2.8 Integration

Standard forms[a]

$$\int u\,\mathrm{d}v = [uv] - \int v\,\mathrm{d}u \qquad (2.353)$$

$$\int uv\,\mathrm{d}x = v\int u\,\mathrm{d}x - \int\left(\int u\,\mathrm{d}x\right)\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x \qquad (2.354)$$

$$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} \quad (n\neq -1) \qquad (2.355)$$

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| \qquad (2.356)$$

$$\int \mathrm{e}^{ax}\,\mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax} \qquad (2.357)$$

$$\int x\mathrm{e}^{ax}\,\mathrm{d}x = \mathrm{e}^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right) \qquad (2.358)$$

$$\int \ln ax\,\mathrm{d}x = x(\ln ax - 1) \qquad (2.359)$$

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln f(x) \qquad (2.360)$$

$$\int x\ln ax\,\mathrm{d}x = \frac{x^2}{2}\left(\ln ax - \frac{1}{2}\right) \qquad (2.361)$$

$$\int b^{ax}\,\mathrm{d}x = \frac{b^{ax}}{a\ln b} \qquad (b>0) \qquad (2.362)$$

$$\int \frac{1}{a+bx}\,\mathrm{d}x = \frac{1}{b}\ln(a+bx) \qquad (2.363)$$

$$\int \frac{1}{x(a+bx)}\,\mathrm{d}x = -\frac{1}{a}\ln\frac{a+bx}{x} \qquad (2.364)$$

$$\int \frac{1}{(a+bx)^2}\,\mathrm{d}x = \frac{-1}{b(a+bx)} \qquad (2.365)$$

$$\int \frac{1}{a^2+b^2x^2}\,\mathrm{d}x = \frac{1}{ab}\arctan\left(\frac{bx}{a}\right) \qquad (2.366)$$

$$\int \frac{1}{x(x^n+a)}\,\mathrm{d}x = \frac{1}{an}\ln\left|\frac{x^n}{x^n+a}\right| \qquad (2.367)$$

$$\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| \qquad (2.368)$$

$$\int \frac{x}{x^2\pm a^2}\,\mathrm{d}x = \frac{1}{2}\ln|x^2\pm a^2| \qquad (2.369)$$

$$\int \frac{x}{(x^2\pm a^2)^n}\,\mathrm{d}x = \frac{-1}{2(n-1)(x^2\pm a^2)^{n-1}} \qquad (2.370)$$

$$\int \frac{1}{(a^2-x^2)^{1/2}}\,\mathrm{d}x = \arcsin\left(\frac{x}{a}\right) \qquad (2.371)$$

$$\int \frac{1}{(x^2\pm a^2)^{1/2}}\,\mathrm{d}x = \ln|x+(x^2\pm a^2)^{1/2}| \qquad (2.372)$$

$$\int \frac{x}{(x^2\pm a^2)^{1/2}}\,\mathrm{d}x = (x^2\pm a^2)^{1/2} \qquad (2.373)$$

$$\int \frac{1}{x(x^2-a^2)^{1/2}}\,\mathrm{d}x = \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) \qquad (2.374)$$

[a] a and b are non-zero constants.

Trigonometric and hyperbolic integrals

$$\int \sin x\,\mathrm{d}x = -\cos x \qquad (2.375)$$

$$\int \sinh x\,\mathrm{d}x = \cosh x \qquad (2.376)$$

$$\int \cos x\,\mathrm{d}x = \sin x \qquad (2.377)$$

$$\int \cosh x\,\mathrm{d}x = \sinh x \qquad (2.378)$$

$$\int \tan x\,\mathrm{d}x = -\ln|\cos x| \qquad (2.379)$$

$$\int \tanh x\,\mathrm{d}x = \ln(\cosh x) \qquad (2.380)$$

$$\int \csc x\,\mathrm{d}x = \ln\left|\tan\frac{x}{2}\right| \qquad (2.381)$$

$$\int \operatorname{csch} x\,\mathrm{d}x = \ln\left|\tanh\frac{x}{2}\right| \qquad (2.382)$$

$$\int \sec x\,\mathrm{d}x = \ln|\sec x + \tan x| \qquad (2.383)$$

$$\int \operatorname{sech} x\,\mathrm{d}x = 2\arctan(\mathrm{e}^x) \qquad (2.384)$$

$$\int \cot x\,\mathrm{d}x = \ln|\sin x| \qquad (2.385)$$

$$\int \coth x\,\mathrm{d}x = \ln|\sinh x| \qquad (2.386)$$

$$\int \sin mx \cdot \sin nx\,\mathrm{d}x = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} \quad (m^2 \neq n^2) \qquad (2.387)$$

$$\int \sin mx \cdot \cos nx\,\mathrm{d}x = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)} \quad (m^2 \neq n^2) \qquad (2.388)$$

$$\int \cos mx \cdot \cos nx\,\mathrm{d}x = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} \quad (m^2 \neq n^2) \qquad (2.389)$$

Named integrals

Error function

$$\operatorname{erf}(x) = \frac{2}{\pi^{1/2}} \int_0^x \exp(-t^2)\,\mathrm{d}t \qquad (2.390)$$

Complementary error function

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\pi^{1/2}} \int_x^\infty \exp(-t^2)\,\mathrm{d}t \qquad (2.391)$$

Fresnel integrals[a]

$$\mathrm{C}(x) = \int_0^x \cos\frac{\pi t^2}{2}\,\mathrm{d}t\,; \quad \mathrm{S}(x) = \int_0^x \sin\frac{\pi t^2}{2}\,\mathrm{d}t \qquad (2.392)$$

$$\mathrm{C}(x) + \mathbf{i}\,\mathrm{S}(x) = \frac{1+\mathbf{i}}{2} \operatorname{erf}\left[\frac{\pi^{1/2}}{2}(1-\mathbf{i})x\right] \qquad (2.393)$$

Exponential integral

$$\mathrm{Ei}(x) = \int_{-\infty}^x \frac{\mathrm{e}^t}{t}\,\mathrm{d}t \quad (x > 0) \qquad (2.394)$$

Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t \quad (x > 0) \qquad (2.395)$$

Elliptic integrals (trigonometric form)

$$\mathrm{F}(\phi,k) = \int_0^\phi \frac{1}{(1-k^2\sin^2\theta)^{1/2}}\,\mathrm{d}\theta \quad \text{(first kind)} \qquad (2.396)$$

$$\mathrm{E}(\phi,k) = \int_0^\phi (1-k^2\sin^2\theta)^{1/2}\,\mathrm{d}\theta \quad \text{(second kind)} \qquad (2.397)$$

[a]See also page 167.

Definite integrals

$$\int_0^\infty \mathrm{e}^{-ax^2}\,\mathrm{d}x = \frac{1}{2}\left(\frac{\pi}{a}\right)^{1/2} \quad (a>0) \tag{2.398}$$

$$\int_0^\infty x\mathrm{e}^{-ax^2}\,\mathrm{d}x = \frac{1}{2a} \quad (a>0) \tag{2.399}$$

$$\int_0^\infty x^n\mathrm{e}^{-ax}\,\mathrm{d}x = \frac{n!}{a^{n+1}} \quad (a>0; n=0,1,2,\ldots) \tag{2.400}$$

$$\int_{-\infty}^\infty \exp(2bx-ax^2)\,\mathrm{d}x = \left(\frac{\pi}{a}\right)^{1/2}\exp\left(\frac{b^2}{a}\right) \quad (a>0) \tag{2.401}$$

$$\int_0^\infty x^n\mathrm{e}^{-ax^2}\,\mathrm{d}x = \begin{cases} 1\cdot 3\cdot 5\cdot\ldots\cdot(n-1)(2a)^{-(n+1)/2}(\pi/2)^{1/2} & n>0 \text{ and even} \\ 2\cdot 4\cdot 6\cdot\ldots\cdot(n-1)(2a)^{-(n+1)/2} & n>1 \text{ and odd} \end{cases} \tag{2.402}$$

$$\int_0^1 x^p(1-x)^q\,\mathrm{d}x = \frac{p!q!}{(p+q+1)!} \quad (p,q \text{ integers} >0) \tag{2.403}$$

$$\int_0^\infty \cos(ax^2)\,\mathrm{d}x = \int_0^\infty \sin(ax^2)\,\mathrm{d}x = \frac{1}{2}\left(\frac{\pi}{2a}\right)^{1/2} \quad (a>0) \tag{2.404}$$

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \int_0^\infty \frac{\sin^2 x}{x^2}\,\mathrm{d}x = \frac{\pi}{2} \tag{2.405}$$

$$\int_0^\infty \frac{1}{(1+x)x^a}\,\mathrm{d}x = \frac{\pi}{\sin a\pi} \quad (0<a<1) \tag{2.406}$$