Chapter 8. Ratio and Proportion

Formulae

1. When two or more ratios are multiplied together, they are said to be compounded.
   Thus if $\frac{a}{b}$ and $\frac{c}{d}$ are any two ratios, then $\frac{ac}{bd}$
   is their compounded ratio.
   $\therefore$ Compounded ratio of $a : b$ and $c : d$ is
   $ac : bd$.
2. A ratio compounded with itself is called duplicate ratio of the given ratio.
   $\therefore$ duplicate ratio of $a : b$ is $a^2 : b^2$.
   Similarly, triplicate ratio of $a : b$ is $a^3 : b^3$.
   Sub-duplicate ratio of $a : b$ is $\sqrt{a} : \sqrt{b}$.
   Sub-triplicate ratio of $a : b$ is $\sqrt[3]{a} : \sqrt[3]{b}$.
3. The reciprocal ratio a : b is b : a.
4. **Proportion:** An equality of two ratios is called a proportion.
   Four (non-zero) quantities a, b, c, d are said to
   be in proportion if $a : b = c : d$ *i.e.*, if $\frac{a}{b} = \frac{c}{d}$.
   We write it as $a : b : : c : d$.
5. The quantities a, b, c and d are called the terms of the proportion; a, b, c and d are the first, second, third and fourth terms respectively. First and fourth terms are called extremes (or extreme terms). Second and third terms are called means (or middle terms).
   If the quantities a, b, c and d are in proportion
   then $\frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc$.
   ⇒ product of extreme terms = product of middle terms.
   Thus, if the four quantities are in proportion then the product of extreme terms = product of middle terms. This is called cross product rule.
6. **Fourth proportional:** If a, b, c and d are in proportion then d is called the fourth proportional.
   if $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} = \frac{e}{f} = \ldots$
7. In particular, three (non-zero) quantities of the same kind, a, b and c are said to be in continued proportion iff the ratio of a to b is equal to the ratio of b to c.

i.e., if $\frac{a}{b} = \frac{b}{c}$.

For example :

2, 4 and 8 are in continued proportion,

since $\frac{2}{4} = \frac{4}{8}$.

8. **First proportional:** If a, b are c are in continued proportion, then a is called the first proportional.
   Third proportional: If a, b and c are in continued proportion, then c is called the third proportional.
   Mean proportional: If a, b and c are in continued proportion, then b is called the mean proportional of a and c.
   Thus, if b is the mean proportional of a and c, then
   $\frac{a}{b} = \frac{b}{c} \Rightarrow b^2 = ac \Rightarrow b = \sqrt{ac}.$
   Hence, the mean proportion between two numbers is the positive square root of their product.

Properties of Ratio & Proportion:

If $\frac{a}{b} = \frac{c}{d} \Rightarrow$

(i) $\frac{b}{a} = \frac{d}{c}$ By Invertendo

(ii) $\frac{a}{c} = \frac{b}{d}$ By Alternendo

(iii) $\frac{a+b}{b} = \frac{c+d}{d}$ By Componendo

(iv) $\frac{a-b}{b} = \frac{c-d}{d}$ By Dividendo

(v) $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

By Componendo and Dividendo

(vi) $\frac{a}{a-b} = \frac{c}{c-d}$ By Convertendo

(vii) If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then each ratio.

$$= \frac{a+c+e}{b+d+f}$$

$$= \frac{\text{sum of antecedents}}{\text{sum of consequents}}$$

Determine the Following

Question 1. Which is greater 4 : 5 or 19 : 25.

Solution : Let $a = 4, b = 5, c = 19$ and $d = 25$

then $ad = 4 \times 25 = 100$

$bc = 5 \times 19 = 95$

Here $100 > 95 \Rightarrow ad > bc$ so $\frac{a}{b} > \frac{c}{d}$

$\Rightarrow$ 4 : 5 is greater.

Hence, $\frac{4}{5}$ is greater.

Question 2. Arrange 5 : 6, 8 : 9, 13 : 18 and 7 : 12 in ascending order of magnitude.

Solution : The given ratios are 5 : 6, 8 : 9, 13 : 18 and 7 : 12.

Now L.C.M., of 6, 9, 18 and 12 = 36

Now $\frac{5}{6} = \frac{5 \times 6}{6 \times 6} = \frac{30}{36}$

$\frac{8}{9} = \frac{8 \times 4}{9 \times 4} = \frac{32}{36}$

$\frac{13}{18} = \frac{13 \times 2}{18 \times 2} = \frac{26}{36}$

Also $\frac{7}{12} = \frac{7 \times 3}{12 \times 3} = \frac{21}{36}$

Here $\frac{21}{36} < \frac{26}{36} < \frac{30}{36} < \frac{32}{36}$

Hence $\frac{7}{12} < \frac{13}{18} < \frac{5}{6} < \frac{8}{9}$

i.e., Ascending order

$7 : 12 < 13 : 18 < 5 : 6 < 8 : 9.$ Ans.

Question 3. Find:

(i) The sub-triplicate ratio of the sub-duplicate ratio of $729x^{18} : 64y^6$.

(ii) The sub-duplicate ratio of the sub-triplicate ratio of $4096x^6 : 729y^{12}$.

Solution : (i) The sub-duplicate ratio of $729x^{18} : 64y^6$

$= \sqrt{729x^{18}} : \sqrt{64y^6} = 27x^9 : 8y^3$

$\therefore$ The sub-triplicate ratio of $27x^9 : 8y^3$

$= \sqrt[3]{27x^9} : \sqrt[3]{8y^3}$

$= 3x^3 : 2y.$ Ans.

(ii) The sub-triplicate ratio of $4096x^6 : 729y^{12}$

$= \sqrt[3]{4096x^6} : \sqrt[3]{729y^{12}} = 16x^2 : 9y^4$

$\therefore$ The sub-duplicate ratio of $16x^2 : 9y^4$

$= \sqrt{16x^2} : \sqrt{9y^4}$

$= 4x : 3y^2.$ Ans.

Question 4. If 3x – 2y – 7z = 0 and 2x + 3y – 5z = 0 find x : y : z.

Solution : Here,

$$3x - 2y - 7z = 0$$
$$2x + 3y - 5z = 0$$

Using method of cross multiplication

$$\therefore \quad \frac{x}{(-2)(-5) - 3(-7)} = \frac{y}{(-7)(2) - (3)(-5)} = \frac{z}{(3)(3) - (2)(-2)}$$

$$\Rightarrow \quad \frac{x}{10 + 21} = \frac{y}{-14 + 15} = \frac{z}{9 + 4}$$

$$\Rightarrow \quad \frac{x}{31} = \frac{y}{1} = \frac{z}{13}$$

$$\Rightarrow \quad x : y : z = 31 : 1 : 13.$$

Question 5. Two numbers are in the ratio of 3 : 5. If 8 is added to each number, the ratio becomes 2 : 3. Find the numbers.

Solution : Let the numbers be $3x$ and $5x$

$$3x + 8 : 5x + 8 = 2 : 3$$

$$\frac{3x + 8}{5x + 8} = \frac{2}{3}$$

$$\Rightarrow \quad 2(5x + 8) = 3(3x + 8)$$

$$\Rightarrow \quad 10x + 16 = 9x + 24$$

$$\Rightarrow \quad 10x - 9x = 24 - 16$$

$$\Rightarrow \quad x = 8$$

$\therefore$ The numbers are 24 and 40.

Question 6. If x : y = 2 : 3, find the value of (3x + 2y) : (2x + 5y).

Solution : We have

$$\frac{x}{y} = \frac{2}{3}$$

Now $$\frac{3x + 2y}{2x + 5y} = \frac{y\left(\frac{3x}{y} + 2\right)}{y\left(\frac{2x}{y} + 5\right)}$$

(Both numerator and denominator are divided by y)

$$= \frac{3 \times \frac{2}{3} + 2}{2 \times \frac{2}{3} + 5} = \frac{4}{\frac{19}{3}}$$

$$= \frac{4 \times 3}{19} = 12 : 19. \qquad \text{Ans.}$$

Question 7. Divide Rs. 720 between Sunil, Sbhil and Akhil. So that Sunil gets 4/5 of Sohil’s and Akhil’s share together and Sohil gets 2/3 of Akhil’s share.

Solution : Let Akhil's share be ₹ x then

$$\text{Sohil's share} = ₹\ \frac{2}{3}x$$

$$\text{and Sunil's share} = ₹\ \frac{4}{5}\left(\frac{2}{3}x + x\right)$$

$$= ₹\ \frac{4}{3}x.$$

According to the question

$$\frac{4}{3}x + \frac{2}{3}x + x = 720$$

$$\Rightarrow \quad \frac{4x + 2x + 3x}{3} = 720$$

$$\Rightarrow \quad 9x = 720 \times 3$$

$$\Rightarrow \quad x = \frac{720 \times 3}{9} = 240$$

∴ Their respective shares are ₹ 320, ₹ 160 and ₹ 240. Ans.

Question 8. The ratio between two numbers is 3 : 4. If their L.C.M., is 180. Find the numbers.

Solution : Let the required numbers be $3x$ and $4x$.

The L.C.M., of $3x$ and $4x = 12x$

then $\quad 12x = 180 \Rightarrow x = 15$

Hence, the required numbers are

$$3x = 3 \times 15 = 45$$

$$4x = 4 \times 15 = 60.$$

Ans.

Question 9. If $(x - 9) : (3x + 6)$ is the duplicate ratio of 4 : 9, find the value of x.

Solution : As given $(x - 9) : (3x + 6)$ is duplicate ratio of 4 : 9.

i.e.,
$$\frac{x-9}{3x+6} = \left(\frac{4}{9}\right)^2$$

$$\frac{x-9}{3x+6} = \frac{16}{81}$$

$$\Rightarrow \quad 81(x - 9) = 16(3x + 6)$$

$$81x - 729 = 48x + 96$$

$$81x - 48x = 96 + 729$$

$$33x = 825$$

$$x = \frac{825}{33} = 25$$

Thus required value of x is 25. Ans.

Question 10. Find the compound ratio of the following:

(i) If A : B = 4 : 5, B : C = 6 : 7 and C : D = 14 : 15. Find A : D.

(ii) If P : Q = 6 : 7, Q : R = 8 : 9 find P : Q : R.

(iii) $(a+b):(a-b)$, $a^2+b^2 : (a+b)^2$ and $(a^2-b^2)^2 : (a^4-b^4)$.

Solution : (i) Given $A : B = 4 : 5$, $B : C = 6 : 7$ and $C : D = 14 : 15$

or $$\frac{A}{B} = \frac{4}{5}, \frac{B}{C} = \frac{6}{7}, \frac{C}{D} = \frac{14}{15}$$

Multiply all $$\frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} = \frac{4}{5} \times \frac{6}{7} \times \frac{14}{15}$$

$$\frac{A}{D} = \frac{16}{25}$$

$\therefore$ A : D = 16 : 25 Ans.

(ii) $P : Q = 6 : 7$ and $Q : R = 8 : 9$

$$\frac{P}{Q} = \frac{6}{7} \times \frac{8}{8}, \frac{Q}{R} = \frac{8}{9} \times \frac{7}{7}$$

$$\frac{P}{Q} = \frac{48}{56}, \frac{Q}{R} = \frac{56}{63}$$

$\therefore$ P : Q : R = 48 : 56 : 63 Ans.

(iii) The compound ratios

$$= \frac{(a+b)}{(a-b)} \times \frac{(a^2+b^2)}{(a+b)^2} \times \frac{(a^2-b^2)^2}{(a^4-b^4)}$$

$$= \frac{(a+b)(a^2+b^2)(a^2-b^2)(a^2-b^2)}{(a-b)(a+b)(a+b)(a^2-b^2)(a^2+b^2)}$$

$$= \frac{(a-b)(a+b)}{(a-b)(a+b)} = 1.$$ Ans.

Question 11. Find:

(i) The duplicate ratio of 7 : 9

(ii) The triplicate ratio of 3 : 7

(iii) The sub-duplicate ratio of 256 : 625

(iv) The sub-triplicate ratio of 216 : 343

(v) The reciprocal ratio of 8 : 15.

Solution : (i) The duplicate ratio of $7 : 9 = 7^2 : 9^2$

$= 49 : 81$. Ans.

(ii) The triplicate ratio of $3 : 7 = 3^3 : 7^3$

$= 27 : 343$. Ans.

(iii) The sub-duplicate ratio of 256 : 625

$= \sqrt{256} : \sqrt{625}$

$= 16 : 25$. Ans.

(iv) The sub-triplicate ratio of 216 : 343

$= \sqrt[3]{216} : \sqrt[3]{343}$

$= 6 : 7$. Ans.

(v) The reciprocal are in the ratio of

8 : 15 = 15 : 8. Ans.

Question 12. If $\frac{x^2+y^2}{x^2-y^2}=\frac{17}{8}$, then find the value of :

(i) $x : y$.

(ii) $\frac{x^3+y^3}{x^3-y^3}$.

Solution : (i) $\frac{x^2+y^2}{x^2-y^2} = \frac{17}{8}$

Applying componendo and dividend rule,

$$\frac{x^2+y^2+x^2-y^2}{x^2+y^2-x^2+y^2} = \frac{17+8}{17-8}$$

$$\frac{2x^2}{2y^2} = \frac{25}{9}$$

$$\frac{x^2}{y^2} = \frac{25}{9}$$

$$\frac{x}{y} = \frac{5}{3}$$

$$x : y = 5 : 3. \qquad \text{Ans.}$$

(ii) $\frac{x}{y} = \frac{5}{3}$

Taking cube on both sides,

$$\frac{x^3}{y^3} = \frac{125}{27}$$

Applying componendo and dividendo rule,

$$\frac{x^3+y^3}{x^3-y^3} = \frac{125+27}{125-27}$$

$$\frac{x^3+y^3}{x^3-y^3} = \frac{152}{98}. \qquad \text{Ans.}$$

Question 13. (i) What number must be added to each of the numbers 6, 15, 20 and 43 to make them proportional ?

(ii) What least number must be added to each of the numbers 5, 11, 19 and 37, so that they are in proportion ?

Solution : (i) Let no. be 'x'

$$\therefore \quad \frac{6+x}{15+x} = \frac{20+x}{43+x}$$

$$\Rightarrow \quad (6+x)(43+x) = (20+x)(15+x)$$

$$\Rightarrow \quad 258 + 49x + x^2 = 300 + 35x + x^2$$

$$\Rightarrow \quad 14x = 42$$

$$x = 3$$

$\therefore$ The required no. is 3. Ans.

(ii) Let x be the number added to 5, 11, 19, 37.

$$\frac{5+x}{11+x} = \frac{19+x}{37+x}$$

App. comp. and divi.

$$\frac{5+x+11+x}{5+x-11-x} = \frac{19+x+37+x}{19+x-37-x}$$

$$\frac{16+2x}{-6} = \frac{56+2x}{-18}$$

$$3(16+2x) = 56+2x$$

$$6x-2x = 56-48$$

$$4x = 8$$

$$x = 2.$$ Ans.

Question 14. What quantity must be added to each term of the ratio $a+b : a-b$ to make it equal to $(a+b)^2 : (a-b)^2$?

Solution : Let the quantity to be added be x. Then

$$\frac{(a+b)+x}{(a-b)+x} = \frac{(a+b)^2}{(a-b)^2}$$

$$\Rightarrow (a+b)(a-b)^2 + (a-b)^2.x = (a+b)^2(a-b) + (a+b)^2.x$$

$$\Rightarrow [(a+b)^2 - (a-b)^2]x = (a^2-b^2)(a-b) - (a^2-b^2)(a+b)$$

$$\Rightarrow (4abx) = (a^2-b^2)[(a-b)-(a+b)]$$

$$\Rightarrow \quad x = \frac{-2b(a^2-b^2)}{4ab} = \frac{b^2-a^2}{2a}.$$ Ans.

Question 15. The work done by (x – 3) men in (2x + 1) days and the work done by (2x + 1) men in (x + 4) days are in the ratio of 3 : 10. Find the value of x.

Solution : $(x-3)$ men do a work in $(2x+1)$ day

$\therefore$ 1 man does it in $(2x+1)(x-3)$ days

$(2x+1)$ men do a work in $(x+4)$ days

$\therefore$ 1 man does it in $(x+4)(2x+1)$ days

$$\therefore \quad \frac{(2x+1)(x-3)}{(x+4)(2x+1)} = \frac{3}{10}$$

$$\frac{x-3}{x+4} = \frac{3}{10}$$

$$10x-30 = 3x+12$$

$$10x-3x = 12+30$$

$$7x = 42$$

$$x = \frac{42}{7}$$

$$x = 6. \qquad \text{Ans.}$$

Question 16. Find the third proportional to:

(i) $x-y, x^2-y^2$ (ii) $\frac{a}{b}+\frac{b}{c}, \sqrt{a^2+b^2}$.

Solution : (i) Let A be the third proportional then

$$(x-y):(x^2-y^2) = (x^2-y^2):\text{A}$$

$$\Rightarrow \quad \frac{x-y}{x^2-y^2} = \frac{x^2-y^2}{\text{A}}$$

$$\Rightarrow \quad \text{A} = \frac{(x^2-y^2)^2}{x-y}$$

$$\Rightarrow \quad \text{A} = (x+y)(x^2-y^2). \qquad \text{Ans.}$$

(ii) Let x be the third proportional then

$$\frac{a}{b}+\frac{b}{a}:\sqrt{a^2+b^2} = \sqrt{a^2+b^2}:x$$

$$\Rightarrow \quad \frac{a^2+b^2}{ab}:\sqrt{a^2+b^2} = \sqrt{a^2+b^2}:x$$

$$\Rightarrow \quad \frac{a^2+b^2}{ab\sqrt{a^2+b^2}} = \frac{\sqrt{a^2+b^2}}{x}$$

$$\Rightarrow \quad x = \frac{ab(a^2+b^2)}{(a^2+b^2)}$$

$$\Rightarrow \quad x = ab. \qquad \text{Ans.}$$

Question 17. Find the fourth proportional to:

(i) $2xy, x^2, y^2$

(ii) $x^3-y^3, x^4+x^2y^2+y^4, x-y$.

Solution : (i) Let A be the fourth proportional then

$$2xy : x^2 = y^2 : A$$

$$\Rightarrow \quad \frac{2xy}{x^2} = \frac{y^2}{A}$$

$$\Rightarrow \quad A = \frac{x^2y^2}{2xy}$$

$$\Rightarrow \quad A = \frac{xy}{2}. \quad \text{Ans.}$$

(ii) Let A be the fourth proportional then

$$x^3 - y^3 : x^4 + x^2y^2 + y^4 = x - y : A$$

$$\Rightarrow \quad \frac{x^3 - y^3}{x^4 + x^2y^2 + y^4} = \frac{x - y}{A}$$

$$\Rightarrow \quad A(x^3 - y^3) = (x - y)(x^4 + x^2y^2 + y^4)$$

$$\Rightarrow \quad A = \frac{(x - y)(x^4 + x^2y^2 + y^4)}{x^3 - y^3}$$

$$\Rightarrow \quad A = \frac{(x - y)(x^2 + y^2 + xy)(x^2 + y^2 - xy)}{(x - y)(x^2 + xy + y^2)}$$

$$\Rightarrow \quad A = x^2 + y^2 - xy. \quad \text{Ans.}$$

Question 18. Find the two numbers such that their mean proprtional is 24 and the third proportinal is 1,536.

Solution : Let x and y be two numbers

$$\text{Mean proprtional} = 24$$

$$\Rightarrow \quad \sqrt{xy} = 24$$

$$\Rightarrow \quad xy = 24 \times 24 = 576$$

$$\Rightarrow \quad x = \frac{576}{y} \quad \ldots(1)$$

Also 1536 is the third proportional then

$$x : y = y : 1{,}536$$

$$\Rightarrow \quad \frac{x}{y} = \frac{y}{1{,}536}$$

From (1), $\quad y^2 = 1{,}536 \times \frac{576}{y}$

$$\Rightarrow \quad y^3 = 1{,}536 \times 576$$

$$\Rightarrow \quad y^3 = 24 \times 24 \times 24 \times 64$$

$$\Rightarrow \quad y = 24 \times 4$$

$$\Rightarrow \quad y = 96$$

Again form (1), we get

$$x = \frac{576}{96} = 6.$$

Hence, the required numbers are 6 and 96.

Question 19. Using componendo and idendo, find the value of x

$$\frac{\sqrt{3x+4}+\sqrt{3x-5}}{\sqrt{3x+4}-\sqrt{3x-5}}=9$$

Solution : $\frac{\sqrt{3x+4}+\sqrt{3x-5}}{\sqrt{3x+4}-\sqrt{3x-5}}=\frac{9}{1}$

Using componendo and dividendo

$$\frac{\sqrt{3x+4}+\sqrt{3x-5}+\sqrt{3x+4}-\sqrt{3x-5}}{\sqrt{3x+4}+\sqrt{3x-5}-\sqrt{3x+4}+\sqrt{3x-5}}$$

$$=\frac{9+1}{9-1}=\frac{10}{8}=\frac{5}{4}$$

$$\frac{2\sqrt{3x+4}}{2\sqrt{3x-5}}=\frac{5}{4}\Rightarrow\frac{3x+4}{3x-5}=\frac{25}{16}$$

(Squaring both sides)

$$48x+64=75x-125$$

$$\Rightarrow \quad 75x-48x=125+64$$

$$27x=189\Rightarrow x=\frac{189}{27}=7$$ Ans.

Question 20. Solve for x:

$$\frac{3x^2+5x+18}{5x^2+6x+12}=\frac{3x+5}{5x+6}.$$

Solution : Multiplying the Numerator and Denominator of R.H.S. by $-x$

$$\frac{3x^2+5x+18}{5x^2+6x+12}=\frac{-3x^2-5x}{-5x^2-6x}$$

Since, each ratio $=\frac{\text{Sum of antecedents}}{\text{Sum of consequents}}$

so $\frac{3x^2+5x+18-3x^2-5x}{5x^2+6x+12-5x^2-6x}$

$$=\frac{-3x^2-5x}{-5x^2-6x}$$

$$\frac{18}{12}=\frac{-3x^2-5x}{-5x^2-6x}$$

$$\Rightarrow \quad \frac{3}{2}=\frac{-3x^2-5x}{-5x^2-6x}$$

$$\Rightarrow \quad \frac{3}{2}=\frac{3x+5}{5x+6}$$

$$\Rightarrow \quad 15x+18=6x+10$$

$$\Rightarrow \quad 9x=-8$$

$$\Rightarrow \quad x=\frac{-8}{9}.$$ Ans.

Question 21. Using the properties of proportion, solve for x, given.

$$\frac{x^4+1}{2x^2} = \frac{17}{8}.$$

Solution : $$\frac{x^4+1}{2x^2} = \frac{17}{8}$$

Using Componendo and Dividendo

$$\frac{x^4+1+2x^2}{x^4+1-2x^2} = \frac{17+8}{17-8}.$$

$$\Rightarrow \quad \frac{(x^2+1)^2}{(x^2-1)^2} = \frac{25}{9}$$

$$\Rightarrow \quad \frac{x^2+1}{x^2-1} = \frac{5}{3}$$

(taking square root on both the sides)

Again applying Componendo and Dividendo

$$\frac{x^2+1+x^2-1}{x^2+1-x^2+1} = \frac{5+3}{5-3}$$

$$\frac{2x^2}{2} = \frac{8}{2}$$

$$\Rightarrow \quad x^2 = 4$$

$$\Rightarrow \quad x = \pm 2$$ Ans.

Question 22. Given that $\frac{a^3+3ab^2}{b^3+3a^2b} = \frac{63}{62}$.

Using Componendo and Dividendo find a : b.

Solution : We have

$$\frac{a^3+3ab^2}{b^3+3a^2b} = \frac{63}{62}$$

App. componendo and dividendo

$$\frac{a^3+3ab^2+b^3+3a^2b}{a^3+3ab^2-b^3-3a^2b} = \frac{63+62}{63-62}$$

$$\frac{a^3+b^3+3ab^2+3a^2b}{a^3+3ab^2-b^3-3a^2b} = \frac{125}{1}$$

$$\frac{(a+b)^3}{(a-b)^3} = \frac{125}{1}$$

$$\frac{a+b}{a-b} = \frac{5}{1}$$

Again Applying Componendo & Dividendo

$$\frac{a+b+a-b}{a+b-a+b} = \frac{5+1}{5-1}$$

$$\frac{2a}{2b} = \frac{6}{4}$$

$$\boxed{a : b = 3 : 2}$$ Ans.

Question 23. If $\frac{3x+5y}{3x-5y}=\frac{7}{3}$, find $x:y$.

Solution : $\frac{3x+5y}{3x-5y}=\frac{7}{3}$

Applying componendo and dividendo

$$\frac{3x+5y+3x-5y}{3x+5y-3x+5y} = \frac{7+3}{7-3}$$

$$\frac{6x}{10y} = \frac{10}{4}$$

$$\frac{x}{y} = \frac{10\times 10}{4\times 6}$$

$$\frac{x}{y} = \frac{25}{6}$$

$\therefore \qquad x:y = 25:6$

Question 24. Solve for x : $\frac{1-px}{1+px}=\sqrt{\frac{1+qx}{1-qx}}$

Solution : $\frac{1-px}{1+px} = \sqrt{\frac{1-qx}{1+qx}}$

Squaring both sides

$$\left(\frac{1-px}{1+px}\right)^2 = \frac{1-qx}{1+qx}$$

$$\Rightarrow \quad \frac{1+p^2x^2-2px}{1+p^2x^2+2px} = \frac{1-qx}{1+qx}$$

Applying componendo and dividendo

$$\frac{1+p^2x^2-2px+1+p^2x^2+2px}{1+p^2x^2-2px-1-p^2x^2-2px} = \frac{1-qx+1+qx}{1-qx-1-qx}$$

$$\Rightarrow \quad \frac{2(1+p^2x^2)}{2(-2px)} = \frac{2}{-2qx}$$

$$\Rightarrow \quad \frac{1+p^2x^2}{2px} = \frac{1}{qx}$$

$$\Rightarrow \quad qx\,(1+p^2x^2) = 2px$$

$$\Rightarrow \quad x(p^2qx^2-2p+q) = 0$$

Either $\qquad x = 0$

or $\qquad p^2qx^2 = 2p-q$

$$x^2 = \frac{2p-q}{p^2q}$$

$$x = 0 \text{ or } x = \pm\frac{1}{p}\sqrt{\frac{2p-q}{q}}.$$

Question 25. Find the value of

$$\frac{x+\sqrt{3}}{x-\sqrt{3}}+\frac{x+\sqrt{2}}{x-\sqrt{2}}, \text{ if } x=\frac{2\sqrt{6}}{\sqrt{3}+\sqrt{2}}.$$

Solution : We have $x = \dfrac{2\sqrt{6}}{\sqrt{3}+\sqrt{2}}$

or $$x = \frac{2\times\sqrt{3}\times\sqrt{2}}{\sqrt{3}+\sqrt{2}}$$

$$\Rightarrow \quad \frac{x}{\sqrt{3}} = \frac{2\sqrt{2}}{\sqrt{3}+\sqrt{2}}$$

Applying Componendo and Dividendo

$$\frac{x+\sqrt{3}}{x-\sqrt{3}} = \frac{2\sqrt{2}+\sqrt{3}+\sqrt{2}}{2\sqrt{2}-\sqrt{3}-\sqrt{2}}$$

$$= \frac{3\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}}$$

$$\frac{x+\sqrt{3}}{x-\sqrt{3}} = \frac{3\sqrt{2}+\sqrt{3}}{-(\sqrt{3}-\sqrt{2})} \quad \ldots\text{(i)}$$

Also $$\frac{x}{\sqrt{2}} = \frac{2\sqrt{3}}{\sqrt{3}+\sqrt{2}}$$

Applying Componendo and Dividendo

$$\frac{x+\sqrt{2}}{x-\sqrt{2}} = \frac{2\sqrt{3}+\sqrt{3}+\sqrt{2}}{2\sqrt{3}-\sqrt{3}-\sqrt{2}}$$

$$\frac{x+\sqrt{2}}{x-\sqrt{2}} = \frac{3\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} \quad \ldots\text{(ii)}$$

Adding (i) and (ii)

$$\frac{x+\sqrt{3}}{x-\sqrt{3}}+\frac{x+\sqrt{2}}{x-\sqrt{2}} = \frac{-3\sqrt{2}-\sqrt{3}+3\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$$

$$= \frac{-2\sqrt{2}+2\sqrt{3}}{\sqrt{3}-\sqrt{2}}$$

$$= \frac{2(\sqrt{3}-\sqrt{2})}{(\sqrt{3}-\sqrt{2})}$$

$$= 2.$$

Prove the Following

Question 1. If (a – x) : (b – x) be the duplicate ratio of a : b show that:

$$\frac{1}{x} = \frac{1}{a}+\frac{1}{b}.$$

Solution : Here $(a-x):(b-x)$ is duplicate ratio of $a:b$

$$\therefore \quad \frac{a^2}{b^2} = \frac{a-x}{b-x}$$

$$\Rightarrow \quad a^2b - a^2x = b^2a - b^2x$$

$$\Rightarrow \quad a^2b - b^2a = a^2x - b^2x$$

$$\Rightarrow \quad ab(a-b) = x(a-b)(a+b)$$

$$\Rightarrow \quad ab = x(a+b)$$

$$\Rightarrow \quad \frac{1}{x} = \frac{a+b}{ab}$$

$$\Rightarrow \quad \frac{1}{x} = \frac{1}{a} + \frac{1}{b} \qquad \text{Hence proved.}$$

Question 2. If a : b with a ≠ b is the duplicate ratio of a + c : b + c, show that c^2 = ab.

Solution : The duplicate ratio of

$$\frac{a}{b} = \frac{(a+c)^2}{(b+c)^2}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{a^2 + c^2 + 2ac}{b^2 + c^2 + 2bc}$$

$$\Rightarrow \quad ab^2 + ac^2 + 2abc = a^2b + bc^2 + 2abc$$

$$\Rightarrow \quad ac^2 - bc^2 = a^2b - ab^2$$

$$\Rightarrow \quad c^2(a-b) = ab(a-b)$$

Hence, $c^2 = ab$. Hence proved.

Question 3. If a : b = 5 : 3, show that (5a + 8b) : (6a – 7b) = 49 : 9.

Solution : Given $a : b = 5 : 3 \Rightarrow \frac{a}{b} = \frac{5}{3}$

$$\therefore \quad \frac{5a+8b}{6a-7b} = \frac{5\left(\frac{a}{b}\right) + 8}{6\left(\frac{a}{b}\right) - 7}$$

Numerator and Denominator is divided by 'b'.

$$= \frac{5 \times \frac{5}{3} + 8}{6 \times \frac{5}{3} - 7}$$

$$= \frac{49}{9} = 49 : 9. \quad \text{Hence proved}$$

Question 4. If b is the mean proprtional between a and c, prove that $\dfrac{a^2-b^2+c^2}{a^{-2}-b^{-2}+c^{-2}}=b^4$.

Solution : Since, b is the mean proportional between a and c. So, $b^2=ac$.

$$\text{L.H.S.} = \frac{a^2-b^2+c^2}{a^{-2}-b^{-2}+c^{-2}}$$

$$= \frac{a^2-b^2+c^2}{\frac{1}{a^2}-\frac{1}{b^2}+\frac{1}{c^2}}$$

$$= \frac{(a^2-b^2+c^2)}{\frac{b^2c^2-a^2c^2+a^2b^2}{a^2b^2c^2}}$$

$$= \frac{a^2b^2c^2\,(a^2-b^2+c^2)}{b^2c^2-b^4+a^2b^2}$$

$$= \frac{b^4\times b^2\,(a^2-b^2+c^2)}{b^2\,(c^2-b^2+a^2)}$$

b^4 = R.H.S. Hence proved.

Question 5. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, Prove that each of these ratios is equal to $\frac{a+c+e}{b+d+f}$.

Solution : Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$ then

$$a=bk,\ c=dk \text{ and } e=fk.$$

Now $\frac{a+c+e}{b+d+f} = \frac{bk+dk+fk}{b+d+f}$

$$= \frac{k\,(b+d+f)}{(b+d+f)}=k$$

Hence, $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{a+c+e}{b+d+f}$.

Hence proved.

Question 6. If $\frac{a}{b+c} = \frac{b}{c+a} = \frac{c}{a+b}$, show that each ratio is equal to $\frac{1}{2}$ or -1.

Solution : Now $\frac{a}{b+c} = \frac{b}{c+a} = \frac{c}{a+b}$

$= \frac{a+b+c}{2(a+b+c)} = \frac{1}{2}$ if $a+b+c \neq 0$.

But if $a+b+c = 0$, (Applying prop. law)

Then $\frac{a}{b+c} = \frac{a}{-a} = -1$

and $\frac{b}{c+a} = \frac{b}{-b} = -1$

Also, $\frac{c}{a+b} = \frac{c}{-c} = -1$.

Hence each of the given ratio is either $\frac{1}{2}$ or -1.

Question 7. If x and y be unequal and x : y is the duplicate ratio of (x + z) and (y + z) prove that z is mean proportional between x and y.

Solution : Since $x : y$ is duplicate ratio of $(x+z)$ and $(y+z)$

$\therefore \quad x : y = (x+z)^2 : (y+z)^2$

$x(y+z)^2 = y(x+z)^2$

On simplifying we get

$xy^2 + xz^2 = yx^2 + yz^2$

$\Rightarrow \quad x^2y - xy^2 = xz^2 - yz^2$

$\Rightarrow \quad xy(x-y) = z^2(x-y)$

$\Rightarrow \quad xy = z^2 \qquad (x \neq y)$

$\Rightarrow x : z :: z : y.$ Hence proved.

Question 8. If ax = by = cz, prove that

$$\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} = \frac{bc}{a^2} + \frac{ca}{b^2} + \frac{ab}{c^2}.$$

Solution : Let $ax = by = cz = k$, then

$x = \frac{k}{a}, y = \frac{k}{b}$ and $z = \frac{k}{c}$

L.H.S. $= \frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy}$

$= \frac{k^2}{a^2 \times \frac{k}{b} \times \frac{k}{c}} + \frac{k^2}{b^2 \times \frac{k}{c} \times \frac{k}{a}} + \frac{k^2}{c^2 \times \frac{k}{a} \times \frac{k}{b}}$

$= \frac{bc}{a^2} + \frac{ca}{b^2} + \frac{ab}{c^2}$

$=$ R.H.S. Hence proved.

Question 9. If a, b, c are in continued proportion, prove that $a : c = (a^2 + b^2) : (b^2 + c^2)$.

Solution : a, b and c are the continuted proportion

$$a : b = b : c$$

$$\Rightarrow \quad \frac{a}{b} = \frac{b}{c}$$

$$\Rightarrow \quad b^2 = ac$$

Now $\quad \frac{a}{c} = \frac{a^2 + b^2}{b^2 + c^2}$

$$\Rightarrow \quad a(b^2 + c^2) = c(a^2 + b^2)$$

L.H.S. $= a(b^2 + c^2) \Rightarrow a(ac + c^2) \Rightarrow ac(a + c)$

R.H.S. $= c(a^2 + b^2) \Rightarrow c(a^2 + ac) \Rightarrow ac(a + c)$

L.H.S. = R.H.S. Hence proved.

Question 10. If $\frac{x}{b+c-a} = \frac{y}{c+a-b} = \frac{z}{a+b-c}$, then show that $(b-c)x + (c-a)y + (a-b)z = 0$.

Solution : Let,

$$\frac{x}{b+c-a} = \frac{y}{c+a-b} = \frac{z}{a+b-c} = k,$$

[By k method]

$$x = (b+c-a)k,$$
$$y = (c+a-b)k,$$
$$z = (a+b-c)k$$

$$\Rightarrow b^2 + bc - ab - bc - c^2 + ca + c^2 + ca - bc - ac - a^2 + ab + a^2 + ab - ac - ab - b^2 + bc = 0$$

Hence proved.

Question 11. If $p + r = 2q$ and $\frac{1}{q} + \frac{1}{s} = \frac{2}{r}$, then prove that

$$p : q = r : s.$$

Solution : $\quad \frac{1}{q} + \frac{1}{s} = \frac{2}{r}$

$$\Rightarrow \quad \frac{s+q}{qs} = \frac{2}{r}$$

$$\Rightarrow \quad 2qs = r(s+q)$$

$$\Rightarrow \quad (p+r)s = r \cdot (s+q)$$

$$\Rightarrow \quad ps + rs = rs + rq$$

$$\Rightarrow \quad ps = rq$$

$$\Rightarrow \quad \frac{p}{q} = \frac{r}{s}.$$

Hence proved.

Question 12. Given : $x = \dfrac{\sqrt{a^2+b^2}+\sqrt{a^2-b^2}}{\sqrt{a^2+b^2}-\sqrt{a^2-b^2}}$

Use componendo and dividendo to prove that $b^2 = \dfrac{2a^2x}{x^2+1}$.

Solution : $$x = \frac{\sqrt{a^2+b^2}+\sqrt{a^2-b^2}}{\sqrt{a^2+b^2}-\sqrt{a^2-b^2}}$$

By componendo and dividendo,

$$\frac{x+1}{x-1} = \frac{2\sqrt{a^2+b^2}}{2\sqrt{a^2-b^2}}$$

Squaring both sides,

$$\frac{x^2+2x+1}{x^2-2x+1} = \frac{a^2+b^2}{a^2-b^2}$$

By componendo and dividendo.

$$\frac{2(x^2+1)}{4x} = \frac{2a^2}{2b^2}$$

$$\Rightarrow \quad \frac{x^2+1}{2x} = \frac{a^2}{b^2}$$

$$\Rightarrow \quad b^2 = \frac{2a^2x}{x^2+1}.$$ Hence proved.

Question 13. If $\frac{a}{b} = \frac{c}{d}$, show that $(9a + 13b)(9c - 13d) = (9c + 13b)(9a - 13d)$.

Solution : We have $\frac{a}{b} = \frac{c}{d}$

$\left[\text{Multiplying both sides by } \frac{9}{13}\right]$

$$\frac{9a}{13b} = \frac{9c}{13d}$$

[By componendo and dividendo]

$$\frac{9a+13b}{9a-13b} = \frac{9c+13d}{9c-13d}$$

(By cross multiplication)

$(9a + 13b)(9c - 13d) = (9a - 13b)(9c + 13d)$.

Hence proved.

Question 14. If $\frac{p}{q} = \frac{r}{s}$, prove that

$$\frac{2p+3q}{2p-3q} = \frac{2r+3s}{2r-3s}.$$

Solution : We have

$$\frac{p}{q} = \frac{r}{s}$$

[Multiplying both side by 2/3]

$$\frac{2p}{3q} = \frac{2r}{3s}$$

[By componendo and dividendo]

$$\therefore \quad \frac{2p+3q}{2p-3q} = \frac{2r+3s}{2r-3s}.$$ Hence proved.

Question 15. If a, b, c, d are in continued proportion, prove that

$$(b-c)^2 + (c-a)^2 + (d-b)^2 = (d-a)^2.$$

Solution : Since a, b, c, d are in continued proportion, we have

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \text{K (say)}$$

$\therefore \quad c = d\text{K}, b = c\text{K} = d\text{K}^2$ and $a = b\text{K} = d\text{K}^3$.

L.H.S.

$= (b-c)^2 + (c-a)^2 + (d-b)^2$

$= (d\text{K}^2 - d\text{K})^2 + (d\text{K} - d\text{K}^3)^2 + (d - d\text{K}^2)^2$

$= d^2\text{K}^2(\text{K}-1)^2 + d^2\text{K}^2(1-\text{K}^2)^2 + d^2(1-\text{K}^2)^2$

$= d^2[\text{K}^2(\text{K}-1)^2 + \text{K}^2(\text{K}^2-1)^2 + d^2(\text{K}^2-1)^2]$

$= d^2[\text{K}^2(\text{K}-1)^2 + \text{K}^2(\text{K}-1)^2(\text{K}+1)^2 + (\text{K}-1)^2(\text{K}+1)^2]$

$= d^2(\text{K}-1)^2[\text{K}^2 + \text{K}^2(\text{K}+1)^2 + (\text{K}+1)^2]$

$= d^2(\text{K}-1)^2[\text{K}^2 + \text{K}^2(\text{K}^2+2\text{K}+1) + \text{K}^2 + 2\text{K} + 1]$

$= d^2(\text{K}-1)^2[\text{K}^4 + 2\text{K}^3 + 3\text{K}^2 + 2\text{K} + 1]$

$= d^2(\text{K}-1)^2(\text{K}^2+\text{K}+1)^2$

$= d^2[(\text{K}-1)(\text{K}^2+\text{K}+1)]^2$

$= d^2(\text{K}^3-1)^2 = (d\text{K}^3 - d)^2 = (a-d)^2 = (d-a)^2$

= R.H.S.

Hence, $(b-c)^2 + (c-a)^2 + (d-b)^2 = (d-a)^2$.

Question 16. If q is the mean proportional between p and r, prove that

$$p^2 - q^2 + r^2 = q^4\left[\frac{1}{p^2} - \frac{1}{q^2} + \frac{1}{r^2}\right].$$

Solution : Since, q is the mean proportional of p and r.

Hence, $q^2 = pr$.

$$\begin{aligned}
\text{R.H.S.} &= q^4\left[\frac{1}{p^2} - \frac{1}{q^2} + \frac{1}{r^2}\right] \\
&= q^4\left[\frac{1}{p^2} - \frac{1}{pr} + \frac{1}{r^2}\right] \\
&= q^4\left[\frac{r^2 - pr + p^2}{p^2r^2}\right] \\
&= q^4\left[\frac{p^2 - pr + r^2}{(pr)^2}\right] \\
&= q^4\left[\frac{p^2 - pr + r^2}{q^4}\right] \\
&= p^2 - pr + r^2 \\
&= p^2 - q^2 + r^2 = \text{L.H.S.}
\end{aligned}$$

Hence proved.

Question 17. If $\frac{a}{b} = \frac{c}{d}$ Show that

$$a + b : c + d = \sqrt{a^2 + b^2} : \sqrt{c^2 + d^2}.$$

Solution : Let $\frac{a}{b} = \frac{c}{d} = k$

$\Rightarrow$ $a = bk$ and $c = dk$

$$\begin{aligned}
\text{L.H.S.} &= \frac{a+b}{c+d} = \frac{bk+b}{dk+d} \\
&= \frac{b\,(k+1)}{d\,(k+1)} = \frac{b}{d} \\
\text{R.H.S.} &= \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} = \frac{\sqrt{b^2k^2+b^2}}{\sqrt{d^2k^2+d^2}} \\
&= \frac{b\,(\sqrt{k^2+1})}{d\,(\sqrt{k^2+1})} = \frac{b}{d}
\end{aligned}$$

L.H.S. = R.H.S. Hence proved.

Question 18. If $a : b = c : d$, show that
$(a-c)\, b^2 : (b-d)\, cd = (a^2 - b^2 - ab) : (c^2 - d^2 - cd)$.

Solution : Let $\frac{a}{b} = \frac{c}{d} = k$

$$\Rightarrow \quad a = bk \text{ and } c = dk$$

$$\text{L.H.S.} = \frac{(a-c)\,b^2}{(b-d)\,cd} = \frac{(bk-dk)\,b^2}{(b-d)\,dk.d}$$

$$= \frac{b^2 \cdot k\,(b-d)}{d^2 k\,(b-d)} = \frac{b^2}{d^2}.$$

$$\text{R.H.S.} = \frac{a^2-b^2-ab}{c^2-d^2-cd}$$

$$= \frac{b^2k^2 - b^2 - bk\cdot b}{d^2k^2 - d^2 - dk\cdot d}$$

$$= \frac{b^2\,(k^2-k-1)}{d^2\,(k^2-k-1)}$$

$$= \frac{b^2}{d^2}$$

L.H.S. = R.H.S. Hence proved.

Question 19. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that
$(ab + cd + ef)^2 = (a^2 + c^2 + e^2)\,(b^2 + d^2 + f^2)$.

Solution : Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$ then

$$a = bk,\ c = dk \text{ and } e = fk$$

$$\text{L.H.S.} = (ab + cd + ef)^2$$

$$= (bk\cdot b + dk\cdot d + fk\cdot f)^2$$

$$= k^2\,(b^2 + d^2 + f^2)^2$$

$$\text{R.H.S.} = (a^2 + c^2 + e^2)(b^2 + d^2 + f^2)$$

$$= (b^2k^2 + d^2k^2 + f^2k^2)(b^2 + d^2 + f^2)$$

$$= k^2\,(b^2 + d^2 + f^2)(b^2 + d^2 + f^2)$$

$$= k^2\,(b^2 + d^2 + f^2)^2.$$

L.H.S. = R.H.S. Hence proved.

Question 20. If $\frac{a}{b} = \frac{c}{d} = \frac{r}{f}$, prove that

$$\left(\frac{a^2b^2 + c^2d^2 + e^2f^2}{ab^3 + cd^3 + ef^3}\right)^{3/2} = \sqrt{\frac{ace}{bdf}}$$

Solution : Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$

$$\therefore \quad a = bk,\ c = dk,\ e = fk$$

$$\text{L.H.S.} = \left(\frac{a^2b^2 + c^2d^2 + e^2f^2}{ab^3 + cd^3 + ef^3}\right)^{3/2}$$

$$= \left(\frac{b^2k^2\cdot b^2 + d^2k^2\cdot d^2 + f^2k^2\cdot f^2}{bk.b^3 + dk\cdot d^3 + fk\cdot f^3}\right)^{3/2}$$

$$= \left[\frac{k^2\,(b^4 + d^4 + f^4)}{k\,(b^4 + d^4 + f^4)}\right]^{3/2}$$

$$= k^{3/2}$$

$$\text{R.H.S.} = \sqrt{\frac{ace}{bdf}} = \sqrt{\frac{bk\cdot dk\cdot fk}{bdf}} = k^{3/2}$$

L.H.S. = R.H.S. Hence proved.

Question 21. If a, b, c, d are in continued proportion, prove that:

(i) $\sqrt{ab} - \sqrt{bc} + \sqrt{cd} = \sqrt{(a-b+c)(b-c+d)}$

(ii) $(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2$.

Solution : (i) Since a, b, c, d are in continued proportion then

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = k$$

$\Rightarrow \quad a = bk, b = ck, c = dk$

$\Rightarrow \quad a = ck^2$

$\Rightarrow \quad a = dk^3, b = dk^2$ and $c = dk$

$$\begin{aligned} \text{L.H.S.} &= \sqrt{ab} - \sqrt{bc} + \sqrt{cd} \\ &= \sqrt{dk^3 \cdot dk^2} - \sqrt{dk^2 \cdot dk} + \sqrt{dk \cdot d} \\ &= d \cdot k^2\sqrt{k} - dk\sqrt{k} + d\sqrt{k} \\ &= (k^2 - k + 1)\, d\sqrt{k}. \end{aligned}$$

$$\begin{aligned} \text{R.H.S.} &= \sqrt{(a-b+c)(b-c+d)} \\ &= \sqrt{(dk^3 - dk^2 + dk)(dk^2 - dk + d)} \\ &= \sqrt{d \times d \times k(k^2-k+1)(k^2-k+1)} \\ &= (k^2 - k + 1)\, d\sqrt{k} \end{aligned}$$

L.H.S = R.H.S. Hence proved.

(ii) $$\begin{aligned} \text{L.H.S.} &= (d^2k^6 + d^2k^4 + d^2k^2)(d^2k^4 + d^2k^2 + d^2) \\ &= d^2k^2(k^4 + k^2 + 1) \cdot d^2(k^4 + k^2 + 1) \\ &= d^4k^2(k^4 + k^2 + 1)^2 \end{aligned}$$

$$\begin{aligned} \text{R.H.S.} &= (ab + bc + cd)^2 \\ &= (dk^3 \cdot dk^2 + dk^2 \cdot dk + dk \cdot d)^2 \\ &= d^4 \cdot k^2(k^4 + k^2 + 1)^2 \end{aligned}$$

L.H.S. = R.H.S. Hence proved.

Question 22. If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, show that

$$\frac{x^3}{a^3} - \frac{y^3}{b^3} + \frac{z^3}{c^3} = \frac{xyz}{abc}.$$

Solution : Let $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = k$

$$x = ak,\ y = bk,\ z = ck$$

$$\text{L.H.S.} = \frac{x^3}{a^3} - \frac{y^3}{b^3} + \frac{z^3}{c^3} \Rightarrow \frac{a^3k^3}{a^3} - \frac{b^3k^3}{b^3} - \frac{c^3k^3}{c^3}$$

$\Rightarrow \quad k^3 - k^3 + k^3 = k^3$

$$\text{R.H.S.} = \frac{xyz}{abc} \Rightarrow \frac{ak \cdot bk \cdot ck}{abc} \Rightarrow \frac{k^3abc}{abc} = k^3$$

L.H.S. = R.H.S. Hence proved.

Question 23. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, prove that

$$\frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2}=\frac{(x+y+z)^3}{(a+b+c)^2}.$$

Solution : Let $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=k,$ [By k method]

$$x=ak,\ y = bk \text{ and } z=ck.$$

$$\text{L.H.S.}=\frac{a^3k^3}{a^2}+\frac{b^3k^3}{b^2}+\frac{c^3k^3}{c^2}\Rightarrow k^3[a+b+c]$$

$$\text{R.H.S.}=\frac{[ak+bk+ck]^3}{[a+b+c]^2}\Rightarrow\frac{k^3[a+b+c]^3}{[a+b+c]^2}$$

$$= k^3(a+b+c)$$

L.H.S. = R.H.S. Hence proved.

Question 24. If $a=\frac{b+c}{2}$, $c=\frac{a+b}{2}$ and b is mean proportional between a and c, prove that

$$\frac{1}{a}+\frac{1}{c}=\frac{2}{b}.$$

Solution : b is the mean proportional of a and c

$$b^2 = ac$$

$$b^2 = \left(\frac{b+c}{2}\right)\cdot\left(\frac{a+b}{2}\right)$$

$$\Rightarrow \quad 4b^2 = ab+ac+b^2+bc$$

$$[\because b^2=ac]$$

$$\Rightarrow \quad 4b^2 = ab+2b^2+bc$$

$$2b^2 = ab+bc,$$

[Dividing both sides by abc]

$$\Rightarrow \quad \frac{2b^2}{abc} = \frac{ab}{abc}+\frac{bc}{abc}$$

$$\Rightarrow \quad \frac{2}{b} = \frac{1}{c}+\frac{1}{a}, \quad [\because b^2=ac]$$

Hence proved.

Question 25. If $\frac{3a+4b}{3c+4d}=\frac{3a-4b}{3c-4d}$ Prove that

$$\frac{a}{b}=\frac{c}{d}$$

Solution :

$$\text{Given } \frac{3a+4b}{3c+4d} = \frac{3a-4b}{3c-4d}$$

$$\text{App. alternendo } = \frac{3a+4b}{3a-4b}=\frac{3c+4d}{3c-4d}$$

App. componendo and dividendo

$$\frac{3a+4b+3a-4b}{3a+4b-3a+4b} = \frac{3c+4d+3c-4d}{3c+4d-3c+4d}$$

$$\therefore \quad \frac{6a}{8b} = \frac{6c}{8d}$$

$$\text{or} \quad \frac{a}{b} = \frac{c}{d}$$ Hence proved.

Question 26. If $a : b = c : d$, show that $(2a-7b)(2c+7d)=(2c-7d)(2a+7b)$.

Solution : We have

$$a : b = c : d$$

$$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{2a}{7b}=\frac{2c}{7d}$$

$$\left[\text{Multiplying both side} \frac{2}{7}\right]$$

[Using componendo and dividendo]

$$\frac{2a+7b}{2a-7b} = \frac{2c+7d}{2c-7d}$$

[By cross multiplication]

$$\Rightarrow (2a-7b)(2c+7d) = (2a+7b)(2c-7d).$$

Hence proved.

Question 27. If $\frac{3x+4y}{3u+4v} = \frac{3x-4y}{3u-4v}$, then show that $\frac{x}{y}=\frac{u}{v}$.

Solution : We have

$$\frac{3x+4y}{3u+4v} = \frac{3x-4y}{3u-4v}$$

[Applying alternendo]

$$\frac{3x+4y}{3x-4y} = \frac{3u+4v}{3u-4v}$$

[By componendo and dividendo]

$$\frac{3x+4y+3x-4y}{3x+4y-3x+4y} = \frac{3u+4v+3u-4v}{3u+4v-3u+4v}$$

$$\Rightarrow \quad \frac{6x}{8y} = \frac{6u}{8v}$$

$$\Rightarrow \quad \frac{x}{y} = \frac{u}{v}.$$ Hence proved