Chapter 5

Magnetostatics

5.1 The Lorentz Force Law

5.1.1 Magnetic Fields

Remember the basic problem of classical electrodynamics: We have a collection of charges q_1, q_2, q_3, ... (the "source" charges), and we want to calculate the force they exert on some other charge Q (the "test" charge). (See Fig. 5.1.) According to the principle of superposition, it is sufficient to find the force of a *single* source charge—the total is then the vector sum of all the individual forces. Up to now we have confined our attention to the simplest case, *electrostatics*, in which the source charge is *at rest* (though the test charge need not be). The time has come to consider the forces between charges *in motion*.

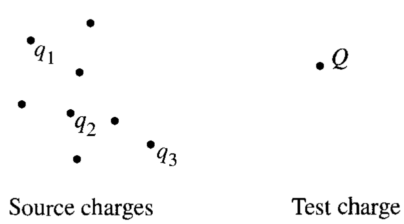
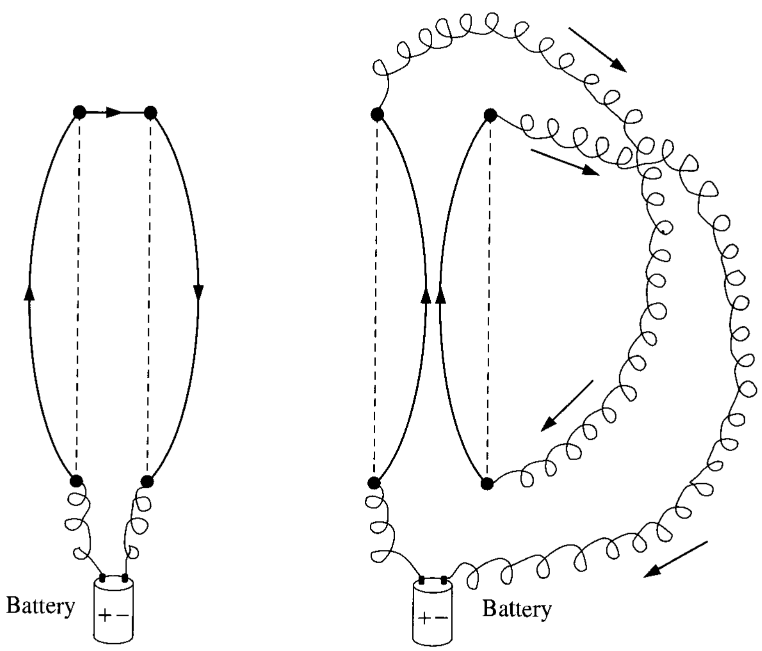


Figure 5.1

To give you some sense of what is in store, imagine that I set up the following demonstration: Two wires hang from the ceiling, a few centimeters apart; when I turn on a current, so that it passes up one wire and back down the other, the wires jump apart—they evidently repel one another (Fig. 5.2(a)). How do you explain this? Well, you might suppose that the battery (or whatever drives the current) is actually charging up the wire, and that the force is simply due to the electrical repulsion of like charges. But this explanation is incorrect. I could hold up a test charge near these wires and there would be *no* force on it.



(a) Currents in opposite directions repel.

(b) Currents in same directions attract.

Figure 5.2

for the wires are in fact electrically neutral. (It's true that electrons are flowing down the line—that's what a current *is*—but there are just as many stationary plus charges as moving minus charges on any given segment.) Moreover, I could hook up my demonstration so as to make the current flow up *both* wires (Fig. 5.2(b)); in this case they are found to *attract!*

Whatever force accounts for the attraction of parallel currents and the repulsion of antiparallel ones is *not* electrostatic in nature. It is our first encounter with a *magnetic* force. Whereas a *stationary* charge produces only an electric field **E** in the space around it, a *moving* charge generates, in addition, a magnetic field **B**. In fact, magnetic fields are a lot easier to detect, in practice—all you need is a Boy Scout compass. How these devices work is irrelevant at the moment; it is enough to know that the needle points in the direction of the local magnetic field. Ordinarily, this means *north*, in response to the *earth's* magnetic field, but in the laboratory, where typical fields may be hundreds of times stronger than that, the compass indicates the direction of whatever magnetic field is present.

Now, if you hold up a tiny compass in the vicinity of a current-carrying wire, you quickly discover a very peculiar thing: The field does not point *toward* the wire, nor *away* from it, but rather it *circles around the wire*. In fact, if you grab the wire with your right

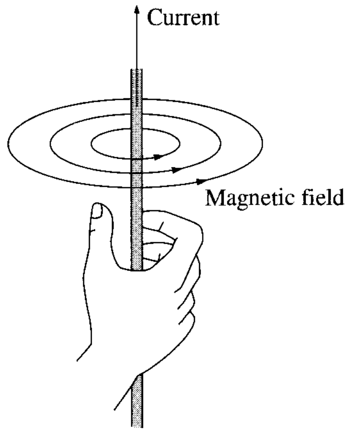


Figure 5.3

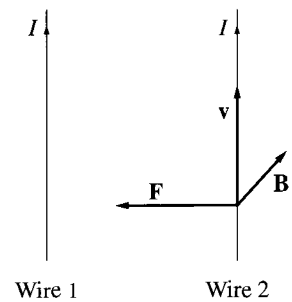


Figure 5.4

hand—thumb in the direction of the current—your fingers curl around in the direction of the magnetic field (Fig. 5.3). How can such a field lead to a force of attraction on a nearby parallel current? At the second wire the magnetic field points *into the page* (Fig. 5.4), the velocity of the charges is *upward*, and yet the resulting force is *to the left*. It's going to take a strange law to account for these directions! I'll introduce this law in the next section. Later on, in Sect. 5.2, we'll return to what is logically the prior question: How do you calculate the magnetic field of the first wire?

5.1.2 Magnetic Forces

It may have occurred to you that the combination of directions in Fig. 5.4 is just right for a cross product. In fact, the magnetic force in a charge Q, moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$, is[1]

$$\boxed{\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B}).} \tag{5.1}$$

This is known as the **Lorentz force law**. In the presence of both electric *and* magnetic fields, the net force on Q would be

$$\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]. \tag{5.2}$$

I do not pretend to have *derived* Eq. 5.1, of course; it is a fundamental axiom of the theory, whose justification is to be found in experiments such as the one I described in Sect. 5.1.1. Our main job from now on is to calculate the magnetic field $\mathbf{B}$ (and for that matter the electric field $\mathbf{E}$ as well, for the rules are more complicated when the source charges are in motion). But before we proceed, it is worthwhile to take a closer look at the Lorentz force law itself; it is a peculiar law, and it leads to some truly bizarre particle trajectories.

[1] Since $\mathbf{F}$ and $\mathbf{v}$ are vectors, $\mathbf{B}$ is actually a *pseudo*vector.

Example 5.1

Cyclotron motion

The archetypical motion of a charged particle in a magnetic field is circular, with the magnetic force providing the centripetal acceleration. In Fig. 5.5, a uniform magnetic field points *into* the page; if the charge Q moves counterclockwise, with speed v, around a circle of radius R, the magnetic force (5.1) points *inward*, and has a fixed magnitude QvB—just right to sustain uniform circular motion:

$$QvB = m\frac{v^2}{R}, \quad \text{or} \quad p = QBR, \tag{5.3}$$

where m is the particle's mass and $p = mv$ is its momentum. Equation 5.3 is known as the **cyclotron formula** because it describes the motion of a particle in a cyclotron—the first of the modern particle accelerators. It also suggests a simple experimental technique for finding the momentum of a particle: send it through a region of known magnetic field, and measure the radius of its circular trajectory. This is in fact the standard means for determining the momenta of elementary particles.

Incidentally, I assumed that the charge moves in a plane perpendicular to **B**. If it starts out with some additional speed $v_{\parallel}$ *parallel* to **B**, this component of the motion is unaffected by the magnetic field, and the particle moves in a *helix* (Fig. 5.6). The radius is still given by Eq. 5.3, but the velocity in question is now the component perpendicular to **B**, $v_{\perp}$.

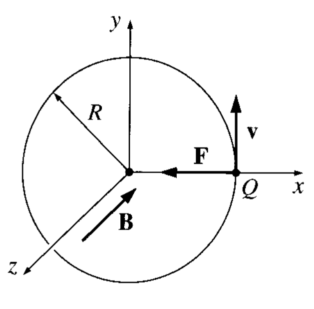


Figure 5.5

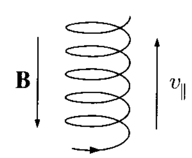


Figure 5.6

Example 5.2

Cycloid Motion

A more exotic trajectory occurs if we include a uniform electric field, at right angles to the magnetic one. Suppose, for instance, that **B** points in the x-direction, and **E** in the z-direction, as shown in Fig. 5.7. A particle at rest is released from the origin; what path will it follow?

Solution: Let's think it through qualitatively, first. Initially, the particle is at rest, so the magnetic force is zero, and the electric field accelerates the charge in the z-direction. As it picks up speed, a magnetic force develops which, according to Eq. 5.1, pulls the charge around

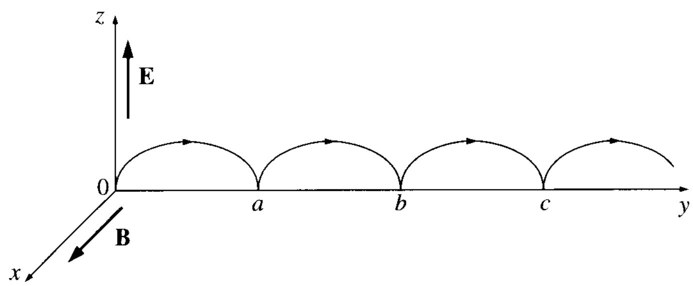


Figure 5.7

to the right. The faster it goes, the stronger F_{mag} becomes; eventually, it curves the particle back around towards the y axis. At this point the charge is moving *against* the electrical force, so it begins to slow down—the magnetic force then decreases, and the electrical force takes over, bringing the charge to rest at point a, in Fig. 5.7. There the entire process commences anew, carrying the particle over to point b, and so on.

Now let's do it quantitatively. There being no force in the x-direction, the position of the particle at any time t can be described by the vector $(0, y(t), z(t))$; the velocity is therefore

$$\mathbf{v} = (0, \dot{y}, \dot{z}),$$

where dots indicate time derivatives. Thus

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} = B\dot{z}\,\hat{\mathbf{y}} - B\dot{y}\,\hat{\mathbf{z}},$$

and hence, applying Newton's second law,

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = Q(E\,\hat{\mathbf{z}} + B\dot{z}\,\hat{\mathbf{y}} - B\dot{y}\,\hat{\mathbf{z}}) = m\mathbf{a} = m(\ddot{y}\,\hat{\mathbf{y}} + \ddot{z}\,\hat{\mathbf{z}}).$$

Or, treating the $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ components separately,

$$QB\dot{z} = m\ddot{y}, \qquad QE - QB\dot{y} = m\ddot{z}.$$

For convenience, let

$$\omega \equiv \frac{QB}{m}. \tag{5.4}$$

(This is the **cyclotron frequency**, at which the particle would revolve in the absence of any electric field.) Then the equations of motion take the form

$$\ddot{y} = \omega\dot{z}, \quad \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right). \tag{5.5}$$

Their general solution[2] is

$$\left.\begin{aligned} y(t) &= C_1 \cos\omega t + C_2 \sin\omega t + (E/B)t + C_3, \\ z(t) &= C_2 \cos\omega t - C_1 \sin\omega t + C_4. \end{aligned}\right\} \tag{5.6}$$

[2]As coupled differential equations, they are easily solved by differentiating the first and using the second to eliminate $\ddot{z}$.

But the particle started from rest ($\dot{y}(0) = \dot{z}(0) = 0$), at the origin ($y(0) = z(0) = 0$); these four conditions determine the constants C_1, C_2, C_3, and C_4:

$$y(t) = \frac{E}{\omega B}(\omega t - \sin \omega t), \quad z(t) = \frac{E}{\omega B}(1 - \cos \omega t). \tag{5.7}$$

In this form the answer is not terribly enlightening, but if we let

$$R \equiv \frac{E}{\omega B}, \tag{5.8}$$

and eliminate the sines and cosines by exploiting the trigonometric identity $\sin^2 \omega t + \cos^2 \omega t = 1$, we find that

$$(y - R\omega t)^2 + (z - R)^2 = R^2. \tag{5.9}$$

This is the formula for a *circle*, of radius R, whose center $(0, R\omega t, R)$ travels in the y-direction at a constant speed,

$$v = \omega R = \frac{E}{B}. \tag{5.10}$$

The particle moves as though it were a spot on the rim of a wheel, rolling down the y axis at speed v. The curve generated in this way is called a **cycloid**. Notice that the overall motion is *not* in the direction of **E**, as you might suppose, but perpendicular to it.

One feature of the magnetic force law (Eq. 5.1) warrants special attention:

Magnetic forces do no work.

For if Q moves an amount $d\mathbf{l} = \mathbf{v}\,dt$, the work done is

$$dW_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\,dt = 0. \tag{5.11}$$

This follows because $(\mathbf{v} \times \mathbf{B})$ is perpendicular to $\mathbf{v}$, so $(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = 0$. Magnetic forces may alter the *direction* in which a particle moves, but they cannot speed it up or slow it down. The fact that magnetic forces do no work is an elementary and direct consequence of the Lorentz force law, but there are many situations in which it *appears* so manifestly false that one's confidence is bound to waver. When a magnetic crane lifts the carcass of a junked car, for instance, *something* is obviously doing work, and it seems perverse to deny that the magnetic force is responsible. Well, perverse or not, deny it we must, and it can be a very subtle matter to figure out what agency *does* deserve the credit in such circumstances. I'll show you several examples as we go along.

Problem 5.1 A particle of charge q enters a region of uniform magnetic field **B** (pointing *into* the page). The field deflects the particle a distance d above the original line of flight, as shown in Fig. 5.8. Is the charge positive or negative? In terms of a, d, B and q, find the momentum of the particle.

Problem 5.2 Find and sketch the trajectory of the particle in Ex. 5.2, if it starts at the origin with velocity

(*a*) $\mathbf{v}(0) = (E/B)\hat{\mathbf{y}}$,

(*b*) $\mathbf{v}(0) = (E/2B)\hat{\mathbf{y}}$,

(*c*) $\mathbf{v}(0) = (E/B)(\hat{\mathbf{y}} + \hat{\mathbf{z}})$.

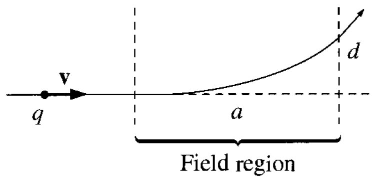


Figure 5.8

Problem 5.3 In 1897 J. J. Thomson "discovered" the electron by measuring the charge-to-mass ratio of "cathode rays" (actually, streams of electrons, with charge q and mass m) as follows:

(a) First he passed the beam through uniform crossed electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ (mutually perpendicular, and both of them perpendicular to the beam), and adjusted the electric field until he got zero deflection. What, then, was the speed of the particles (in terms of E and B)?

(b) Then he turned off the electric field, and measured the radius of curvature, R, of the beam, as deflected by the magnetic field alone. In terms of E, B, and R, what is the charge-to-mass ratio (q/m) of the particles?

5.1.3 Currents

The **current** in a wire is the *charge per unit time* passing a given point. By definition, negative charges moving to the left count the same as positive ones to the right. This conveniently reflects the *physical* fact that almost all phenomena involving moving charges depend on the *product* of charge and velocity—if you change the sign of q *and* $\mathbf{v}$, you get the same answer, so it doesn't really matter which you have. (The Lorentz force law is a case in point; the Hall effect (Prob. 5.39) is a notorious exception.) In practice, it is ordinarily the negatively charged electrons that do the moving—in the direction *opposite* the electric current. To avoid the petty complications this entails, I shall often pretend it's the positive charges that move, as in fact everyone assumed they did for a century or so after Benjamin Franklin established his unfortunate convention.[3] Current is measured in coulombs-per-second, or **amperes** (A):

$$1\ \mathrm{A} = 1\ \mathrm{C/s}. \tag{5.12}$$

A line charge λ traveling down a wire at speed v (Fig. 5.9) constitutes a current

$$I = \lambda v, \tag{5.13}$$

because a segment of length $v\Delta t$, carrying charge $\lambda v \Delta t$, passes point P in a time interval Δt. Current is actually a *vector:*

$$\mathbf{I} = \lambda \mathbf{v}; \tag{5.14}$$

[3] If we called the electron plus and the proton minus, the problem would never arise. In the context of Franklin's experiments with cat's fur and glass rods, the choice was completely arbitrary.

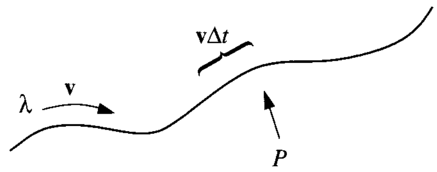


Figure 5.9

since the path of the flow is dictated by the shape of the wire, most people don't bother to display the vectorial character of **I** explicitly, but when it comes to surface and volume currents we cannot afford to be so casual, and for the sake of notational consistency it is a good idea to acknowledge this right from the start. A neutral wire, of course, contains as many stationary positive charges as mobile negative ones. The former do not contribute to the current—the charge density λ in Eq. 5.13 refers only to the *moving* charges. In the unusual situation where *both* types move, $\mathbf{I} = \lambda_+\mathbf{v}_+ + \lambda_-\mathbf{v}_-$.

The magnetic force on a segment of current-carrying wire is evidently

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B})\lambda\, dl = \int (\mathbf{I} \times \mathbf{B})\, dl. \tag{5.15}$$

Inasmuch as **I** and $d\mathbf{l}$ both point in the same direction, we can just as well write this as

$$\boxed{\mathbf{F}_{\text{mag}} = \int I\,(d\mathbf{l} \times \mathbf{B}).} \tag{5.16}$$

Typically, the current is constant (in magnitude) along the wire, and in that case I comes outside the integral:

$$\mathbf{F}_{\text{mag}} = I \int (d\mathbf{l} \times \mathbf{B}). \tag{5.17}$$

Example 5.3

A rectangular loop of wire, supporting a mass m, hangs vertically with one end in a uniform magnetic field **B**, which points into the page in the shaded region of Fig. 5.10. For what current I, in the loop, would the magnetic force upward exactly balance the gravitational force downward?

Solution: First of all, the current must circulate clockwise, in order for $(\mathbf{I} \times \mathbf{B})$ in the horizontal segment to point upward. The force is

$$F_{\text{mag}} = IBa,$$

where a is the width of the loop. (The magnetic forces on the two vertical segments cancel.) For F_{mag} to balance the weight (mg), we must therefore have

$$I = \frac{mg}{Ba}. \tag{5.18}$$

The weight just *hangs* there, suspended in mid-air!

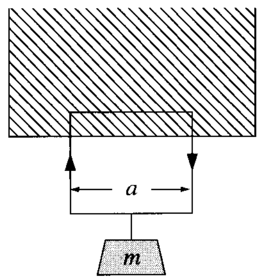


Figure 5.10

What happens if we now *increase* the current? Then the upward magnetic force *exceeds* the downward force of gravity, and the loop rises, lifting the weight. *Somebody's* doing work, and it sure looks as though the magnetic force is responsible. Indeed, one is tempted to write

$$W_{\text{mag}} = F_{\text{mag}}h = IBah, \tag{5.19}$$

where h is the distance the loop rises. But we know that magnetic forces *never* do work. What's going on here?

Well, when the loop starts to rise, the charges in the wire are no longer moving horizontally—their velocity now acquires an upward component u, the speed of the loop (Fig. 5.11), in addition to the horizontal component w associated with the current ($I = \lambda w$). The magnetic force, which is always perpendicular to the velocity, no longer points straight up, but tilts back. It is perpendicular to the *net* displacement of the charge (which is in the direction of **v**), and therefore *it does no work on* q. It does have a vertical *component* (qwB); indeed, the net vertical force on all the charge (λa) in the upper segment of the loop is

$$F_{\text{vert}} = \lambda awB = IBa \tag{5.20}$$

(as before); but now it also has a *horizontal* component (quB), which opposes the flow of current. Whoever is in charge of maintaining that current, therefore, must now *push* those charges along, against the backward component of the magnetic force.

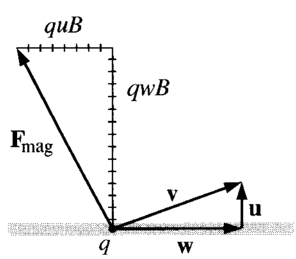


Figure 5.11

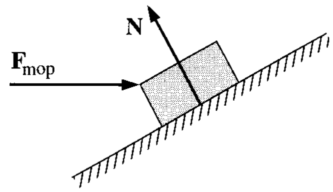


Figure 5.12

The total horizontal force on the top segment is evidently

$$F_{\text{horiz}} = \lambda a u B. \tag{5.21}$$

In a time dt the charges move a (horizontal) distance $w\,dt$, so the work done by this agency (presumably a battery or a generator) is

$$W_{\text{battery}} = \lambda a B \int uw\,dt = IBah,$$

which is precisely what we naïvely attributed to the *magnetic* force in Eq. 5.19. Was work done in this process? Absolutely! Who *did* it? The battery! What, then, was the role of the magnetic force? Well, it *redirected* the horizontal force of the battery into the *vertical* motion of the loop and the weight.

It may help to consider a mechanical analogy. Imagine you're pushing a trunk up a frictionless ramp, by pushing on it horizontally with a mop (Fig. 5.12). The normal force (**N**) does no work, because it is perpendicular to the displacement. But it *does* have a vertical component (which in fact is what lifts the trunk), and a (backward) horizontal component (which you have to overcome by pushing on the mop). Who is doing the work here? *You* are, obviously—and yet your *force* (which is purely horizontal) is not (at least, not directly) what lifts the box. The normal force plays the same passive (but crucial) role as the magnetic force in Ex. 5.3: while doing no work itself, it *redirects* the efforts of the active agent (you, or the battery, as the case may be), from horizontal to vertical.

When charge flows over a *surface*, we describe it by the **surface current density**, **K**, defined as follows: Consider a "ribbon" of infinitesimal width $dl_\perp$, running parallel to the flow (Fig. 5.13). If the current in this ribbon is $d\mathbf{I}$, the surface current density is

$$\mathbf{K} \equiv \frac{d\mathbf{I}}{dl_\perp}. \tag{5.22}$$

In words, K is the *current per unit width-perpendicular-to-flow*. In particular, if the (mobile) surface charge density is σ and its velocity is $\mathbf{v}$, then

$$\mathbf{K} = \sigma\mathbf{v}. \tag{5.23}$$

In general, **K** will vary from point to point over the surface, reflecting variations in σ and/or $\mathbf{v}$. The magnetic force on the surface current is

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\sigma\,da = \int (\mathbf{K} \times \mathbf{B})\,da. \tag{5.24}$$

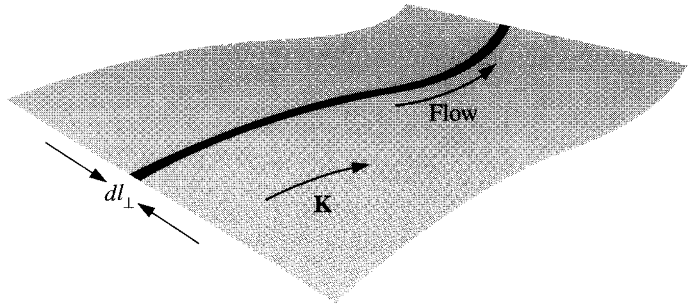


Figure 5.13

Caveat: Just as **E** suffers a discontinuity at a surface *charge*, so **B** is discontinuous at a surface *current*. In Eq. 5.24, you must be careful to use the *average* field, just as we did in Sect. 2.5.3.

When the flow of charge is distributed throughout a three-dimensional region, we describe it by the **volume current density**, **J**, defined as follows: Consider a "tube" of infinitesimal cross section $da_\perp$, running parallel to the flow (Fig. 5.14). If the current in this tube is $d\mathbf{I}$, the volume current density is

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_\perp}. \tag{5.25}$$

In words, J is the *current per unit area-perpendicular-to-flow*. If the (mobile) volume charge density is ρ and the velocity is $\mathbf{v}$, then

$$\mathbf{J} = \rho\mathbf{v}. \tag{5.26}$$

The magnetic force on a volume current is therefore

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\rho\, d\tau = \int (\mathbf{J} \times \mathbf{B})\, d\tau. \tag{5.27}$$

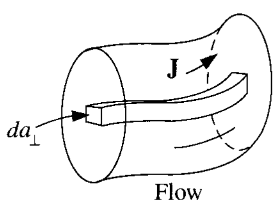


Figure 5.14

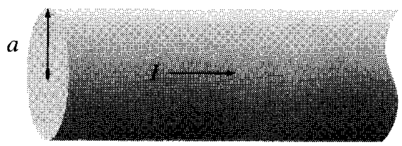


Figure 5.15

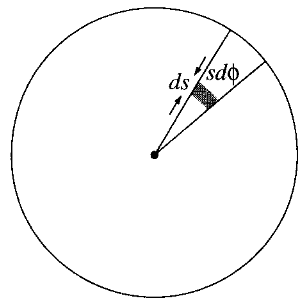


Figure 5.16

Example 5.4

(a) A current I is uniformly distributed over a wire of circular cross section, with radius a (Fig. 5.15). Find the volume current density J.

Solution: The area-perpendicular-to-flow is πa^2, so

$$J = \frac{I}{\pi a^2}.$$

This was trivial because the current density was uniform.

(b) Suppose the current density in the wire is proportional to the distance from the axis,

$$J = ks$$

(for some constant k). Find the total current in the wire.

Solution: Because J varies with s, we must *integrate* Eq. 5.25. The current in the shaded patch (Fig. 5.16) is $J da_\perp$, and $da_\perp = s\,ds\,d\phi$. So,

$$I = \int (ks)(s\,ds\,d\phi) = 2\pi k \int_0^a s^2\,ds = \frac{2\pi k a^3}{3}.$$

According to Eq. 5.25, the current crossing a surface $\mathcal{S}$ can be written as

$$I = \int_{\mathcal{S}} J\,da_\perp = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}. \tag{5.28}$$

(The dot product serves neatly to pick out the appropriate component of $d\mathbf{a}$.) In particular, the total charge per unit time leaving a volume $\mathcal{V}$ is

$$\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{J})\,d\tau.$$

Because charge is conserved, whatever flows out through the surface must come at the expense of that remaining inside:

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{J})\, d\tau = -\frac{d}{dt}\int_{\mathcal{V}} \rho\, d\tau = -\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t}\right) d\tau.$$

(The minus sign reflects the fact that an *outward* flow *decreases* the charge left in $\mathcal{V}$.) Since this applies to *any* volume, we conclude that

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.} \tag{5.29}$$

This is the precise mathematical statement of local charge conservation; it is called the **continuity equation**.

For future reference, let me summarize the "dictionary" we have implicitly developed for translating equations into the forms appropriate to point, line, surface, and volume currents:

$$\sum_{i=1}^{n} (\;\;) q_i \mathbf{v}_i \sim \int_{\text{line}} (\;\;) \mathbf{I}\, dl \sim \int_{\text{surface}} (\;\;) \mathbf{K}\, da \sim \int_{\text{volume}} (\;\;) \mathbf{J}\, d\tau. \tag{5.30}$$

This correspondence, which is analogous to $q \sim \lambda\, dl \sim \sigma\, da \sim \rho\, d\tau$ for the various charge distributions, generates Eqs. 5.15, 5.24, and 5.27 from the original Lorentz force law (5.1).

Problem 5.4 Suppose that the magnetic field in some region has the form

$$\mathbf{B} = kz\,\hat{\mathbf{x}}$$

(where k is a constant). Find the force on a square loop (side a), lying in the yz plane and centered at the origin, if it carries a current I, flowing counterclockwise, when you look down the x axis.

Problem 5.5 A current I flows down a wire of radius a.

(a) If it is uniformly distributed over the surface, what is the surface current density K?

(b) If it is distributed in such a way that the volume current density is inversely proportional to the distance from the axis, what is J?

Problem 5.6

(a) A phonograph record carries a uniform density of "static electricity" σ. If it rotates at angular velocity ω, what is the surface current density K at a distance r from the center?

(b) A uniformly charged solid sphere, of radius R and total charge Q, is centered at the origin and spinning at a constant angular velocity ω about the z axis. Find the current density $\mathbf{J}$ at any point (r, θ, ϕ) within the sphere.

Problem 5.7 For a configuration of charges and currents confined within a volume $\mathcal{V}$, show that

$$\int_{\mathcal{V}} \mathbf{J}\, d\tau = d\mathbf{p}/dt,$$

where $\mathbf{p}$ is the total dipole moment. [*Hint:* evaluate $\int_{\mathcal{V}} \nabla \cdot (x\mathbf{J})\, d\tau$.]

5.2 The Biot-Savart Law

5.2.1 Steady Currents

Stationary charges produce electric fields that are constant in time; hence the term **electrostatics.**[4] *Steady currents* produce magnetic fields that are constant in time; the theory of steady currents is called **magnetostatics.**

Stationary charges	$\Rightarrow$	**constant electric fields: electrostatics.**
Steady currents	$\Rightarrow$	**constant magnetic fields: magnetostatics.**

By **steady current** I mean a continuous flow that has been going on forever, without change and without charge piling up anywhere. (Some people call them "stationary currents"; to my ear, that's a contradiction in terms.) Of course, there's no such thing in practice as a *truly* steady current, any more than there is a *truly* stationary charge. In this sense both electrostatics and magnetostatics describe artificial worlds that exist only in textbooks. However, they represent suitable *approximations* as long as the actual fluctuations are reasonably slow; in fact, for most purposes magnetostatics applies very well to household currents, which alternate 60 times a second!

Notice that a moving *point* charge *cannot possibly constitute a steady current.* If it's here one instant, it's gone the next. This may seem like a minor thing to *you*, but it's a major headache for *me*. I developed each topic in electrostatics by starting out with the simple case of a point charge at rest; then I generalized to an arbitrary charge distribution by invoking the superposition principle. This approach is not open to us in magnetostatics because a moving point charge does not produce a static field in the first place. We are *forced* to deal with extended current distributions, right from the start, and as a result the arguments are bound to be more cumbersome.

When a steady current flows in a wire, its magnitude I must be the same all along the line; otherwise, charge would be piling up somewhere, and it wouldn't be a steady current. By the same token, $\partial\rho/\partial t = 0$ in magnetostatics, and hence the continuity equation (5.29) becomes

$$\nabla \cdot \mathbf{J} = 0. \tag{5.31}$$

5.2.2 The Magnetic Field of a Steady Current

The magnetic field of a steady line current is given by the **Biot-Savart law**:

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\boldsymbol{\imath}}}{\imath^2}\,dl' = \frac{\mu_0}{4\pi}I\int \frac{d\mathbf{l}'\times\hat{\boldsymbol{\imath}}}{\imath^2}.} \tag{5.32}$$

[4] Actually, it is not necessary that the charges be stationary, but only that the charge *density* at each point be constant. For example, the sphere in Prob. 5.6b produces an electrostatic field $1/4\pi\epsilon_0(Q/r^2)\hat{\mathbf{r}}$, even though it is rotating, because ρ does not depend on t.

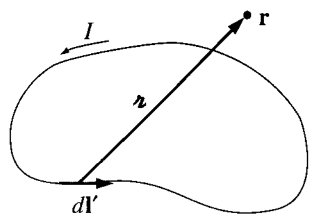


Figure 5.17

The integration is along the current path, in the direction of the flow; $d\mathbf{l}'$ is an element of length along the wire, and $\mathscr{r}$, as always, is the vector from the source to the point $\mathbf{r}$ (Fig. 5.17). The constant μ_0 is called the **permeability of free space**:[5]

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}. \tag{5.33}$$

These units are such that $\mathbf{B}$ itself comes out in newtons per ampere-meter (as required by the Lorentz force law), or **teslas** (T):[6]

$$1\ \mathrm{T} = 1\ \mathrm{N/(A \cdot m)}. \tag{5.34}$$

As the starting point for magnetostatics, the Biot-Savart law plays a role analogous to Coulomb's law in electrostatics. Indeed, the $1/\mathscr{r}^2$ dependence is common to both laws.

Example 5.5

Find the magnetic field a distance s from a long straight wire carrying a steady current I (Fig. 5.18).

Solution: In the diagram, $(d\mathbf{l}' \times \hat{\mathscr{r}})$ points *out* of the page, and has the magnitude

$$dl' \sin\alpha = dl' \cos\theta.$$

Also, $l' = s\tan\theta$, so

$$dl' = \frac{s}{\cos^2\theta}\, d\theta,$$

and $s = \mathscr{r}\cos\theta$, so

$$\frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}.$$

[5]This is an exact number, not an empirical constant. It serves (via Eq. 5.37) to define the ampere, and the ampere in turn defines the coulomb.

[6]For some reason, in this one case the cgs unit (the **gauss**) is more commonly used than the SI unit: 1 tesla = 10^4 gauss. The earth's magnetic field is about half a gauss; a fairly strong laboratory magnetic field is, say, 10,000 gauss.

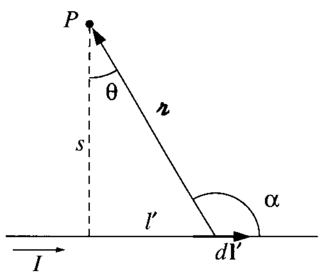


Figure 5.18

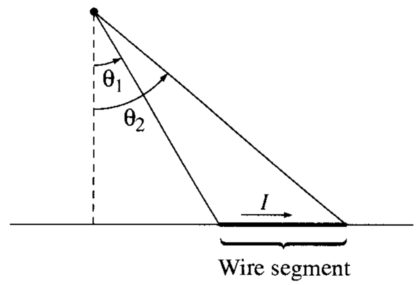


Figure 5.19

Thus

$$\begin{aligned} B &= \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right)\cos\theta\,d\theta \\ &= \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1). \end{aligned} \tag{5.35}$$

Equation 5.35 gives the field of any straight segment of wire, in terms of the initial and final angles θ_1 and θ_2 (Fig. 5.19). Of course, a finite segment by itself could never support a steady current (where would the charge go when it got to the end?), but it might be a *piece* of some closed circuit, and Eq. 5.35 would then represent its contribution to the total field. In the case of an infinite wire, $\theta_1 = -\pi/2$ and $\theta_2 = \pi/2$, so we obtain

$$B = \frac{\mu_0 I}{2\pi s}. \tag{5.36}$$

Notice that the field is inversely proportional to the distance from the wire—just like the electric field of an infinite line charge. In the region *below* the wire, **B** points *into* the page, and in general, it "circles around" the wire, in accordance with the right-hand rule stated earlier (Fig. 5.3).

As an application, let's find the force of attraction between two long, parallel wires a distance d apart, carrying currents I_1 and I_2 (Fig. 5.20). The field at (2) due to (1) is

$$B = \frac{\mu_0 I_1}{2\pi d},$$

and it points into the page. The Lorentz force law (in the form appropriate to line currents, Eq. 5.17) predicts a force directed towards (1), of magnitude

$$F = I_2\left(\frac{\mu_0 I_1}{2\pi d}\right)\int dl.$$

The *total* force, not surprisingly, is infinite, but the force per unit length is

$$f = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}. \tag{5.37}$$

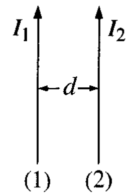


Figure 5.20

If the currents are antiparallel (one up, one down), the force is repulsive—consistent again with the qualitative observations in Sect. 5.1.1.

Example 5.6

Find the magnetic field a distance z above the center of a circular loop of radius R, which carries a steady current I (Fig. 5.21).

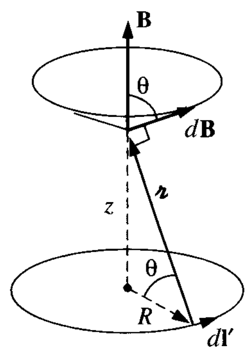


Figure 5.21

Solution: The field $d\mathbf{B}$ attributable to the segment $d\mathbf{l}'$ points as shown. As we integrate $d\mathbf{l}'$ around the loop, $d\mathbf{B}$ sweeps out a cone. The horizontal components cancel, and the vertical components combine to give

$$B(z) = \frac{\mu_0}{4\pi} I \int \frac{dl'}{\mathscr{r}^2} \cos\theta.$$

(Notice that $d\mathbf{l}'$ and $\hat{\boldsymbol{\mathscr{r}}}$ are perpendicular, in this case; the factor of $\cos\theta$ projects out the vertical component.) Now, $\cos\theta$ and $\mathscr{r}^2$ are constants, and $\int dl'$ is simply the circumference, $2\pi R$, so

$$B(z) = \frac{\mu_0 I}{4\pi}\left(\frac{\cos\theta}{\mathscr{r}^2}\right) 2\pi R = \frac{\mu_0 I}{2} \frac{R^2}{(R^2+z^2)^{3/2}}. \tag{5.38}$$

For surface and volume currents the Biot-Savart law becomes

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,da' \quad \text{and} \quad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau', \tag{5.39}$$

respectively. You might be tempted to write down the corresponding formula for a moving point charge, using the "dictionary" 5.30:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{q\mathbf{v} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}, \tag{5.40}$$

but this is simply *wrong*.[7] As I mentioned earlier, a point charge does not constitute a steady current, and the Biot-Savart law, which only holds for steady currents, does *not* correctly determine its field.

Incidentally, the superposition principle applies to magnetic fields just as it does to electric fields: If you have a *collection* of source currents, the net field is the (vector) sum of the fields due to each of them taken separately.

Problem 5.8

(a) Find the magnetic field at the center of a square loop, which carries a steady current I. Let R be the distance from center to side (Fig. 5.22).

(b) Find the field at the center of a regular n-sided polygon, carrying a steady current I. Again, let R be the distance from the center to any side.

(c) Check that your formula reduces to the field at the center of a circular loop, in the limit $n \to \infty$.

Problem 5.9 Find the magnetic field at point P for each of the steady current configurations shown in Fig. 5.23.

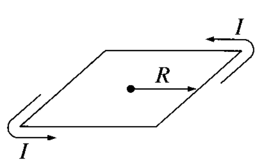


Figure 5.22

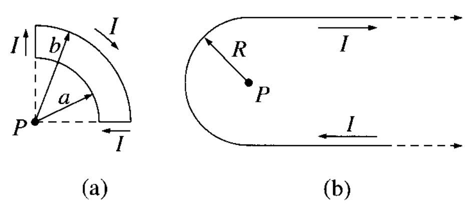


Figure 5.23

[7] I say this loud and clear to emphasize the point of principle; actually, Eq. 5.40 is *approximately* right for nonrelativistic charges ($v \ll c$), under conditions where retardation can be neglected (see Ex. 10.4).

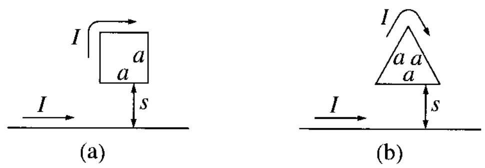


Figure 5.24

Problem 5.10

(a) Find the force on a square loop placed as shown in Fig. 5.24(a), near an infinite straight wire. Both the loop and the wire carry a steady current I.

(b) Find the force on the triangular loop in Fig. 5.24(b).

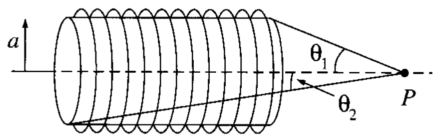


Figure 5.25

Problem 5.11 Find the magnetic field at point P on the axis of a tightly wound solenoid (helical coil) consisting of n turns per unit length wrapped around a cylindrical tube of radius a and carrying current I (Fig. 5.25). Express your answer in terms of θ_1 and θ_2 (it's easiest that way). Consider the turns to be essentially circular, and use the result of Ex. 5.6. What is the field on the axis of an *infinite* solenoid (infinite in both directions)?

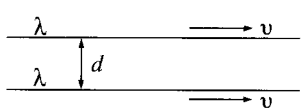


Figure 5.26

Problem 5.12 Suppose you have two infinite straight line charges λ, a distance d apart, moving along at a constant speed v (Fig. 5.26). How great would v have to be in order for the magnetic attraction to balance the electrical repulsion? Work out the actual number... Is this a reasonable sort of speed?[8]

---

[8]If you've studied special relativity, you may be tempted to look for complexities in this problem that are not really there—λ and v are both measured *in the laboratory frame*, and this is *ordinary electrostatics* (see footnote 4).

5.3 The Divergence and Curl of B

5.3.1 Straight-Line Currents

The magnetic field of an infinite straight wire is shown in Fig. 5.27 (the current is coming *out* of the page). At a glance, it is clear that this field has a nonzero curl (something you'll *never* see in an *electro*static field); let's *calculate* it.

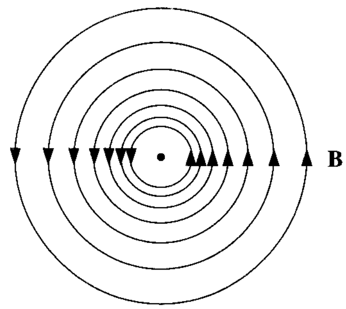


Figure 5.27

According to Eq. 5.36, the integral of **B** around a circular path of radius s, centered at the wire, is

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s}\, dl = \frac{\mu_0 I}{2\pi s} \oint dl = \mu_0 I.$$

Notice that the answer is independent of s; that's because B *de*creases at the same rate as the circumference *in*creases. In fact, it doesn't have to be a circle; *any* old loop that encloses the wire would give the same answer. For if we use cylindrical coordinates (s, ϕ, z), with the current flowing along the z axis,

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}, \tag{5.41}$$

and $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$, so

$$\oint \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi} \oint \frac{1}{s}\, s\, d\phi = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \mu_0 I.$$

This assumes the loop encircles the wire exactly once; if it went around twice, the ϕ would run from 0 to 4π, and if it didn't enclose the wire at all, then ϕ would go from ϕ_1 to ϕ_2 and back again, with $\int d\phi = 0$ (Fig. 5.28).

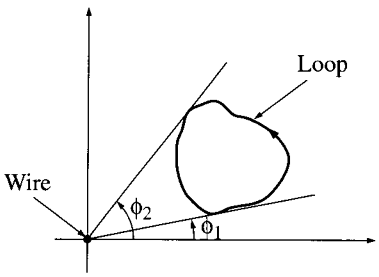


Figure 5.28

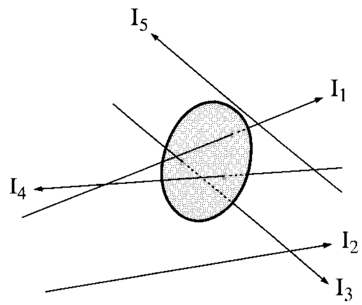


Figure 5.29

Now suppose we have a *bundle* of straight wires. Each wire that passes through our loop contributes $\mu_0 I$, and those outside contribute nothing (Fig. 5.29). The line integral will then be

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}, \tag{5.42}$$

where I_{enc} stands for the total current enclosed by the integration path. If the flow of charge is represented by a volume current density $\mathbf{J}$, the enclosed current is

$$I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a}, \tag{5.43}$$

with the integral taken over the surface bounded by the loop. Applying Stokes' theorem to Eq. 5.42, then,

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a},$$

and hence

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{5.44}$$

With minimal labor we have actually obtained the general formula for the curl of $\mathbf{B}$. But our derivation is seriously flawed by the restriction to infinite straight line currents (and combinations thereof). Most current configurations *cannot* be constructed out of infinite straight wires, and we have no right to assume that Eq. 5.44 applies to them. So the next section is devoted to the formal derivation of the divergence and curl of $\mathbf{B}$, starting from the Biot-Savart law itself.

5.3.2 The Divergence and Curl of B

The Biot-Savart law for the general case of a volume current reads

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d\tau'. \tag{5.45}$$

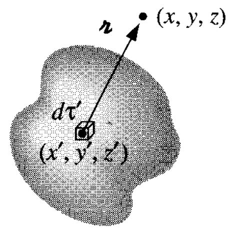


Figure 5.30

This formula gives the magnetic field at a point $\mathbf{r} = (x, y, z)$ in terms of an integral over the current distribution $\mathbf{J}(x', y', z')$ (Fig. 5.30). It is best to be absolutely explicit at this stage:

$$\mathbf{B} \text{ is a function of } (x, y, z),$$

$$\mathbf{J} \text{ is a function of } (x', y', z'),$$

$$\boldsymbol{\mathscr{r}} = (x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}},$$

$$d\tau' = dx'\,dy'\,dz'.$$

The integration is over the *primed* coordinates; the divergence and the curl are to be taken with respect to the *unprimed* coordinates.

Applying the divergence to Eq. 5.45, we obtain:

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) d\tau'. \tag{5.46}$$

Invoking product rule number (6),

$$\nabla \cdot \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left(\nabla \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right). \tag{5.47}$$

But $\nabla \times \mathbf{J} = 0$, because $\mathbf{J}$ doesn't depend on the unprimed variables (x, y, z), whereas $\nabla \times (\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2) = 0$ (Prob. 1.62), so

$$\boxed{\nabla \cdot \mathbf{B} = 0.} \tag{5.48}$$

Evidently, the *divergence* of the magnetic field is *zero*.

Applying the curl to Eq. 5.45, we obtain:

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) d\tau'. \tag{5.49}$$

Again, our strategy is to expand the integrand, using the appropriate product rule—in this case number 8:

$$\nabla \times \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = \mathbf{J} \left(\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) - (\mathbf{J} \cdot \nabla) \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}. \tag{5.50}$$

(I have dropped terms involving derivatives of $\mathbf{J}$, because $\mathbf{J}$ does not depend on x, y, z.) The second term integrates to zero, as we'll see in the next paragraph. The first term involves the divergence we were at pains to calculate in Chapter 1 (Eq. 1.100):

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}}). \tag{5.51}$$

Thus

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')4\pi\delta^3(\mathbf{r}-\mathbf{r}')\,d\tau' = \mu_0\mathbf{J}(\mathbf{r}),$$

which confirms that Eq. 5.44 is not restricted to straight-line currents, but holds quite generally in magnetostatics.

To complete the argument, however, we must check that the second term in Eq. 5.50 integrates to zero. Because the derivative acts only on $\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$, we can switch from ∇ to ∇' at the cost of a minus sign:[9]

$$-(\mathbf{J}\cdot\nabla)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = (\mathbf{J}\cdot\nabla')\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}. \tag{5.52}$$

The x component, in particular, is

$$(\mathbf{J}\cdot\nabla')\left(\frac{x-x'}{\mathscr{r}^3}\right) = \nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right] - \left(\frac{x-x'}{\mathscr{r}^3}\right)(\nabla'\cdot\mathbf{J})$$

(using product rule 5). Now, for *steady* currents the divergence of $\mathbf{J}$ is zero (Eq. 5.31), so

$$\left[-(\mathbf{J}\cdot\nabla)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right]_x = \nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right],$$

and therefore this contribution to the integral (5.49) can be written

$$\int_{\mathcal{V}} \nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right]d\tau' = \oint_{\mathcal{S}} \frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\cdot d\mathbf{a}'. \tag{5.53}$$

(The reason for switching from ∇ to ∇' was precisely to permit this integration by parts.) But what region are we integrating over? Well, it's the volume that appears in the Biot-Savart law (5.45)—large enough, that is, to include all the current. You can make it *bigger* than that, if you like; $\mathbf{J} = 0$ out there anyway, so it will add nothing to the integral. The essential point is that *on the boundary* the current is *zero* (all current is safely *inside*) and hence the surface integral (5.53) vanishes.[10]

[9]The point here is that $\boldsymbol{\mathscr{r}}$ depends only on the *difference* between the coordinates, and $(\partial/\partial x)f(x-x') = -(\partial/\partial x')f(x-x')$.

[10]If $\mathbf{J}$ itself extends to infinity (as in the case of an infinite straight wire), the surface integral is still typically zero, though the analysis calls for greater care.

5.3.3 Applications of Ampère's Law

The equation for the curl of **B**

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J},} \tag{5.54}$$

is called **Ampère's law** (in differential form). It can be converted to integral form by the usual device of applying one of the fundamental theorems—in this case Stokes' theorem:

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}.$$

Now, $\int \mathbf{J} \cdot d\mathbf{a}$ is the total current passing through the surface (Fig. 5.31), which we call I_{enc} (the **current enclosed** by the **amperian loop**). Thus

$$\boxed{\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}.} \tag{5.55}$$

This is the integral version of Ampère's law; it generalizes Eq. 5.42 to *arbitrary* steady currents. Notice that Eq. 5.55 inherits the sign ambiguity of Stokes' theorem (Sect. 1.3.5): Which *way* around the loop am I supposed to go? And which *direction* through the surface corresponds to a "positive" current? The resolution, as always, is the right-hand rule: If the fingers of your right hand indicate the direction of integration around the boundary, then your thumb defines the direction of a positive current.

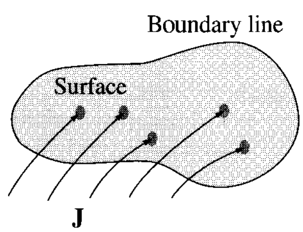


Figure 5.31

Just as the Biot-Savart law plays a role in magnetostatics that Coulomb's law assumed in electrostatics, so Ampère's plays the role of Gauss's:

$$\left\{ \begin{array}{lll} \text{Electrostatics}: & \text{Coulomb} & \rightarrow \text{Gauss}, \\ \text{Magnetostatics}: & \text{Biot-Savart} & \rightarrow \text{Ampère}. \end{array} \right.$$

In particular, for currents with appropriate symmetry, Ampère's law in integral form offers a lovely and extraordinarily efficient means for calculating the magnetic field.

Example 5.7

Find the magnetic field a distance s from a long straight wire (Fig. 5.32), carrying a steady current I (the same problem we solved in Ex. 5.5, using the Biot-Savart law).

Solution: We know the direction of **B** is "circumferential," circling around the wire as indicated by the right hand rule. By symmetry, the magnitude of **B** is constant around an amperian loop of radius s, centered on the wire. So Ampère's law gives

$$\oint \mathbf{B} \cdot d\mathbf{l} = B \oint dl = B 2\pi s = \mu_0 I_{\text{enc}} = \mu_0 I,$$

or

$$B = \frac{\mu_0 I}{2\pi s}.$$

This is the same answer we got before (Eq. 5.36), but it was obtained this time with far less effort.

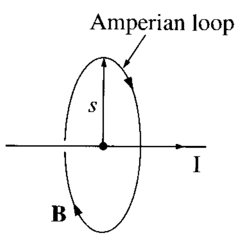


Figure 5.32

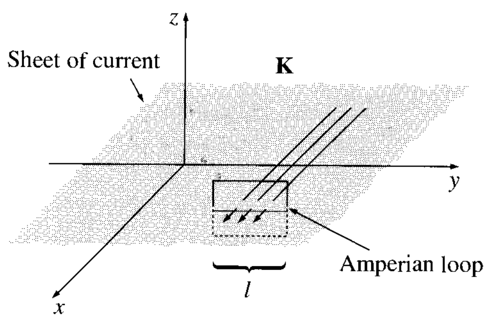


Figure 5.33

Example 5.8

Find the magnetic field of an infinite uniform surface current $\mathbf{K} = K\,\hat{\mathbf{x}}$, flowing over the xy plane (Fig. 5.33).

Solution: First of all, what is the *direction* of **B**? Could it have any x-component? *No*: A glance at the Biot-Savart law (5.39) reveals that **B** is *perpendicular* to **K**. Could it have a z-component? *No again*. You could confirm this by noting that any vertical contribution from a filament at $+y$ is canceled by the corresponding filament at $-y$. But there is a nicer argument: Suppose the field pointed *away* from the plane. By reversing the direction of the current, I could make it point *toward* the plane (in the Biot-Savart law, changing the sign of the current switches the sign of the field). But the z-component of **B** cannot possibly depend on the *direction* of the current in the xy plane. (Think about it!) So **B** can only have a y-component, and a quick check with your right hand should convince you that it points to the *left* above the plane and to the *right* below it.

With this in mind we draw a rectangular amperian loop as shown in Fig. 5.33, parallel to the yz plane and extending an equal distance above and below the surface. Applying Ampère's law, we find

$$\oint \mathbf{B} \cdot d\mathbf{l} = 2Bl = \mu_0 I_{\text{enc}} = \mu_0 K l,$$

(one Bl comes from the top segment, and the other from the bottom), so $B = (\mu_0/2)K$, or, more precisely,

$$\mathbf{B} = \begin{cases} +(\mu_0/2)K\,\hat{\mathbf{y}} & \text{for } z < 0, \\ -(\mu_0/2)K\,\hat{\mathbf{y}} & \text{for } z > 0. \end{cases} \tag{5.56}$$

Notice that the field is independent of the distance from the plane, just like the *electric* field of a uniform surface *charge* (Ex. 2.4).

Example 5.9

Find the magnetic field of a very long solenoid, consisting of n closely wound turns per unit length on a cylinder of radius R and carrying a steady current I (Fig. 5.34). [The point of making the windings so close is that one can then pretend each turn is circular. If this troubles you (after all, there is a net current I in the direction of the solenoid's axis, no matter *how* tight the winding), picture instead a sheet of aluminum foil wrapped around the cylinder, carrying the equivalent uniform surface current $K = nI$ (Fig. 5.35). Or make a double winding, going up to one end and then—always in the same sense—going back down again, thereby eliminating the net longitudinal current. But, in truth, this is all unnecessary fastidiousness, for the field inside a solenoid is huge (relatively speaking), and the field of the longitudinal current is at most a tiny refinement.]

Solution: First of all, what is the *direction* of **B**? Could it have a radial component? *No.* For suppose B_s were *positive*; if we reversed the direction of the current, B_s would then be *negative*. But switching I is physically equivalent to turning the solenoid upside down, and

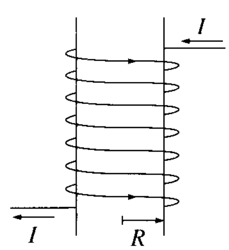


Figure 5.34

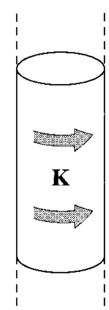


Figure 5.35

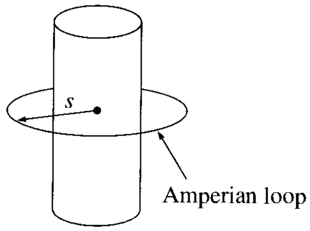


Figure 5.36

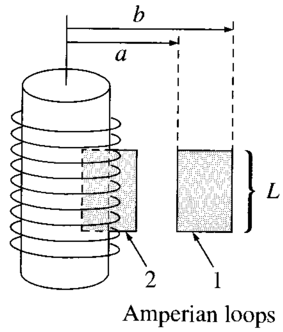


Figure 5.37

that certainly should not alter the radial field. How about a "circumferential" component? *No.* For B_ϕ would be constant around an amperian loop concentric with the solenoid (Fig. 5.36), and hence

$$\oint \mathbf{B} \cdot d\mathbf{l} = B_\phi (2\pi s) = \mu_0 I_{\text{enc}} = 0,$$

since the loop encloses no current.

So the magnetic field of an infinite, closely wound solenoid runs *parallel to the axis.* From the right hand rule, we expect that it points upward inside the solenoid and downward outside. Moreover, it certainly approaches zero as you go very far away. With this in mind, let's apply Ampère's law to the two rectangular loops in Fig. 5.37. Loop 1 lies entirely outside the solenoid, with its sides at distances a and b from the axis:

$$\oint \mathbf{B} \cdot d\mathbf{l} = [B(a) - B(b)]L = \mu_0 I_{\text{enc}} = 0,$$

so

$$B(a) = B(b).$$

Evidently the *field outside does not depend on the distance from the axis.* But we know that it goes to *zero* for large s. It must therefore be zero *everywhere*! (This astonishing result can also be derived from the Biot-Savart law, of course, but it's much more difficult. See Prob. 5.44.)

As for loop 2, which is half inside and half outside, Ampère's law gives

$$\oint \mathbf{B} \cdot d\mathbf{l} = BL = \mu_0 I_{\text{enc}} = \mu_0 n I L,$$

where B is the field inside the solenoid. (The right side of the loop contributes nothing, since $B = 0$ out there.) *Conclusion:*

$$\mathbf{B} = \begin{cases} \mu_0 n I\,\hat{\mathbf{z}}, & \text{inside the solenoid,} \\ 0, & \text{outside the solenoid.} \end{cases} \tag{5.57}$$

Notice that the field inside is *uniform*; in this sense the solenoid is to magnetostatics what the parallel-plate capacitor is to electrostatics: a simple device for producing strong uniform fields.

Like Gauss's law, Ampère's law is always *true* (for steady currents), but it is not always *useful*. Only when the symmetry of the problem enables you to pull B outside the integral $\oint \mathbf{B} \cdot d\mathbf{l}$ can you calculate the magnetic field from Ampère's law. When it *does* work, it's by far the fastest method; when it doesn't, you have to fall back on the Biot-Savart law. The current configurations that can be handled by Ampère's law are

1. Infinite straight lines (prototype: Ex. 5.7).
2. Infinite planes (prototype: Ex. 5.8).
3. Infinite solenoids (prototype: Ex. 5.9).
4. Toroids (prototype: Ex. 5.10).

The last of these is a surprising and elegant application of Ampère's law; it is treated in the following example. As in Exs. 5.8 and 5.9, the hard part is figuring out the *direction* of the field (which we will now have done, once and for all, for each of the four geometries); the actual application of Ampère's law takes only one line.

Example 5.10

A toroidal coil consists of a circular ring, or "donut," around which a long wire is wrapped (Fig. 5.38). The winding is uniform and tight enough so that each turn can be considered a closed loop. The cross-sectional shape of the coil is immaterial. I made it rectangular in Fig. 5.38 for the sake of simplicity, but it could just as well be circular or even some weird asymmetrical form, as in Fig. 5.39, just as long as the shape remains the same all the way around the ring. In that case it follows that the *magnetic field of the toroid is circumferential at all points, both inside and outside the coil.*

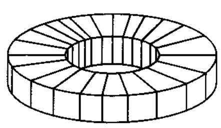


Figure 5.38

Proof: According to the Biot-Savart law, the field at $\mathbf{r}$ due to the current element at $\mathbf{r}'$ is

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{\mathbf{I} \times \boldsymbol{\mathscr{r}}}{\mathscr{r}^3}\,dl'.$$

We may as well put $\mathbf{r}$ in the xz plane (Fig. 5.39), so its Cartesian components are $(x, 0, z)$, while the source coordinates are

$$\mathbf{r}' = (s'\cos\phi', s'\sin\phi', z').$$

Then

$$\boldsymbol{\mathscr{r}} = (x - s'\cos\phi', -s'\sin\phi', z - z').$$

Since the current has no ϕ component, $\mathbf{I} = I_s\,\hat{\mathbf{s}} + I_z\,\hat{\mathbf{z}}$, or (in Cartesian coordinates)

$$\mathbf{I} = (I_s\cos\phi', I_s\sin\phi', I_z).$$

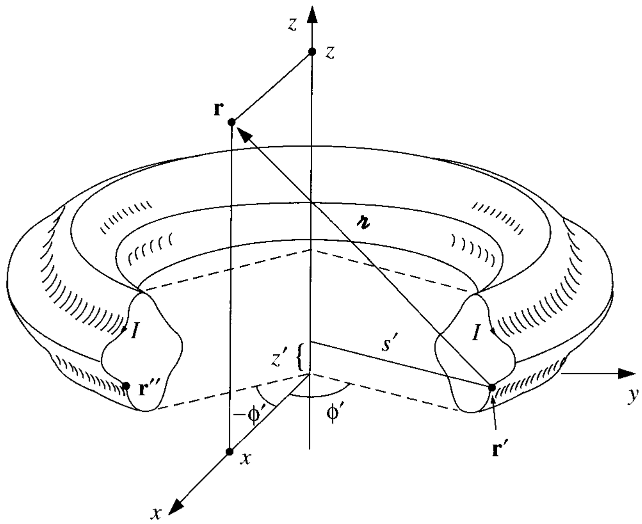


Figure 5.39

Accordingly,

$$\mathbf{I}\times\boldsymbol{\mathscr{r}} = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ I_s\cos\phi' & I_s\sin\phi' & I_z \\ (x-s'\cos\phi') & (-s'\sin\phi') & (z-z') \end{bmatrix}$$

$$= [\sin\phi'(I_s(z-z')+s'I_z)]\,\hat{\mathbf{x}}$$

$$+\,[I_z(x-s'\cos\phi')-I_s\cos\phi'(z-z')]\,\hat{\mathbf{y}}+[-I_s x\sin\phi']\,\hat{\mathbf{z}}.$$

But there is a symmetrically situated current element at $\mathbf{r}''$, with the same s', the same $\mathscr{r}$, the same dl', the same I_s, and the same I_z, *but negative* ϕ' (Fig. 5.39). Because $\sin\phi'$ changes sign, the $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ contributions from $\mathbf{r}'$ and $\mathbf{r}''$ cancel, leaving only a $\hat{\mathbf{y}}$ term. Thus the field at $\mathbf{r}$ is in the $\hat{\mathbf{y}}$ direction, and in general the field points in the $\hat{\boldsymbol{\phi}}$ direction. qed

Now that we know the field is circumferential, determining its magnitude is ridiculously easy. Just apply Ampère's law to a circle of radius s about the axis of the toroid:

$$B2\pi s = \mu_0 I_{\text{enc}},$$

and hence

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 N I}{2\pi s}\hat{\boldsymbol{\phi}}, & \text{for points inside the coil,} \\ 0, & \text{for points outside the coil,} \end{cases} \qquad (5.58)$$

where N is the total number of turns.

Problem 5.13 A steady current I flows down a long cylindrical wire of radius a (Fig. 5.40). Find the magnetic field, both inside and outside the wire, if

(a) The current is uniformly distributed over the outside surface of the wire.

(b) The current is distributed in such a way that J is proportional to s, the distance from the axis.

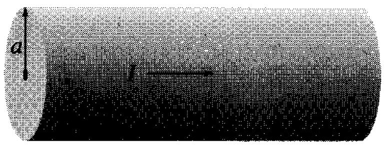


Figure 5.40

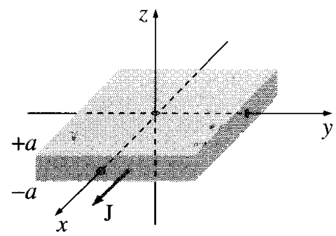


Figure 5.41

Problem 5.14 A thick slab extending from $z = -a$ to $z = +a$ carries a uniform volume current $\mathbf{J} = J\,\hat{\mathbf{x}}$ (Fig. 5.41). Find the magnetic field, as a function of z, both inside and outside the slab.

Problem 5.15 Two long coaxial solenoids each carry current I, but in opposite directions, as shown in Fig. 5.42. The inner solenoid (radius a) has n_1 turns per unit length, and the outer one (radius b) has n_2. Find $\mathbf{B}$ in each of the three regions: (i) inside the inner solenoid, (ii) between them, and (iii) outside both.

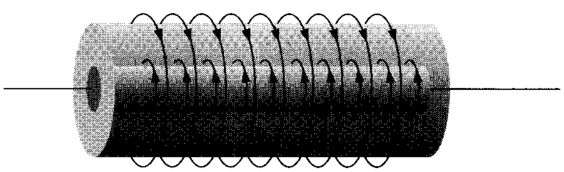


Figure 5.42

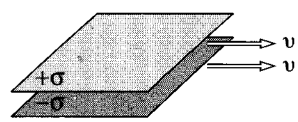


Figure 5.43

Problem 5.16 A large parallel-plate capacitor with uniform surface charge σ on the upper plate and $-\sigma$ on the lower is moving with a constant speed v, as shown in Fig. 5.43.

(a) Find the magnetic field between the plates and also above and below them.

(b) Find the magnetic force per unit area on the upper plate, including its direction.

(c) At what speed v would the magnetic force balance the electrical force?[11]

[11] See footnote 8.

! **Problem 5.17** Show that the magnetic field of an infinite solenoid runs parallel to the axis, *regardless of the cross-sectional shape of the coil*, as long as that shape is constant along the length of the solenoid. What is the magnitude of the field, inside and outside of such a coil? Show that the toroid field (5.58) reduces to the solenoid field, when the radius of the donut is so large that a segment can be considered essentially straight.

Problem 5.18 In calculating the current enclosed by an amperian loop, one must, in general, evaluate an integral of the form

$$I_{enc} = \int_S \mathbf{J} \cdot d\mathbf{a}.$$

The trouble is, there are infinitely many surfaces that share the same boundary line. Which one are we supposed to use?

5.3.4 Comparison of Magnetostatics and Electrostatics

The divergence and curl of the *electrostatic* field are

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho, & \text{(Gauss's law);} \\ \nabla \times \mathbf{E} = 0, & \text{(no name).} \end{cases}$$

These are **Maxwell's equations** for electrostatics. Together with the boundary condition $\mathbf{E} \to 0$ far from all charges, Maxwell's equations determine the field, if the source charge density ρ is given; they contain essentially the same information as Coulomb's law plus the principle of superposition. The divergence and curl of the *magnetostatic* field are

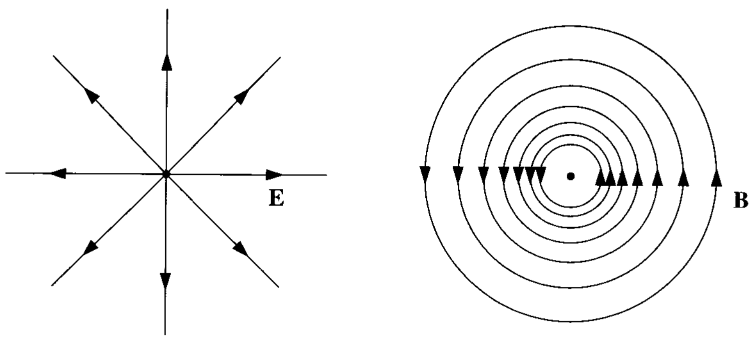
$$\begin{cases} \nabla \cdot \mathbf{B} = 0, & \text{(no name);} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, & \text{(Ampère's law).} \end{cases}$$

These are Maxwell's equations for magnetostatics. Again, together with the boundary condition $\mathbf{B} \to 0$ far from all currents, Maxwell's equations determine the magnetic field; they are equivalent to the Biot-Savart law (plus superposition). Maxwell's equations and the force law

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

constitute the most elegant formulation of electrostatics and magnetostatics.

The electric field *diverges away from* a (positive) charge; the magnetic field line *curls around* a current (Fig. 5.44). Electric field lines originate on positive charges and terminate on negative ones; magnetic field lines do not begin or end anywhere—to do so would require a nonzero divergence. They either form closed loops or extend out to infinity. To put it another way, *there are no point sources for* **B**, as there are for **E**; there exists no magnetic analog to electric charge. This is the physical content of the statement $\nabla \cdot \mathbf{B} = 0$. Coulomb and others believed that magnetism was produced by **magnetic charges** (**magnetic monopoles**, as we would now call them), and in some older books you will still



(a) Electrostatic field of a point charge

(b) Magnetostatic field of a long wire

Figure 5.44

find references to a magnetic version of Coulomb's law, giving the force of attraction or repulsion between them. It was Ampère who first speculated that all magnetic effects are attributable to *electric* charges *in motion* (currents). As far as we know, Ampère was right; nevertheless, it remains an open experimental question whether magnetic monopoles exist in nature (they are obviously pretty *rare*, or somebody would have found one[12]), and in fact some recent elementary particle theories *require* them. For our purposes, though, **B** is divergenceless and there are no magnetic monopoles. It takes a *moving* electric charge to *produce* a magnetic field, and it takes another moving electric charge to "feel" a magnetic field.

Typically, electric forces are enormously larger than magnetic ones. That's not something you can tell from the theory as such; it has to do with the sizes of the fundamental constants ϵ_0 and μ_0. In general, it is only when both the source charges and the test charge are moving at velocities comparable to the speed of light that the magnetic force approaches the electric force in strength. (Problems 5.12 and 5.16 illustrate this rule.) How is it, then, that we ever notice magnetic effects at all? The answer is that both in the production of a magnetic field (Biot-Savart) and in its detection (Lorentz) it is the *current* (charge times velocity) that enters, and we can compensate for a smallish velocity by pouring huge quantities of charge down the wire. Ordinarily, this charge would simultaneously generate so large an *electric* force as to swamp the magnetic one. But if we arrange to keep the wire *neutral*, by embedding in it an equal amount of opposite charge at rest, the electric field cancels out, leaving the magnetic field to stand alone. It sounds very elaborate, but of course this is precisely what happens in an ordinary current carrying wire.

[12]An apparent detection (B. Cabrera, *Phys. Rev. Lett.* **48**, 1378 (1982)) has never been reproduced—and not for want of trying. For a delightful brief history of ideas about magnetism, see Chapter 1 in D. C. Mattis, *The Theory of Magnetism* (New York: Harper and Row, 1965).

Problem 5.19

(a) Find the density ρ of mobile charges in a piece of copper, assuming each atom contributes one free electron. [Look up the necessary physical constants.]

(b) Calculate the average electron velocity in a copper wire 1 mm in diameter, carrying a current of 1 A. [*Note*: this is literally a *snail's* pace. How, then, can you carry on a long distance telephone conversation?]

(c) What is the force of attraction between two such wires, 1 cm apart?

(d) If you could somehow remove the stationary positive ions, what would the electrical repulsion force be? How many times greater than the magnetic force is it?

Problem 5.20 Is Ampère's law consistent with the general rule (Eq. 1.46) that divergence-of-curl is always zero? Show that Ampère's law *cannot* be valid, in general, outside magnetostatics. Is there any such "defect" in the other three Maxwell equations?

Problem 5.21 Suppose there *did* exist magnetic monopoles. How would you modify Maxwell's equations and the force law, to accommodate them? If you think there are several plausible options, list them, and suggest how you might decide experimentally which one is right.

5.4 Magnetic Vector Potential

5.4.1 The Vector Potential

Just as $\nabla \times \mathbf{E} = 0$ permitted us to introduce a scalar potential (V) in electrostatics,

$$\mathbf{E} = -\nabla V,$$

so $\nabla \cdot \mathbf{B} = 0$ invites the introduction of a *vector* potential $\mathbf{A}$ in magnetostatics:

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}.} \tag{5.59}$$

The former is authorized by Theorem 1 (of Sect. 1.6.2), the latter by Theorem 2 (the proof of Theorem 2 is developed in Prob. 5.30). The potential formulation automatically takes care of $\nabla \cdot \mathbf{B} = 0$ (since the divergence of a curl is *always* zero); there remains Ampère's law:

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}. \tag{5.60}$$

Now, the electric potential had a built-in ambiguity: you can add to V any function whose gradient is zero (which is to say, any *constant*), without altering the *physical* quantity $\mathbf{E}$. Likewise, you can add to the magnetic potential any function whose *curl* vanishes (which is to say, the *gradient of any scalar*), with no effect on $\mathbf{B}$. We can exploit this freedom to eliminate the divergence of $\mathbf{A}$:

$$\boxed{\nabla \cdot \mathbf{A} = 0.} \tag{5.61}$$

To prove that this is always possible, suppose that our original potential, $\mathbf{A}_0$, is *not* divergenceless. If we add to it the gradient of λ ($\mathbf{A} = \mathbf{A}_0 + \nabla\lambda$), the new divergence is

$$\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}_0 + \nabla^2\lambda.$$

We can accommodate Eq. 5.61, then, if a function λ can be found that satisfies

$$\nabla^2\lambda = -\nabla \cdot \mathbf{A}_0.$$

But this is *mathematically* identical to Poisson's equation (2.24),

$$\nabla^2 V = -\frac{\rho}{\epsilon_0},$$

with $\nabla \cdot \mathbf{A}_0$ in place of ρ/ϵ_0 as the "source." And we *know* how to solve Poisson's equation—that's what electrostatics is all about ("given the charge distribution, find the potential"). In particular, if ρ goes to zero at infinity, the solution is Eq. 2.29:

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{\mathscr{r}}\,d\tau',$$

and by the same token, if $\nabla \cdot \mathbf{A}_0$ goes to zero at infinity, then

$$\lambda = \frac{1}{4\pi}\int \frac{\nabla \cdot \mathbf{A}_0}{\mathscr{r}}\,d\tau'.$$

If $\nabla \cdot \mathbf{A}_0$ does *not* go to zero at infinity, we'll have to use other means to discover the appropriate λ, just as we get the electric potential by other means when the charge distribution extends to infinity. But the *essential* point remains: *It is always possible to make the vector potential divergenceless.* To put it the other way around: The definition $\mathbf{B} = \nabla \times \mathbf{A}$ specifies the *curl* of $\mathbf{A}$, but it doesn't say anything about the *divergence*—we are at liberty to pick that as we see fit, and zero is ordinarily the simplest choice.

With this condition on $\mathbf{A}$, Ampère's law (5.60) becomes

$$\boxed{\nabla^2\mathbf{A} = -\mu_0\mathbf{J}.} \tag{5.62}$$

This *again* is nothing but Poisson's equation—or rather, it is *three* Poisson's equations, one for each Cartesian[13] component. Assuming $\mathbf{J}$ goes to zero at infinity, we can read off the solution:

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'.} \tag{5.63}$$

[13]In Cartesian coordinates, $\nabla^2\mathbf{A} = (\nabla^2 A_x)\hat{\mathbf{x}} + (\nabla^2 A_y)\hat{\mathbf{y}} + (\nabla^2 A_z)\hat{\mathbf{z}}$, so Eq. 5.62 reduces to $\nabla^2 A_x = -\mu_0 J_x$, $\nabla^2 A_y = -\mu_0 J_y$, and $\nabla^2 A_z = -\mu_0 J_z$. In curvilinear coordinates the unit vectors *themselves* are functions of position, and must be differentiated, so it is *not* the case, for example, that $\nabla^2 A_r = -\mu_0 J_r$. The safest way to calculate the Laplacian of a *vector*, in terms of its curvilinear components, is to use $\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A})$. Remember also that even if you *calculate* integrals such as 5.63 using curvilinear coordinates, you must first express $\mathbf{J}$ in terms of its *Cartesian* components (see Sect. 1.4.1).

For line and surface currents,

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}}{\mathscr{r}}\,dl' = \frac{\mu_0 I}{4\pi}\int \frac{1}{\mathscr{r}}\,d\mathbf{l}'; \qquad \mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}}{\mathscr{r}}\,da'. \tag{5.64}$$

(If the current does *not* go to zero at infinity, we have to find other ways to get **A**; some of these are explored in Ex. 5.12 and in the problems at the end of the section.)

It must be said that **A** is not as *useful* as V. For one thing, it's still a *vector*, and although Eqs. 5.63 and 5.64 are somewhat easier to work with than the Biot-Savart law, you still have to fuss with components. It would be nice if we could get away with a *scalar* potential,

$$\mathbf{B} = -\nabla U, \tag{5.65}$$

but this is incompatible with Ampère's law, since the curl of a gradient is always zero. (A **magnetostatic scalar potential** *can* be used, if you stick scrupulously to simply-connected. current-free regions, but as a theoretical tool it is of limited interest. See Prob. 5.28.) Moreover, since magnetic forces do no work, **A** does not admit a simple physical interpretation in terms of potential energy per unit charge. (In some contexts it can be interpreted as *momentum* per unit charge.[14]) Nevertheless, the vector potential has substantial theoretical importance, as we shall see in Chapter 10.

Example 5.11

A spherical shell, of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity $\boldsymbol{\omega}$. Find the vector potential it produces at point **r** (Fig. 5.45).

Solution: It might seem natural to align the polar axis along $\boldsymbol{\omega}$, but in fact the integration is easier if we let **r** lie on the z axis, so that $\boldsymbol{\omega}$ is tilted at an angle ψ. We may as well orient the x axis so that $\boldsymbol{\omega}$ lies in the xz plane, as shown in Fig. 5.46. According to Eq. 5.64,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}')}{\mathscr{r}}\,da',$$

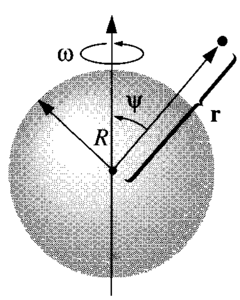


Figure 5.45

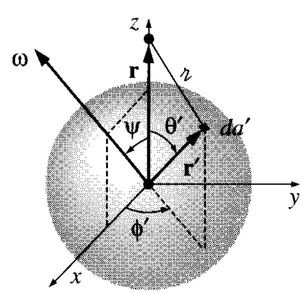


Figure 5.46

[14]M. D. Semon and J. R. Taylor, *Am. J. Phys.* **64**, 1361 (1996).

where $\mathbf{K} = \sigma \mathbf{v}$, $\imath = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$, and $da' = R^2 \sin\theta'\, d\theta'\, d\phi'$. Now the velocity of a point $\mathbf{r}'$ in a rotating rigid body is given by $\boldsymbol{\omega} \times \mathbf{r}'$; in this case,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}' = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega \sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$= R\omega[-(\cos\psi \sin\theta' \sin\phi')\,\hat{\mathbf{x}} + (\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta')\,\hat{\mathbf{y}} + (\sin\psi\sin\theta'\sin\phi')\,\hat{\mathbf{z}}].$$

Notice that each of these terms, save one, involves either $\sin\phi'$ or $\cos\phi'$. Since

$$\int_0^{2\pi} \sin\phi'\, d\phi' = \int_0^{2\pi} \cos\phi'\, d\phi' = 0,$$

such terms contribute nothing. There remains

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 R^3 \sigma \omega \sin\psi}{2} \left(\int_0^{\pi} \frac{\cos\theta' \sin\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}}\, d\theta' \right) \hat{\mathbf{y}}.$$

Letting $u \equiv \cos\theta'$, the integral becomes

$$\int_{-1}^{+1} \frac{u}{\sqrt{R^2 + r^2 - 2Rru}}\, du = -\frac{(R^2 + r^2 + Rru)}{3R^2r^2} \sqrt{R^2 + r^2 - 2Rru}\,\Bigg|_{-1}^{+1}$$

$$= -\frac{1}{3R^2r^2}\left[(R^2 + r^2 + Rr)|R - r| - (R^2 + r^2 - Rr)(R + r)\right].$$

If the point $\mathbf{r}$ lies *inside* the sphere, then $R > r$, and this expression reduces to $(2r/3R^2)$; if $\mathbf{r}$ lies *outside* the sphere, so that $R < r$, it reduces to $(2R/3r^2)$. Noting that $(\boldsymbol{\omega} \times \mathbf{r}) = -\omega r \sin\psi\, \hat{\mathbf{y}}$, we have, finally,

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 R\sigma}{3}(\boldsymbol{\omega} \times \mathbf{r}), & \text{for points } \textit{inside} \text{ the sphere,} \\[2ex] \dfrac{\mu_0 R^4 \sigma}{3r^3}(\boldsymbol{\omega} \times \mathbf{r}), & \text{for points } \textit{outside} \text{ the sphere.} \end{cases} \tag{5.66}$$

Having evaluated the integral, I revert to the "natural" coordinates of Fig. 5.45, in which $\boldsymbol{\omega}$ coincides with the z axis and the point $\mathbf{r}$ is at (r, θ, ϕ):

$$\mathbf{A}(r, \theta, \phi) = \begin{cases} \dfrac{\mu_0 R\omega\sigma}{3} r\sin\theta\, \hat{\boldsymbol{\phi}}, & (r \le R), \\[2ex] \dfrac{\mu_0 R^4 \omega\sigma}{3} \dfrac{\sin\theta}{r^2}\, \hat{\boldsymbol{\phi}}, & (r \ge R). \end{cases} \tag{5.67}$$

Curiously, the field inside this spherical shell is *uniform*:

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta\, \hat{\mathbf{r}} - \sin\theta\, \hat{\boldsymbol{\theta}}) = \frac{2}{3}\mu_0 \sigma R\omega\, \hat{\mathbf{z}} = \frac{2}{3}\mu_0 \sigma R\boldsymbol{\omega}. \tag{5.68}$$

Example 5.12

Find the vector potential of an infinite solenoid with n turns per unit length, radius R, and current I.

Solution: This time we cannot use Eq. 5.64, since the current itself extends to infinity. But here's a cute method that does the job. Notice that

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi, \tag{5.69}$$

where Φ is the flux of $\mathbf{B}$ through the loop in question. This is reminiscent of Ampère's law in the integral form (5.55),

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}.$$

In fact, it's the same equation, with $\mathbf{B} \to \mathbf{A}$ and $\mu_0 I_{\text{enc}} \to \Phi$. If symmetry permits, we can determine $\mathbf{A}$ from Φ in the same way we got $\mathbf{B}$ from I_{enc}, in Sect. 5.3.3. The present problem (with a uniform longitudinal magnetic field $\mu_0 n I$ inside the solenoid and no field outside) is analogous to the Ampère's law problem of a fat wire carrying a uniformly distributed current. The vector potential is "circumferential" (it mimics the magnetic field of the wire); using a circular "amperian loop" at radius s *inside* the solenoid, we have

$$\oint \mathbf{A} \cdot d\mathbf{l} = A(2\pi s) = \int \mathbf{B} \cdot d\mathbf{a} = \mu_0 n I (\pi s^2),$$

so

$$\mathbf{A} = \frac{\mu_0 n I}{2} s \hat{\boldsymbol{\phi}}, \quad \text{for } s < R. \tag{5.70}$$

For an amperian loop *outside* the solenoid, the flux is

$$\int \mathbf{B} \cdot d\mathbf{a} = \mu_0 n I (\pi R^2),$$

since the field only extends out to R. Thus

$$\mathbf{A} = \frac{\mu_0 n I}{2} \frac{R^2}{s} \hat{\boldsymbol{\phi}}, \quad \text{for } s > R. \tag{5.71}$$

If you have any doubts about this answer, *check* it: Does $\nabla \times \mathbf{A} = \mathbf{B}$? Does $\nabla \cdot \mathbf{A} = 0$? If so, we're done.

Typically, the direction of $\mathbf{A}$ will mimic the direction of the current. For instance, both were azimuthal in Exs. 5.11 and 5.12. Indeed, if all the current flows in *one* direction, then Eq. 5.63 suggests that $\mathbf{A}$ *must* point that way too. Thus the potential of a finite segment of straight wire (Prob. 5.22) is in the direction of the current. Of course, if the current extends to infinity you can't use Eq. 5.63 in the first place (see Probs. 5.25 and 5.26). Moreover, you can always add an arbitrary constant vector to $\mathbf{A}$—this is analogous to changing the reference point for V, and it won't affect the divergence or curl of $\mathbf{A}$, which is all that matters (in Eq. 5.63 we have chosen the constant so that $\mathbf{A}$ goes to zero at infinity). In principle you could even use a vector potential that is not divergenceless, in which case all bets are off. Despite all these caveats, the essential point remains: *Ordinarily* the direction of $\mathbf{A}$ will match the direction of the current.

Problem 5.22 Find the magnetic vector potential of a finite segment of straight wire, carrying a current I. [Put the wire on the z axis, from z_1 to z_2, and use Eq. 5.64.] Check that your answer is consistent with Eq. 5.35.

Problem 5.23 What current density would produce the vector potential, $\mathbf{A} = k\hat{\boldsymbol{\phi}}$ (where k is a constant), in cylindrical coordinates?

Problem 5.24 If $\mathbf{B}$ is *uniform*, show that $\mathbf{A}(\mathbf{r}) = -\frac{1}{2}(\mathbf{r} \times \mathbf{B})$ works. That is, check that $\nabla \cdot \mathbf{A} = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}$. Is this result unique, or are there other functions with the same divergence and curl?

Problem 5.25

(a) By whatever means you can think of (short of looking it up), find the vector potential a distance s from an infinite straight wire carrying a current I. Check that $\nabla \cdot \mathbf{A} = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}$.

(b) Find the magnetic potential *inside* the wire, if it has radius R and the current is uniformly distributed.

Problem 5.26 Find the vector potential above and below the plane surface current in Ex. 5.8.

Problem 5.27

(a) Check that Eq. 5.63 is consistent with Eq. 5.61, by applying the *divergence*.

(b) Check that Eq. 5.63 is consistent with Eq. 5.45, by applying the *curl*.

(c) Check that Eq. 5.63 is consistent with Eq. 5.62, by applying the *Laplacian*.

Problem 5.28 Suppose you want to define a magnetic scalar potential U (Eq. 5.65), in the vicinity of a current-carrying wire. First of all, you must stay away from the wire itself (there $\nabla \times \mathbf{B} \neq 0$); but that's not enough. Show, by applying Ampère's law to a path that starts at $\mathbf{a}$ and circles the wire, returning to $\mathbf{b}$ (Fig. 5.47), that the scalar potential cannot be single-valued (that is, $U(\mathbf{a}) \neq U(\mathbf{b})$, even if they represent the same physical point). As an example, find the scalar potential for an infinite straight wire. (To avoid a multivalued potential, you must restrict yourself to simply-connected regions that remain on one side or the other of every wire, never allowing you to go all the way around.)

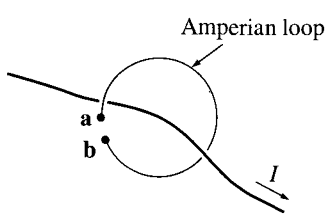


Figure 5.47

Problem 5.29 Use the results of Ex. 5.11 to find the field inside a uniformly charged sphere, of total charge Q and radius R, which is rotating at a constant angular velocity $\boldsymbol{\omega}$.

Problem 5.30

(a) Complete the proof of Theorem 2, Sect. 1.6.2. That is, show that any divergenceless vector field $\mathbf{F}$ can be written as the curl of a vector potential $\mathbf{A}$. What you have to do is find A_x, A_y, and A_z such that: (i) $\partial A_z/\partial y - \partial A_y/\partial z = F_x$; (ii) $\partial A_x/\partial z - \partial A_z/\partial x = F_y$; and (iii) $\partial A_y/\partial x - \partial A_x/\partial y = F_z$. Here's one way to do it: Pick $A_x = 0$, and solve (ii) and (iii) for A_y and A_z. Note that the "constants of integration" here are themselves functions of y and z—they're constant only with respect to x. Now plug these expressions into (i), and use the fact that $\nabla \cdot \mathbf{F} = 0$ to obtain

$$A_y = \int_0^x F_z(x', y, z)\, dx'; \quad A_z = \int_0^y F_x(0, y', z)\, dy' - \int_0^x F_y(x', y, z)\, dx'.$$

(b) By direct differentiation, check that the $\mathbf{A}$ you obtained in part (a) satisfies $\nabla \times \mathbf{A} = \mathbf{F}$. Is $\mathbf{A}$ divergenceless? [This was a very asymmetrical construction, and it would be surprising if it *were*—although we know that there *exists* a vector whose curl is $\mathbf{F}$ *and* whose divergence is zero.]

(c) As an example, let $\mathbf{F} = y\,\hat{\mathbf{x}} + z\,\hat{\mathbf{y}} + x\,\hat{\mathbf{z}}$. Calculate $\mathbf{A}$, and confirm that $\nabla \times \mathbf{A} = \mathbf{F}$. (For further discussion see Prob. 5.51.)

5.4.2 Summary; Magnetostatic Boundary Conditions

In Chapter 2, I drew a triangular diagram to summarize the relations among the three fundamental quantities of electrostatics: the charge density ρ, the electric field $\mathbf{E}$, and the potential V. A similar diagram can be constructed for magnetostatics (Fig. 5.48), relating

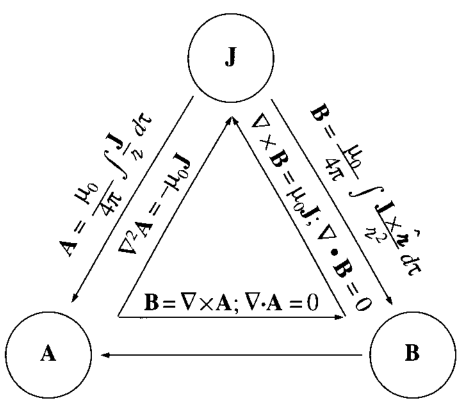


Figure 5.48

the current density **J**, the field **B**, and the potential **A**. There is one "missing link" in the diagram: the equation for **A** in terms of **B**. It's unlikely you would ever need such a formula, but in case you are interested, see Probs. 5.50 and 5.51.

Just as the electric field suffers a discontinuity at a surface *charge*, so the magnetic field is discontinuous at a surface *current*. Only this time it is the *tangential* component that changes. For if we apply Eq. 5.48, in the integral form

$$\oint \mathbf{B} \cdot d\mathbf{a} = 0,$$

to a wafer-thin pillbox straddling the surface (Fig. 5.49), we get

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}. \qquad (5.72)$$

As for the tangential components, an amperian loop running perpendicular to the current (Fig. 5.50) yields

$$\oint \mathbf{B} \cdot d\mathbf{l} = \left(B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}}\right) l = \mu_0 I_{\text{enc}} = \mu_0 K l,$$

or

$$B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K. \qquad (5.73)$$

Thus the component of **B** that is parallel to the surface but perpendicular to the current is discontinuous in the amount $\mu_0 K$. A similar amperian loop running *parallel* to the current reveals that the *parallel* component is *continuous*. These results can be summarized in a single formula:

$$\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}}), \qquad (5.74)$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the surface, pointing "upward."

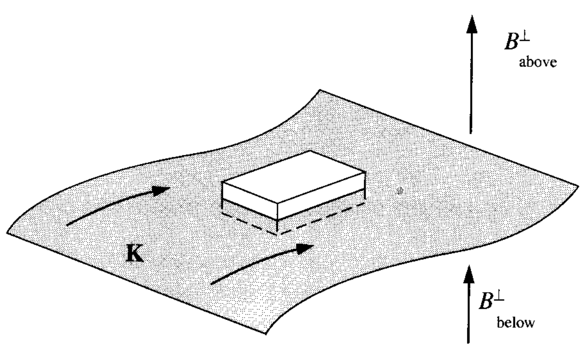


Figure 5.49

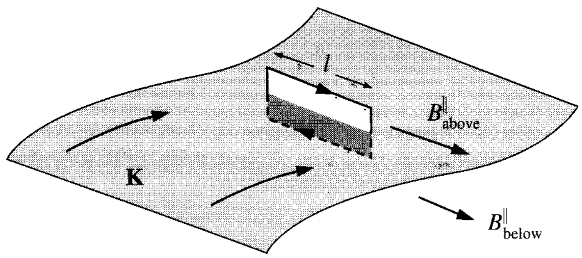


Figure 5.50

Like the scalar potential in electrostatics, the vector potential is continuous across any boundary:

$$\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}, \tag{5.75}$$

for $\nabla \cdot \mathbf{A} = 0$ guarantees[15] that the *normal* component is continuous, and $\nabla \times \mathbf{A} = \mathbf{B}$, in the form

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi,$$

means that the tangential components are continuous (the flux through an amperian loop of vanishing thickness is zero). But the *derivative* of $\mathbf{A}$ inherits the discontinuity of $\mathbf{B}$:

$$\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}. \tag{5.76}$$

Problem 5.31

(a) Check Eq. 5.74 for the configuration in Ex. 5.9.

(b) Check Eqs. 5.75 and 5.76 for the configuration in Ex. 5.11.

Problem 5.32 Prove Eq. 5.76, using Eqs. 5.61, 5.74, and 5.75. [*Suggestion:* I'd set up Cartesian coordinates at the surface, with z perpendicular to the surface and x parallel to the current.]

5.4.3 Multipole Expansion of the Vector Potential

If you want an approximate formula for the vector potential of a localized current distribution, valid at distant points, a multipole expansion is in order. Remember: the idea of a multipole expansion is to write the potential in the form of a power series in $1/r$, where r is the distance to the point in question (Fig. 5.51); if r is sufficiently large, the series will be

[15] Note that Eqs. 5.75 and 5.76 presuppose that $\mathbf{A}$ is divergenceless.

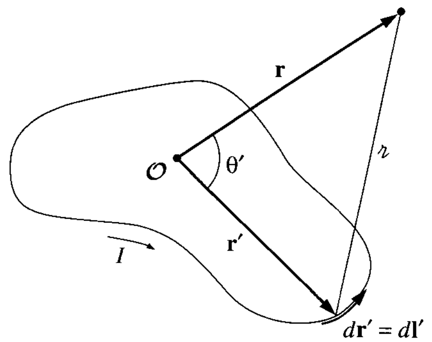


Figure 5.51

dominated by the lowest nonvanishing contribution, and the higher terms can be ignored. As we found in Sect. 3.4.1 (Eq. 3.94),

$$\frac{1}{\imath} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr'\cos\theta'}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta'). \tag{5.77}$$

Accordingly, the vector potential of a current loop can be written

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{1}{\imath}\, d\mathbf{l}' = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\theta')\, d\mathbf{l}', \tag{5.78}$$

or, more explicitly:

$$\begin{aligned}\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\bigg[&\frac{1}{r}\oint d\mathbf{l}' + \frac{1}{r^2}\oint r'\cos\theta'\, d\mathbf{l}' \\ &+ \frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right) d\mathbf{l}' + \cdots\bigg].\end{aligned} \tag{5.79}$$

As in the multipole expansion of V, we call the first term (which goes like $1/r$) the **monopole** term, the second (which goes like $1/r^2$) **dipole**, the third **quadrupole**, and so on.

Now, it happens that the *magnetic monopole term is always zero*, for the integral is just the total vector displacement around a closed loop:

$$\oint d\mathbf{l}' = 0. \tag{5.80}$$

This reflects the fact that there are (apparently) no magnetic monopoles in nature (an assumption contained in Maxwell's equation $\nabla \cdot \mathbf{B} = 0$, on which the entire theory of vector potential is predicated).

In the absence of any monopole contribution, the dominant term is the dipole (except in the rare case where it, too, vanishes):

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\theta' \, d\mathbf{l}' = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}'. \tag{5.81}$$

This integral can be rewritten in a more illuminating way if we invoke Eq. 1.108, with $\mathbf{c} = \hat{\mathbf{r}}$:

$$\oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}' = -\hat{\mathbf{r}} \times \int d\mathbf{a}'. \tag{5.82}$$

Then

$$\boxed{\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2},} \tag{5.83}$$

where $\mathbf{m}$ is the **magnetic dipole moment**:

$$\boxed{\mathbf{m} \equiv I \int d\mathbf{a} = I\mathbf{a}.} \tag{5.84}$$

Here $\mathbf{a}$ is the "vector area" of the loop (Prob. 1.61); if the loop is *flat*, $\mathbf{a}$ is the ordinary area enclosed, with the direction assigned by the usual right hand rule (fingers in the direction of the current).

Example 5.13

Find the magnetic dipole moment of the "bookend-shaped" loop shown in Fig. 5.52. All sides have length w, and it carries a current I.

Solution: This wire could be considered the superposition of two plane square loops (Fig. 5.53). The "extra" sides (AB) cancel when the two are put together, since the currents flow in opposite directions. The net magnetic dipole moment is

$$\mathbf{m} = I w^2 \, \hat{\mathbf{y}} + I w^2 \, \hat{\mathbf{z}};$$

its magnitude is $\sqrt{2} I w^2$, and it points along the 45° line $z = y$.

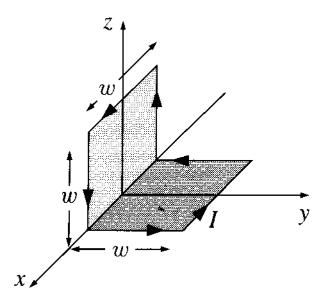


Figure 5.52

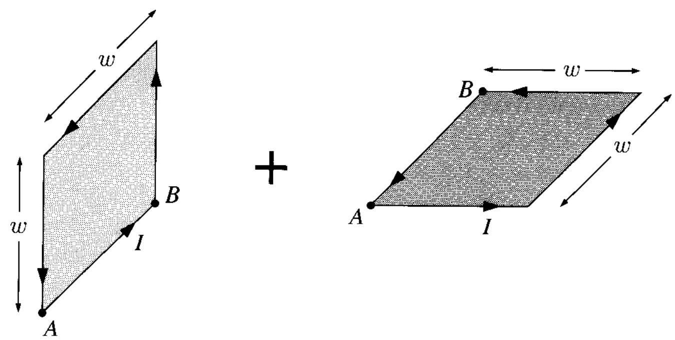


Figure 5.53

It is clear from Eq. 5.84 that the magnetic dipole moment is independent of the choice of origin. You may remember that the *electric* dipole moment is independent of the origin only when the total charge vanishes (Sect. 3.4.3). Since the *magnetic* monopole moment is *always* zero, it is not really surprising that the magnetic dipole moment is always independent of origin.

Although the dipole term *dominates* the multipole expansion (unless $\mathbf{m} = 0$), and thus offers a good approximation to the true potential, it is not ordinarily the *exact* potential; there will be quadrupole, octopole, and higher contributions. You might ask, is it possible to devise a current distribution whose potential is "pure" dipole—for which Eq. 5.83 is *exact*? Well, yes and no: like the electrical analog, it can be done, but the model is a bit contrived. To begin with, you must take an *infinitesimally small* loop at the origin, but then, in order to keep the dipole moment finite, you have to crank the current up to infinity, with the product $m = Ia$ held fixed. In practice, the dipole potential is a suitable approximation whenever the distance r greatly exceeds the size of the loop.

The magnetic *field* of a (pure) dipole is easiest to calculate if we put $\mathbf{m}$ at the origin and let it point in the z-direction (Fig. 5.54). According to Eq. 5.83, the potential at point

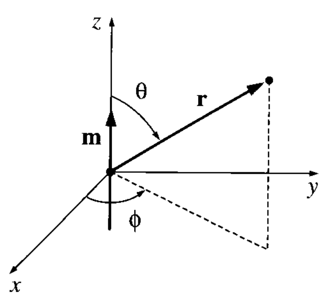
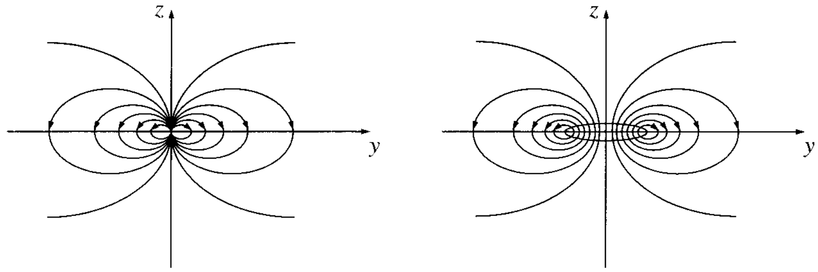


Figure 5.54



(a) Field of a "pure" dipole (a) Field of a "physical" dipole

Figure 5.55

(r, θ, ϕ) is

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{m \sin\theta}{r^2} \hat{\boldsymbol{\phi}}, \tag{5.85}$$

and hence

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \nabla \times \mathbf{A} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}). \tag{5.86}$$

Surprisingly, this is *identical* in structure to the field of an *electric* dipole (Eq. 3.103)! (Up close, however, the field of a *physical* magnetic dipole—a small current loop—looks quite different from the field of a physical electric dipole—plus and minus charges a short distance apart. Compare Fig. 5.55 with Fig. 3.37.)

• **Problem 5.33** Show that the magnetic field of a dipole can be written in coordinate-free form:

$$\boxed{\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}].} \tag{5.87}$$

Problem 5.34 A circular loop of wire, with radius R, lies in the xy plane, centered at the origin, and carries a current I running counterclockwise as viewed from the positive z axis.

(a) What is its magnetic dipole moment?

(b) What is the (approximate) magnetic field at points far from the origin?

(c) Show that, for points on the z axis, your answer is consistent with the *exact* field (Ex. 5.6), when $z \gg R$.

Problem 5.35 A phonograph record of radius R, carrying a uniform surface charge σ, is rotating at constant angular velocity ω. Find its magnetic dipole moment.

Problem 5.36 Find the magnetic dipole moment of the spinning spherical shell in Ex. 5.11. Show that for points $r > R$ the potential is that of a perfect dipole.

Problem 5.37 Find the exact magnetic field a distance z above the center of a square loop of side w, carrying a current I. Verify that it reduces to the field of a dipole, with the appropriate dipole moment, when $z \gg w$.

More Problems on Chapter 5

Problem 5.38 It may have occurred to you that since parallel currents attract, the current within a single wire should contract into a tiny concentrated stream along the axis. Yet in practice the current typically distributes itself quite uniformly over the wire. How do you account for this? If the positive charges (density ρ_+) are at rest, and the negative charges (density ρ_-) move at speed v (and none of these depends on the distance from the axis), show that $\rho_- = -\rho_+\gamma^2$, where $\gamma \equiv 1/\sqrt{1-(v/c)^2}$ and $c^2 = 1/\mu_0\epsilon_0$. If the wire as a whole is neutral, where is the compensating charge located?[16] [Notice that for typical velocities (see Prob. 5.19) the two charge densities are essentially unchanged by the current (since $\gamma \approx 1$). In **plasmas**, however, where the positive charges are *also* free to move, this so-called **pinch effect** can be very significant.]

Problem 5.39 A current I flows to the right through a rectangular bar of conducting material, in the presence of a uniform magnetic field $\mathbf{B}$ pointing out of the page (Fig. 5.56).

(a) If the moving charges are *positive*, in which direction are they deflected by the magnetic field? This deflection results in an accumulation of charge on the upper and lower surfaces of the bar, which in turn produces an electric force to counteract the magnetic one. Equilibrium occurs when the two exactly cancel. (This phenomenon is known as the **Hall effect.**)

(b) Find the resulting potential difference (the **Hall voltage**) between the top and bottom of the bar, in terms of B, v (the speed of the charges), and the relevant dimensions of the bar.[17]

(c) How would your analysis change if the moving charges were *negative*? [The Hall effect is the classic way of determining the sign of the mobile charge carriers in a material.]

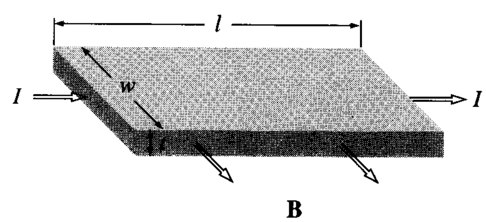


Figure 5.56

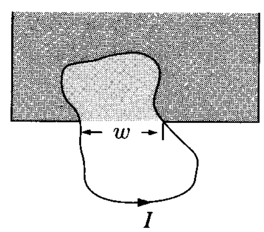


Figure 5.57

Problem 5.40 A plane wire loop of irregular shape is situated so that part of it is in a uniform magnetic field $\mathbf{B}$ (in Fig. 5.57 the field occupies the shaded region, and points perpendicular to the plane of the loop). The loop carries a current I. Show that the net magnetic force on the loop is $F = IBw$, where w is the chord subtended. Generalize this result to the case where the magnetic field region itself has an irregular shape. What is the direction of the force?

[16] For further discussion, see D. C. Gabuzda, *Am. J. Phys.* **61**, 360 (1993).

[17] The potential *within* the bar makes an interesting boundary-value problem. See M. J. Moelter, J. Evans, and G. Elliot, *Am. J. Phys.* **66**, 668(1998).

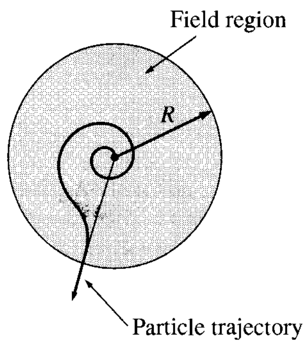


Figure 5.58

Problem 5.41 A circularly symmetrical magnetic field (**B** depends only on the distance from the axis), pointing perpendicular to the page, occupies the shaded region in Fig. 5.58. If the total flux ($\int \mathbf{B} \cdot d\mathbf{a}$) is zero, show that a charged particle that starts out at the center will emerge from the field region on a *radial* path (provided it escapes at all—if the initial velocity is too great, it may simply circle around forever). On the reverse trajectory, a particle fired at the center from outside will hit its target, though it may follow a weird route getting there. [*Hint:* Calculate the total angular momentum acquired by the particle, using the Lorentz force law.]

Problem 5.42 Calculate the magnetic force of attraction between the northern and southern hemispheres of a spinning charged spherical shell (Ex. 5.11). [*Answer:* $(\pi/4)\mu_0\sigma^2\omega^2 R^4$.]

! **Problem 5.43** Consider the motion of a particle with mass m and electric charge q_e in the field of a (hypothetical) stationary magnetic *monopole* q_m at the origin:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q_m}{r^2}\,\hat{\mathbf{r}}.$$

(a) Find the acceleration of q_e, expressing your answer in terms of q, q_m, m, $\mathbf{r}$ (the position of the particle), and $\mathbf{v}$ (its velocity).

(b) Show that the speed $v = |\mathbf{v}|$ is a constant of the motion.

(c) Show that the vector quantity

$$\mathbf{Q} \equiv m(\mathbf{r} \times \mathbf{v}) - \frac{\mu_0 q_e q_m}{4\pi}\,\hat{\mathbf{r}}$$

is a constant of the motion. [*Hint:* differentiate it with respect to time, and prove—using the equation of motion from (a)—that the derivative is zero.]

(d) Choosing spherical coordinates (r, θ, ϕ), with the polar (z) axis along $\mathbf{Q}$,

(i) calculate $\mathbf{Q} \cdot \hat{\boldsymbol{\phi}}$, and show that θ is a constant of the motion (so q_e moves on the surface of a cone—something Poincaré first discovered in 1896)[18];

[18] In point of fact the charge follows a *geodesic* on the cone. The original paper is H. Poincaré, *Comptes rendus de l'Academie des Sciences* **123**, 530 (1896); for a more modern treatment see B. Rossi and S. Olbert, *Introduction to the Physics of Space* (New York: McGraw-Hill, 1970).

(ii) calculate $\mathbf{Q} \cdot \hat{\mathbf{r}}$, and show that the magnitude of $\mathbf{Q}$ is

$$Q = -\frac{\mu_0 q_e q_m}{4\pi \cos\theta};$$

(iii) calculate $\mathbf{Q} \cdot \hat{\boldsymbol{\theta}}$, show that

$$\frac{d\phi}{dt} = \frac{k}{r^2},$$

and determine the constant k.

(e) By expressing v^2 in spherical coordinates, obtain the equation for the trajectory, in the form

$$\frac{dr}{d\phi} = f(r)$$

(that is: determine the function $f(r)$).

(f) Solve this equation for $r(\phi)$.

! **Problem 5.44** Use the Biot-Savart law (most conveniently in the form 5.39 appropriate to surface currents) to find the field inside and outside an infinitely long solenoid of radius R, with n turns per unit length, carrying a steady current I.

Problem 5.45 A semicircular wire carries a steady current I (it must be hooked up to some *other* wires to complete the circuit, but we're not concerned with them here). Find the magnetic field at a point P on the other semicircle (Fig. 5.59). [*Answer:* $(\mu_0 I/8\pi R)\ln\{\tan\left(\frac{\theta+\pi}{4}\right)/\tan\left(\frac{\theta}{4}\right)\}$]

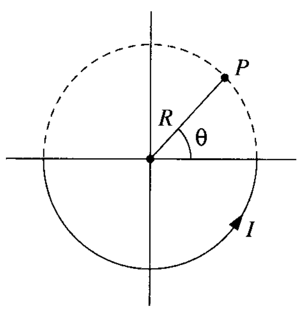


Figure 5.59

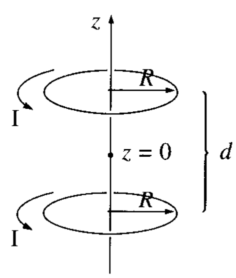


Figure 5.60

Problem 5.46 The magnetic field on the axis of a circular current loop (Eq. 5.38) is far from uniform (it falls off sharply with increasing z). You can produce a more nearly uniform field by using *two* such loops a distance d apart (Fig. 5.60).

(a) Find the field (B) as a function of z, and show that $\partial B/\partial z$ is zero at the point midway between them ($z = 0$). Now, if you pick d just right the *second* derivative of B will *also* vanish at the midpoint. This arrangement is known as a **Helmholtz coil**; it's a convenient way of producing relatively uniform fields in the laboratory.

(b) Determine d such that $\partial^2 B/\partial z^2 = 0$ at the midpoint, and find the resulting magnetic field at the center. [*Answer:* $8\mu_0 I/5\sqrt{5}R$]

! **Problem 5.47** Find the magnetic field at a point $z > R$ on the axis of (a) the rotating disk and (b) the rotating sphere, in Prob. 5.6.

Problem 5.48 Suppose you wanted to find the field of a circular loop (Ex. 5.6) at a point **r** that is *not* directly above the center (Fig. 5.61). You might as well choose your axes so that **r** lies in the yz plane at $(0, y, z)$. The source point is $(R\cos\phi', R\sin\phi', 0)$, and ϕ' runs from 0 to 2π. Set up the integrals from which you could calculate B_x, B_y, and B_z, and evaluate B_x explicitly.

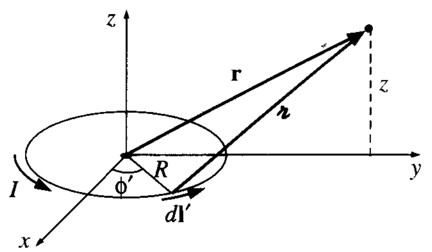


Figure 5.61

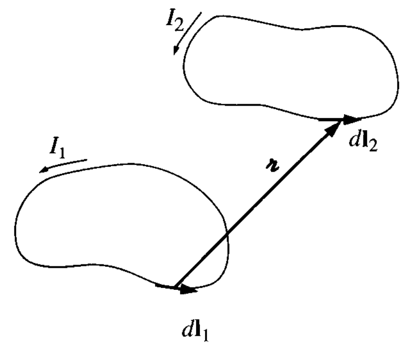


Figure 5.62

Problem 5.49 Magnetostatics treats the "source current" (the one that sets up the field) and the "recipient current" (the one that experiences the force) so asymmetrically that it is by no means obvious that the magnetic force between two current loops is consistent with Newton's third law. Show, starting with the Biot-Savart law (5.32) and the Lorentz force law (5.16), that the force on loop 2 due to loop 1 (Fig. 5.62) can be written as

$$\mathbf{F}_2 = -\frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\mathbf{l}_1 \cdot d\mathbf{l}_2. \tag{5.88}$$

In this form it is clear that $\mathbf{F}_2 = -\mathbf{F}_1$, since $\hat{\boldsymbol{\mathscr{r}}}$ changes direction when the roles of 1 and 2 are interchanged. (If you seem to be getting an "extra" term, it will help to note that $d\mathbf{l}_2 \cdot \hat{\boldsymbol{\mathscr{r}}} = d\mathscr{r}$.)

Problem 5.50

(a) One way to fill in the "missing link" in Fig. 5.48 is to exploit the analogy between the defining equations for **A** ($\nabla \cdot \mathbf{A} = 0$, $\nabla \times \mathbf{A} = \mathbf{B}$) and Maxwell's equations for **B** ($\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$). Evidently **A** depends on **B** in exactly the same way that **B** depends on $\mu_0\mathbf{J}$ (to wit: the Biot-Savart law). Use this observation to write down the formula for **A** in terms of **B**.

(b) The electrical analog to your result in (a) is

$$V(\mathbf{r}) = -\frac{1}{4\pi}\int \frac{\mathbf{E}(\mathbf{r}') \cdot \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'.$$

Derive it, by exploiting the appropriate analogy.

! **Problem 5.51** Another way to fill in the "missing link" in Fig. 5.48 is to look for a magnetostatic analog to Eq. 2.21. The obvious candidate would be

$$\mathbf{A}(\mathbf{r}) = \int_{\mathcal{O}}^{\mathbf{r}} (\mathbf{B} \times d\mathbf{l}).$$

(a) Test this formula for the simplest possible case—uniform **B** (use the origin as your reference point). Is the result consistent with Prob. 5.24? You could cure this problem by throwing in a factor of $\frac{1}{2}$, but the flaw in this equation runs deeper.

(b) Show that $\int(\mathbf{B} \times d\mathbf{l})$ is *not* independent of path, by calculating $\oint(\mathbf{B} \times d\mathbf{l})$ around the rectangular loop shown in Fig. 5.63.

As far as I know[19] the best one can do along these lines is the pair of equations

(i) $V(\mathbf{r}) = -\mathbf{r} \cdot \int_0^1 \mathbf{E}(\lambda \mathbf{r})\, d\lambda$,

(ii) $\mathbf{A}(\mathbf{r}) = -\mathbf{r} \times \int_0^1 \lambda \mathbf{B}(\lambda \mathbf{r})\, d\lambda$.

[Equation (i) amounts to selecting a *radial* path for the integral in Eq. 2.21; equation (ii) constitutes a more "symmetrical" solution to Prob. 5.30.]

(c) Use (ii) to find the vector potential for *uniform* **B**.

(d) Use (ii) to find the vector potential of an infinite straight wire carrying a steady current I. Does (ii) automatically satisfy $\nabla \cdot \mathbf{A} = 0$? [*Answer:* $(\mu_0 I/2\pi s)(z\,\hat{\mathbf{s}} - s\,\hat{\mathbf{z}})$]

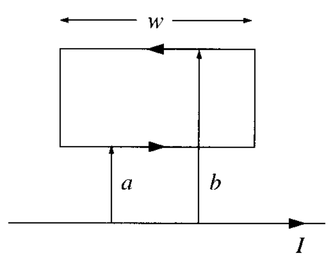


Figure 5.63

Problem 5.52

(a) Construct the scalar potential $U(\mathbf{r})$ for a "pure" magnetic dipole **m**.

(b) Construct a scalar potential for the spinning spherical shell (Ex. 5.11). [*Hint:* for $r > R$ this is a pure dipole field, as you can see by comparing Eqs. 5.67 and 5.85.]

(c) Try doing the same for the interior of a *solid* spinning sphere. [*Hint:* if you solved Prob. 5.29, you already know the *field*; set it equal to $-\nabla U$, and solve for U. What's the trouble?]

[19] R. L. Bishop and S. I. Goldberg, *Tensor Analysis on Manifolds*, Section 4.5 (New York: Macmillan, 1968).

Problem 5.53 Just as $\nabla \cdot \mathbf{B} = 0$ allows us to express $\mathbf{B}$ as the curl of a vector potential ($\mathbf{B} = \nabla \times \mathbf{A}$), so $\nabla \cdot \mathbf{A} = 0$ permits us to write $\mathbf{A}$ itself as the curl of a "higher" potential: $\mathbf{A} = \nabla \times \mathbf{W}$. (And this hierarchy can be extended ad infinitum.)

(a) Find the general formula for $\mathbf{W}$ (as an integral over $\mathbf{B}$), which holds when $\mathbf{B} \to 0$ at ∞.

(b) Determine $\mathbf{W}$ for the case of a *uniform* magnetic field $\mathbf{B}$. [*Hint:* see Prob. 5.24.]

(c) Find $\mathbf{W}$ inside and outside an infinite solenoid. [*Hint:* see Ex. 5.12.]

Problem 5.54 Prove the following uniqueness theorem: If the current density $\mathbf{J}$ is specified throughout a volume $\mathcal{V}$, and *either* the potential $\mathbf{A}$ *or* the magnetic field $\mathbf{B}$ is specified on the surface $\mathcal{S}$ bounding $\mathcal{V}$, then the magnetic field itself is uniquely determined throughout $\mathcal{V}$. [*Hint:* First use the divergence theorem to show that

$$\int \{(\nabla \times \mathbf{U}) \cdot (\nabla \times \mathbf{V}) - \mathbf{U} \cdot [\nabla \times (\nabla \times \mathbf{V})]\}\, d\tau = \oint [\mathbf{U} \times (\nabla \times \mathbf{V})] \cdot d\mathbf{a},$$

for arbitrary vector functions $\mathbf{U}$ and $\mathbf{V}$.]

Problem 5.55 A magnetic dipole $\mathbf{m} = -m_0\,\hat{\mathbf{z}}$ is situated at the origin, in an otherwise uniform magnetic field $\mathbf{B} = B_0\,\hat{\mathbf{z}}$. Show that there exists a spherical surface, centered at the origin, through which no magnetic field lines pass. Find the radius of this sphere, and sketch the field lines, inside and out.

Problem 5.56 A thin uniform donut, carrying charge Q and mass M, rotates about its axis as shown in Fig. 5.64.

(a) Find the ratio of its magnetic dipole moment to its angular momentum. This is called the **gyromagnetic ratio** (or **magnetomechanical ratio**).

(b) What is the gyromagnetic ratio for a uniform spinning sphere? [This requires no new calculation; simply decompose the sphere into infinitesimal rings, and apply the result of part (a).]

(c) According to quantum mechanics, the angular momentum of a spinning electron is $\frac{1}{2}\hbar$, where $\hbar$ is Planck's constant. What, then, is the electron's magnetic dipole moment, in $\mathrm{A} \cdot m^2$? [This semiclassical value is actually off by a factor of almost exactly 2. Dirac's relativistic electron theory got the 2 right, and Feynman, Schwinger, and Tomonaga later calculated tiny further corrections. The determination of the electron's magnetic dipole moment remains the finest achievement of quantum electrodynamics, and exhibits perhaps the most stunningly precise agreement between theory and experiment in all of physics. Incidentally, the quantity $(e\hbar/2m)$, where e is the charge of the electron and m is its mass, is called the **Bohr magneton**.]

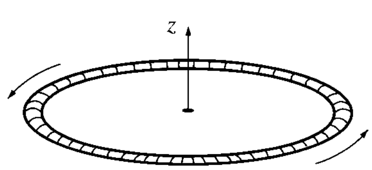


Figure 5.64

• **Problem 5.57**

(a) Prove that the average magnetic field, over a sphere of radius R, due to steady currents within the sphere, is

$$\mathbf{B}_{\text{ave}} = \frac{\mu_0}{4\pi}\frac{2\mathbf{m}}{R^3}, \tag{5.89}$$

where $\mathbf{m}$ is the total dipole moment of the sphere. Contrast the electrostatic result, Eq. 3.105. [This is tough, so I'll give you a start:

$$\mathbf{B}_{\text{ave}} = \frac{1}{\frac{4}{3}\pi R^3}\int \mathbf{B}\, d\tau.$$

Write $\mathbf{B}$ as $(\nabla \times \mathbf{A})$, and apply Prob. 1.60b. Now put in Eq. 5.63, and do the surface integral first, showing that

$$\int \frac{1}{\mathscr{r}}\, d\mathbf{a} = \frac{4}{3}\pi \mathbf{r}'$$

(see Fig. 5.65). Use Eq. 5.91, if you like.]

(b) Show that the average magnetic field due to steady currents *outside* the sphere is the same as the field they produce at the center.

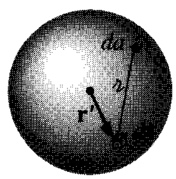


Figure 5.65

Problem 5.58 A uniformly charged solid sphere of radius R carries a total charge Q, and is set spinning with angular velocity ω about the z axis.

(a) What is the magnetic dipole moment of the sphere?

(b) Find the average magnetic field within the sphere (see Prob. 5.57).

(c) Find the approximate vector potential at a point (r, θ) where $r \gg R$.

(d) Find the *exact* potential at a point (r, θ) outside the sphere, and check that it is consistent with (c). [*Hint:* refer to Ex. 5.11.]

(e) Find the magnetic field at a point (r, θ) *inside* the sphere, and check that it is consistent with (b).

Problem 5.59 Using Eq. 5.86, calculate the average magnetic field of a dipole over a sphere of radius R centered at the origin. Do the angular integrals first. Compare your answer with the general theorem in Prob. 5.57. Explain the discrepancy, and indicate how Eq. 5.87 can be corrected to resolve the ambiguity at $r = 0$. (If you get stuck, refer to Prob. 3.42.)

Evidently the *true* field of a magnetic dipole is

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] + \frac{2\mu_0}{3}\mathbf{m}\delta^3(\mathbf{r}). \tag{5.90}$$

Compare the electrostatic analog, Eq. 3.106. [Incidentally, the delta-function term is responsible for the **hyperfine splitting** in atomic spectra—see, for example, D. J. Griffiths, *Am. J. Phys.* **50**, 698 (1982).]

Problem 5.60 I worked out the multipole expansion for the vector potential of a *line* current because that's the most common type, and in some respects the easiest to handle. For a *volume* current **J**:

(a) Write down the multipole expansion, analogous to Eq. 5.78.

(b) Write down the monopole potential, and prove that it vanishes.

(c) Using Eqs. 1.107 and 5.84, show that the dipole moment can be written

$$\mathbf{m} = \frac{1}{2}\int (\mathbf{r}\times\mathbf{J})\,d\tau. \tag{5.91}$$

Problem 5.61 A thin glass rod of radius R and length L carries a uniform surface charge σ. It is set spinning about its axis, at an angular velocity ω. Find the magnetic field at a distance $s \gg R$ from the center of the rod (Fig. 5.66). [*Hint:* treat it as a stack of magnetic dipoles.] [*Answer:* $\mu_0\omega\sigma L R^3/4[s^2 + (L/2)^2]^{3/2}$]

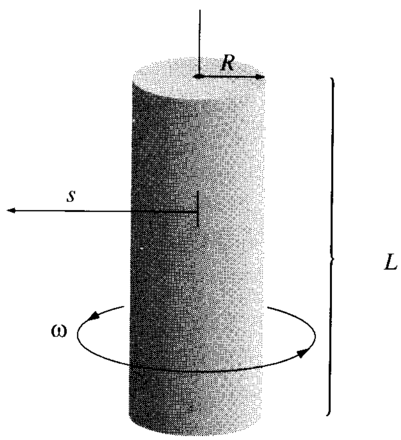


Figure 5.66