Chapter 11

MISCELLANEOUS EXAMPLES I

1. A train of mass 300 tonnes is originally at rest upon a level track. It is acted on by a horizontal force which increases uniformly with the time in such a way that $F = 0$ when $t = 0$, and $F = 5$ when $t = 15$, F being measured in tonnes wt. and t in seconds. When in motion the train may be assumed to be acted upon by a constant frictional force equal to 3 tonnes wt. Find the instant of starting, and show that, when $t = 15$, the speed of the train is about 0.196 metres per second, whilst the horse-power required at this instant is about 13 (1 tonne = 10^3 kg.) and (1 metric horse-power = 75 kg. f. metres of work per second).
2. In starting a train the pull of the engine on the rails is at first constant and equal to P; and after the speed attains a certain value u the engine works at a constant rate R $(= Pu)$. When the engine has attained a speed v greater than u, show that the time t and space x from the start are given by

$$t = \frac{M}{2R}(v^2 + u^2) \quad \text{and} \quad x = \frac{M}{3R}(v^3 + \frac{1}{2}u^2),$$

   where M is the combined mass of the engine and train.
   Calculate the time occupied at the space described, in attaining a speed of 72 km. per hour when the total mass is 300 tonnes, if the engine has 420 H.P. and can exert a pull equal to 12 tonnes wt.

3. A unit particle is attracted by two centres of force, A and B, each of which attracts it with a force $\dfrac{\mu}{r^3}$ at distance r. show that, if the particle is initially at rest at a point in AB produced distant $\sqrt{3}.a$ from the middle point of AB, it will arrive at B after a time

$$\frac{a^2}{\sqrt{\mu}}\left[1-\frac{1}{\sqrt{6}}\log_e(\sqrt{3}+m\sqrt{2})\right],$$

where $2a$ is the distance AB.

4. A heavy particle, of mass m, is fastened at the middle point of an elastic string, of natural length $2a$, and the string is stretched between two points, $2l$ apart, in the same vertical line. If the particle starts from rest at a point midway between the two points, find the time of oscillation if the modulus of elasticity $\lambda \not< \dfrac{mga}{l-a}$. What happens if $\lambda < \dfrac{mga}{l-a}$?

5. A plank, of length $2a$ and mass m, is placed with one end against a smooth vertical wall and the other end upon a smooth horizontal plane, its inclination to the horizontal being α. The plank is initially at rest and a monkey, of mass m', runs down it in such a way that the plank always remains at rest; show that the square of his velocity when he has gone a distance x is

$$\frac{gx}{2\sin\alpha}\left[\frac{2(m+2m')}{m'}-\frac{x}{a}\right],$$

and that the time he takes to get to the bottom of the plank is

$$\sqrt{\frac{2a\sin\alpha}{a}}\cos^{-1}\frac{m}{m+2m'}.$$

6. A plank, of mass m, is placed on a rough plane inclined to the horizon at an angle α. A man of mass M runs down it. If the plank is not to slip, show that the acceleration of the man must not be

less than

$$\frac{M+m}{M}(\sin\alpha - \mu\cos\alpha)g$$

nor greater than

$$\frac{M+m}{M}(\sin\alpha + \mu\cos\alpha)g.$$

7. A chain, of length l, is placed along a line of greatest slope of a smooth plane whose inclination to the horizontal is α. If initially an end of the chain just hangs over the lower edge of the plane, prove that the chain will finally leave the plane in time

$$\sqrt{\frac{1}{g(1-\sin\alpha)}}.\log\cot\frac{\alpha}{2}.$$

8. Referred to fixed axes the path of a particle is given by the equations $x = a\cos\omega t, y = b\sin\omega t$. Show that, relatively to axes rotating with angular velocity ω, the path of the particle is a circle.
9. The greatest and least velocities of a planet in its orbit round the sun, which may be regarded as fixed, are 30 and 29.2 kilometres per second respectively. Show that the eccentricity of the orbit is $\frac{1}{74}$.
10. A particle describes an ellipse with an acceleration which is always directed towards its centre; show that the average value of its kinetic energy, taken with regard to the time, is equal to half the sum of its greatest and least kinetic energies.
11. A particle, of mass m, is held on a smooth table. A string attached to this particle passes through a hole in the table and supports a particle of mass $3m$. Motion is started by the particle on the table being projected with velocity V at right angles to the string. If a is the original length of the string on the table, show that when the

hanging weight has descended a distance $\frac{a}{2}$ (assuming this to be possible) its velocity will be

$$\frac{\sqrt{3}}{2}\sqrt{ga - V^2}.$$

12. A straight smooth tube is at rest in a horizontal position and contains a particle at A. The tube is rigidly attached to a point O vertically above A, and is made to rotate about O with constant angular velocity ω, so as to move in a vertical plane. If $OA = a$, show that the distance of the particle from A at time t is

$$a \sinh \omega t + \frac{g}{2\omega^2}(\sinh \omega t - \sin \omega t).$$

13. A particle is projected vertically upwards with a velocity which would carry it to a height of 120 metres if there were no resistance; if the resistance varies as the square of the velocity, and the terminal velocity is 90 metres per sec., show that the height to which it actually rises is about 107 metres, that its velocity on reaching the ground again is 43 metres per sec., and that the total time of its flight is about 9.3 seconds.

14. A chain rests upon a smooth circular cylinder, whose radius is a and whose axis is horizontal; the length of the chain is equal to the semi-circumference of the cylinder. If the chain be slightly displaced, show that its acceleration when a length x has slipped off the cylinder is

$$\frac{g}{\pi a}\left[x + a \sin \frac{x}{a}\right].$$

15. Two particles, of masses m and m', are joined by an elastic string of natural length a and of modulus λ; they are at rest with the string just tight when a force F begins and continues to act on the

particle m in the direction away from m'. Show that at time t the distance between the particles is

$$a+\frac{2F}{mp^2}\sin^2\frac{pt}{2},\ \text{where}\ p^2=\lambda\frac{m+m'}{amm'}.$$

Find also the displacement of m at this time.

16. A safety device for lifts consists of an extension of the lift shaft below ground level; the floor of the lift is made to fit this well closely so that a pneumatic buffer is thus provided. A lift weighting 1300 kg. falls from a height of 9 metres above ground level into such a safety pit 3 metres deep, the base of the lift being 2.5 metres by 1.5 metres. Show that the distance x through which the lift will descend before it is stopped is given by the equation

$$89.4\log_e\frac{3-x}{3}+9+30.8x=0,$$

and $x=1.215$ metres approx. Neglect air leakage, and assume that the pressure of the air varies inversely as its volume, and that atmospheric pressure is 1.033 kg.f. per cm^2.

17. A heavy uniform string, of length l and mass $3m$, passes over a smooth horizontal peg and supports at one end a mass m and at the other end a mass $2m$. When there is equilibrium the mass m is pulled slowly downwards through a space $\frac{l}{9}$, and the system is then left to itself. Prove that, until the mass $2m$ reaches the peg, the space passed over by any point of the system at the end of time t is

$$\frac{l}{9}\left\{\cosh\sqrt{\frac{g}{l}}t-1\right\}$$

and find the time in which the mass $2m$ will reach the peg and its velocity then.

18. A four-wheeled carriage is propelled by a force acting horizontally at a height h above the centre of gravity; the back and front axles are respectively at distance d_1 behind and d_2 in front of the centre of gravity. Neglecting the inertia of the wheels, show that the greatest possible acceleration of the carriage is $\frac{gd_2}{h}$, and that the greatest retardation is $\frac{gd_1}{h}$; whilst, if the forces act at a depth h below the centre of gravity, the greatest acceleration is $\frac{gd_1}{h}$ and the greatest retardation is $\frac{gd_2}{h}$.
19. A hydrometer floats in a liquid with a volume V immersed; if the area of the cross-section of its stem is A, show that the time of its oscillation about its position of equilibrium is $2\pi\sqrt{\frac{V}{Ag}}$.
20. A horizontal shelf is given a horizontal simple harmonic motion. The amplitude of the motion is a and n complete oscillations are performed per second. A particle of mass m is placed on the shelf at the instant when it is at extremity of its motion. Show that, if μ is less than $\frac{4\pi^2n^2a}{g}$, slipping between the particle and shelf will occur for a period t given by the equation

$$\frac{\sin 2\pi nt}{2\pi nt} = \frac{\mu g}{4\pi^2n^2a}.$$

Show that, if for a particular case this value of t is $\frac{1}{6n}$, the distance through which particle moves relative to the shelf in this time is

$$\frac{a}{2}\left(1 - \frac{\mu\sqrt{3}}{6}\right).$$

21. In sinking a caisson in a muddy river bed, the resistance is found to increase in direct proportion to the depth in the mud.

    A caisson, weighing 6 tonnes, sinks 1.2 metres under its own weight before coming to rest. Show that if a load of 8 tonnes is then suddenly added it will sink 40 cm. farther. (1 tonne is 10^3 kg.)

22. A uniform iron rod, of mass M, length a and specific gravity σ, hangs vertically just immersed in water from a light inextensible string which passes over a smooth peg and carries a counterpoise that maintains equilibrium.

    A mass μM is gently added to the counterpoise; show that, if μ exceeds a certain value, the rod will emerge from the water after a time

$$\sqrt{\frac{4a}{g}\{(\mu+2)\,\sigma-1\}}.\sin^{-1}\sqrt{\frac{1}{2\mu\sigma}}.$$

    Discuss in general terms the subsequent motion.

    [The counterpoise is quite clear of the water and motion of the water is neglected.]

23. A weightless string AB consists of two positions AC, CB of unequal lengths and elasticities. The composite string is stretched and held in a vertical position with the ends A and B secured. A particle is attached to C and the steady displacement of C is found to be δ. Show that a further small vertical displacement of C will cause the particle to execute a simple harmonic motion, and that the length of the simple equivalent pendulum is δ.

24. A particle moves under forces whose components parallel to a pair of fixed rectangular axes OX, OY are $-2k^2x+k\dot{y}$ and $-2k^2y+k\dot{x}$ per unit of mass. Interpret the equations giving the motion.

Show that the path, relative to a second pair of rectangular axes rotating about the same origin with constant angular velocity k or $-2k$, is a circle.

25. A particle moves along a plane curve; v is its velocity when its distance from the origin is r, and ρ is the corresponding radius of curvature of its path; show that the velocity of the foot of the perpendicular drawn from the origin upon the tangent to its path $\frac{r}{\rho}v$.
26. A particle moves under a central attractive force which varies as the distance, and there is also a resiting fore proportional to the velocity. Show that the path may be an equiangular spiral.
27. A particle moves with a central acceleration $\mu u^2 + \nu u^3$; find the orbit. If ν be small, show that the path may approximately be represented by an ellipse whose axis revolves round the focus with a small angular velocity.
28. A straight tube, without mass, which moves on a horizontal table and contains a particle of mass m, is started with an angular velocity ω; find the position of the particle at the end of time t, and show that, if θ be the angle turned through in that time, then $\tan\theta = \omega t$.
29. The angular displacement of a pendulum is given by

$$\theta = \theta_0 e^{-kt} \sin nt.$$

Show that the successive maximum values of θ form a series in geometrical progression.

If the time of a complete oscillation is one second, and if the ratio of the first and fifth angular displacements on the same side is 4 : 1, show that the time in swinging out from the position not equilibrium to an extreme displacement is 0.241 sec.

30. The horse-power required to propel a steamer of M tonnes displacement at its maximum speed of V metres per second is H. The resistance is proportional to the square of the speed, and the engine exerts a constant propeller thrust at all speeds. In time t from rest the steamer describes s metres and acquires a velocity of v metres per second. Show that
$t = \frac{500}{75}\frac{MV^2}{Hg}\log_e \frac{V+v}{V-v}$, $s = \frac{500}{75}\frac{MV^3}{Hg}\log_e \frac{V^2}{V^2-v^2}$, and

$$s = \frac{1000}{75}\frac{MV^3}{Hg}\log\cosh\left(\frac{75}{1000}\frac{Hgt}{MV^2}\right).$$

31. A loader motor-car of 50 H.P. weights 2300 kg. and its full speed is 120 km. per hour; it is driven by a constant force at all speeds and the air resistance varies as the square of the velocity; show that it acquires a speed of 72 km. per hour from rest in 48.1 seconds, and that it has then described a distance of 516.5 metres.

32. The horse-power required to propel a steamer of 10,000 tonnes displacement at a steady speed of 20 knots is 15,000. If the resistance is proportional to the square of the speed, and the engines exert a constant propeller thrust at all speeds, find the acceleration when the speed is 15 knots.
Show that the time taken from rest to acquire a speed of 15 knots is a little over $1\frac{1}{2}$ minutes, given that $\log_e 7 = 1.946$ and that 1 knot = 30.9 metres per minute.

33. A train of total mass M is drawn by an engine which exerts a constant pull P at all speeds and the total resistances to the motion of the train are equal to $\mu \times$ (velocity)^2 per unit of its mass.
If M=300 tonnes, if the maximum speed on the level is 90 km. per hour, and if the horse-power then developed is 1500, show that

when climbing a slope of 1 in 100 the maximum speed is nearly 52 km. per hour.

34. The constant propelling force of the engines upon a ship of M tonnes is equal to P tonnes wt; the resistance to the motion varies as the square of the velocity and the limiting velocity is k. If, when the ship is going at full speed, the engines are reversed, show that the ship is brought to rest in time $\frac{\pi}{4}\frac{Mk}{Pg}$ secs. after describing a distance $\frac{Mk^2}{2Pg}\log_e 2$.

35. An engine draws a total mass of M tonnes on the level and works at constant horse-power, overcoming a resistance to motion which varies as the square of the velocity. When the speed is u km. per hour, the tractive force is P kg. wt. and the limiting speed is v km. per hour; show that it reaches a speed of V km. per hour $(V < v)$ from the speed of u km. per hour in a distance.

$$0.00262.\frac{Mv^2}{Pu}\log_e \frac{v^3 - u^3}{v^3 - V^3} \text{ kilometres.}$$

If M = 264 tonnes, P = 9000 kg. u = 24, v = 96, and V = 72 km. per hours, show that the distance is about 1508 metres.

36. A ship of 1680 tonnes and of 72 metres in length is travelling at full speed ahead 18 knots; the effective horse-power is then 2500. Show that, if the engines are reversed, the ship can be stopped in about 7 lengths, assuming that the resistance is proportional to the square of the speed, and that the effective propeller thrust developed by the engines reversed is one-third of that at full speed ahead. (1 knot = 1852 metres per hours; $\log_e 4 = 1.386$.)

37. The resistance to the motion of a train for speeds between 20 and 30 km. per hour may be taken to be $\frac{V^2}{800} + 4.5$ in kg. wt. per tonne, where V is the velocity in km. per hour. Steam is shut off when the speed is 30 km. per hour, and the train slows down under the given resistance. In what time will the speed fall to 20 km. per hour and what distance will the train have described in that time?

38. The effective horse-power required to drive a ship of 15,000 tonnes at a steady speed of 20 knots is 25,000. Assuming the resistance to consist of two parts, one constant and one proportional to the square of the speed, these parts being equal at 20 knots, and that the propeller thrust is the same at all speeds, find the initial acceleration when starting from rest, and the acceleration when a speed of 10 knots is obtained.

    Show that this speed is attained from rest in about 90 seconds, and the distance traversed is about 235 metres. (One knot = 0.5 metres per second, approximately.)

39. A spherical rain-drop falls through a cloud consisting of minute drops of water floating in air and occupying $\frac{a}{n}$th of the whole volume of the cloud; it is assumed that the rain-drop starts from rest, its radius being c, and that as it falls it picks up all the drops of water with which it comes into contact, its shape remaining spherical throughout. If, when it has fallen through a distance x, its radius is a and its velocity is v, show that

$$x = 4n(a-c) \quad \text{and} \quad v^2 = \frac{8}{7}ng\left(a - \frac{c^7}{a^6}\right).$$

40. A uniform chain lies in a coil upon a smooth table, and a force equal to the weight of a length a of the chain is applied to one end.

Show that the length uncoiled in time t is $t\sqrt{ga}$. Show also that the kinetic energy of the moving part of the chain at any time is equal to half the work done by the force.

41. A particle is projected horizontally with velocity $\sqrt{2ga}$ along the smooth surface of a sphere, of radius a, at the level of the centre; prove that the motion is confined between two horizontal planes at a distance $\frac{1}{2}(\sqrt{5}-1)a$ apart.

42. A particle moves under gravity on the surface of a smooth sphere of radius one metre; if the horizontal circles between which its motion is confined are at depths 40 and 50 centimetres below the centre of the sphere, show that the velocity of the particle ranges between 404 and 428 centimetres per second.

43. A particle is projected horizontally under gravity with velocity V from a point on the inner surface of a smooth sphere at an angular distance α from the lowest point. Show that, whatever be the value of V, this angular distance of the particle will not exceed $\pi - \alpha$ in the subsequent motion, and that the particle will not leave the surface if $3 \sin\alpha > 1$.

    Prove that in the subsequent motion the particle will leave the surface if $3\sin\alpha < 1$ and $\dfrac{2V^2}{ag} - 7\cos\alpha$ lies between $\pm 3\sqrt{1-9\sin^2\alpha}$.

44. The bob of a simple pendulum of length a is projected in a horizontal direction at right angles to the string with velocity $2\sqrt{ga}$ when the string is inclined at an angle α to the downward vertical. Show that, if $4\sin^2\dfrac{\alpha}{2} + 6\sin\dfrac{\alpha}{2} - 1$ is positive, the string will not become slack during the ensuing motion.

45. A particle is free to move within a smooth circular tube whose radius is a, which is compelled to rotate with constant angular ve-

locity ω about a vertical axis in it own plane, whose distance is $b(>a)$ from its centre. Show that the period of a small oscillation about the position of relative equilibrium is

$$\frac{2\pi}{\omega}\sqrt{\frac{a\sin\alpha}{b+a\sin^3\alpha}},$$

where α is the angle between the vertical and the radius to the particle when it is in equilibrium.

46. A simple pendulum, of length b, is initially at rest when the point of support is suddenly made to describe a vertical circle, of radius a, with uniform angular velocity ω, starting at the lowest point of the circle. Form the differential equation to give the inclination of the string to the vertical. Integrate it in the case when $\frac{a}{b}$ is small, and show that in this case the inclination of the string will never exceed

$$\frac{a\omega}{b(n\sim\omega)}, \text{ where } n^2b=g.$$

47. A railway carriage, of mass M, impinges with velocity v on a carriage of mass M' at rest. The force necessary to compress a buffer through the full extent l is equal to the weight of a mass m. Assuming that the compression is proportional to the force, show that the buffers will not be completely compressed if

$$v^2<2mgl\left(\frac{1}{M}+\frac{1}{M'}\right).$$

If v exceeds this limits, and the backing against which the buffers are driven is inelastic, the ratio of the final velocities of the carriages is

$$Mv - \left\{2mM'gl\left(1+\frac{M'}{M}\right)\right\}^{1/2} : MV + \left\{2mMgl\left(1+\frac{M}{M'}\right)\right\}^{1/2}.$$

48. A motor car is driven and braked by the back wheels. The centre of gravity is at a height h above the ground and the back and front axles are respectively at horizontal distance d_1 behind and d_2 in front of the centre of gravity. Show that, however great the horse-power, the maximum possible acceleration is

$$\frac{\mu g d_2}{d_1 + d_2 - \mu h},$$

and the maximum retardation that can be produced by the brake is

$$\frac{\mu g d_2}{d_1 + d_2 + \mu h},$$

where μ is the coefficient of friction.

If the car is driven and braked by the front wheels, show that these quantities are respectively

$$\frac{\mu g d_1}{d_1 + d_2 + \mu h} \text{ and } \frac{\mu g d_1}{d_1 + d_2 - \mu h}.$$

[The inertia of the wheels and driving gear is neglected.]

49. Two particles, of masses M and $2M$, are connected by an inextensible string passing over a smooth peg. From the particle of mass M another equal particle hangs by an elastic string, of natural length a and modulus λ equal to Mg. The system is released from rest in this position. Show that, provided the first string be sufficiently long, the motion will be simple harmonic with period $\pi\sqrt{\frac{3a}{g}}$.

Show also that the extension of the second string after time t is

$$a\left[1-\cos\left(2t\sqrt{\frac{g}{3a}}.\right)\right].$$

[Treat the strings as weightless.]

50. Two particles, of masses m_1 and m_2, are connected by a fine elastic string whose modulus of elasticity is λ and whose natural length is l. They are placed on a smooth table at a distance l apart, and equal impulses I in opposite directions in the line of the string act simultaneously on them, so that the string extends. Show that in the ensuing motion the greatest extension is

$$I\sqrt{\frac{(m_1+m_2)l}{m_1m_2\lambda}},$$

and that this value is attained in time

$$\frac{\pi}{2}\sqrt{\frac{m_1m_2l}{(m_1+m_2)\lambda}}.$$

51. A circular disc, of mass M, lies on a smooth horizontal table; if a particle, of mass m, resting on the disc is attached to the centre by a spring which exerts a force μx when extended a length x, prove that the period of oscillations when the spring is extended and then set free is

$$2\pi\sqrt{\frac{Mm}{(M+m)\mu}}.$$

52. The component accelerations of a particle referred to axes, revolving with constant angular velocity ω, are $-4\omega v$ and $4\omega u$, where u and v are the component velocities parallel to these axes. Initially the particle is at the point $(0,-4b)$, and is at rest relative to the moving axes is a four-cusped hypocycloid and that is path in space is a circle.

53. A particle is moving in a circle of radius a under the action of a force to the centre varying inversely as the fourth power of the distance; prove that, if slightly disturbed, it will ultimately be found on one of the curves

$$\frac{r}{a} = \frac{\cosh\theta + 1}{\cosh\theta - 2} \quad \text{or} \quad \frac{r}{a} = \frac{\cosh\theta - 1}{\cosh\theta + 2}.$$

If the force vary as the fifth power of the distance, show that the corresponding curves are

$$\frac{r}{a} = \coth\frac{\theta}{\sqrt{2}} \quad \text{and} \quad \frac{r}{a} = \tanh\frac{\theta}{\sqrt{2}}.$$

54. A particle is projected towards the origin from infinity with any velocity and is acted upon by a force μu^3 at right angles to the radius vector; show that it will describe a curve of the family

$$u = a\theta^{1/4} J_{1/4}(\theta),$$

where $J_n(x)$ is the Bessel's function of the nth order, and find the velocity of projection in order that a particular curve may be described.

55. A particle is attached to a fixed point by a slightly elastic string and is projected at right angles to the string; show that the polar equation of the path is approximately

$$r = c + c' \sin^2\left[\theta\sqrt{\frac{c}{2c'}}\right],$$

where c is the natural length of the string which is supposed to be unstretched when the motion begins, and $c + c'$ is the greatest length it attains during the motion.

56. A fine straight true, of length l, whose inner surface is smooth, is made to rotate in a vertical plane with uniform angular velocity

ω about its middle point. At an instant when the tube is vertical a particle is dropped into it with negligible vertical velocity; prove that the particle will leave the tube by the end at which it enters, or the opposite end, according at l is greater, or less than $\frac{g}{\omega^2}$. Discuss the motion of the particle when l is equal to $\frac{g}{\omega^2}$.

57. One end of a light string, of length $\pi a + b$, is tied to a point of the circumference of a circle which is fixed to a horizontal table. The string is wrapped round the semi-circumference of the circle, and a length b of the string is straight and tangential to the circle. At the end of the straight portion is attached particle of mass m which is projected with velocity V in a direction perpendicular to the straight portion. Show that the string is completely unwound at the end of time $\frac{\pi^2 a + 2\pi b}{2V}$, and that the tension of the string during the unwinding at time t from the commencement of the motion is

$$\frac{mV^2}{\sqrt{b^2 + 2Vat}}.$$

58. A smooth circular wire, of radius a, is constrained to rotate about a vertical diameter with constant angular velocity ω, and a small bead rests on the wire at the lowest point. Show that, if $a\omega^2 > g$, the relative equilibrium is unstable and that, if the bead is slightly displaced, it will rise to a point whose vertical depth below the highest point of the wire is $\frac{g}{\omega^2}$. Show further that the work done by the constraining couple during the time occupied by the rise is twice the work done against gravity.

59. In the case of a nearly flat trajectory, with initial velocity V and a resistance equal to $\mu(\text{velocity})^2$, show that the path of the projectile is approximately

$$y = x\left(\tan\alpha + \frac{g}{2\mu V^2}\right) + \frac{g}{4\mu^2 V^2}(1 - e^2\mu^x)$$
$$= x\tan\alpha - \frac{gx^2}{2V^2} - \frac{\mu g}{3V^2}x^3 - \cdots,$$

where α is the (small) inclination to the horizontal of the path initially.

60. A golf ball owing to undercut is acted on at each point of its path by a force producing an acceleration $\mu v g \sin\alpha$ along the upward drawn normal and a retardation $\mu v g \cos\alpha$ along the tangent, where v is the velocity at the point. Show that, at time t, the horizontal and vertical components of the velocity are
$V\cos\beta - \mu g(x\cos\alpha + y\sin\alpha)$, and

$V\sin\beta - gt + \mu g(x\sin\alpha - y\cos\alpha)$,

where x and y are the horizontal and vertical coordinates, the motion, being in two dimensions; and express these coordinates in terms of the time.

61. A particle is moving in a straight line under the action of a force towards a fixed point C in the line and proportional to the distance from C, in a medium whose resistance is proportional to the velocity. It makes damped oscillations with three consecutive positions of rest at distances, a, b, c from a given point O on the line; show that the distances from O of C and of the next position of rest are respectively

$$\frac{ac - b^2}{a - 2b + c} \quad \textit{and} \quad \frac{ac + bc - b^2 - c^2}{a - b}.$$

62. A particle moving in a straight line is subject to a resistance which produces a retardation kv^3, where v is the velocity and k is a constant. Show that v and the time t are given in terms of s, the distance described by the equations

$$v = \frac{u}{1+ksu}, \text{ and } t = \frac{s}{u} + \frac{1}{2}ks^2,$$

where u is the initial velocity.

A bullet left the rifle with a velocity 740 metres per sec., and had its velocity reduced to 720 metres per second when it had described a distance of 100 metres.

Assuming that the air resistance varied as v^3, find the time taken in traversing 1000 metres, gravity being neglected.

63. An insect, of mass m, alights perpendicularly on one end of a flexible string, of mass M and length l, which is laid in a straight line on a smooth horizontal table, and proceeds to crawl with uniform velocity along the string. When it reaches the other end of the string, show that end will have moved through a distance

$$\frac{ml}{M}\log\left(1+\frac{M}{m}\right).$$

64. A weightless string, passing over a smooth peg, connects a weight P with a uniform string of weight $2P$ hanging vertically with its lower end just in contact with a horizontal table. When motion is allowed to take place, prove that weight P ascends with uniform acceleration $\frac{g}{3}$, until the whole chain is coiled up on the table.

65. A driving belt, which weights m per units length, is moving at a uniform speed. Show that the form assumed by the belt is a catenary whose shape does not depend on the particular speed of the belt. If the speed is altered from v_1 to v_2, show that the tension of

the belt is everywhere increased by an amount equal to

$$m.\frac{v_2^2 - v_1^2}{g}.$$

66. Show that a uniform chain, of density m per unit of length, which is subject to no external forces, can run with constant velocity v in the form of any given curve provided that its tension is equal to mv^2.

67. A smooth surface has the form of a prolate spheroid of major axis (which is vertical) $2a$ and eccentricity e. A particle is describing on the inside of the spheroid a horizontal circle, whose plane is at a distance $a\cos\alpha$ below the centre of the spheroid; prove that the time of a small oscillation about the steady motion is

$$2\pi\sqrt{\frac{a\cos\alpha(1-e^2\cos^2\alpha)}{g(1+3\cos^2\alpha)}}.$$

68. Two particles are connected by an elastic spring. If they vibrate freely in a straight line their period is $\frac{2\pi}{n}$. If they are set to rotate about one another with angular velocity ω, show that the period of a small oscillation is

$$\frac{2\pi}{\sqrt{n^2+3\omega^2}}.$$

69. The motion of a system depends on a single coordinate x; its energy at any instant is $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}ex^2$, and the time-rate of frictional damping of its energy is $\frac{1}{2}m\dot{x}^2$. Show that the period of τ of its free oscillation is

$$2\pi\left(\frac{e}{m} - \frac{1}{16}\frac{k^2}{m^2}\right)^{-1/2}.$$

Show that the forced oscillation sustained by a force of type $A\cos pt$ is at its maximum when $p^2 = \frac{e}{m} - \frac{k^2}{8m^2}$, that the amplitude of this oscillation is then $\frac{A\tau}{\pi k}$, and that its phase lags behind that of the force by the amount $\tan^{-1}\frac{4mp}{k}$.

70. A particle, of mass m', is attached by a light inextensible string of length l to a ring of mass m which is free to slide on a smooth horizontal rod. Initially the masses are held with the string taut along the rod, and they are then set free. Prove that the greatest angular velocity of the string is $\{2g(m+m')/lm\}^{1/2}$. Also show that the time of a small oscillation about the vertical is $2\pi\{lm/g(m+m')\}^{1/2}$.

71. A mass of m attached to a fixed point by a light spring and its time of oscillation vertically is $\frac{2\pi}{p_1}$. If a mass m' is suspended from m by a second spring and the period of m' when m is held fixed is $\frac{2\pi}{p_2}$, show that, when both masses are free, the periods $\frac{2\pi}{n}$ of the normal modes of vertical vibrations of the system are given by the equation

$$n^4 - \left\{p_1^2 + \left(1+\frac{m'}{m}\right)p_2^2\right\}n^2 + p_1^2 p_2^2 = 0.$$

72. A particle, of mass m, moves in a resisting medium under a central attraction $m.P$; show that the equation to the orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2},$$

where $h = h_0 e^{-\int \frac{R}{v}dt}$, and R is the resistance of the medium per unit of mass.

73. In a long railway journey performed with average velocity V, if the actual velocity $v = V + U\sin nt$ and if the resistances vary as the square of the velocity, show that the average H.P. required is increased by $\frac{3}{2}\frac{U^2}{V^2}$ of what is required for uniform velocity V.

74. A particle moves from rest at the distance a towards a centre of force whose acceleration is μ times the distance; if the resistance to the motion is equal to kv^4, where v is the velocity, show that if squares of k are neglected, the time of falling to the centre of force is greater by $\frac{k\sqrt{\mu}a^3}{5}$ than it would be if there were no resisting medium, and that the amplitude of the swing is diminished by $\frac{16k\mu a^4}{5}$.

75. A particle moves in a straight line under a retardation kv^{m+1}, where v is the velocity at time t. Show that, if u be the starting velocity, then

$$kt = \frac{1}{m}\left(\frac{1}{v^m} - \frac{1}{u^m}\right) \text{ and } ks = \frac{1}{m-1}\left(\frac{1}{v^{m-1}} - \frac{1}{u^{m-1}}\right).$$

A bullet fired with a horizontal velocity 729 metres per second is travelling with a velocity of 484 metres per second at the end of one second. Assuming that $m = \frac{1}{2}$, find the value of k, and show that the space described in one second is 594 metres, neglecting the effect of gravity.

76. A particle moves on the surface of a rough circular cone under the action of no forces. It is projected with velocity V at right angles to a generator at a distance d from the vertex. Show that, when its has moved through a distance s, its velocity v is given by

$$\log\frac{V}{v} = \frac{\mu s \cot\alpha}{\sqrt{s^2 + d^2}},$$

where μ is the coefficient of friction and α is the half angle of the cone.

77. A particle moves on the surface of a sphere being acted upon by attractive forces to the ends of the polar axis each equal at distance r to $\frac{\mu m}{r^3}$; if it be projected with moment of momentum about that axis equal to $m\sqrt{\mu}$, its latitude increases uniformly with the time.

78. A smooth cup is formed by the revolution of the parabola $z^2 = 4ax$ about the axis of z, which is vertical. A particle is projected horizontally on the inner surface at a height z_0 with velocity $\sqrt{2kgz_0}$. Prove that, if $k = \frac{1}{4}$, the particle will describe a horizontal circle; but if $k = \frac{1}{30}$, its path will lie between the two planes $z = z_0$ and $z = \frac{1}{2}z_0$.

79. A train in the Northern hemisphere is travelling southward along a meridian of the Earth with velocity V; show that, in latitude λ, it presses on the western rail with a force equal to $\frac{2V\omega}{g}\sin\lambda$ times its own weight, were ω is the angular velocity of the Earth about its axis.

80. A smooth cone, of vertical angle 2α, has its axis vertical and vertex upwards. A heavy particle moving on the outer surface is projected horizontally from a point at a distance R from the vertex with velocity $\sqrt{2gh}$. Show that the particle goes to infinity, and that, for contact to be preserved, $h \not> \frac{1}{2}R\sin\alpha\tan\alpha$.

81. A particle is projected along the surface of a smooth right circular cone, whose axis is vertical vertex upwards, with a velocity due to the depth below the vertex. Show that the equation to the path on the cone, when developed into a plane, is of the form

$$r^{3/2}\cos\frac{3\theta}{2} = a^{3/2}.$$

82. In latitude 45^o N gun is fired due north at an object distant 20 kilometres, this being the maximum range of the gun. Show that if the Earth's rotation has not been allowed for in aiming, the shell should fall about 44 metres east of the mark. Show also that, if the shell is fired due south under similar conditions, the deviation will be twice as great and towards the west.
[Air resistance is neglected.]

ANSWERS WITH HINTS

MISCELLANEOUS EXAMPLES I

1. $300\ddot{x} = \left(\frac{t}{3} - 3\right) g$;
2. About 201 secs.;
27. The path is very nearly that of a conic, focus the origin, whose apse line revolves around the origin with small angular velocity $\frac{v}{2h^2}\dot{\theta}$;
28. $\tan^{-1}\omega t$;
32. 99/640;
38. about 97 secs., 1.08 km.
54. $u = \sqrt{\frac{2}{\mu}}V.\theta^{\frac{1}{4}}J_{\frac{1}{4}}(\theta)$.
62. 1.4 secs.
75. $v\frac{dv}{ds} = \frac{dv}{dt} = kv^{m+1}. \therefore \quad kt = -\int \frac{dv}{v^{m+1}} = \frac{1}{m}\left[\frac{1}{v^m} - \frac{1}{u^m}\right]$, and $ks = -\int \frac{dv}{v^m} = \frac{1}{m-1}\left[\frac{1}{v^{m-1}} - \frac{1}{u^{m-1}}\right]$. If $m = \frac{1}{2}$, $u = 729$, $v = 484$, and $t = 1$ then find k and s.

Appendix A

ON THE SOLUTION OF SOME OF THE MORE COMMON FORMS OF DIFFERENTIAL EQUATIONS

I. $\dfrac{dy}{dx} + Py = Q$, where P and Q are functions of x.
[Linear equation of the first order.]
Multiply the equation by $e^{\int pdx}$, and it becomes

$$\frac{d}{dx}[ye^{\int pdx}] = Qe^{\int pdx}.$$

Hence $ye^{\int pdx} = \int Qe^{\int pdx}dx + a$ constant.
Ex. $\dfrac{dy}{dx} + y\tan x = \sec x$.
Here $e^{\int pdx} = e^{\int \tan x dx} = e^{-\log\cos x} = \dfrac{1}{\cos x}$.
Hence the equation becomes
$\dfrac{1}{\cos x}\dfrac{dy}{dx} + y\dfrac{\sin x}{\cos^x} = \sec^2 x$.

$$\therefore \quad \frac{y}{\cos x} = \tan x + C.$$

II. $\dfrac{d^2y}{dx^2} + P\left(\dfrac{dy}{dx}\right)^2 = Q$, where P and Q are functions of y.
On putting $\left(\dfrac{dy}{dx}\right)^2 = T$, we have $\dfrac{dy}{dx}\,\dfrac{d^2y}{dx^2} = \dfrac{dT}{dx}$,

so that $\dfrac{d^2y}{dx^2} = \dfrac{1}{2}\,\dfrac{dT}{dy}$.

The equation then becomes $\quad \dfrac{dT}{dy} + 2P.T = 2Q,$

a linear equation between T and y, and is thus reduced to the form I.

III. $\dfrac{d^2y}{dx^2} = n^2y$. Multiplying by $\dfrac{dy}{dx}$ and integrating, we have

$\left(\dfrac{dy}{dx}\right)^2 = -n^2y^2 + \text{ const. } = n^2(C^2 - y^2)$

$$\therefore \quad nx = \int \frac{dy}{\sqrt{C^2 - y^2}} = \sin^{-1}\frac{y}{C} + \text{ const.}$$

$$\therefore \quad y = C\sin(nx + D) = L\sin nx + M\cos nx,$$

where C, D, L and M are arbitrary constants.

IV. $\dfrac{d^2y}{dx^2} = n^2y.$

We obtain, as in III,

$\left(\dfrac{dy}{dx}\right)^2 = n^2y^2 + \text{ a const. } = n^2(y^2 - C^2)$

$$\therefore \quad nx = \int \frac{dy}{\sqrt{y^2 - C^2}} = \cosh^{-1}\frac{y}{C} + \text{ const.}$$

$$\therefore \quad y = C\cosh(nx + D) = Le^{nx} + Me^{nx},$$

where C, D, L and M are arbitrary constants.

V. $\dfrac{d^2y}{dx^2} = f(y).$

Similarly, we have in this case

$$\left(\frac{dy}{dx}\right)^2 = 2\int f(y)\frac{dy}{dx}dx = 2f(y)dy.$$

VI. Linear equation with constant coefficients, such as

$$\frac{d^3y}{dx^3} + a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = (x).$$

[The methods which follow are the same, whatever be the order of the equation.]

Let η be *any* solution of this equation, so that

$$(D^3 + aD^2 + bD + c)\eta = f(x) \qquad ...(1).$$

On putting $y = Y + \eta$, we then have

$$(D^3 + aD^2 + bD + c)Y = 0 \qquad ...(2).$$

To solve (2), put $Y = e^{px}$, and we have

$$p^3 + ap^2 + bp + c = 0 \qquad ...(3).$$

an equation whose roots are p_1, p_2 and p_3.

Hence Ae^{p_1x},Be^{p_2x},Ce^{p_3x} (where A, B and C are arbitrary constants) are solutions of (2), and hence $Ae^{p_1x} + Be^{p_2x} + Ce^{p_3x}$ is solution also.

This solution, since it contains three arbitrary and independent constants, is the most general solution that an equation of the third order, such as (2), can have.

Hence

$$Y = Ae^{p_1x} + Be^{p_2x} + Ce^{p_3x} \qquad ...(4).$$

This part of the solution is called the **Complementary Function.**

If some of the roots of equation (3) are imaginary, the equation (4) takes another form.

For let $\alpha + \beta\sqrt{-1}$, $\alpha - \beta\sqrt{-1}$, and p_3 be the roots. Then

$$
\begin{aligned}
y &= Ae^{(\alpha+\beta\sqrt{-1})x} + Be^{(\alpha-\beta\sqrt{-1})x} Ce^{p_3x} \\
&= Ae^{ax}[\cos\beta x + i\sin\beta x] + Be^{ax}[\cos\beta x - i\sin\beta x] + Ce^{p_3x} \\
&= e^{ax}[A_1\cos\beta x + B_1\sin\beta x] + Ce^{p_3x}
\end{aligned}
$$

where A_1 and B_1 are new arbitrary constants.

In some cases two of the quantities p_1, p_2, p_3 are equal, and then the form (4) for the Complementary Function must be modified.

Let $p_2 = p_1 + \gamma$, where γ is ultimately to be zero.

Then the form (4)

$$
\begin{aligned}
&= Ae^{p_1x} + Be^{(p_1+\gamma)x} + Ce^{p_2x} \\
&= Ae^{p_1x} + Be^{p_1x}\left[1 + \gamma x + \frac{\gamma^2x^2}{2!} + \cdots\right] + Ce^{p_3x} \\
&= A_1e^{p_1x} + B_1e^{p_1x}\left[x + \frac{\gamma x^2}{2!} + \cdots\right] + Ce^{p_3x}
\end{aligned}
$$

where A_1, B_1 are fresh arbitrary constants.

If γ be now made equal to zero, this becomes

$$(A_1 + B_1x)e^{p_1x} + Ce^{p_3x}.$$

If three roots p_1, p_2, p_3 are all equal, we have, similarly,

$$(A_1 + B_1x + C_1x^2)e^{p_1x}$$

as the form of the Complementary Function.

The value of η given by (1) is called the **Particular Integral.**

The method of obtaining η depends on the form of $f(x)$. The only forms we need consider are

$$x^n, e^{\lambda x}\begin{pmatrix}\sin\\ \cos\end{pmatrix}\lambda x \text{ and } e^{\mu x}\begin{pmatrix}\sin\\ \cos\end{pmatrix}\lambda x.$$

(i) $f(x) = x^n$.

Here, by the principles of operators,

$$\eta = \frac{1}{D^3 + aD^2 + bD + c}.x^n$$
$$= [A_0 + A_1 D + A_2 D^2 + \cdots + A_n D^n + \cdots].x^n,$$

on expanding the operator in powers of D.

Every term is now known, and hence

$$\eta = A_0 x^n + A_1.nx^{n-1} + A_2.n(n-1)x^{n-2} + \cdots + A_n 1.2\ldots n.$$

(ii) $f(x) = e^{\lambda x}$.

We easily see that $D^r e^{\lambda x} = \lambda^r e^{\lambda x}$.

$$\therefore\ \ \eta = \frac{1}{D^3 + aD^2 + bD + c}.e^{\lambda x}$$
$$= (A_0 + A_1 D + A_2 D^2 + \cdots + \cdots)e^{\lambda x}$$
$$= (A_0 + A_1 \lambda + A_2 \lambda^2 + \cdots + \cdots)e^{\lambda x}$$
$$= \frac{1}{\lambda^3 + a\lambda^2 + b\lambda + c}e^{\lambda x},$$

so that in this case η is obtained by substituting λ for D.

(iii) $f(x) = \sin\lambda x$

We know that $D^2 \sin\lambda x = -\lambda^2 \sin\lambda x$, and that

$$D^{2r} \sin\lambda x = (-\lambda^2)^r \sin\lambda x,$$

and in general that $F(D^2)\sin\lambda x = f(-\lambda^2)\sin\lambda x$.

Hence

$$\begin{aligned}
\eta &= \frac{1}{D^3+aD^2+bD+c}\sin\lambda x \\
&= (D^3-aD^2+bD-c).\frac{1}{D^2(D^2+b)^2-(aD^2+c)^2}\sin\lambda x \\
&= (D^3-aD^2+bD-c).\frac{1}{-\lambda^2(b-\lambda^2)^2-(-a\lambda^2+c)^2}\sin\lambda x \\
&= -\frac{1}{\lambda^2(\lambda^2-b)^2+(a\lambda^2-c)^2}. \\
&\qquad (-\lambda^3\cos\lambda x+a\lambda^2\sin\lambda x+b\lambda\cos\lambda x-c\sin\lambda x) \\
&= \frac{(\lambda^3-b\lambda)\cos\lambda x-(a\lambda^2-c)\sin\lambda x}{\lambda^2(\lambda^2-b)^2+(a\lambda^2-c)^2}.
\end{aligned}$$

(iv) $f(x) = e^{\mu x}\sin\lambda x$

We easily obtain

$$D(e^{\mu x}\sin\lambda x) = e^{\mu x}(D+\mu)\sin\lambda x,$$

$$D^2(e^{\mu x}\sin\lambda x) = e^{\mu x}(D+\mu)^2\sin\lambda x,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$D^r(e^{\mu x}\sin\lambda x) = e^{\mu x}(D+\mu)^r\sin\lambda x,$$

and generally, $\quad F(D)(e^{\mu x}\sin\lambda x) = e^{\mu x}F(D+\mu)\sin\lambda x$

Hence

$$\begin{aligned}
\eta &= \frac{1}{D^3+aD^2+bD+c}e^{\mu x}\sin\lambda x \\
&= e^{\mu x}\frac{1}{(D+\mu)^3+a(D+\mu)^2+b(D+\mu)+c}\sin\lambda x
\end{aligned}$$

the value of which is obtained as in (iii).

In some cases we have to adjust the form of the Particular Integral.

Thus, in the equation $(D-1)(D-2)(D-3)y = e^{2x}$,
the particular integral obtained as above becomes infinite; to get the corrected form we may proceed as follows:

$$\begin{aligned}
\eta &= \frac{1}{(D-1)(D-2)(D-3)} e^{2x} \\
&= \frac{1}{D-2} \cdot \frac{1}{(D-1)(D-3)} e^{2x} \\
&= \frac{1}{D-2} \cdot \frac{1}{1.(-1)} e^{2x}, \quad \text{by the result of (ii),} \\
&= -\lim_{\gamma=0} \frac{1}{D-2} e^{(2+\gamma)x} \\
&= -\lim_{\gamma=0} \frac{1}{\gamma} e^{2x}.e^{\gamma x} \\
&= -e^{2x} \lim_{\gamma=0} \frac{1}{\gamma} \left[1 + \gamma x + \frac{\gamma^2 x^2}{1.2} + \cdots \right] \\
&= \text{ something infinite which may be included in the}
\end{aligned}$$

$$\text{Complementary Function } -xe^{2x}.$$

Hence the complete solution is

$$y = Ae^x + Be^x + Ce^x - xe^{2x}.$$

As another example take the equation

$$(D^2+4)(D-3)y = \cos 2x.$$

The Complementary Function = $A\cos 2x + B\sin 2x + Ce^{3x}$.
The Particular Integral as found by the rule of (iii) becomes infinite.
But we may write

$$\eta = \frac{1}{D^2+4}.\frac{D+3}{D^2+9}\cos 2x$$
$$= -\frac{1}{13}\frac{1}{D^2+4}[3\cos 2x - 2\sin 2x]$$
$$= -\frac{1}{13}\lim_{\gamma=0}\frac{1}{D^2+4}[3\cos(2+\gamma)x - 2\sin(2+\gamma)x]$$
$$= -\frac{1}{13}\lim_{\gamma=0}\frac{1}{4-(2+\gamma)^2}[(3\cos 2x - 2\sin 2x)\cos\gamma x$$
$$-(3\sin 2x + 2\cos 2x)\sin\gamma x]$$
$$= -\frac{1}{13}\lim_{\gamma=0}\frac{1}{-4\gamma-\gamma^2}\left[(3\cos 2x - 2\sin 2x)\left(1-\frac{\gamma^2x^2}{2!}+\cdots\right)\right.$$
$$\left.-(3\sin 2x + 2\cos 2x)\left(\gamma x - \frac{\gamma^3x^3}{3!}+\cdots\right)\right]$$

= something infinite included in the

Complementary Function $-\frac{1}{52}(3\sin 2x + 2\cos 2x).x.$

VII. Linear equations with two independent variables, *e.g.*

$$f_1(D)y + f_2(D)z = 0 \qquad ...(1),$$
$$F_1(D)y + F_2(D)z = 0 \qquad ...(2),$$

where $D \equiv \frac{d}{dx}$

Perform the operation $F_2(D)$ on (1) and $f_2(D)$ on (2) and subtract; we thus have

$$\{f_1(D).F_2(D) - f_2(D)F_1(D)\}y = 0,$$

a linear equation which is soluble as in VI.

Substitute the solution for y thus obtained in (1), and we have a linear equation for z.

Ex.

$$\frac{d^2y}{dx^2}+y+6\frac{dz}{dx}=0 \qquad ...(1),$$

and

$$\frac{dy}{dx}+\frac{d^2z}{dx^2}+2\frac{dz}{dx}=0 \qquad ...(2),$$

i.e. $\quad (D^2+1)y+6Dz=0, \quad$ and $\quad Dy+(D^2+2)z=0.$

$$\therefore \ [(D^2+2)(D^2+1)-D.6D]y=0, \ \ i.e. \ (D^2-1)(D^2-2)y=0.$$

$$\therefore \ y=Ae^x+Be^{-x}+Ce^{\sqrt{2x}}+De^{-\sqrt{2x}}.$$

Hence (1) gives

$$6\frac{dz}{dx}+2Ae^x+2Be^{-x}+3Ce^{\sqrt{2x}}+3De^{-\sqrt{2x}}=0,$$

and hence we have the value of z, viz.

$$z=-\frac{A}{3}e^x+\frac{B}{3}e^{-x}-\frac{C}{2\sqrt{2}}e^{\sqrt{2x}}+\frac{D}{2\sqrt{2}}e^{-\sqrt{2x}}+E.$$

On substituting in (2), we find that $E=0$.

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About the Author

Sidney Luxton Loney popularly known as **S. L. Loney** or simply **Loney**. He was born on 16 March 1860 and death at the age of seventy-nine years occurred on May 16 1939. He was educated at Maidstone Grammar School, in Tonbridge and at Sidney Sussex College, Cambridge, where he graduated B.A. as 3rd Wrangler in 1882 and then M.A.

He was a Professor of Mathematics at the Royal Holloway College, Egham, Surrey, and a fellow of Sidney Sussex College, Cambridge. He belonged to the older generation of Cambridge mathematicians, whose training was broad as well as specialized, and his position as third Wrangler in the then undivided Tripos justified the high opinion formed of his abilities at Tonbridge School by his teacher, Mr. Hilary, one of the most remarkable mathematical teachers of his generation.

He authored a number of mathematics text books, some of which have been reprinted numerous times. He is known as an early influence on Srinivasa Ramanujan.

His mathematics books are widely in use in India. His book, *The Elements of Statics and Dynamics*, published in 1897, is used across various universities in South India. His books, *The Elements of Coordinate Geometry and Plane Trigonometry*, are also popular in India among students preparing for competitive engineering examinations. Som of his popular books are (1) *A treatise on elementary dynamics*, (1889), Cambridge: University Press, (2) *Mechanics and*

Chapter 11

MISCELLANEOUS EXAMPLES I

1. $300\,\ddot{x} = \left(\frac{t}{3} - 3\right) g$, so that $\dot{x} = \frac{g}{1800}(t-9)^2$.

When $t = 15$, $\dot{x} = \frac{32}{1800} \times 6^2 = \cdot 64$.

Also the H.P. then $= \frac{15}{3} \times 2240g \times \frac{32}{50} \div 550g = 13 \cdot 03$.

2. $u^2 = \frac{2P}{M} s_1$ and $u = \frac{P}{M} t_1$. After this $v\frac{dv}{ds} = \frac{dv}{dt} = \frac{Pu}{Mv}$;

$$\therefore\ v^3 = \frac{3Pus}{M} + u^3, \text{ and } v^2 = \frac{2Put}{M} + u^2.$$

$$\therefore \text{ total time} = \frac{Mu}{P} + \frac{M(v^2 - u^2)}{2Pu} = \frac{M(v^2 + u^2)}{2R},$$

and the total space $= \frac{Mu^2}{2P} + \frac{M(v^3 - u^3)}{3Pu} = \frac{M}{3R}\left(v^3 + \frac{1}{2}u^3\right)$.

Given $\frac{M}{R} = \frac{300 \times 2240}{420 \times 550g} = \frac{1}{11}$, $u = \frac{R}{P} = \frac{420 \times 550g}{12 \times 2240g} = \frac{275}{32}$; and $v = 66$,

then $t = \frac{1}{22}\left[66^2 + \left(\frac{275}{32}\right)^2\right] = \frac{11}{2} \times \left[36 + \left(\frac{25}{32}\right)^2\right] =$ about 201 secs.,

and $s = \frac{1}{33}\left[66^3 + \frac{1}{2}\left(\frac{275}{32}\right)^3\right] = \frac{121}{6}\left[2 \times 6^3 + \left(\frac{25}{32}\right)^3\right] =$ about 8722 feet.

3. $\ddot{x} = -\frac{\mu}{(x+a)^2} - \frac{\mu}{(x-a)^2}$, so that $\dot{x}^2 = \frac{\mu}{(x+a)^2} + \frac{\mu}{(x-a)^2} - \frac{2\mu}{a^2}$,

since $\dot{x} = 0$ when $x = \sqrt{3}\,a$. $\therefore\ \dot{x}^2 = \frac{2\mu}{a^2}\frac{x^2(3a^2 - x^2)}{(x^2 - a^2)^2}$.

$$\therefore\ t\sqrt{\frac{2\mu}{a^2}} = -\int_{\sqrt{3}a}^{a} \frac{x^2 - a^2}{x\sqrt{3a^2 - x^2}}\,dx \qquad [\text{Put } x = \sqrt{3}a\sin\theta]$$

$$= \frac{a}{\sqrt{3}}\int_{\sin^{-1}\frac{1}{\sqrt{3}}}^{\frac{\pi}{2}} \left(3\sin\theta - \frac{1}{\sin\theta}\right) d\theta = -\frac{a}{\sqrt{3}}\left[3\cos\theta + \log\frac{\sin\theta}{1+\cos\theta}\right]_{\sin^{-1}\frac{1}{\sqrt{3}}}^{\frac{\pi}{2}}$$

$$= \frac{a}{\sqrt{3}}\left(\sqrt{6} + \log\frac{1}{\sqrt{3}+\sqrt{2}}\right) = \sqrt{2}\,a\left[1 - \frac{1}{\sqrt{6}}\log_e(\sqrt{3}+\sqrt{2})\right].$$

4. When the particle has descended a distance x, we have

$$m\ddot{x} = \lambda\frac{l - x - a}{a} - \lambda\frac{l + x - a}{a} + mg = -\frac{2\lambda}{a}(x - c), \text{ where } c = \frac{amg}{2\lambda}.$$

$\therefore\ \dot{x}^2 = -\frac{2\lambda}{am}\left[(x-c)^2 - c^2\right]$, and hence $\dot{x}$ vanishes again when $x = 2c$.

The lower string is then still stretched if $l-2c>a$,

i.e. if $\lambda>\frac{mga}{l-a}$, and the time of oscillation $=2\pi\sqrt{\frac{am}{2\lambda}}$.

If λ is $<\frac{mga}{l-a}$, the lower string is just unstretched when $x=l-a$, and then $\dot{x}^2=\frac{2\lambda}{am}[c^2-(l-a-c)^2]$. The equation of motion now becomes

$$m\ddot{x}=mg-\lambda\frac{l+x-a}{a}, \text{ etc.}$$

5. If S is the vertical force at the lower end of the plank, and F and R the forces between the plank and monkey along and perpendicular to the plank, we have, by resolving vertically and taking moments about the upper end,

$$S=R\cos\alpha-F\sin\alpha+mg, \text{ and } S.2a\cos\alpha=R.x+mga\cos\alpha.$$

$$\therefore\ F\sin\alpha=\frac{mg}{2}+R\left(\cos\alpha-\frac{x}{2a\cos\alpha}\right)=\frac{mg}{2}+m'g.\frac{2a\cos^2\alpha-x}{2a}.$$

Hence $$m'\ddot{x}=m'g\sin\alpha+F=\frac{(2m'+m)a-m'x}{2a\sin\alpha}g.$$

$$\therefore\ \ddot{x}=-\frac{g}{2a\sin\alpha}[x-\lambda], \text{ where } \lambda=\frac{2m'+m}{m'}a,$$

$$\therefore\ \dot{x}^2=\frac{g}{2a\sin\alpha}[\lambda^2-(\lambda-x)^2],$$

and hence $$t\sqrt{\frac{g}{2a\sin\alpha}}=\int_0^{2a}\frac{dx}{\sqrt{\lambda^2-(\lambda-x)^2}}=\left[\cos^{-1}\frac{\lambda-x}{\lambda}\right]_0^{2a}$$

$$=\cos^{-1}\frac{\lambda-2a}{\lambda}=\cos^{-1}\frac{m}{m+2m'}.$$

6. Let R_1 and μR_1 be the normal pressure and the friction between the plank and the plane, and F the friction between the man's feet and the plank. Then, if the plank be on the point of sliding down,

$$R_1=mg\cos\alpha+Mg\cos\alpha, \text{ and } F+\mu R_1=mg\sin\alpha.$$

The acceleration of the man $=\frac{F+Mg\sin\alpha}{M}=\frac{(M+m)g(\sin\alpha-\mu\cos\alpha)}{M}$.

If the plank is on the point of sliding upwards, we change the sign of μ, and the corresponding limiting acceleration of the man

$$=\frac{(M+m)g(\sin\alpha+\mu\cos\alpha)}{M}.$$

7. $M\ddot{x}=\frac{Mxg}{l}+\frac{M(l-x)g\sin\alpha}{l}$, so that $\frac{l}{g}\ddot{x}=(1-\sin\alpha)(x+\lambda)$,

where $\lambda=\frac{l\sin\alpha}{1-\sin\alpha}$. $\therefore\ \frac{l}{g}\dot{x}^2=(1-\sin\alpha)[(x+\lambda)^2-\lambda^2]$, and hence

$$t\sqrt{\frac{g}{l}(1-\sin\alpha)}=\int_0^l\frac{dx}{\sqrt{(x+\lambda)^2-\lambda^2}}=\log\left[x+\lambda+\sqrt{(x+\lambda)^2-\lambda^2}\right]_0^l$$

$$=\log\frac{l+\lambda+\sqrt{(l+\lambda)^2-\lambda^2}}{\lambda}=\log\frac{1+\cos\alpha}{\sin\alpha}=\log\cot\frac{\alpha}{2}.$$

8. If ξ, η are the coordinates referred to the revolving axes,

$$a \cos \omega t = \xi \cos \omega t - \eta \sin \omega t, \text{ and } b \sin \omega t = \xi \sin \omega t + \eta \cos \omega t.$$

$$\therefore \xi = a\cos^2 \omega t + b \sin^2 \omega t = \frac{a+b}{2} + \frac{a-b}{2}\cos 2\omega t, \text{ and } \eta = \frac{b-a}{2}\sin 2\omega t.$$

Hence
$$\left(\xi - \frac{a+b}{2}\right)^2 + \eta^2 = \left(\frac{a-b}{2}\right)^2, \text{ etc.}$$

9. $30^2 = \mu\left[\frac{2}{a(1-e)} - \frac{1}{a}\right] = \frac{\mu}{a}\frac{1+e}{1-e}$,

and
$$(29{\cdot}2)^2 = \mu\left[\frac{2}{a(1+e)} - \frac{1}{a}\right] = \frac{\mu}{a}\frac{1-e}{1+e}.$$

$$\therefore \left(\frac{1+e}{1-e}\right)^2 = \left(\frac{30}{29{\cdot}2}\right)^2 = \left(\frac{75}{73}\right)^2. \quad \therefore e = \frac{1}{74}.$$

10. As in Art. 40, $v = \sqrt{\mu}\,.\,CD$, $CD = \sqrt{a^2 \sin^2\theta + b^2\cos^2\theta}$,

$$ds = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta, \quad h = pv, \text{ and } hT = 2\pi ab.$$

Hence the average kinetic energy

$$= \frac{\int_0^T \frac{1}{2} m v^2 dt}{T} = \frac{m}{2T}\int \frac{\mu CD^2 . p\,ds}{h} = \frac{m\mu}{4\pi ab}\int_0^{2\pi} ab\,(a^2\cos^2\theta + b^2 \sin^2\theta)\,d\theta$$

$$= \frac{m\mu}{4\pi}\,.\,4\,.\left[\frac{1}{2}a^2 + \frac{1}{2}b^2\right].\frac{\pi}{2} = \frac{1}{2}m\mu\left(\frac{a^2+b^2}{2}\right) = \text{as given.}$$

11. $m(\ddot{r} - r\dot{\theta}^2) = -T$, $\frac{1}{r}\frac{d}{dt}(r^2\dot\theta) = 0$, and $3m(-\ddot{r}) = 3mg - T$.

$$\therefore 4\ddot{r} - r\dot\theta^2 = -3g, \text{ and } r^2\dot\theta = \text{const.} = aV.$$

$$\therefore 4\ddot{r} - \frac{a^2V^2}{r^3} = -3g, \text{ so that } 2\dot{r}^2 + \frac{1}{2}\frac{a^2V^2}{r^2} = -3gr + \frac{1}{2}V^2 + 3ga.$$

Hence, when
$$r = \frac{a}{2}, \quad \dot{r} = -\frac{\sqrt{3}}{2}\sqrt{ga - V^2}, \text{ etc.}$$

12. $AP = x$, at time t, when OA has rotated through the angle ωt.

$$\therefore \ddot{x} = \omega^2 . OP \sin AOP - g \sin \omega t = \omega^2 x - g \sin \omega t.$$

$$\therefore x = C\cosh \omega t + D \sinh \omega t + \frac{g}{2\omega^2}\sin \omega t,$$

where
$$0 = C \text{ and } -a\omega = [\dot{x}]_{t=0} = D\omega + \frac{g}{2\omega}.$$

$$\therefore x = -\left[a \sinh \omega t + \frac{g}{2\omega^2}(\sinh \omega t - \sin \omega t)\right].$$

13. $V = \sqrt{2g . 400} = 160$, and $k = 300$. Hence, by Art. 107,

$$\frac{2gx_1}{300^2} = \log_e \frac{160^2 + 300^2}{300^2} = \log_e\left(\frac{34}{30}\right)^2.$$

$$\therefore x_1 = \frac{90000}{32}[1{\cdot}22378 - 1{\cdot}09861] = 352 \text{ ft. approx.}$$

Also $\frac{gt_1}{300} = \tan^{-1}\frac{160}{300} = \tan^{-1}({\cdot}5333) = \frac{28^\circ\,4'}{180^\circ}\pi = \frac{421}{2700}\pi$, so that $t_1 = 4{\cdot}593$.

Also, by Art. 106,

$$v^2 = 300^2\left[1 - e^{\frac{-2gx}{300^2}}\right] = 300^2\left[1 - \frac{30^2}{34^2}\right], \text{ so that } v = \frac{300 \times 16}{34} = 141{\cdot}176.$$

Also $$\cosh\frac{gt_2}{300} = e^{\frac{gx_1}{k^2}} = \frac{34}{30} = 1{\cdot}1333 = \cosh({\cdot}5108).$$

$$\therefore\ t_2 = \frac{300}{32} \times {\cdot}5108 = 4{\cdot}787, \text{ and the total time} = 9{\cdot}38 \text{ secs.}$$

14. At time t, let $x = a\theta$ and f be the acceleration. Then, for the vertical part, we have $mxf = mxg - T$, and for the part on the cylinder

$$dT - ma\,d\phi\, g\cos\phi = f\,.\,ma\,d\phi.$$

$$\therefore\ T = ma\left[g\sin\phi + f\phi\right]_\theta^\pi = ma\left[-g\sin\frac{x}{a} + f\left(\pi - \frac{x}{a}\right)\right].$$

Hence f is as given.

15. At time t let m' have moved through a distance x and let ξ be the length of the string, so that

$$m'\ddot{x} = \lambda\frac{\xi - a}{a}, \text{ and } m(\ddot{x} + \ddot{\xi}) = -\lambda\frac{\xi - a}{a} + F.$$

Hence $$\ddot{\xi} = -p^2\left[\xi - a - \frac{F}{mp^2}\right].$$

$$\therefore\ \xi - a - \frac{F}{mp^2} = A\cos(pt + B) = -\frac{F}{mp^2}\cos pt.$$

Also $\ddot{x} = \dfrac{\lambda}{am'}\cdot\dfrac{F}{mp^2}(1 - \cos pt)$, and hence $x = \dfrac{\lambda F}{amm'p^2}\left[\dfrac{t^2}{2} + \dfrac{1}{p^2}\cos pt - \dfrac{1}{p^2}\right]$.

16. When the lift has penetrated a distance x into the well, the pressure of the air in it is $\Pi\,.\,\dfrac{10}{10 - x}$.

Hence $$3000\ddot{x} = 3000g + \left[\Pi - \Pi\frac{10}{10 - x}\right] \times 8 \times 5, \text{ and } \Pi = 15 \times 144g.$$

$$\therefore\ \ddot{x} = g\left[\frac{149}{5} - \frac{288}{10 - x}\right], \text{ so that } \dot{x}^2 = 2g\left[\frac{149}{5}x + 288\log\frac{(10 - x)}{10}\right] + 2g\,.\,30.$$

The velocity vanishes when $30 + \dfrac{149}{5}x = 288\log_e\dfrac{10}{10 - x}$.

On reference to the tables it is easily seen that $x = 4{\cdot}1$ nearly.

17. There is equilibrium when a length $\dfrac{2l}{3}$ of the string is on the same side as m. Hence, when this length is x,

$$6m\ddot{x} = mg + \frac{x}{l}\,.\,3mg - 2mg - \frac{l - x}{l}\,.\,3mg, \text{ i.e. } \ddot{x} = \frac{g}{l}\left(x - \frac{2l}{3}\right).$$

$$\therefore\ x - \frac{2l}{3} = \frac{l}{9}\cosh\left(\sqrt{\frac{g}{l}}\,t\right), \text{ since initially } \dot{x} = 0, \text{ and } x = \frac{7l}{9}, \text{ etc.}$$

Also $$x = l \text{ when } \cosh\left(\sqrt{\frac{g}{l}}\,t\right) = 3,$$

and then the velocity $= \dfrac{\sqrt{gl}}{9}\sinh\left(\sqrt{\dfrac{g}{l}}\,t\right) = \dfrac{2}{9}\sqrt{2gl}$.

18. Let R and S be the vertical forces on the back and front wheels. The frictional reactions of the ground must be zero; for otherwise the angular accelerations of the wheels would be infinite, since their inertia is neglected.

Resolving vertically and taking moments, we have, if F is the force applied,

$$R(d_1+d_2)=Mgd_2-Fh, \text{ and } S(d_1+d_2)=Mgd_1+Fh.$$

Now $R \geqq 0$; for otherwise the back wheel would leave the ground; hence the maximum value of F is $\frac{Mgd_2}{h}$ and the corresponding acceleration $=\frac{gd_2}{h}$.

For a retardation, change the sign of F; then since $S \geqq 0$, we have the maximum value of $F=\frac{Mgd_1}{h}$, and the maximum retardation $=\frac{gd_1}{h}$.

If the force acts below the centre of gravity, we change the sign of h, and for an acceleration have

$$R(d_1+d_2)=Mgd_2+Fh, \text{ and } S(d_1+d_2)=Mgd_1-Fh.$$

Hence the maximum value of F is $\frac{Mgd_1}{h}$, etc.

Similarly for the retardation as before.

19. The mass of the hydrometer $=V.\rho$. When it has been displaced vertically through a distance x, we have

$$V\rho.\ddot{x}=V\rho g-(V+Ax)\rho g=-Ax\rho g, \text{ etc.}$$

20. The displacement of the shelf being ξ at time t, we have

$$\xi=a\cos 2\pi nt.$$

Also, if the distance of the particle from the origin at this time is x, then

$$\ddot{x}=-\mu g, \text{ and } \dot{x}=-\mu gt.$$

Slipping ceases when $\dot{x}=\dot{\xi}$, *i.e.* when

$$\frac{\sin 2\pi nt}{2\pi nt}=\frac{\mu g}{4\pi^2 n^2 a}.$$

Since $\sin\theta$ is always $<\theta$, we must have $\mu<\frac{4\pi^2 n^2 a}{g}$.

In the particular case given, we have $\sin\frac{\pi}{3}=\frac{\pi}{3}\frac{\mu g}{4\pi^2 n^2 a}$,

i.e. $\mu g=6\sqrt{3}\pi n^2 a$, and $\therefore\ \dot{x}=-6\sqrt{3}\pi n^2 at$.

$$\therefore\ x=-3\sqrt{3}\pi n^2 at^2+x_0, \text{ so that, when } t=\frac{1}{6n},\ x_0-x=\frac{\sqrt{3}\pi a}{12}.$$

Also when $t=\frac{1}{6n}$, $\xi=\frac{a}{2}$. Hence the distance slipped through

$$=(x-\xi)-(x_0-a)=\text{etc.}$$

21. When the depth of immersion is x, let the resistance be $\lambda.2240gx$.

Hence $$6\times 2240\ddot{x}=6\times 2240g-\lambda.2240gx.$$

$$\therefore\ \dot{x}^2=2gx-\frac{\lambda g}{6}x^2. \text{ Since } \dot{x}=0 \text{ when } x=4,\ \therefore\ \lambda=3.$$

Again $\quad 14 \times 2240\ddot{y} = 14 \times 2240g - \lambda \,.\, 2240gy,$

so that $\ddot{y} = g - \frac{3}{14}gy$, and hence $\dot{y}^2 = 2gy - \frac{3}{14}gy^2 - \left[2g\,.\,4 - \frac{3}{14}g\,.\,4^2\right]$.

$\therefore\ \dot{y}$ is zero when $2(y-4) = \frac{3}{14}(y^2 - 4^2)$, *i.e.* when $y = 5\frac{1}{3}$ feet. Hence, etc.

22. The counterpoise $P = M - \frac{M}{\sigma}$. When a length x is out of the water

$$(M + \mu M + P)\ddot{x} = (\mu M + P)g + \frac{Mg(a-x)}{\sigma a} - Mg.$$

$$\therefore\ \ddot{x} = -\lambda^2(x - a\mu\sigma), \text{ where } \lambda^2 = \frac{g}{a\{(\mu+2)\sigma - 1\}}.$$

$\therefore\ x - a\mu\sigma = A\cos[\lambda t + B] = -a\mu\sigma\cos\lambda t$, since $x = 0$ and $\dot{x} = 0$ when $t = 0$.

The rod will emerge when $a - a\mu\sigma = -a\mu\sigma\cos\lambda t$,

i.e. when $t = \frac{2}{\lambda}\sin^{-1}\frac{1}{\sqrt{2\mu\sigma}}$, provided that then $\dot{x} \geqq 0$,

i.e. if $\quad a\lambda\mu\sigma\sin\lambda t$ is then $\geqq 0$, *i.e.* if $0 < \lambda t < \pi$,

i.e. if $\quad 0 < \sin^{-1}\frac{1}{\sqrt{2\mu\sigma}} < \frac{\pi}{2}$, *i.e.* if $\mu > \frac{1}{2\sigma}$.

On emergence the weight on one side $= (P + \mu M)g = \left(M - \frac{M}{\sigma} + \mu M\right)g$, and that on the other is Mg.

If $\mu < \frac{1}{\sigma}$, the mass M will be brought to rest, and then fall back into the water.

If $\mu > \frac{1}{\sigma}$, the mass M will continue to rise, and finally be pulled over the peg.

23. Let x, y be the original stretched lengths of AC, CB and l, l' their unstretched lengths, so that

$\lambda\frac{x-l}{l} = \lambda'\frac{y-l'}{l'}$, and $\lambda\frac{x+\delta-l}{l} = \lambda'\frac{y-\delta-l'}{l'} + mg$, and thus $\frac{\lambda}{l} + \frac{\lambda'}{l'} = \frac{mg}{\delta}$.

When the further displacement of the particle is ξ, then

$$m\ddot{\xi} = mg - \lambda\frac{x+\delta+\xi-l}{l} + \lambda'\frac{y-\delta-\xi-l'}{l'} = -\xi\left[\frac{\lambda}{l} + \frac{\lambda'}{l'}\right] = -\frac{mg}{\delta}\xi, \text{ etc.}$$

24. The forces acting on the body are; one, equal to $2k^2 \times$ distance acting towards the origin, and the other, equal to $k \times$ velocity acting in a direction perpendicular to the direction of motion.

The equations of motion are

$$(D^2 + 2k^2)x - kDy = 0, \text{ and } (D^2 + 2k^2)y + kDx = 0,$$

$$\therefore\ (D^2 + kiD + 2k^2)(x + yi) = 0.$$

Putting $x + yi = e^{pt}$, we obtain $p = ki$ or $-2ki$, so that

$$x + yi = (A + Bi)(\cos kt + i\sin kt) + (C + Di)(\cos 2kt - i\sin 2kt),$$

and $\quad \therefore\ x = A\cos kt - B\sin kt + C\cos 2kt + D\sin 2kt,$

and $\quad y = A\sin kt + B\cos kt - C\sin 2kt + D\cos 2kt.$

If ξ, η are the coordinates of the point referred to the first pair of revolving axes, then

$$\xi = x\cos kt + y\sin kt = A + C\cos 3kt + D\sin 3kt,$$

and
$$\eta = y\cos kt - x\sin kt = B - C\sin 3kt + D\cos 3kt,$$

and hence
$$(\xi - A)^2 + (\eta - B)^2 = C^2 + D^2.$$

So, for the second pair of revolving axes,

$$\xi = x\cos 2kt - y\sin 2kt = A\cos 3kt - B\sin 3kt + C,$$
$$\eta = y\cos 2kt + x\sin 2kt = A\sin 3kt + B\cos 3kt + D,$$

and hence
$$(\xi - C)^2 + (\eta - D)^2 = A^2 + B^2.$$

25. The polar coordinates of the foot of the perpendicular are p and ψ, so that $V^2 = \left(\frac{dp}{dt}\right)^2 + p^2\left(\frac{d\psi}{dt}\right)^2$, and hence $\frac{V^2}{v^2} = \frac{dp^2 + p^2 d\psi^2}{ds^2}$.

Now $\left(\frac{dp}{d\psi}\right)^2 = PY^2 = r^2 - p^2$, so that $\frac{V^2}{v^2} = \frac{r^2 d\psi^2}{ds^2} = \frac{r^2}{\rho^2}$.

26. $v\frac{dv}{ds} = -\mu r\frac{dr}{ds} - \lambda v$, and $\frac{v^2}{\rho} = \mu r\sin\phi$.

If the curve is the equiangular spiral $r = ae^{\theta\cot\alpha}$, then $\phi = \alpha$,

$$\rho = \frac{r}{\sin\alpha},\ v^2 = \mu r^2,\ v\frac{dv}{ds} = \mu r\cos\alpha,$$

and the first equation is satisfied if $\cos\alpha = -\frac{\lambda}{2\sqrt{\mu}}$.

27. $\frac{d^2u}{d\theta^2} + u = \frac{\mu + \nu u}{h^2}$, so that $\frac{d^2u}{d\theta^2} + u\left(1 - \frac{\nu}{h^2}\right) = \frac{\mu}{h^2}$.

Put $1 - \frac{\nu}{h^2} = m^2$. Hence

$$u = \frac{\mu}{m^2h^2} + A\cos(m\theta + B) = \frac{\mu}{m^2h^2} + A\cos[\theta - \{(1-m)\theta - B\}].$$

Since ν is small, m is nearly unity, so that $(1-m)\theta$ is small. Hence the path is very nearly that of a conic, focus the origin, whose apse line revolves round the origin with a small angular velocity, which

$$= \frac{d}{dt}\{(1-m)\theta\} = \dot\theta\left[1 - \left(1 - \frac{\nu}{2h^2}\right)\right] = \frac{\nu}{2h^2}\dot\theta \text{ approx.}$$

28. Let a be the initial distance of the particle from the point about which the tube started to turn. Then $\ddot r - r\dot\theta^2 = 0$, and $\frac{1}{r}\frac{d}{dt}(r^2\dot\theta) = 0$, since the reaction of the tube is zero; for otherwise the tube would have an infinite angular acceleration, since its mass is zero.

$$\therefore\ r^2\dot\theta = \text{const.} = a^2\omega. \quad \text{Hence } \ddot r = \frac{a^4\omega^2}{r^3},\ \dot r^2 = a^2\omega^2 - \frac{a^4\omega^2}{r^2},$$

and
$$a\omega t = \int\frac{r\,dr}{\sqrt{r^2 - a^2}} = \sqrt{r^2 - a^2}.$$

$$\therefore\ \dot\theta = \frac{\omega}{1 + \omega^2t^2}, \text{ and } \theta = \tan^{-1}\omega t.$$

29. $\dot\theta$ is zero, when $-k\sin nt+n\cos nt=0$, *i.e.* when $\tan nt=\frac{n}{k}=\tan\alpha$, *i.e.* when $nt=\alpha,\ \pi+\alpha,\ 2\pi+\alpha$, and then the values of θ form a G. P. with constant ratio $e^{\frac{-k\pi}{n}}$.

If $\frac{2\pi}{n}=1$, and $\frac{1}{4}=e^{\frac{-8k\pi}{n}}=e^{-4k}$, then $k=\frac{1}{2}\log_e 2=\cdot 3466$.

$$\therefore\ \tan 2\pi t=\frac{n}{k}=\frac{2\pi}{\cdot 3466}=18\cdot 1=\tan 86°\ 50',$$

$$\therefore\ t=\frac{1}{2\pi}\times\frac{86°\ 50'}{90°}\cdot\frac{\pi}{2}=\frac{1}{4}\times\frac{521'}{540}=\cdot 241\text{ sec.}$$

30. $\mu MV^2 . V=H\times 550g$, where μV^2 is the resistance per ton.

Also $\quad 2240Mv\frac{dv}{ds}=2240M\frac{dv}{dt}=C-\mu Mv^2=\mu M(V^2-v^2)$.

$$\therefore\ \frac{\mu t}{2240}=\int\frac{dv}{V^2-v^2}=\frac{1}{2V}\log\frac{V+v}{V-v},$$

giving
$$t=\frac{112}{55}\frac{MV^2}{Hg}\log\frac{V+v}{V-v}.\ \ldots\ldots\ldots\ldots\ldots(1)$$

Also $\quad s=\frac{2240}{\mu}\int\frac{vdv}{V^2-v^2}=-1120\times\frac{MV^3}{550Hg}\log(V^2-v^2)+\text{const.}$

$$=\frac{112}{55}\frac{MV^3}{Hg}\log\frac{V^2}{V^2-v^2}.$$

Also if $\lambda=\frac{55}{112}\frac{Hg}{MV^3}$, then (1) gives $\frac{ds}{dt}=v=V\frac{e^{\lambda t}-1}{e^{\lambda t}+1}$.

$$\therefore\ s=V\int\tanh\frac{\lambda t}{2}=\frac{2V}{\lambda}\log\cosh\frac{\lambda t}{2}=\text{etc., as given.}$$

31. With the notation of the last example, $M=\frac{5000}{2240}=\frac{125}{56}$, $V=110$, $v=66$ and $H=50$. Hence

$$t=\frac{112}{55}\cdot\frac{125}{56}\cdot\frac{110^2}{50 . 32}\log\frac{110+66}{110-66}=\frac{275}{8}\log_e 4=\frac{275}{8}\times 1\cdot 38629=47\cdot 7\text{ secs.},$$

and $s=\frac{112}{55}\cdot\frac{125}{56}\cdot\frac{110^3}{50 . 32}\log_e\frac{110^2}{110^2-66^2}=\frac{15125}{4}\log_e\frac{25}{16}=\frac{15125}{2}\log_e 1\cdot 25$

$$=\frac{15125}{2}\times\cdot 22314=1687\tfrac{1}{2}\text{ feet.}$$

32. $M=10000$, $V=\frac{20\times 100}{60}=\frac{100}{3}$, $v=\frac{15\times 100}{60}=25$, and $H=15000$.

Hence, as in Ex. 30, we have

$$t=\frac{112}{55}\cdot\frac{10000\times 10000}{15000\times 9g}\log\frac{\frac{100}{3}+25}{\frac{100}{3}-25}=\frac{14000}{297}\times\log_e 7=91\cdot 7\text{ secs.}$$

Also, by the same example, $\frac{dv}{dt}=\frac{550Hg}{2240MV^2}(V^2-v^2)$,

so that in this case the acceleration, when $v=25$,

$$=\frac{550}{2240}\cdot\frac{15000\,.\,32\,.\,27}{10^{10}}\left(\frac{10^4}{9}-25^2\right)=\frac{99}{640}.$$

33. $P=\mu\,.\,88^2$, and $\mu\,.\,88^2\times 88=1500\times 550g$. Also $P=\mu v^2+\frac{Mg}{100}$.

$$\therefore\ \frac{300\times 2240g}{100}=\mu\,(88^2-v^2)=\frac{88^2-v^2}{88^3}\times 1500\times 550g.$$

Hence

$\frac{v^2}{88^2}=\frac{708}{2500}$, so that $v=\frac{88}{25}\sqrt{177}$ ft. per sec. $=\frac{60}{25}\sqrt{177}$ miles per hour

$$=\frac{12}{5}\times 13{\cdot}3=32 \text{ miles per hour nearly.}$$

34. $\lambda k^2=P\,.\,2240g$. Also

$$2240M\frac{dv}{dt}=2240M\frac{v\,dv}{dx}=-P\,.\,2240g-\lambda v^2=-P\,.\,2240g\,.\,\frac{v^2+k^2}{k^2}.$$

$$\therefore\ \frac{Pg}{Mk^2}t=-\int_k^0\frac{dv}{v^2+k^2}=\left[\frac{1}{k}\tan^{-1}\frac{v}{k}\right]_0^k=\frac{\pi}{4k},$$

and $$x\,.\,\frac{Pg}{Mk^2}=-\int_k^0\frac{v\,dv}{v^2+k^2}=\left[\frac{1}{2}\log(v^2+k^2)\right]_0^k=\frac{1}{2}\log_e 2.$$

35. Take a mile and an hour as the units of space and time, so that

$$g=32\times(60^2)^2\div 5280=\frac{32\times 27\times 10^3}{11}.$$

We then have $\mu Mv^2\,.\,v=\text{rate of working}=\frac{Pg}{2240}\,.\,u$,

and $$M\frac{V\,dV}{ds}=\frac{Pug}{2240\,.\,V}-\mu MV^2=\frac{Pug}{2240}\,\frac{v^3-V^3}{v^3V}.$$

$$\therefore\ \frac{Pug\,.\,s}{2240Mv^3}=\int_u^V\frac{V^2dV}{v^3-V^3}=\frac{1}{3}\log\frac{v^3-u^3}{v^3-V^3}.$$

$$\therefore\ s=\frac{Mv^3}{Pu}\times\frac{2240\times 11}{3\times 32\times 27\times 10^3}\log_e\frac{v^3-u^3}{v^3-V^3}=\text{as stated.}$$

With the numbers given, this

$=\frac{77}{8100}\times\frac{264\times 60^2}{20000\times 15}\log_e\frac{63}{37}$ miles $=\frac{77\times 704\times 176}{1000}(1{\cdot}84055-1{\cdot}30833)$ feet $=$ about 5080 feet.

36. $V=\text{full speed}=\frac{18\times 6080}{60^2}=\frac{152}{5}=$ nearly 30 and the propeller thrust then $=M\,.\,\mu V^2$.

$$2240\,.\,M\,.\,v\frac{dv}{ds}=-\frac{1}{3}M\,.\,\mu V^2-\mu Mv^2,$$

so that $$\frac{\mu s}{1120}=-6\int_V^0\frac{v\,dv}{V^2+3v^2}=\log_e\frac{4V^2}{V^2+3v^2}.$$

Hence, when $v=0$,

$$s=\frac{1120\times MV^2}{2500\times 550g}\times\log_e 4=\frac{1120\times 1680\times 30^2}{2500\times 550\,g}\times(1{\cdot}386)\text{ nearly}$$
$$=1601\text{ ft.}=\text{about 7 lengths.}$$

37. Take a mile and an hour as the units of space and time, so that

$$g=32\times(60^2)^2\div 5280=\frac{32\times 27\times 10^3}{11}.$$

Then $$M.\,2240\frac{dv}{dt}=M.\,2240\frac{v\,dv}{ds}=-Mg\left[\frac{v^2}{400}+9\right].$$

Hence

$$\frac{gt}{400\times 2240}=-\int_{30}^{20}\frac{dv}{v^2+3600}=-\frac{1}{60}\left[\tan^{-1}\frac{v}{60}\right]_{30}^{20}=\left(\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{3}\right)\times\frac{1}{60}$$
$$=\frac{26^\circ\ 34'-18^\circ\ 26'}{180^\circ}\cdot\frac{\pi}{60}=\frac{61\pi}{1350}\cdot\frac{1}{60}\text{ hours.}$$
$$\therefore\ t=\frac{400\times 2240\times 11}{32\times 27\times 10^3}\times\frac{61\pi}{1350}\text{ minutes}=\text{about 97 secs.}$$

Also $$\frac{2gs}{400\times 2240}=-\int_{30}^{20}\frac{2v\,dv}{v^2+3600}=-\left[\log_e(v^2+3600)\right]_{30}^{20}=\log_e\frac{9}{8}.$$
$$\therefore\ s=\frac{200\times 2240\times 11}{32\times 27\times 10^3}[2{\cdot}19722-2{\cdot}07944]\text{ miles}=\text{about 1180 yards.}$$

38. The resistance $=A+Bv^2$, where $A=B\times\left(\frac{100}{3}\right)^2$. Also

$$25000\times 550\,g=\left[A+B\left(\frac{100}{3}\right)^2\right]\times\frac{100}{3}=2B\frac{10^6}{27},\text{ so that }B=54\times 110,$$

and the propeller thrust $=A+B\left(\frac{100}{3}\right)^2=132\times 10^5$.

The initial acceleration

$$=\left[132\times 10^5-\frac{10^4}{9}\times 54\times 110\right]\div[15000\times 2240]=\frac{11}{56}\text{ ft./sec.}^2$$

The acceleration at 10 knots

$$=\left[132\times 10^5-\left\{\left(\frac{100}{3}\right)^2+\left(\frac{50}{3}\right)^2\right\}\times 54\times 110\right]\div(15000\times 2240)=\frac{33}{224}.$$

The general equation of motion is

$$15000\times 2240\times\left[v\frac{dv}{ds}\right]=15000\times 2240\times\frac{dv}{dt}=132\times 10^5-54\times 110\left[\frac{10^4}{9}+v^2\right].$$
$$\therefore\ s\times\frac{11}{56\times 10^4}=\int_0^{\frac{50}{3}}\frac{v\,dv}{10^4-9v^2}=-\frac{1}{18}\left[\log(10^4-9v^2)\right]_0^{\frac{50}{3}}=\frac{1}{18}\log_e\frac{4}{3}.$$
$$\therefore\ s=\frac{56\times 10^4}{18\times 11}.[{\cdot}2877]=813{\cdot}7\text{ ft.}=271\text{ yds. approx.}$$

Also $$t\times\frac{11}{56\times 10^4}=\int_0^{\frac{50}{3}}\frac{dv}{10^4-9v^2}=\frac{1}{600}\left[\log\frac{100+3v}{100-3v}\right]_0^{\frac{50}{3}}=\frac{1}{600}\log_e 3,\text{ etc.}$$

39. $\frac{d}{dx}\left(\frac{4\pi}{3}a^3\right) = \frac{d}{dx}[\text{volume}] = \frac{\pi a^2}{n}$, so that $dx = 4n\,da$.

The equation of motion is

$$\frac{d}{dt}\left[\frac{4}{3}\pi a^3 \rho \frac{dx}{dt}\right] = \frac{4\pi}{3}a^3 \rho g, \quad i.e.\ 4n\frac{d}{dt}\left[a^3\frac{da}{dt}\right] = a^3 g.$$

Hence
$$2n\left(a^3\frac{da}{dt}\right)^2 = \int a^6 \frac{da}{dt} g = \frac{g}{7}(a^7 - c^7).$$

$$\therefore\ v^2 = 16n^2\left(\frac{da}{dt}\right)^2 = \frac{8ng}{7}\left(a - \frac{c^7}{a^6}\right).$$

Also
$$x = 4na + \text{const.} = 4n(a - c).$$

40. $\frac{d}{dt}(mx\dot{x}) = mga$, so that $x^2\dot{x}^2 = gax^2$. $\therefore\ x = \sqrt{ga}\,t$.

Also the kinetic energy of the moving part

$$= \tfrac{1}{2}mx\,.\,\dot{x}^2 = \tfrac{1}{2}mga\,.\,x = \tfrac{1}{2}\text{ work done by the force.}$$

41. By Art. 127, $\dot{\theta}^2 + \frac{2g}{a\sin^2\theta} = \frac{2g}{a}\cos\theta + A = \frac{2g}{a}\cos\theta + \frac{2g}{a}$.

$\dot{\theta}$ is zero again when $\cos^2\theta + \cos\theta = 1$, *i.e.* when $\cos\theta = \frac{\sqrt{5}-1}{2}$.
The required distance $= a\cos\theta$.

42. With the notation of Art. 127,

$$\cos\alpha = \frac{40}{100} = \frac{2}{5},\quad \cos\beta = \frac{50}{100} = \frac{1}{2},$$

and
$$\dot{\theta}^2 = \frac{V^2}{l^2}\left[1 - \frac{21}{25\sin^2\theta}\right] - \frac{2g}{l}\left[\frac{2}{5} - \cos\theta\right].$$

Also $\dot{\theta}$ is given to vanish again when $\theta = \beta$, so that

$$V^2\left[\frac{84}{75} - 1\right] = 2gl\left[\frac{1}{2} - \frac{2}{5}\right],\quad i.e.\ V = \sqrt{\frac{500g}{3}} = 10\times\sqrt{1635} = 404{\cdot}3\ldots$$

Also $V_1^2 = V^2 + 2g\,.\,10 = \frac{560\,g}{3} = \frac{560\times 981}{3}$, giving $V_1 = 427{\cdot}9$ cms. per sec.

43. With the notation of Art. 127,

$$\theta_1 > \pi - \alpha,\ \text{if}\ \cos\theta_1 < \cos(\pi - \alpha) < -\cos\alpha,$$

and then $\sqrt{1 - 2n^2\cos\alpha + n^4} < n^2 - \cos\alpha$, and hence $\cos^2\alpha > 1$, which is impossible.

Also $\frac{R}{mg} = 4n^2 + 3\cos\theta - 2\cos\alpha$, and R vanishes when $\cos\theta_2 = \frac{2}{3}\cos\alpha - \frac{4n^2}{3}$.

The particle will leave the surface if R vanishes before the velocity vanishes, *i.e.* if $\theta_2 < \theta_1$, *i.e.* if $\cos\theta_2 > \cos\theta_1$,

i.e. if $\frac{2\cos\alpha}{3} - \frac{n^2}{3} > \sqrt{1 - 2n^2\cos\alpha + n^4}$, *i.e.* if $8n^4 - 14n^2\cos\alpha < 4\cos^2\alpha - 9$,

i.e. if
$$\left(n^2 - \frac{7}{8}\cos\alpha\right)^2 < \frac{81\cos^2\alpha - 72}{64} < \frac{9(1 - 9\sin^2\alpha)}{64},$$

i.e. if
$$\frac{2V^2}{ga} - 7\cos\alpha \text{ lies between } \pm 3\sqrt{1 - 9\sin^2\alpha}.$$

If $3 \sin \alpha > 1$, this gives imaginary values of V, so that in this case the particle never leaves the surface.

If $3 \sin \alpha < 1$, we get real values for V.

44. Here $n^2 = 1$, and $\cos \theta_1 = -1 + \sqrt{2 - 2\cos\alpha} = -1 + 2 \sin \frac{\alpha}{2}$.

Thus, when $\theta = \theta_1$, $\quad \frac{T}{mg} = 4 + 3\cos\theta_1 - 2\cos\alpha$

$= 1 - 2\cos\alpha + 6\sin\frac{\alpha}{2} = 4\sin^2\frac{\alpha}{2} + 6\sin\frac{\alpha}{2} - 1 =$ positive. Hence the result.

45. For the relative motion we give an acceleration $m\omega^2(b + a\sin\theta)$ to the particle away from the vertical axis, and have

$a\ddot{\theta} = -g\sin\theta + \omega^2(b + a\sin\theta)\cos\theta$, where $0 = -g\sin\alpha + \omega^2(b + a\sin\alpha)\cos\alpha$.

For a small oscillation, put $\theta = \alpha + \psi$, where ψ is small.

Then $a\ddot{\psi} = -g\cos\alpha \cdot \psi + \omega^2[-b\sin\alpha + a(\cos^2\alpha - \sin^2\alpha)]\psi$

$$= -\frac{\omega^2\psi}{\sin\alpha}(b + a\sin^3\alpha), \text{ etc.}$$

46. $-T\sin\theta = \frac{d^2}{dt^2}[a\sin\omega t + b\sin\theta] = -a\omega^2\sin\omega t + b(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2)$,

and $mg - T\cos\theta = \frac{d^2}{dt^2}[a\cos\omega t + b\cos\theta] = -a\omega^2\cos\omega t - b(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2)$.

$\therefore$ Eliminating T, we have $b\ddot{\theta} + a\omega^2\sin(\theta - \omega t) = -g\sin\theta = -n^2 b\sin\theta$,

or, since θ is small, $\quad b\ddot{\theta} + a\omega^2\theta \cdot \cos\omega t - a\omega^2\sin\omega t = -bn^2\theta$. ....(1)

The first approximation, on neglecting a, is

$\theta = A\sin(nt + B) = -\frac{a\omega}{nb}\sin nt$, since $\theta = 0$ and $b\dot{\theta} = -a\omega$ initially.

Hence the second approximation is given by

$$b(\ddot{\theta} + n^2\theta) = a\omega^2\left[\sin\omega t + \frac{a\omega}{nb}\sin nt\cos\omega t\right] = a\omega^2\sin\omega t.^*$$

$$\therefore\ \theta = C\sin(nt + D) + \frac{a\omega^2}{b(n^2 - \omega^2)}\sin\omega t$$

$$= \frac{a\omega}{b(n^2 - \omega^2)}[\omega\sin\omega t - n\sin nt], \text{ from the initial conditions.}$$

Whatever values ω and n have, it follows that θ cannot exceed

$$\frac{a\omega}{b(n^2 - \omega^2)}[\omega + n], \text{ i.e. } \frac{a\omega}{b(n - \omega)}.$$

47. If ξ is the length of each buffer-spring at time t after the collision, and x the distance moved through by the first carriage in this time,

$M\ddot{x} = -\lambda\frac{l_0 - \xi}{l_0} = -2mg\frac{l_0 - \xi}{l}$, where l_0 is the original length of the spring,

and $$M'(\ddot{x} + \ddot{\xi}) = 2mg\frac{l_0 - \xi}{l}.$$

$$\therefore\ \ddot{\xi} = -\frac{2mg}{l}\left(\frac{1}{M} + \frac{1}{M'}\right)(\xi - l_0) = -\mu^2(\xi - l_0).$$

$\therefore\ \xi - l_0 = C\cos[\mu t + D] = -\frac{v}{\mu}\sin \mu t$, since $\xi = l_0$ and $\dot{\xi} = -v$ when $t=0$.

$\therefore\ \dot{\xi} = -v\cos\mu t$, and vanishes when $t = \frac{\pi}{2\mu}$ and then $l_0 - \xi = \frac{v}{\mu}$. The buffer is then not wholly compressed if $\frac{v}{\mu} < l$, *i.e.* if $v^2 < \mu^2 l^2$.

Since the total momentum is unaltered, the velocity of each carriage at the moment of final compression $= \frac{Mv}{M+M'}$.

Let V and V' be the final velocities. Then $MV + M'V' = Mv$, ...(1)

and $\frac{1}{2}MV^2 + \frac{1}{2}M'V'^2 = \frac{1}{2}(M+M')\frac{M^2v^2}{(M+M')^2}$ + work done by the spring in expanding

$$= \frac{1}{2}\frac{(MV+M'V')^2}{M+M'} + \frac{1}{2}\cdot 2mgl, \text{ so that } V - V' = -\sqrt{2mgl\frac{M+M'}{MM'}}, \quad ...(2)$$

Hence the given result, on solving (1) and (2).

48. Let R and S be the normal reactions on the back and front wheels, and F the friction on the back wheel on which the engine acts. [There can be no friction on the front wheel ; for, if there were, its acceleration would be infinite, since its mass is zero.]

Resolving vertically and taking moments about the centre of gravity, we have

$$R + S = Mg \text{ and } d_2S - d_1R + Fh = 0.$$

Hence $(d_1+d_2)R = Mgd_2 + Fh$. Now $F \overline{\gtrless} \mu R$.

$$\therefore\ (d_1+d_2)F \overline{\gtrless} \mu[Mgd_2 + Fh].$$

Hence the acceleration $\left(=\frac{F}{M}\right) \overline{\gtrless} \frac{\mu g d_2}{d_1+d_2-\mu h}$.

For a retardation, change the sign of F, and we have

$$R(d_1+d_2) = Mgd_2 - Fh, \text{ and } F(d_1+d_2+\mu h) \overline{\gtrless} \mu Mgd_2,$$

and hence the maximum retardation.

If the car is braked from the front wheels, let F be the friction on the front wheels. Then as before

$$R+S = Mg,\ d_2S - d_1R + Fh = 0, \text{ so that } (d_1+d_2)S = Mgd_1 - Fh.$$

But $F \overline{\gtrless} \mu S$, so that $(d_1+d_2+\mu h)F \overline{\gtrless} \mu Mgd_1$,

and hence the maximum acceleration.

Similarly, for a retardation, changing the sign of F, we have

$$(d_1+d_2)S = Mgd_1 + Fh.$$

$\therefore\ (d_1+d_2-\mu h)F \overline{\gtrless} \mu Mgd_1$, and hence the maximum retardation.

49. At time t, let x be the length of the elastic string, and y the depth of the upper particle M. Then

$$(M+2M)\ddot{y} = Mg - 2Mg + \lambda\frac{x-a}{a}, \text{ and } M(\ddot{x}+\ddot{y}) = Mg - \lambda\frac{x-a}{a}.$$

$$\therefore\ \ddot{x} = \frac{4g}{3} - \frac{4\lambda}{3Ma}(x-a) = -\frac{4g}{3a}(x-2a), \text{ since } \lambda = Mg.$$

$$\therefore\ x-2a=A\cos 2\sqrt{\frac{g}{3a}}t+B\sin 2\sqrt{\frac{g}{3a}}t=-a\cos 2\sqrt{\frac{g}{3a}}t,$$

since $x=a$ and $\dot{x}=0$ initially.

Also the periodic time $=2\pi\div 2\sqrt{\frac{g}{3a}}=\pi\sqrt{\frac{3a}{g}}$.

50. At time t, let m_1 have moved through a distance ξ, and let the length of the string be then x, so that

$$m_1\ddot{\xi}=\lambda\frac{x-l}{l},\ \text{and}\ m_2(\ddot{\xi}+\ddot{x})=-\lambda\frac{x-l}{l}.$$

Hence $$\ddot{x}=-\lambda\frac{m_1+m_2}{m_1 m_2}\frac{x-l}{l}.\ \ldots\ldots\ldots\ldots(1)$$

Now, initially, $\dot{\xi}=-\frac{I}{m_1}$ and $\dot{\xi}+\dot{x}=\frac{I}{m_2}$, so that then $\dot{x}=I\frac{m_1+m_2}{m_1 m_2}$.

Hence (1) gives $\dot{x}^2=-\lambda\frac{m_1+m_2}{m_1 m_2}\frac{(x-l)^2}{l}+\left(I\frac{m_1+m_2}{m_1 m_2}\right)^2$,

so that $\dot{x}$ is zero when $x-l$ is as given.

Also, by (1), the required time $=\frac{1}{4}$. periodic time $=$ as stated.

51. At time t, let the centre of the disc have moved through a distance ξ, and let x be the extension of the spring. Then

$$M\ddot{\xi}=T=\mu x,\ \text{and}\ m(\ddot{\xi}+\ddot{x})=-\mu x.$$

$\therefore\ \ddot{x}=-\mu\frac{M+m}{Mm}x$, and the time of oscillation is as given.

We easily obtain $x=a\cos\lambda t$, and $\xi=\frac{am}{M+m}(1-\cos\lambda t)$, where

$\lambda^2=\mu\frac{M+m}{Mm}$; and a is the initial extension of the spring.

52. By Art. 51 we have

$$\ddot{x}-\omega^2x-2\omega\dot{y}=-4\omega(\dot{y}+x\omega),\ \text{and}\ \ddot{y}-\omega^2y+2\omega\dot{x}=4\omega(\dot{x}-y\omega).$$

$$\therefore\ (D^2+3\omega^2)x+2\omega Dy=0,\ \text{and}\ (D^2+3\omega^2)y-2\omega\,.\,Dx=0.$$

$$\therefore\ (D^2+3\omega^2)(x+yi)-2\omega iD(x+yi)=0.$$

On putting $x+yi=e^{pt}$ we obtain $p=3\omega i$ or $-\omega i$, so that

$$x+yi=(A+Ci)e^{3\omega it}+(B+Ei)e^{-\omega it}.$$

$$\therefore\ x=A\cos 3\omega t-C\sin 3\omega t+B\cos\omega t+E\sin\omega t,$$

and $$y=A\sin 3\omega t+C\cos 3\omega t-B\sin\omega t+E\cos\omega t.$$

Now when $t=0$, $x=0$, $\dot{x}=0$, $y=-4b$ and $\dot{y}=0$. We thus obtain

$$x=b[\sin 3\omega t-3\sin\omega t]=-4b\sin^3\omega t,$$

and $$y=-b[\cos 3\omega t+3\cos\omega t]=-4b\cos^3\omega t.$$

Hence $x^{\frac{2}{3}}+y^{\frac{2}{3}}=(4b)^{\frac{2}{3}}$, a four-cusped hypocycloid.

Also, if X, Y are the coordinates referred to fixed axes coinciding with the moving axes at time $t=0$, then

$$X=x\cos\omega t-y\sin\omega t=b\sin 4\omega t,$$

and $$Y=x\sin\omega t+y\cos\omega t=-b\cos 4\omega t-3b.$$

$$\therefore\ X^2+(Y+3b)^2=b^2,\ \text{which is a circle.}$$

53. $v^2 = h^2(u^2 + \dot{u}^2) = \int 2\mu u^2 du = \frac{2\mu u^3}{3} + C$, where $\frac{\mu}{a^3} = \frac{h^2}{a^2} = \frac{2\mu}{3a^3} + C$.

Hence $$\frac{du}{d\theta} = \pm\frac{1}{a\sqrt{3}}(au-1)\sqrt{2au+1}.$$

Case I. $au > 1$, so that $\frac{du}{d\theta}$ is positive. Hence

$$\theta = \int \frac{\sqrt{3}a\,du}{(au-1)\sqrt{2au+1}} \quad [\text{Put } 2au+1=\xi^2, \text{ so that } \xi^2 > 3.]$$

$$= \int \frac{2\sqrt{3}d\xi}{\xi^2-3} = \log\frac{\xi-\sqrt{3}}{\xi+\sqrt{3}} + \gamma.$$

$$\therefore\ 2au+1 = \xi^2 = 3\frac{\cosh^2\frac{1}{2}(\theta-\gamma)}{\sinh^2\frac{1}{2}(\theta-\gamma)} = 3\times\frac{\cosh(\theta-\gamma)+1}{\cosh(\theta-\gamma)-1}, \text{ etc.}$$

Case II. $au < 1$, so that $\frac{du}{d\theta}$ is negative. Hence

$$-\theta = \int \frac{\sqrt{3}a\,du}{(1-au)\sqrt{2au+1}} \quad [\text{Put } 2au+1=\xi^2, \text{ so that } \xi^2 < 3.]$$

$$= \int \frac{2\sqrt{3}a\,d\xi}{3-\xi^2} = \log\frac{\sqrt{3}+\xi}{\sqrt{3}-\xi} - \delta.$$

$$\therefore\ 2au+1 = \xi^2 = 3\frac{\sinh^2\frac{1}{2}(\theta-\delta)}{\cosh^2\frac{1}{2}(\theta-\delta)} = 3\frac{\cosh(\theta-\delta)-1}{1+\cosh(\theta-\delta)}, \text{ etc.}$$

In the second part we similarly obtain $\frac{du}{d\theta} = \pm\frac{1}{a\sqrt{2}}(a^2u^2-1)$.

Case I. $au > 1$, and $\frac{du}{d\theta}$ positive. Then

$$\frac{\theta}{\sqrt{2}} = \int \frac{a\,du}{a^2u^2-1} = \frac{1}{2}\log\frac{au-1}{au+1} + \frac{\gamma}{\sqrt{2}}, \text{ giving } au = -\coth\frac{\theta-\gamma}{\sqrt{2}}.$$

Case II. $au < 1$, and $\frac{du}{d\theta}$ negative.

$$\therefore\ -\frac{\theta}{\sqrt{2}} = \int \frac{a\,du}{1-a^2u^2} = \frac{1}{2}\log\frac{1+au}{1-au} + \frac{\delta}{\sqrt{2}}, \text{ giving } au = -\tanh\frac{\theta-\delta}{\sqrt{2}}$$

54. $\ddot{r} - r\dot{\theta}^2 = 0$, and $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{\mu}{r^3}$, so that $(r^2\dot{\theta})^2 = 2\mu\theta$,

if the initial line is in the direction of the initial velocity.

Then $\dot{r} = -\frac{1}{u^2}\frac{du}{d\theta}\cdot\dot{\theta} = -\sqrt{2\mu\theta}\,\frac{du}{d\theta}$, and $\ddot{r} = -2\mu\left[\ddot{u}u^2\theta + \frac{1}{2}u^2\frac{du}{d\theta}\right]$.

Hence the first equation gives $2\theta(\ddot{u}+u) + \frac{du}{d\theta} = 0$.

Put $u = \xi\theta^{\frac{1}{4}}$, and we have $\ddot{\xi} + \frac{1}{\theta}\dot{\xi} + \xi\left(1 - \frac{1}{16\theta^2}\right) = 0$.

$$\therefore\ \xi = AJ_{\frac{1}{4}}(\theta) + BJ_{-\frac{1}{4}}(\theta),$$

so that $$u = A\theta^{\frac{1}{2}}[1 - L\theta^2 + M\theta^4 - \dots] + B[1 - L_1\theta^2 + M_1\theta^4 - \dots],$$
where $L, M, \dots$ are known constants.

Now when $\theta = 0$, $r = \infty$, and therefore $u = 0$, so that $B = 0$.

Also when $\theta = 0$, $\frac{dr}{dt} = -V$, *i.e.* $\frac{du}{dt} = Vu^2$.

$$\therefore V = \left[\frac{du}{d\theta}\sqrt{2\mu\theta}\right]_{\theta=0} = \left[A \cdot \sqrt{2\mu\theta} \cdot \left(\frac{1}{2}\theta^{-\frac{1}{2}} + \ldots\right)\right]_{\theta=0} = A\sqrt{\frac{\mu}{2}}.$$

$$\therefore u = \xi\theta^{\frac{1}{2}} = \sqrt{\frac{2}{\mu}}\, V \cdot \theta^{\frac{1}{2}} J_{\frac{1}{2}}(\theta).$$

55. $u^2 + \dot{u}^2 = \int \frac{2\lambda}{h^2} \frac{\frac{1}{u} - c}{cu^2}\, du = \frac{1}{c^2} - \frac{\lambda}{ch^2}\left(\frac{1}{u} - c\right)^2,$

so that $$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^2(r^2 - c^2)}{c^2} - \frac{\lambda}{ch^2} r^4 (r - c)^2.$$

Now $\dot{r} = 0$ when $r = c + c'$, so that $\frac{\lambda}{ch^2} = \frac{2c + c'}{c^2 c' (c + c')^2}$.

Put $$\theta = \sqrt{\frac{2c'}{c}}\,\phi, \text{ and } r = c + c'\xi.$$

$\therefore \frac{cc'}{2}\left(\frac{d\xi}{d\phi}\right)^2 = 2cc'\xi - 2cc'\xi^2$, where the squares of c' are omitted, since c' is small, the string being only slightly elastic.

$\therefore 2\phi = \int \frac{d\xi}{\sqrt{\xi - \xi^2}} = \sin^{-1}(2\xi - 1) + \frac{\pi}{2}$. $\therefore \xi = \frac{1}{2}[1 - \cos 2\phi] = \sin^2\phi$, etc.

56. $\ddot{x} - x\omega^2 = -g\cos\omega t$.

$$\therefore x = \frac{g}{2\omega^2}\cos\omega t + A\cosh(\omega t + B) = \frac{g}{2\omega^2}[\cos\omega t + n\cosh\omega t],$$

where $l = \frac{g}{\omega^2}(1 + n)$; for, when $t = 0$, $x = \frac{l}{2}$ and $\dot{x} = 0$.

Initially, $\ddot{x} = \frac{l}{2}\omega^2 - g$ and is therefore positive, and the particle at once leaves the tube, if $l > \frac{2g}{\omega^2}$.

Case I. $l > \frac{g}{\omega^2}$ and $< \frac{2g}{\omega^2}$, *i.e.* n positive and < 1.

Then $x = \frac{l}{2}$ when $1 + n = \cos\omega t + n\cosh\omega t$, *i.e.* when $\sin^2\frac{\omega t}{2} = n\sinh^2\frac{\omega t}{2}$. This always has a real root, where $\omega t < 2\pi$, if n is positive, as is clear from a graph.

It is also clear from a graph that $\dot{x}$ vanishes for a value of ωt between 0 and π, and then becomes positive and remains so at least until $\omega t = 2\pi$. The particle thus leaves at the end where it came in.

Case II. $l < \frac{g}{\omega^2}$, *i.e.* n negative and numerically < 1.

Then $$x = -\frac{l}{2} \text{ when } -1 - n = \cos\omega t + n\cosh\omega t,$$

i.e. when $$0 = \cos^2\frac{\omega t}{2} + n\cosh^2\frac{\omega t}{2}.$$

This has a root for ωt, which is $<\pi$, since n is negative. Also it is easily seen that $\dot{x}$ is then negative. The particle thus goes out at the other end of the tube.

Case III. $l=\frac{g}{\omega^2}$. Here $n=0$ and $x=\frac{l}{2}\cos\omega t$, *i.e.* the particle continually oscillates about the centre of the tube, its velocity just vanishing when it reaches the ends of the tube.

57. When the string has turned through an angle θ, so that the part unwound $s=a\theta+b$, we have as in Ex. 25, Page 115,

$$-s\dot{\theta}^2=-\frac{T}{m}, \text{ and } s\ddot{\theta}+\dot{s}\dot{\theta}=0, \ i.e.\ s\dot{\theta}=\text{const.}=V.$$

$$\therefore\ (a\theta+b)\,\dot{\theta}=V, \text{ so that } (a\theta+b)^2=2Vat+b^2.$$

The string is completely unwound when $\theta=\pi$, *i.e.* when $t=\frac{\pi^2 a+2\pi b}{2V}$.

Also
$$T=m\,(a\theta+b)\,\dot{\theta}^2=\frac{mV^2}{\sqrt{b^2+2Vat}}.$$

58. $a\ddot{\theta}=\omega^2 a\sin\theta\,.\cos\theta-g\sin\theta.$ ....(1)

When θ is small, this gives $a\ddot{\theta}=(\omega^2 a-g)\,\theta$. Hence $\ddot{\theta}$ is positive, and therefore the equilibrium unstable, if $a\omega^2>g$.

Integrating (1), we have

$$a\dot{\theta}^2=\omega^2 a\sin^2\theta+2g\cos\theta-2g.$$

$\dot{\theta}$ is zero again when $a\,(1+\cos\theta_1)=\frac{2g}{\omega^2}$, etc.

The work done against gravity $=mg\,(a-a\cos\theta_1)=2mga\left(1-\frac{g}{\omega^2 a}\right)$.

The total work done $=2mga\left(1-\frac{g}{\omega^2 a}\right)+\int_0^{\theta_1} m\omega^2 a\sin\theta\,.\,d\,(a\sin\theta)$

$$=2mga\left(1-\frac{g}{\omega^2 a}\right)+\frac{m\omega^2 a^2}{2}\sin^2\theta_1=4mga\left(1-\frac{g}{\omega^2 a}\right).$$

59. In the work of Art. 110, we have $\phi=90°$ nearly, and we can replace s by x, so that

$$v\frac{dv}{dx}=-\mu v^2, \text{ and } v^2=g\rho=-g\div\frac{d^2y}{dx^2} \text{ approx.}$$

$$\therefore\ v=Ve^{-\mu x}, \text{ and } \frac{d^2y}{dx^2}=-\frac{g}{V^2}e^{2\mu x}. \quad \therefore\ \frac{dy}{dx}=\frac{-g}{2\mu V^2}\left[e^{2\mu x}-1\right]+\tan\alpha.$$

$$\therefore\ y=x\left(\tan\alpha+\frac{g}{2\mu V^2}\right)+\frac{g}{4\mu^2 V^2}(1-e^{2\mu x})=\text{etc., on expansion.}$$

60. $\ddot{x}=-\mu vg\cos\alpha\cos\psi-\mu vg\sin\alpha\sin\psi=-\mu g\,[\cos\alpha\dot{x}+\sin\alpha\dot{y}]$,

and $\ddot{y}=\mu vg\sin\alpha\cos\psi-\mu vg\cos\alpha\sin\psi-g=\mu g\,[\sin\alpha\dot{x}-\cos\alpha\dot{y}]-g.$

$\therefore\ \dot{x}=-\mu g\,(x\cos\alpha+y\sin\alpha)+V\cos\beta,$ ....(1)

and $\dot{y}=\mu g\,(x\sin\alpha-y\cos\alpha)-gt+V\sin\beta.$ ....(2)

$$\therefore\ (D+\mu g\cos\alpha)\,x+\mu g\sin\alpha y=V\cos\beta,$$

and
$$-\mu g\sin\alpha x+(D+\mu g\cos\alpha)\,y=V\sin\beta-gt.$$

$$\therefore\ (D^2+2\mu gD\cos\alpha+\mu^2 g^2)\,x=\mu gV\cos(\alpha+\beta)+\mu g^2\sin\alpha t.$$

$\therefore\ x = e^{-\mu g \cos\alpha . t}[A \cos(\mu g \sin \alpha t) + B \sin(\mu g \sin \alpha t)]$

$$+\frac{F}{mg}\cos(\alpha+\beta) + \frac{t \sin\alpha}{\mu} - \frac{2 \sin\alpha \cos\alpha}{\mu^2 g}.$$

Substituting in (1) we have y. The arbitrary constants can be found since x and y both vanish with t.

61. Let O be the origin, and $OC = k$. Then

$$\ddot{x} = -\lambda(x-k) - \mu\dot{x}, \text{ so that } x - k = Ae^{-\frac{\mu t}{2}} \cos\left[\sqrt{\lambda - \frac{\mu^2}{4}}\, t + B\right],$$

as in Art. 117, and, as in that article, the particle is at rest when

$$\sqrt{\lambda - \frac{\mu^2}{4}}\, t + B = \alpha,\ \pi + \alpha,\ 2\pi + \alpha,\ \dots, \text{ etc.},$$

so that $a - k = Ae^{-\frac{\mu}{2}t_1} \cos\alpha$, $b - k = -Ae^{-\frac{\mu}{2}t_2} \cos\alpha$, and $c - k = Ae^{-\frac{\mu}{2}t_3} \cos\alpha$.

$$\therefore\ \frac{b-k}{a-k} = \frac{c-k}{b-k} = \frac{d-k}{c-k}, \text{ where } d \text{ is the required distance.}$$

$$\therefore\ OC = k = \frac{ac - b^2}{a + c - 2b}, \text{ and } d = \frac{ac + bc - b^2 - c^2}{a - b}.$$

62. $v\dfrac{dv}{ds} = \dfrac{dv}{dt} = -kv^3$. Hence $\dfrac{1}{v} = ks + A = ks + \dfrac{1}{u}$,

and $\dfrac{1}{v^2} = 2kt + B = 2kt + \dfrac{1}{u^2}$. $\therefore\ v = \dfrac{u}{1 + ksu}$, and $2kt = \left(\dfrac{1+ksu}{u}\right)^2 - \dfrac{1}{u^2} =$ etc.

If $u = 2400$, $v = 2350$, and $s = 300$, then $k = \dfrac{1}{47 \times 300 \times 2400}$.

Hence when $s = 3000$, $t = \dfrac{3000}{2400} + \dfrac{1}{2}\dfrac{3000^2}{47 \times 300 \times 2400} = 1\frac{18}{47} = 1{\cdot}4$ secs.

63. At time t let the string have moved through x, and the insect have crawled a distance y along the string. Then

$$\frac{d}{dt}\left[\left(M - M\frac{y}{l}\right)\dot{x}\right] = F,\ m(\ddot{x} - \ddot{y}) = -F, \text{ and } \dot{y} = u, \text{ so that } y = ut.$$

$$\therefore\ \left(M - M\frac{y}{l}\right)\dot{x} + m\dot{x} = mu, \text{ since initially } (M+m)\dot{x} = mu.$$

$$\therefore\ x = \int_0^{\frac{l}{u}} \frac{mu\,dt}{M + m - M\frac{u}{l}t} = -\frac{ml}{M}\left[\log\left(M + m - M\frac{u}{l}t\right)\right]_0^{\frac{l}{u}} = \text{etc.}$$

64. $\dfrac{d}{dt}\left[\left\{P + \dfrac{2P}{l}(l - x)\right\}\dot{x}\right] = \left\{\dfrac{2P}{l}(l - x) - P\right\}g.$

$$\therefore\ (3l - 2x)\ddot{x} - 2\dot{x}^2 = g(l - 2x).$$

$$\therefore\ \dot{x}^2(3l - 2x)^2 = \int 2g(l - 2x)(3l - 2x)\,dx = 2g\left[3l^2x - 4lx^2 + \frac{4x^3}{3}\right]$$

$$= \frac{2gx}{3}(3l - 2x)^2, \text{ i.e. } \dot{x}^2 = \frac{2g}{3}x, \text{ etc.}$$

65. If v is the velocity of the element which is inclined at ψ to the horizon, then $mv\frac{dv}{ds}=\frac{dT}{ds}-mg\sin\psi$, and $m\frac{v^2}{\rho}=\frac{T}{\rho}-mg\cos\psi$. ......(1)

If v is constant, these give $mg\sin\psi=\frac{d}{ds}[mg\rho\cos\psi+mv^2]$.

$$\therefore\ \rho\sin\psi=\frac{d}{d\psi}(\rho\cos\psi).\quad \therefore\ \frac{1}{\rho}\frac{d\rho}{d\psi}=2\tan\psi,\ i.e.\ \rho\cos^2\psi=\text{const.},$$

i.e. $s=c\tan\psi$, a catenary whose shape does not depend on v.

Also, from (1), if the velocity changes from v_1 to v_2, the tension is increased by $m(v_2^2-v_1^2)$, etc.

66. If v is constant and there are no external forces, then

$$0=\frac{dT}{ds}\text{ and }m\frac{v^2}{\rho}=\frac{T}{\rho},\text{ so that }T=mv^2,$$

which is independent of the particular curve in which the chain runs.

67. If V be the velocity in the steady motion, then

$$\frac{V^2}{b\sin\alpha}=\frac{R}{m}\cos\psi=g\cot\psi=g\cdot\frac{a}{b}\tan\alpha,\text{ so that }V^2=ga\frac{\sin^2\alpha}{\cos\alpha}.$$

When the particle is at a point whose eccentric angle is θ, the equations of Art. 133 give

$$\rho^2\dot\phi=V.b\sin\alpha\text{ and }\dot s^2+\rho^2\dot\phi^2=V^2+2ga(\cos\theta-\cos\alpha),$$

i.e. $$(a^2\sin^2\theta+b^2\cos^2\theta)\dot\theta^2=V^2\left(1-\frac{\sin^2\alpha}{\sin^2\theta}\right)+2ga(\cos\theta-\cos\alpha).$$

Hence, on differentiation,

$$(a^2\sin^2\theta+b^2\cos^2\theta)\ddot\theta+2\dot\theta^2(a^2-b^2)\sin\theta\cos\theta=\frac{V^2\sin^2\alpha\cos\theta}{\sin^3\theta}-ga\sin\theta.$$

For a small oscillation put $\theta=\alpha+\phi$, where ϕ is small, and neglect $\dot\phi^2$; then

$$(a^2\sin^2\alpha+b^2\cos^2\alpha)\ddot\phi=ga[\sin\alpha(1+\phi\cot\alpha)^{-3}(1-\phi\tan\alpha)-\sin\alpha-\phi\cos\alpha]$$
$$=-\frac{ga}{\cos\alpha}(1+3\cos^2\alpha)\phi.\quad\text{Hence, etc.}$$

68. Reduce m to rest by giving to the whole system the acceleration $\frac{T}{m}$ along the line joining the particles.

In the first case $\ddot\xi=-\frac{\lambda}{m}\frac{\xi-a}{a}-\frac{\lambda}{m'}\frac{\xi-a}{a}$ so that $n^2=\frac{\lambda}{a}\frac{m+m'}{mm'}$.

In the second case $\ddot r-r\dot\theta^2=-\lambda\frac{m+m'}{mm'}\frac{r-a}{a}=-n^2(r-a)$,

and $$r^2\dot\theta=\text{const.}=b^2\omega,\text{ where }-b\omega^2=-n^2(b-a).$$

$$\therefore\ \ddot r-\frac{b^4}{r^3}\omega^2=-n^2(r-a).$$

Put $r=b+\xi$, where ξ is small. Then

$$\ddot\xi=b\omega^2\left(1-\frac{3\xi}{b}\right)-n^2(b-a+\xi)=-(3\omega^2+n^2)\xi.$$

$$\therefore\ \text{required period}=2\pi\div\sqrt{n^2+3\omega^2}.$$

69. $\frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2+\frac{1}{2}ex^2\right]=-\frac{1}{2}k\dot{x}^2. \quad \therefore \; m\ddot{x}+\frac{1}{2}k\dot{x}+ex=0.$

Put $x=e^{pt}$, and we have $p=-\frac{k}{4m}\pm i\sqrt{\frac{e}{m}-\frac{k^2}{16m^2}}$.

$$\therefore \; x=Ae^{-\frac{kt}{4m}}\cos\left[\sqrt{\frac{e}{m}-\frac{k^2}{16m^2}}\cdot t+B\right].$$

Hence the period is as stated.

For the forced oscillation, $m\ddot{x}+\frac{1}{2}k\dot{x}+ex=A\cos pt$.

$$\therefore \; x=\frac{1}{mD^2+\frac{1}{2}kD+e}\cdot A\cos pt=\frac{(e-mp^2)-\frac{1}{2}kD}{(e-mp^2)^2+\frac{1}{4}k^2p^2}\cdot A\cos pt$$

$$=\frac{A}{\sqrt{(e-mp^2)^2+\frac{1}{4}k^2p^2}}\cos(pt-\epsilon), \quad\text{.................................(1)}$$

where $\frac{\cos\epsilon}{e-mp^2}=\frac{\sin\epsilon}{\frac{1}{2}kp}=\frac{1}{\sqrt{(e-mp^2)^2+\frac{1}{4}k^2p^2}}$.

The denominator of (1) is least when $-4mp(e-mp^2)+\frac{k^2}{2}p=0$,

i.e. when $p^2=\frac{e}{m}-\frac{k^2}{8m^2}$, and then it $=\sqrt{\frac{k^2}{4}\left(\frac{e}{m}-\frac{k^2}{16m^2}\right)}=\frac{k\pi}{\tau}$,

and the amplitude of the oscillation $=\frac{A\tau}{k\pi}$.

Also $\tan\epsilon=\frac{1}{2}kp\div(e-mp^2)=\frac{1}{2}kp\div\left(\frac{k^2}{8m}\right)=\frac{4mp}{k}$.

70. At time t, let the ring m have moved through x, and the string be inclined at θ to the horizontal.

Since there is no horizontal force acting on the system, the total horizontal momentum is constant. Hence

$$m\dot{x}+m'(\dot{x}-l\sin\theta\dot{\theta})=0, \quad i.e.\; (m+m')\dot{x}=m'l\sin\theta\dot{\theta}. \quad\text{.........(1)}$$

Also the Principle of Energy gives

$$\frac{1}{2}m\dot{x}^2+\frac{1}{2}m'[(\dot{x}-l\dot{\theta}\sin\theta)^2+l^2\dot{\theta}^2\cos^2\theta]=m'gl\sin\theta,$$

i.e. $(m+m')\dot{x}^2-2m'\dot{x}l\dot{\theta}\sin\theta+m'l^2\dot{\theta}^2=2m'gl\sin\theta$,

or, by (1), $\dot{\theta}^2(m+m'\cos^2\theta)=\frac{2g}{l}(m+m')\sin\theta. \quad\text{..................(2)}$

The greatest value of $\dot{\theta}$ is thus as given.

Differentiate (2), put $\theta=90°-\phi$, where ϕ is small, and neglect $\dot{\phi}^2$, and we have $\ddot{\phi}=-\frac{m+m'}{m}\frac{g}{l}\phi$, etc.

71. $m\ddot{x}=mg-\lambda_1\frac{x-l}{l}$, so that $\frac{\lambda_1}{ml_1}=p_1^2$ and similarly $\frac{\lambda_2}{m'l_2}=p_2^2$.

When both masses are free, let x and $x+\xi$ be their depths at time t.

Then $m\ddot{x}=\lambda_2\frac{\xi-l_2}{l_2}-\lambda_1\frac{x-l_1}{l_1}+mg=m'p_2^2(\xi-l_2)-mp_1^2(x-l_1)+mg$,

and $m'(\ddot{x}+\ddot{\xi})=m'g-m'p_2^2(\xi-l_2)$.

The equilibrium positions (x_0 and ξ_0) are given by

$$0 = mg + m'p_2^2(\xi_0 - l_2) - mp_1^2(x_0 - l_1), \text{ and } 0 = m'g - m'p_2^2(\xi_0 - l_2).$$

Put $x = x_0 + y$ and $\xi = \xi_0 + \eta$, and we have

$$m\ddot{y} = m'p_2^2\eta - mp_1^2 y, \text{ and } \ddot{y} + \ddot{\eta} = -p_2^2\eta,$$

i.e. $$(D^2 + p_1^2)y - \frac{m'}{m}p_2^2\eta = 0, \text{ and } D^2y + (D^2 + p_2^2)\eta = 0.$$

$$\therefore \left\{(D^2 + p_1^2)(D^2 + p_2^2) + \frac{m'}{m}p_2^2 D^2\right\} y = 0.$$

If we put $y = A\cos(nt + B)$, we have the given equation for n.

72. $\ddot{r} - r\dot{\theta}^2 = -P - R\dfrac{dr}{ds}$, and $\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right) = -R\cdot\dfrac{r\,d\theta}{ds} = -\dfrac{R}{v}\cdot r\dot{\theta}$.

$$\therefore r^2\dot{\theta} = h_0 e^{-\int \frac{R}{v}dt} = h.$$

Hence $$\dot{\theta} = hu^2, \quad \dot{r} = -\frac{du}{d\theta}h,$$

and $$\ddot{r} = -\frac{d^2u}{d\theta^2}\cdot h^2u^2 - \frac{du}{d\theta}\frac{dh}{dt} = -h^2u^2\frac{d^2u}{d\theta^2} + \frac{R}{v}h\frac{du}{d\theta}.$$

Hence the first equation gives

$$-h^2u^2\frac{d^2u}{d\theta^2} + \frac{Rh}{v}\frac{du}{d\theta} - h^2u^3 = -P - \frac{R}{v}\cdot\left(-h\frac{du}{d\theta}\right), \text{ etc.}$$

73. $P - \lambda(V + U\sin nt)^2 = \dfrac{dv}{dt} = nU\cos nt$.

$$\therefore \text{Work done} = \int P\,ds = \int_0^{\frac{2\pi}{n}} Pv\,dt$$

$$= \int_0^{2\pi}[nU\cos\theta(V + U\sin\theta) + \lambda(V + U\sin\theta)^3]\frac{d\theta}{n},$$

(where $nt = \theta$)

$$= \frac{\lambda}{n}\int_0^{2\pi}[V^3 + 3VU^2\sin^2\theta]\,d\theta = \frac{\lambda}{n}\left[V^3.2\pi + 4\times 3VU^2\times\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{2\pi\lambda V}{n}\left[V^2 + \frac{3}{2}U^2\right].$$

If the velocity were constant, and $= V$, the work done would

$$= \lambda V^2.\,V.\frac{2\pi}{n}, \text{ and the ratio required} = 1 + \frac{3}{2}\frac{U^2}{V^2}.$$

74. $\ddot{x} = -\mu x + k\dot{x}^4$. If k were zero, the solution would be $x = a\cos(\sqrt{\mu}t)$. Put this value in the small term $k\dot{x}^4$ of the above equation, and we have

$$\ddot{x} + \mu x = ka^4\mu^2\sin^4\sqrt{\mu}t = \frac{ka^4\mu^2}{8}[3 - 4\cos(2\sqrt{\mu}t) + \cos(4\sqrt{\mu}t)].$$

$$\therefore x = A\cos[\sqrt{\mu}t + B] + \frac{ka^4\mu}{8}\left[3 + \frac{4}{3}\cos(2\sqrt{\mu}t) - \frac{1}{15}\cos(4\sqrt{\mu}t)\right].$$

The initial conditions give $a = A\cos B + \frac{8}{15}ka^4\mu$, and $0 = A\sin B$.

$\therefore\ x = \left(a - \frac{8}{15}k\mu a^4\right)\cos\sqrt{\mu}t + \frac{k\mu a^4}{8}\left[3 + \frac{4}{3}\cos(2\sqrt{\mu}t) - \frac{1}{15}\cos(4\sqrt{\mu}t)\right]$.

When $x=0$ let $\sqrt{\mu}t = \frac{\pi}{2} + \psi$, where ψ is small, and we have

$$0 = -\left(a - \frac{8}{15}k\mu a^4\right)\psi + \frac{k\mu a^4}{8}\left(3 - \frac{4}{3} - \frac{1}{15}\right),$$

so that $\psi = \frac{k\mu a^3}{5}$, and the increase in the time $= \frac{\psi}{\sqrt{\mu}} = \frac{k\sqrt{\mu}a^3}{5}$.

When $\sqrt{\mu}t = \pi$, $x = -\left(a - \frac{8k\mu a^4}{15}\right) + \frac{k\mu a^4}{8}\left[3 + \frac{4}{3} - \frac{1}{15}\right] = -\left[a - \frac{16k\mu a^4}{15}\right]$,

so that the amplitude of the swing is diminished as stated.

75. $v\frac{dv}{ds} = \frac{dv}{dt} = -kv^{m+1}$.

$\therefore\ kt = -\int\frac{dv}{v^{m+1}} = \frac{1}{m}\left[\frac{1}{v^m} - \frac{1}{u^m}\right]$, and $ks = -\int\frac{dv}{v^m} = \frac{1}{m-1}\left[\frac{1}{v^{m-1}} - \frac{1}{u^{m-1}}\right]$.

If $m = \frac{1}{2}$, $u = 2500$, $v = 1600$, and $t = 1$, then $k = 2\left[\frac{1}{\sqrt{1600}} - \frac{1}{\sqrt{2500}}\right] = \frac{1}{100}$,

and $\frac{s}{100} = -2[\sqrt{1600} - \sqrt{2500}] = 20$, so that $s = 2000$.

76. $v\frac{dv}{ds} = \frac{dv}{dt} = -\mu R$. Also the equations of Art. 125 give

$$\ddot{r} - r\sin^2\alpha\dot{\phi}^2 = -\mu R\cdot\frac{\dot{r}}{v} = \frac{\dot{r}}{v}\frac{dv}{dt}, \qquad \text{......(1)}$$

$$r\sin\alpha\cos\alpha\dot{\phi}^2 = R = -\frac{1}{\mu}\frac{dv}{dt}, \qquad \text{......(2)}$$

and $$\frac{\sin\alpha}{r}\frac{d}{dt}(r^2\dot{\phi}) = -\mu R\cdot\frac{r\sin\alpha\dot{\phi}}{v} = \frac{r\sin\alpha\dot{\phi}}{v}\frac{dv}{dt}. \qquad \text{......(3)}$$

(1) and (2) give $$\frac{\ddot{r}}{v} - \frac{\dot{r}}{v^2}\frac{dv}{dt} = -\frac{\tan\alpha}{\mu v}\frac{dv}{dt},$$

so that $\frac{\dot{r}}{v} = -\frac{\tan\alpha}{\mu}\log\frac{v}{V}$, since $\dot{r} = 0$ when $v = V$.

(3) gives $r^2\dot{\phi} = Av = v\frac{d}{\sin\alpha}$, since initially $r\sin\alpha\dot{\phi} = V$.

$$\therefore\ 1 = \frac{\dot{r}^2 + r^2\sin^2\alpha\dot{\phi}^2}{v^2} = \frac{\tan^2\alpha}{\mu^2}\left(\log\frac{v}{V}\right)^2 + \frac{d^2}{r^2}.$$

$\therefore\ v\frac{dv}{ds} = -\mu R = -\mu r\sin\alpha\cos\alpha\dot{\phi}^2 = -\frac{\mu d^2v^2\cot\alpha}{r^3}$

$$= -\frac{\mu v^2\cot\alpha}{d}\left[1 - \frac{\tan^2\alpha}{\mu^2}\left(\log\frac{v}{V}\right)^2\right]^{\frac{3}{2}}. \quad \left[\text{Put } \log\frac{v}{V} = \frac{\mu}{\tan\alpha}\cdot\sin\theta\right]$$

$$\therefore\ \frac{s}{d} = -\int\frac{d\theta}{\cos^2\theta} = -\tan\theta.$$

$$\therefore\ \log\frac{v}{V} = \frac{\mu}{\tan\alpha}\left(\frac{-s}{\sqrt{s^2+d^2}}\right), \text{ i.e. } \log\frac{V}{v} = \frac{\mu s\cot\alpha}{\sqrt{s^2+d^2}}.$$

77. The equations of Art. 125 give

$$a(\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2) = -\frac{\mu}{8a^2\sin^3\frac{\theta}{2}}\cos\frac{\theta}{2} + \frac{\mu}{8a^3\cos^3\frac{\theta}{2}}\sin\frac{\theta}{2},$$

and $$\frac{1}{\sin\theta}\frac{d}{dt}(a\sin^2\theta\dot{\phi}) = 0, \text{ so that } a\sin^2\theta\dot{\phi} = \frac{1}{a}\sqrt{\mu}.$$

$$\therefore \frac{8a^4}{\mu}\ddot{\theta} - 8\frac{\cos\theta}{\sin^3\theta} = \frac{\sin^4\frac{\theta}{2} - \cos^4\frac{\theta}{2}}{\cos^3\frac{\theta}{2}\sin^3\frac{\theta}{2}} = -\frac{8 \cdot \cos\theta}{\sin^3\theta}.$$

$$\therefore \ddot{\theta} = 0, \text{ and } \dot{\theta} = \text{const. Hence, etc.}$$

78. As in Art. 133, $x^2\dot{\phi} = \text{const.} = \frac{z_0^2}{4a}\sqrt{2kgz_0}$,

and $\dot{s}^2 + x^2\dot{\phi}^2 = 2kgz_0 + 2g(z_0 - z)$. Also $\dot{s}^2 = \dot{x}^2 + \dot{z}^2 = \frac{z^2 + 4a^2}{4a^2}\dot{z}^2$.

Hence $$\frac{z^2 + 4a^2}{4a^2}\dot{z}^2 = 2g(z_0 - z) + 2gkz_0\left(1 - \frac{z_0^4}{z^4}\right).$$

If $k = \frac{1}{4}$, $\dot{z} = 0$, in addition to the starting point, when

$4z^4 = z_0(z^3 + z^2z_0 + zz_0^2 + z_0^3)$, *i.e.* when $(z - z_0)(4z^3 + 3z^2z_0 + 2zz_0^2 + z_0^3) = 0$,
the only solution of which is $z = z_0$.

If $k = \frac{1}{30}$, then $\dot{z} = 0$ again when $30z^4 = z_0(z^3 + z^2z_0 + zz_0^2 + z_0^3)$,
i.e. when $$(2z - z_0)(15z^3 + 7z^2z_0 + 3zz_0^2 + z_0^3) = 0,$$
so that the motion is confined between $z = z_0$ and $z = \frac{1}{2}z_0$.

79. The third equation of Art. 125 gives

$$\frac{R}{m} = \frac{a}{\sin\theta}\frac{d}{dt}(\sin^2\theta\dot{\phi}) = \frac{a\omega}{\sin\theta} \times 2\sin\theta\cos\theta\dot{\theta} = 2\omega\cos\theta V.$$

$$\therefore R = \frac{2V\omega\sin\lambda}{g} \times \text{weight.}$$

80. From Art. 125 we have

$$\ddot{r} - r\sin^2\alpha\dot{\phi}^2 = g\cos\alpha, \quad -r\sin\alpha\cos\alpha\dot{\phi}^2 = \frac{S}{m} - g\sin\alpha,$$

and $$\frac{1}{r\sin\alpha}\frac{d}{dt}(r^2\sin^2\alpha\dot{\phi}) = 0, \text{ so that } r^2\sin^2\alpha\dot{\phi} = \sqrt{2gh}\,R\sin\alpha.$$

$\therefore \ddot{r} = g\cos\alpha + \frac{2ghR^2}{r^3} =$ positive always, so that r continually increases.

Also $\frac{S}{m} = g\sin\alpha - \frac{2ghR^2\cot\alpha}{r^3}$. Hence S increases with r.

It is positive originally if $h < \frac{1}{2}R\sin\alpha\tan\alpha$.

81. As in the previous question, $r^2\sin^2\alpha\dot{\phi} = R\sin\alpha\sqrt{2gR\cos\alpha}$

Also $\dot{r}^2 + r^2\sin^2\alpha\dot{\phi}^2 = 2gr\cos\alpha$, so that $\dot{r}^2 = \frac{2g\cos\alpha}{r^2}(r^3 - R^3)$.

$$\therefore \frac{dr}{d\phi \cdot \sin\alpha} \cdot \frac{1}{r} = \sqrt{\frac{r^3 - R^3}{R^3}}.$$

If $d\theta$ be the angle between two successive generators, we have

$$r\,d\theta = r\sin\alpha\,.\,d\phi, \text{ so that } d\theta = d\phi\sin\alpha.$$

$$\therefore\ \theta = \int \frac{dr}{r}\sqrt{\frac{R^3}{r^3-R^3}} = \int \frac{1}{3}\cdot\frac{d\xi}{\xi\sqrt{\xi-1}}, \text{ if } \frac{r}{R}=\xi^{\frac{1}{3}},$$

$$\therefore\ 3\theta = 2\tan^{-1}\sqrt{\xi-1}. \quad \therefore\ \xi = \sec^2\frac{3\theta}{2}.$$

$$\therefore\ r = R\sec^{\frac{2}{3}}\left(\frac{3\theta}{2}\right), \text{ i.e. } r^{\frac{3}{2}}\cos\frac{3\theta}{2} = R^{\frac{3}{2}}.$$

82. Let the starting point P be in latitude l, and the initial velocity be V at an angle α to the horizon measured towards the North. Let the axis of z be the line through the centre O of the earth in a northward direction, let Ox be the line in which the plane through the shell and Oz cuts the equator, and let Oy be perpendicular to Ox and Oz, and towards the East.

Then ω being the angular velocity of the earth, and ω^2 being neglected, the equations of Art. 51 give

$$\ddot{x} - 2\omega\dot{y} = -g\cos\lambda = -g\cos l, \text{ nearly}, \quad\quad(1)$$

$$\ddot{y} + 2\omega\dot{x} = 0, \quad\quad(2)$$

and

$$\ddot{z} = -g\sin\lambda = -g\sin l. \quad\quad(3)$$

Now $$[\dot{x}]_0 = V\cos\left(l+\frac{\pi}{2}-\alpha\right) = V\sin(\alpha - l),$$

$$[\dot{y}]_0 = 0, \text{ and } [\dot{z}]_0 = V\sin\left(l+\frac{\pi}{2}-\alpha\right) = V\cos(\alpha-l).$$

Then (1) gives $\dot{x} - 2\omega y = -g\cos l\,.\,t + V\sin(\alpha - l)$.

Hence (2) is $\ddot{y} = 2\omega g\cos l t - 2V\omega\sin(\alpha - l)$, on neglecting ω^2.

$$\therefore\ \dot{y} = \omega g\cos l t^2 - 2V\omega\sin(\alpha - l)\,.\,t,$$

and $$y = \tfrac{1}{3}\omega g\cos l t^3 - V\omega\sin(\alpha-l)\,t^2.$$

When $t = \dfrac{2V\sin\alpha}{g}$, $y = \dfrac{4V^3\omega\sin^2\alpha}{g^2}\left[\dfrac{2}{3}\sin\alpha\cos l - \sin(\alpha - l)\right]$.

In the first case we are given $\alpha = 45°$ and $l = 45°$, so that $y_1 = \dfrac{2}{3}\dfrac{V^3\omega}{g^2}$.

In the second case, $\alpha = 135°$ and $l = 45°$, so that $y_2 = -\dfrac{4}{3}\dfrac{V^3\omega}{g^2}$.

Now we are given $\dfrac{V^2}{g} = 20$ kilometres $= 2\times 10^6$ cms.

$$\therefore\ \frac{V^3\omega}{g^2} = 2\times10^6\times\sqrt{\frac{2\times10^6}{981}}\times\frac{2\pi}{24\times60\times60}\text{ cms.} = \frac{10^5}{216}\times\pi\times\sqrt{\frac{2}{981}}\text{ metres}$$

$$= \frac{10^3}{216}\times\pi\times\sqrt{20}\text{ metres nearly} = \frac{10^3}{216}\times\frac{22}{7}\times4\tfrac{1}{2}\text{ metres nearly}$$

$$= 65\tfrac{1}{2}\text{ metres nearly.}$$

$$\therefore\ y_1 = \text{about 44 metres towards the East,}$$

and $$y_2 = \text{about 88 metres towards the West.}$$