at)^2}\right] = a^2u_{xx}$

Hence the given function (4) is the solution of equation $u_{tt} = a^2u_{xx}$

Answer 79E.

The given function is

$$u(x,t) = f(x+at) + g(x-at) \qquad \text{----------------- (1)}$$

Take $u = x + at$ and $v = x - at$

Then $\dfrac{\partial u}{\partial x} = 1$, $\dfrac{\partial u}{\partial t} = a$

$\dfrac{\partial v}{\partial t} = 1$, $\dfrac{\partial v}{\partial t} = -a$

On differentiating (1) partially with respect to x

$$\frac{\partial u}{\partial x} = \frac{df}{du}\cdot\frac{\partial u}{\partial x} + \frac{dg}{dv}\cdot\frac{\partial u}{\partial x}$$

i.e. $u_x = f'(u)(1) + g'(v)(1)$

i.e. $u_x = f'(x+at) + g'(x-at)$ ------------- (2)

Now differentiating (1) partially with respect to t

$$\frac{\partial u}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t} + \frac{dg}{dv}\cdot\frac{\partial v}{\partial t}$$

i.e. $u_t = f'(u)(a) + g'(v)(-a)$

i.e. $u_t = a\,f'(x+at) - a\,g'(x-at)$ ----------- (3)

And then $u_{tt} = a\dfrac{df'}{du}\dfrac{\partial u}{\partial t} - a\dfrac{dg'}{dv}\dfrac{\partial v}{\partial t}$

$= a\,f''(u)(a) - a\,g''(v)(-a)$

$= a^2 f''(x+at) + a^2 g''(x-at)$

$= a^2\left[f''(x+at) + g''(x-at)\right]$

Also differentiating (2) partially with respect to x

$$u_{xx} = \frac{df'}{du}\frac{\partial u}{\partial x} + \frac{dg'}{dv}\frac{\partial v}{\partial x}$$

$= f''(u)(1) + g''(v)(1)$

$= f''(x+at) + g''(x-at)$

Now $u_{tt} = a^2\left[f''(x+at) + g''(x-at)\right] = a^2 u_{xx}$

Hence the given function is a solution of the equation $u_{tt} = a^2 u_{xx}$

Answer 80E.

$u = e^{a_1x_1+a_2x_2+\ldots\ldots+a_nx_n}$ ------------- (1)

Differentiating (1) partially with respect to x_1

$$\frac{\partial u}{\partial x_1} = a_1\, e^{a_1x_1+a_2x_2+\ldots\ldots\ldots+a_nx_n}$$

Again differentiating partially with respect to x_1

$$\frac{\partial^2 u}{\partial x_1^2} = a_1^2\, e^{a_1x_1+a_2x_2+\ldots\ldots a_nx_n}$$

Similarly differentiating (1), partially twice with respect to x_2

$$\frac{\partial^2 u}{\partial x_2^2} = a_2^2 e^{a_1x_1+a_2x_2+\ldots\ldots+a_nx_n}$$

Therefore in general

$$\frac{\partial^2 u}{\partial x_i^2} = a_i^2\, e^{a_1x_1+a_2x_2+\ldots\ldots\ldots+a_nx_n}$$

Then $$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots\ldots + \frac{\partial^2 u}{\partial x_n^2}$$
$$= \left(a_1^2 + a_2^2 + \ldots\ldots + a_n^2\right) e^{a_1x_1+a_2x_2+\ldots\ldots+a_nx_n}$$

Since $a_1^2 + a_2^2 + \ldots\ldots\ldots + a_n^2 = 1$ (given)

Therefore $$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots\ldots\ldots + \frac{\partial^2 u}{\partial x_n^2} = e^{a_1x_1+a_2x_2+\ldots\ldots\ldots+a_nx_n}$$

i. e. $$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots\ldots\ldots + \frac{\partial^2 u}{\partial x_n^2} = u$$

Answer 81E.

Given function is $z=\ln\left(e^x+e^y\right)$

Differentiate z partially with respect to x, we have

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left[\ln\left(e^x+e^y\right)\right]$$

$$=\frac{1}{\left(e^x+e^y\right)}\frac{\partial}{\partial x}\left(e^x+e^y\right)$$

Thus $\frac{\partial z}{\partial x}=\frac{e^x}{e^x+e^y}$

now differentiate $\frac{\partial z}{\partial x}$ partially with respect to x, we have

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{e^x}{e^x+e^y}\right)$$

$$=\frac{\left(e^x+e^y\right)\frac{\partial}{\partial x}\left(e^x\right)-e^x\frac{\partial}{\partial x}\left(e^x+e^y\right)}{\left(e^x+e^y\right)^2}$$

$$=\frac{\left(e^x+e^y\right)\left(e^x\right)-e^x\left(e^x\right)}{\left(e^x+e^y\right)^2}$$

$$=\frac{e^{x+y}}{\left(e^x+e^y\right)^2}$$

Thus $\frac{\partial^2 z}{\partial x^2}=\frac{e^{x+y}}{\left(e^x+e^y\right)^2}$

Differentiate z partially with respect to y, we have

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[\ln\left(e^x + e^y\right)\right]$$

$$= \frac{1}{\left(e^x + e^y\right)} \frac{\partial}{\partial y}\left(e^x + e^y\right)$$

Thus $\frac{\partial z}{\partial y} = \frac{e^y}{e^x + e^y}$

now differentiate $\frac{\partial z}{\partial y}$ partially with respect to y, we have

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{e^y}{e^x + e^y}\right)$$

$$= \frac{\left(e^x + e^y\right)\frac{\partial}{\partial y}\left(e^y\right) - e^y \frac{\partial}{\partial y}\left(e^x + e^y\right)}{\left(e^x + e^y\right)^2}$$

$$= \frac{\left(e^x + e^y\right)\left(e^y\right) - e^y\left(e^y\right)}{\left(e^x + e^y\right)^2}$$

$$= \frac{e^{x+y}}{\left(e^x + e^y\right)^2}$$

Thus $\frac{\partial^2 z}{\partial y^2} = \frac{e^{x+y}}{\left(e^x + e^y\right)^2}$

now differentiate $\frac{\partial z}{\partial y}$ partially with respect to x, we have

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{e^y}{e^x+e^y}\right)$$

$$=\frac{\left(e^x+e^y\right)\frac{\partial}{\partial x}\left(e^y\right)-e^y\frac{\partial}{\partial x}\left(e^x+e^y\right)}{\left(e^x+e^y\right)^2}$$

$$=\frac{\left(e^x+e^y\right)(0)-e^y\left(e^x\right)}{\left(e^x+e^y\right)^2}$$

$$=\frac{-e^{x+y}}{\left(e^x+e^y\right)^2}$$

Thus $\frac{\partial^2 z}{\partial x\partial y}=\frac{-e^{x+y}}{\left(e^x+e^y\right)^2}$

consider $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\frac{e^x}{e^x+e^y}+\frac{e^y}{e^x+e^y}$

$$=\frac{e^x+e^y}{e^x+e^y}$$

$$=1$$

Therefore $z=\ln\left(e^x+e^y\right)$ is a solution to the differential equation $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=1$

consider $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2=\frac{e^{x+y}}{\left(e^x+e^y\right)^2}+\frac{e^{x+y}}{\left(e^x+e^y\right)^2}-\left[\frac{-e^{x+y}}{\left(e^x+e^y\right)^2}\right]^2$

$$=\frac{e^{2(x+y)}}{\left(e^x+e^y\right)^2}-\frac{e^{2(x+y)}}{\left(e^x+e^y\right)^2}$$

$$=0$$

Therefore $z=\ln\left(e^x+e^y\right)$ is a solution to the differential equation $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2=0$

Answer 82E.

$$T(x,y)=\frac{60}{1+x^2+y^2} \quad ------------ (1)$$

(A)

Differentiating (1) partially with respect to x

$$\frac{\partial T}{\partial x} = \frac{60(-2x)}{\left(1+x^2+y^2\right)^2}$$

$$= \frac{-120\,x}{\left(1+x^2+y^2\right)^2}$$

Then $\left(\frac{\partial T}{\partial x}\right)_{(2,1)} = \frac{-120(2)}{\left(1+2^2+1^2\right)^2}$

$$= \frac{-240}{36}$$

$$= \frac{-20}{3}$$

Then the rate of change of temperature with respect to distance at the point (2, 1) in x – direction is -20/3

(B)

Now differentiating (1) partially with respect to y

$$\frac{\partial T}{\partial y} = \frac{60(-2y)}{\left(1+x^2+y^2\right)^2}$$

$$= \frac{-120y}{\left(1+x^2+y^2\right)^2}$$

Then $\left(\frac{\partial T}{\partial y}\right)_{(2,1)} = \frac{-120(1)}{\left(1+2^2+1^2\right)^2}$

$$= \frac{-120}{36}$$

$$= \frac{-10}{3}$$

Hence the rate of change of temperature with respect to distance at the point (2, 1) in y direction is -10/3

Answer 83E.

Differentiate $R = \frac{R_1R_2R_3}{R_2R_3 + R_1R_3 + R_1R_2}$ on both sides partially with respect to R_1.

$$\frac{\partial R}{\partial R_1} = \frac{\partial}{\partial R_1}\left(\frac{R_1R_2R_3}{R_2R_3 + R_1R_3 + R_1R_2}\right)$$

$$= \frac{(R_2R_3 + R_1R_3 + R_1R_2)\frac{\partial}{\partial R_1}(R_1R_2R_3) - R_1R_2R_3\frac{\partial}{\partial R_1}(R_2R_3 + R_1R_3 + R_1R_2)}{(R_2R_3 + R_1R_3 + R_1R_2)^2}$$

$$= \frac{(R_2R_3 + R_1R_3 + R_1R_2)R_2R_3 \cdot 1 - R_1R_2R_3(R_3 \cdot 1 + R_2 \cdot 1)}{(R_2R_3 + R_1R_3 + R_1R_2)^2}$$

$$= \frac{(R_2R_3 + R_1R_3 + R_1R_2)R_2R_3 - R_1R_2R_3(R_3 + R_2)}{(R_2R_3 + R_1R_3 + R_1R_2)^2}$$

$$= \frac{R_2^2R_3^2 + R_1R_2R_3^2 + R_1R_2^2R_3 - R_1R_2R_3^2 - R_1R_2^2R_3}{(R_2R_3 + R_1R_3 + R_1R_2)^2}$$

$$= \frac{R_2^2R_3^2}{(R_2R_3 + R_1R_3 + R_1R_2)^2}$$

Therefore, $\frac{\partial R}{\partial R_1} = \boxed{\frac{R_2^2R_3^2}{(R_2R_3 + R_1R_3 + R_1R_2)^2}}$.

Answer 84E.

The given function is

$$P = bL^{\alpha}K^{\beta} \quad \text{-------------} \quad (1)$$

Differentiating (1) partially with respect to L

$$\frac{\partial P}{\partial L} = b\alpha L^{\alpha-1}K^{\beta}$$

Differentiating (1) partially with respect to K

$$\frac{\partial P}{\partial K} = b\beta L^{\alpha}K^{\beta-1}$$

Consider $L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K}$

$$= -L\left(b\alpha L^{\alpha-1}K^{\beta}\right) + K\left(b\beta L^{\alpha}K^{\beta-1}\right)$$

$$= b\alpha L^{\alpha}K^{\beta} + b\beta L^{\alpha}K^{\beta}$$

$$= (\alpha + \beta)\left(bL^{\alpha}K^{\beta}\right)$$

$$= (\alpha + \beta)P$$

Hence $L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = (\alpha + \beta)P$

Answer 85E.

The Cobb – Douglas production function is

$$P = bL^{\alpha}K^{\beta}$$

If we keep K constant $(K = K_0)$, $P = bL^{\alpha}K_0^{\beta}$

Then $\dfrac{dP}{dL} = b\alpha L^{\alpha-1}K_0^{\beta}$

Or $\dfrac{dP}{dL} = \dfrac{\alpha b L^{\alpha} K_0^{\beta}}{L}$

Or $\dfrac{dP}{dL} = \dfrac{\alpha P}{L}$

It is a first order differential equation of variable separable form

Then on separating variables

$$\frac{dP}{P} = \frac{\alpha\, dL}{L}$$

Integrating both sides

$$\ln P = \alpha \ln L + C(K_0)$$

Where C is a constant of integration depending upon K_0

Then $\ln P - \alpha \ln L = C(K_0)$

Or $\ln \dfrac{P}{L^{\alpha}} = C(K_0)$

Or $\dfrac{P}{L^{\alpha}} = e^{C(K_0)}$

Put $e^{c} = C_1(K_0)$ is another arbitrary

Therefore $P = L^{\alpha}\, C(K_0)$

Or $P(L, K_0) = L^{\alpha}C(K_0)$

Answer 86E.

(a)Let us differentiate $P(L, K)$ with respect to L, keeping K as a constant to determine $P_L(L, K)$.

$$\begin{aligned} P_L(L, K) &= \frac{\partial}{\partial L}\left(1.01L^{0.75}K^{0.25}\right) \\ &= (1.01)(0.75)L^{0.75-1}K^{0.25} \\ &= 0.7575L^{-0.25}K^{0.25} \\ &= 0.7575\left(\frac{K}{L}\right)^{0.25} \end{aligned}$$

Similarly, find $P_K(L, K)$.

$$\begin{aligned} P_K(L, K) &= \frac{\partial}{\partial K}\left(1.01L^{0.75}K^{0.25}\right) \\ &= (1.01)(0.25)L^{0.75}K^{0.25-1} \\ &= 0.2525L^{0.75}K^{-0.75} \\ &= 0.2525\left(\frac{L}{K}\right)^{0.75} \end{aligned}$$

Thus, we get $P_L(L, K) = 0.7575\left(\frac{K}{L}\right)^{0.25}$ and

$$P_K(L, K) = 0.2525\left(\frac{L}{K}\right)^{0.75}.$$

(b)The marginal productivity of labor in 1920 is given by $P_L(194, 407)$.

Replace L with 194 and K with 407 in $P_L(L, K) = 0.7575\left(\frac{K}{L}\right)^{0.25}$.

$$\begin{aligned} P_L(194, 407) &= 0.7575\left(\frac{407}{194}\right)^{0.25} \\ &= 0.9116 \end{aligned}$$

Now, find the marginal productivity of capital in 1920 given by $P_K(194, 407)$.

$$\begin{aligned} P_K(194, 407) &= 0.7575\left(\frac{194}{407}\right)^{0.75} \\ &= 0.1448 \end{aligned}$$

These results show that the marginal productivity of labor has increased from 0.7575 in 1899 to 0.9116 in 1920 and the marginal productivity of capital has decreased from 0.2525 in 1899 to 0.1448 in 1920.

(c)From the result obtained in part (b), we can say that the productivity of 1920 obtained through labor is 0.9116 and the productivity obtained by increasing the capital investment is 0.1448. Thus, we can say that increase in spending on labor would have benefited the productivity more.

Answer 87E.

We have $\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$.

Start by finding $\frac{\partial T}{\partial P}$.

$$(1 + 0)(V - nb) = nR\frac{\partial T}{\partial P}$$

$$\frac{\partial T}{\partial P} = \frac{V - nb}{nR}$$

$$\frac{\partial T}{\partial P} = \frac{V}{nR} - \frac{b}{R}$$

Thus, we get $\frac{\partial T}{\partial P} = \frac{V}{nR} - \frac{b}{R}$.

We have $\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$ or $P = \frac{nRT}{V - nb} - \frac{n^2 a}{V^2}$.

Now, let us find $\frac{\partial P}{\partial V}$.

$$\frac{\partial P}{\partial V} = \frac{-nRT}{(V - nb)^2} + \frac{2n^2 a}{V^3}$$

Thus, we get $\frac{\partial P}{\partial V} = \frac{2n^2 a}{V^3} - \frac{nRT}{(V - nb)^2}$.

Answer 88E.

The given relation is

$$PV = mRT \qquad \text{--------------- (1)}$$

Or $\quad P = \frac{mRT}{V}$

Differentiating partially with respect to V

$$\frac{\partial P}{\partial V} = -\frac{mRT}{V^2} \quad \text{-------------- (2)}$$

Also from (1) $V = \frac{mRT}{P}$

Differentiating partially with respect to T

$$\frac{\partial V}{\partial T} = \frac{mR}{P} \quad \text{--------------- (3)}$$

Also from (1) $T = \frac{PV}{mR}$

Differentiating partially with respect to P

$$\frac{\partial T}{\partial P} = \frac{V}{mR} \quad \text{--------------- (4)}$$

Multiplying (2), (3) and (4)

$$\frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial P} = -\frac{mRT}{V^2} \cdot \frac{mR}{P} \cdot \frac{V}{mR}$$

$$= -\frac{mRT}{PV}$$

$$= -1$$

Hence $\frac{\partial P}{\partial V} \frac{\partial V}{\partial T} \frac{\partial T}{\partial P} = -1$

Answer 89E.

The gas law of an ideal gas is

$$PV = mRT \quad \text{------------- (1)}$$

Where m is fixed mass of the gas and R is gas constant

From (1) $P = \frac{mRT}{V}$

Differentiating partially with respect to T

$$\frac{\partial P}{\partial T} = \frac{mR}{V}$$

Also from (1), $V = \frac{mRT}{P}$

Differentiating partially with respect to T,

$$\frac{\partial V}{\partial T} = \frac{mR}{P}$$

Then consider

$$T\frac{\partial P}{\partial T}\frac{\partial V}{\partial T} = T\frac{mR}{V}.\frac{mR}{P}$$
$$= \left(\frac{mRT}{PV}\right)mR$$
$$= (1)mR$$

Hence $\boxed{T\frac{\partial P}{\partial T}.\frac{\partial V}{\partial T} = mR}$

Answer 90E.

It is given that the wind chill index is

$$w = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16} \qquad \text{------------- (1)}$$

Then differentiating partially with respect to T

$$\frac{\partial w}{\partial T} = 0.6215 + 0.3965v^{0.16} \qquad \text{--------------- (2)}$$

Which denotes the rate of change of wind – chill index s wind speed remains constant and temperature varies

And now differentiating (1) partially with respect to v

$$\frac{\partial w}{\partial v} = \left[-11.37(0.16) + 0.3965(0.16)T\right]v^{-0.84} \text{ --------------- (3)}$$

Now when $T = -15°C$ and $v = 30\, {}^{km}/_{h}$

Then $w_T(-15, 30) = 0.6215 + 0.3965(30)^{0.16}$
$$= 1.3$$

Hence we see that if actual temperature decreases by 1°C then the apparent temperature decreases by $\boxed{1.3°C}$

And $w_v(-15, 30) = \left[-1.8192 + 0.06347(-15)\right](30)^{-0.84}$
$$= -0.15$$

Since $\frac{\partial w}{\partial v}$ denotes the rate of change of wind – chill index as wind speed varies and temperature remains constant, therefore we see that if the wind speed increases by 1 km/hr then the wind – chill index decreases by $\boxed{0.15°C}$

Answer 91E.

It is given that

$$K = \frac{1}{2}mv^2 \qquad \text{------------ (1)}$$

Differentiating partially with respect to m

$$\frac{\partial K}{\partial m} = \frac{1}{2}v^2$$

And differentiating (1) partially twice with respect to v

$$\frac{\partial K}{\partial v} = mv$$

And $$\frac{\partial^2 K}{\partial v^2} = m$$

Consider $\frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2}$

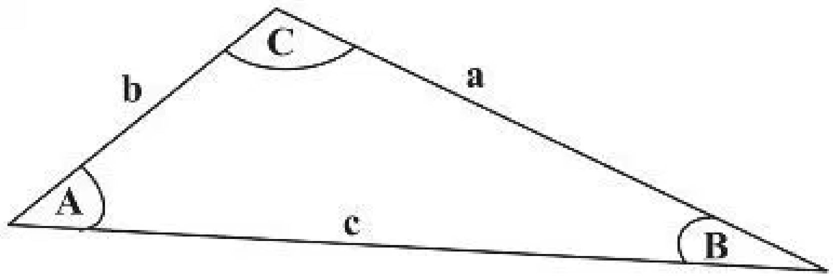
$$= \left(\frac{1}{2}v^2\right)(m)$$

$$= \frac{1}{2}mv^2$$

$$= K$$

Hence $$\boxed{\frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2} = K}$$

Answer 92E.



Now differentiating (1) partially with respect to a

$$2a = -2bc(-\sin A)\frac{\partial A}{\partial a}$$

i.e. $$\boxed{\frac{\partial A}{\partial a} = \frac{a}{bc\sin A}}$$

Now differentiating (1) partially with respect to a

$$2a = -2bc(-\sin A)\frac{\partial A}{\partial a}$$

i.e. $$\boxed{\frac{\partial A}{\partial a} = \frac{a}{bc\sin A}}$$

Now differentiating (1) partially with respect to b

$$0 = 2b + 0 - 2c\cos A + 2bc\sin A\frac{\partial A}{\partial b}$$

i.e. $$\boxed{\frac{\partial A}{\partial b} = \frac{c\cos A - b}{bc\sin A}}$$

And differentiating (1) partially with respect to c

$$0 = 0 + 2c - 2b\cos A + 2bc\sin A\frac{\partial A}{\partial c}$$

i.e. $$\boxed{\frac{\partial A}{\partial c} = \frac{b\cos A - c}{bc\sin A}}$$

Answer 93E.

Let us first consider $f_x(x, y) = x + 4y$

Then $f_{xy}(x, y) = 4$

Now $f_y(x, y) = 3x - y$

Then $f_{yx}(x, y) = 3$

Since $f_{yx}(x, y)$ and $f_{xy}(x, y)$ both are continuous everywhere

But $f_{yx}(x, y) \neq f_{xy}(x, y)$

So this is the contradiction with Clairaut's Theorem

Hence there is no function $f(x, y)$ for which

$f_x(x, y) = x + 4y$, and $f_y(x, y) = 3x - y$

Answer 94E.

Consider the paraboloid $z = 6 - x - x^2 - 2y^2$.

The paraboloid intersects the plane $x = 1$ in a parabola.

It is required to find the parametric equations for the tangent line to this parabola at the point $(1, 2, -4)$.

Since the tangent line lies in the plane $x = 1$, the parameterization of x is $x = 1$.

Since y is an independent variable, the parameterization of y is $y = 2 + t$.

Since $x=1$, the equation of the parabola in terms of z and y is as follows:

$$z=6-x-x^2-2y^2$$

$$z=6-(1)-(1)^2-2y^2$$ Substitute 1 for x

$$=6-1-1-2y^2$$

$$z=4-2y^2$$

The slope of the tangent line to this parabola is equal to the derivative of the equation of the parabola. So, differentiate $z=4-2y^2$ partially with respect to y.

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\left(4-2y^2\right)$$

$$=\frac{\partial}{\partial y}(4)-2\frac{\partial}{\partial y}\left(y^2\right)$$ Since $\frac{\partial}{\partial y}(\text{Constant})=0$ and $\frac{\partial}{\partial y}\left(y^n\right)=n\cdot y^{n-1}$

$$=(0)-2(2y)$$

$$\frac{\partial z}{\partial y}=-4y$$

Calculate the slope of the tangent at the point $(1,2,-4)$.

For this replace y by 2.

$$\frac{\partial z}{\partial y}(1,2,-4)=-4(2)$$

$$=-8$$

$$\frac{\partial z}{\partial y}(1,2,-4)=-8$$

Therefore, the equation of the tangent line in the $x=1$ plane is $z=-8y+b$.

Since the tangent line passes through $(1,2,-4)$, substitute the coordinates of the point to find b.

$$z=-8y+b$$
$$(-4)=-8(2)+b$$
$$-4=-16+b$$
$$12=b$$

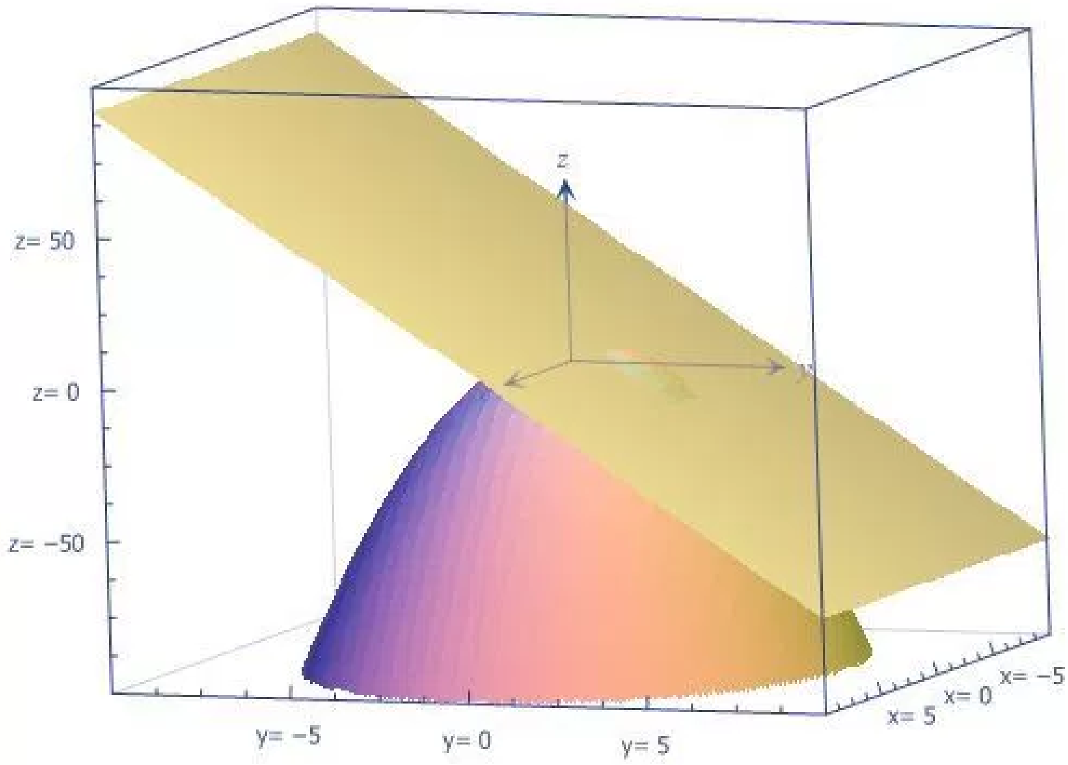
Therefore, $z=-8y+12$

Find the parameterization of z.

$$z=-8y+b$$
$z=-8(2+t)+12$ Substitute $y=2+t$
$$z=-8t-16+12$$
$$z=-8t-4$$

Hence, the equation of the tangent line is $\boxed{r(t)=\langle 1,2+t,-8t-4\rangle}$.

The graph of the paraboloid, the parabola, and the tangent line is as shown below.



Answer 95E.

Plane $y=2$ cuts the given ellipsoid $4x^2+2y^2+z^2=16$ in the ellipse

$$4x^2+2y^2+z^2=16; \quad y=2$$

The slope of tangent line at $(1,2,2)$ is the partial derivative of z with respect to x at $(1,2,2)$ i.e. $\left(\frac{\partial z}{\partial x}\right)_{(1,2,2)}$

Differentiating $4x^2+z^2=16$ partially with respect to x,

$$\frac{\partial}{\partial x}\left(4x^2+z^2\right)=\frac{\partial}{\partial x}(16)$$

$$\frac{\partial}{\partial x}\left(4x^2\right)+\frac{\partial}{\partial x}\left(z^2\right)=0$$

$$8x+2z\frac{\partial z}{\partial x}=0$$

$$\Rightarrow \quad \frac{\partial z}{\partial x}=-\frac{4x}{z}$$

$$\Rightarrow \quad \left(\frac{\partial z}{\partial x}\right)_{at(1,2,2)}=-2$$

Equation of tangent line at $(1,2,2)$ is $z-2=\left(\frac{\partial z}{\partial x}\right)_{(1,2,2)}(x-1);\qquad y=2$

$$z-2=-2(x-1);\qquad y=2$$

$$\frac{z-2}{-2}=\frac{x-1}{1};\quad y=2$$

Taking t as parameter

$$\frac{x-1}{1}=\frac{z-2}{-2}=t;\qquad y=2$$

$$x=1+t;\qquad y=2,\quad z=2-2t.$$

Hence,

Equation of tangent line is
$x=1+t;\qquad y=2;\quad z=2-2t$

Answer 96E.

Given $T = T_0 + T_1 e^{-\lambda x} \sin(\omega t - \lambda x)$

Differentiating T partially with respect to x,

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left[T_0 + T_1 e^{-\lambda x} \sin(\omega t - \lambda x)\right]$$

$$= \frac{\partial T_0}{\partial x} + T_1 \frac{\partial}{\partial x} e^{-\lambda x} \sin(\omega t - \lambda x)$$

$$= 0 + T_1\left[e^{-\lambda x} \frac{\partial}{\partial x} \sin(\omega t - \lambda x) + \sin(\omega t - \lambda x) \frac{\partial}{\partial x} e^{-\lambda x}\right]$$

$$= T_1\left[e^{-\lambda x} \cos(\omega t - \lambda x)\cdot(-\lambda) + \sin(\omega t - \lambda x)\cdot e^{-\lambda x}\cdot(-\lambda)\right]$$

$$= -\lambda T_1 e^{-\lambda x}\left[\cos(\omega t - \lambda x) + \sin(\omega t - \lambda x)\right]$$

Hence,

$$\boxed{\frac{\partial T}{\partial x} = -\lambda T_1 e^{-\lambda x}\left[\cos(\omega t - \lambda x) + \sin(\omega t - \lambda x)\right]}$$

Physical significance: $\frac{\partial T}{\partial x}$ tells us the rate of change in temperature at different depth on a particular day at given time t.

(B) Given $T = T_0 + T_1 e^{-\lambda x} \sin(\omega t - \lambda x)$

Differentiating T partially with respect to t,

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}\left[T_0 + T_1 e^{-\lambda x} \sin(\omega t - \lambda x)\right]$$

$$= \frac{\partial}{\partial t} T_0 + T_1 e^{-\lambda x} \frac{\partial}{\partial t} \sin(\omega t - \lambda x)$$

$$= 0 + T_1 e^{-\lambda x} \cos(\omega t - \lambda x)\cdot(\omega)$$

$$= \omega T_1 e^{-\lambda x} \cos(\omega t - \lambda x)$$

Hence,

$$\boxed{\frac{\partial T}{\partial t} = \omega T_1 e^{-\lambda x} \cos(\omega t - \lambda x)}$$

Physical significance: $\frac{\partial T}{\partial t}$ represents the rate of change of temperature with respect to time t when the depth x is fixed.

(C) We have found in part (a)

$$T_x = -\lambda T_1 e^{-\lambda x}\left[\cos(\omega t-\lambda x)+\sin(\omega t-\lambda x)\right]$$

Differentiating T_x partially with respect to x,

$$T_{xx} = \frac{\partial}{\partial x}T_x$$

$$= -\lambda T_1 \frac{\partial}{\partial x} e^{-\lambda x}\left[\cos(\omega t-\lambda x)+\sin(\omega t-\lambda x)\right]$$

$$= -\lambda T_1 \begin{bmatrix} e^{-\lambda x}\frac{\partial}{\partial x}\left\{\cos(\omega t-\lambda x)+\sin(\omega t-\lambda x)\right\}+ \\ \left\{\cos(\omega t-\lambda x)+\sin(\omega t-\lambda x)\right\}\frac{\partial}{\partial x}e^{-\lambda x}\end{bmatrix}$$

$$= -\lambda T_1 \begin{bmatrix} e^{-\lambda x}\left\{-\sin(\omega t-\lambda x)\cdot(-\lambda)+\cos(\omega t-\lambda x)\cdot(-\lambda)\right\}+ \\ \left\{\cos(\omega t-\lambda x)+\sin(\omega t-\lambda x)\right\}e^{-\lambda x}(-\lambda)\end{bmatrix}$$

$$= \lambda^2 T_1 e^{-\lambda x}\begin{bmatrix}-\sin(\omega t-\lambda x)+\cos(\omega t-\lambda x) \\ +\cos(\omega t-\lambda x)+\sin(\omega t-\lambda x)\end{bmatrix}$$

$$= 2\lambda^2 T_1 e^{-\lambda x}\cos(\omega t-\lambda x)$$

Also, we have found in part (b)

$$T_t = \frac{\partial T}{\partial t} = \omega T_1 e^{-\lambda x}\cos(\omega t-\lambda x)$$

$$= \frac{\omega}{2\lambda^2}\left[2\lambda^2 T_1 e^{-\lambda x}\cos(\omega t-\lambda x)\right]$$

$$= KT_{xx}$$

Where $K = \frac{\omega}{2\lambda^2}$ is a constant.

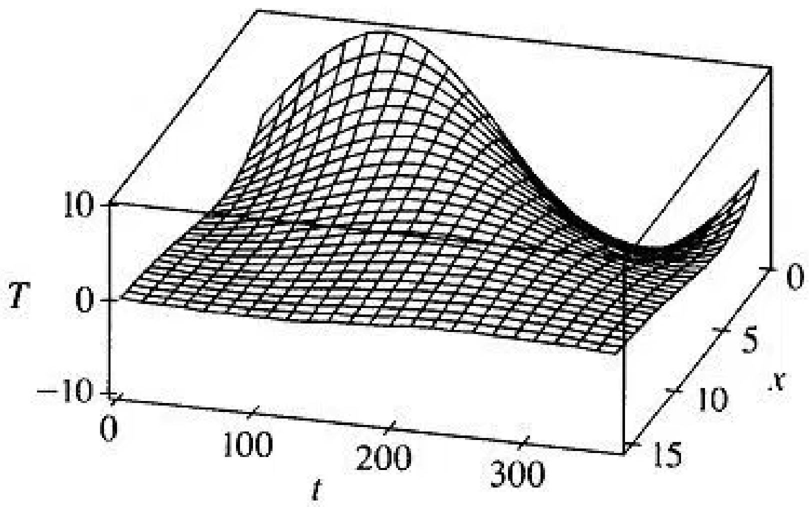
Hence T satisfies the heat equation $T_t = KT_{xx}$ for a certain constant.

(d) We have $\lambda = 0.2,\ T_0 = 0,\ T_1 = 10$

Then

$$T(x,t) = 10e^{-0.2x}\sin(\omega t - 0.2x)$$

Now we treat ω as constant



(e)

The term $-\lambda x$ in the expression $\sin(\omega t - \lambda x)$ is the phase shift, it represents that the heat passes slowly through soil and at the deeper point it takes time for changes in surface temperature

Answer 97E.

It is given that the third order partial derivatives are continuous. This implies that the second order derivatives f_{xy} and f_{yx} are also continuous

Then by Clairaut's theorem

$$f_{xy} = f_{yx} \qquad \text{--------------- (1)}$$

Differentiating both the sides partially with respect to y

$$\frac{\partial}{\partial y}(f_{xy}) = \frac{\partial}{\partial y}(f_{yx})$$

$$f_{yxy} = f_{yyx} \qquad \text{------------- (2)}$$

Now $f_{yxy} = (f_{yx})_y$

$$= (f_{xy})_y$$

$$= f_{xyy} \qquad \text{---------- (3)}$$

Hence equations (2) and (3) combine to give

$$\boxed{f_{xyy} = f_{yxy} = f_{yyx}}$$

Answer 98E.

(A)

For a function f of two variables the second order partial derivatives are

$f_{xy}, f_{yx}, f_{xx}, f_{yy}$

i.e. 2^2

And the third order partial derivatives are

$f_{xxx}, f_{yyy}, f_{xyx}, f_{xxy}, f_{xyy}, f_{yxy}, f_{yyx}, f_{yxx}$

i.e. 2^3

And the fourth order partial derivatives of f are

$f_{xxxx}, f_{yyyy}, f_{xyxx}, f_{xxyx}, f_{xxxy}, f_{xxyy}, f_{yyxx}, f_{xyyy}, f_{yxyy}, f_{yyxy}, f_{yyyx}$

$f_{yxxx}, f_{yxyx}, f_{xyxy}, f_{xyyx}, f_{yxxy}$

i.e. 2^4

Proceeding in the same manner we find that there are 2^n n^{th} order partial derivatives of a function f of two variables

(B)

If all of these partial derivatives are continuous then by Clairaut's theorem we see that only $r_{xxx.......x}$ times and $f_{yyy.....x}$ times are distinct and rest are all equal. That is only two of the 2^n derivatives are distinct

(C)

If f is a function of three variables say x, y, z then the first order partial derivatives are

f_x, f_y, f_z

So the number of 1^{st} order partial derivatives $= 3 = 3^1$

And the second order partial derivatives are

$f_{xx}, f_{yy}, f_{zz}, f_{xy}, f_{xz}, f_{yz}, f_{zy}, f_{yx}, f_{zx}$

So the second order partial derivatives are $9 = 3^2$

Let us now write down the third order partial derivatives. These are

$f_{xxx}, f_{yyy}, f_{zzz}, f_{xyy}, f_{yxy}, f_{zyy}, f_{xzz}, f_{yzz}, f_{xxy}, f_{zxy}, f_{xxz}, f_{yxz}, f_{zxz}, f_{xyz}, f_{yyz}$

$f_{zyz}, f_{xzy}, f_{yzy}, f_{zzy}, f_{xyx}, f_{yyx}, f_{zyx}, f_{xzx}, f_{yzx}, f_{zzx}, f_{yxx}, f_{zxx}$

The total numbers of 3^{rd} order partial derivatives are $27 = 3^3$

If we wish to find the 4^{th} order partial derivatives, then each of the 27 derivatives of 3^{rd} order will give rise to 3 derivatives. So the number of 4^{th} order partial derivatives will be $27 \times 3 = 81 = 3^4$

Hence for a function of 3 variables there will be 3^n partial derivatives

Answer 99E.

$$f(x,y) = x\left(x^2+y^2\right)^{-3/2} e^{\sin(x^2 y)}$$

From definition of partial derivatives we know

$$f_x(1,0)=\lim_{h\to 0}\frac{f(1+h,0)-f(1,0)}{h}$$

$$=\lim_{h\to 0}\frac{(1+h)(1+h)^{-3}e^0-1(1+0)^{-3/2}e^0}{h}$$

$$=\lim_{h\to 0}\frac{(1+h)^{-2}-1}{h}$$

$$i.e. f_x(1,0)=\lim_{h\to 0}\frac{1-(1+h)^2}{h(1+h)^2}$$

$$=\lim_{h\to 0}\frac{1-1-h^2-2h}{h(1+h)^2}$$

$$=\lim_{h\to 0}\frac{-(h+2)h}{(1+h)^2 h}$$

$$=\lim_{h\to 0}\frac{-(h+2)}{(1+h)^2}$$

$$=-2$$

Hence $f_x(1,0)=\boxed{-2}$

Answer 100E.

$$f(x,y)=\sqrt[3]{x^3+y^3}$$

By the definition of partial derivatives

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt[3]{(0+h)^3+0}-0}{h}$$

$$=\lim_{h\to 0}\frac{h}{h}$$

$$=\lim_{h\to 0}1$$

$$=1$$

Hence $f_x(0,0)=\boxed{1}$

Answer 101E.

$$f(x,y)=\begin{cases}\dfrac{x^3y-xy^3}{x^2+y^2} & if \quad (x,y)\neq(0,0)\\ 0 & if \quad (x,y)=(0,0)\end{cases}$$

(B)

When $(x,y) \neq (0,0)$

Then $$f_x(x,y) = \frac{\partial}{\partial x}\left(x^3y - xy^3\right)\left(x^2+y^2\right)^{-1}$$

$$= \frac{\left(3x^2y - y^3\right)\left(x^2+y^2\right) - 2x\left(x^3y - xy^3\right)}{\left(x^2+y^2\right)^2}$$

$$= \frac{3x^4y + 3x^2y^3 - x^2y^3 - y^5 - 2x^4y + 2x^2y^3}{\left(x^2+y^2\right)^2}$$

$$= \frac{x^4y + 4x^2y^3 - y^5}{\left(x^2+y^2\right)^2}$$

And $$f_y(x,y) = \frac{\partial}{\partial y}\left(x^3y - xy^3\right)\left(x^2+y^2\right)^{-1}$$

$$= \frac{\left(x^3 - 3xy^2\right)\left(x^2+y^2\right) - 2y\left(x^3y - xy^3\right)}{\left(x^2+y^2\right)^2}$$

$$= \frac{x^5 + x^3y^2 - 3x^3y^2 - 3xy^4 - 2x^2y^2 + 2xy^4}{\left(x^2+y^2\right)^2}$$

$$= \frac{x^5 - 4x^3y^2 - xy^4}{\left(x^2+y^2\right)^2}$$

Hence $$f_x(x,y) = \frac{x^4y + 4x^2y^3 - y^5}{\left(x^2+y^2\right)^2}$$

$$f_y(x,y) = \frac{x^5 - 4x^3y^2 - xy^4}{\left(x^2+y^2\right)^2}$$

(C)

As we know (from definition)

$$f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0}\frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{h^3(0) - h(0)^3}{h^2+0^2} - 0}{h}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

i.e. $f_x(0,0) = 0$

And $f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(0)^3 h - 0(h)^3}{0^2 + h^2} - 0}{h}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

i.e. $f_y(0,0) = 0$

(D)

Now $f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,0+h) - f_x(0,0)}{h}$

$$= \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{0+0-h^5}{h^4} - 0}{h} \quad \text{\{Using parts (B) and (C)\}}$$

$$= \lim_{h\to 0} \frac{-h^5}{h^5}$$

$$= \lim_{h\to 0} (-1)$$

$$= -1$$

And $f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h}$

$$= \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h^5 - 0 - 0}{h^4} - 0}{h} \quad \text{\{Using parts (B) and (C)\}}$$

$$= \lim_{h\to 0} \frac{h^5}{h^5}$$

$$= \lim_{h\to 0} (1)$$

$$= 1$$

Hence $f_{xy}(0,0) = -1$, and $f_{yx}(0,0) = 1$