Thermal Radiation. Quantum Nature of Light (Part - 1)

Q.246. Using Wien's formula, demonstrate that
(a) the most probable radiation frequency $\omega_{pr} \propto T$;
(b) the maximum spectral density of thermal radiation
$(u_\omega)_{max} \propto \propto T^3$
(c) the radiosity $M_e \propto T^4$.

Ans. (a) The most probable radiation frequency ω_{pr} is the frequency for which

$$\frac{d}{d\omega} u_\omega = 3\,\omega^2 F(\omega/T) + \frac{\omega^3}{T} F'(\omega/T) = 0$$

The maximum frequency is the root other than ω = 0 of this equation. It is

$$\omega = -\frac{3\,T F(\omega/T)}{F'(\omega/T)}$$

$\omega_{pr} = x_0 T$ where x_0 is the solution of the transcendental equation

$$3F(x_0) + x_0 F'(x_0) = 0$$

(b) The maximum spectral density is the density corresponding to most probable frequency. It is

$$(u_\omega)_{max} = x_0^3 F(x_0)\, T^3 \;\alpha\; T^3$$

where x_0 is defined above,

(c) The radiosity is

$$M_e = \frac{c}{4}\int_0^\infty \omega^3 F\left(\frac{\omega}{T}\right) d\omega = T^4 \left[\frac{c}{4}\int_0^\infty x^3 F(x)\, dx\right] \alpha\, T^4$$

Q.247. The temperature of one of the two heated black bodies is T_1 = 2500 K. Find the temperature of the other body if the wavelength corresponding to its maximum emissive capacity exceeds by Δλ = 0.50 gm the wavelength corresponding to the maximum emissive capacity of the first black body.

Ans. **F**or the first black body

$$(\lambda_m)_1 = \frac{b}{T_1}$$

Then

$$(\lambda_m)_2 = \frac{b}{T_1} + \Delta\lambda = \frac{b}{T_2}$$

Hence

$$T_2 = \frac{b}{\frac{b}{T_1} + \Delta\lambda} = \frac{b\,T_1}{b + T_1\,\Delta\lambda} = 1{\cdot}747\,kK$$

Q.248. The radiosity of a black body is M_e =3.0 W/cm^2. Find the wavelength corresponding to the maximum emissive capacity of that body.

Ans. From the radiosity we get the temperature of the black body. It is

$$T = \left(\frac{M_e}{\sigma}\right)^{1/4} = \left(\frac{3{\cdot}0 \times 10^4}{5{\cdot}67 \times 10^{-8}}\right)^{1/4} = 852{\cdot}9\,K$$

Hence the wavelength corresponding to the maximum emissive capacity of the body is

$$\frac{b}{T} = \frac{0{\cdot}29}{852{\cdot}9}\ \text{cm} = 3{\cdot}4 \times 10^{-4}\ \text{cm} = 3{\cdot}4\ \mu\text{m}$$

(Note that $3{\cdot}0\ \text{W/cm}^2 = 3{\cdot}0 \times 10^4\ \text{W/m}^2$.)

Q.249. The spectral composition of solar radiation is much the same as that of a black body whose maximum emission corresponds to the wavelength 0.48µm. Find the mass lost by the Sun every second due to radiation. Evaluate the time interval during which the mass of the Sun diminishes by 1 per cent.

Ans. The black body temperature of the sun maybe taken as

$$T_\odot = \frac{0{\cdot}29}{0{\cdot}48 \times 10^{-4}} = 6042\ \text{K}$$

Thus the radiosity is

$$M_{e\odot} = 5{\cdot}67 \times 10^{-8} (6042)^4 = 0{\cdot}7555 \times 10^8 \text{ W/m}^2$$

Energy lost by sun is

$$4\pi (6{\cdot}95)^2 \times 10^{16} \times 0{\cdot}7555 \times 10^8 = 4{\cdot}5855 \times 10^{26} \text{ watt}$$

This corresponds to a mass loss of

$$\frac{4{\cdot}5855 \times 10^{26}}{9 \times 10^{16}} \text{ kg/sec} = 5{\cdot}1 \times 10^9 \text{ kg/sec}$$

The sun loses 1 % of its mass in

$$\frac{1{\cdot}97 \times 10^{30} \times 10^{-2}}{5{\cdot}1 \times 10^9} \text{ sec} = 1{\cdot}22 \times 10^{11} \text{ years}.$$

Q.250. Find the temperature of totally ionized hydrogen plasma of density p = 0.10 g/cm^3 at which the thermal radiation pressure is equal to the gas kinetic pressure of the particles of plasma. Take into account that the thermal radiation pressure p = u/3, where u is the space density of radiation energy, and at high temperatures all substances obey the equation of state of an ideal gas.

Ans. For an ideal gas p = n k T where n = number density of the particles and $k = \frac{R}{N_A}$ is

Boltzman constant In a fully ionized hydrogen plasma, both H ions (protons) and electrons contribute to pressure but since the mass of electrons is quite

small $(= m_p/1836)$ only protons contribute to mass density. Thus

$$n = \frac{2\rho}{m_H}$$

and $$p = \frac{2\rho R}{N_A m_H} T$$

Where $m_H = m_p$ is the proton or hydrogen mass.

Equating this to thermal radiation pressure

$$\frac{2\rho R}{N_A m_H} T = \frac{u}{3} = \frac{M_e}{3} \times \frac{4}{c} = \frac{4\sigma T^4}{3c}$$

Then $$T^3 = \frac{3c\rho R}{2\sigma N_A m_H} = \frac{3c\rho R}{\sigma M}$$

where $M = 2N_A m_H =$ molecular weight of hydrogen = 2 x10^{-3} kg .

Thus $$T = \left(\frac{3c\rho R}{\sigma M}\right)^{1/3} = 1\cdot89 \times 10^7 \text{ K}$$

Q.251. A copper ball of diameter d = 1.2 cm was placed in an evacuated vessel whose walls are kept at the absolute zero temperature. The initial temperature of the ball is T_o = 300 K. Assuming the surface of the ball to be absolutely black, find how soon its temperature decreases η = 2.0 times.

Ans. In time dt after the instant t when the temperature of the ball is T, it loses

$$\pi d^2 \sigma T^4 dt$$

Joules of energy. As & result its temperature falls by - dT and

$$\pi d^2 \sigma T^4 dt = -\frac{\pi}{6} d^3 \rho C dT$$

Where ρ = density of copper, C = its sp.heat

Thus $$dt = -\frac{C\rho d}{6\sigma}\frac{dT}{T^4}$$

or $$t_0 = \frac{C\rho d}{6\sigma}\int_{T_0}^{T_0/\eta} -\frac{dT}{T^4} = \frac{C\rho d}{18\sigma T_0^3}(\eta^3 - 1) = 2\cdot94 \text{ hours}.$$

Q.252. There are two cavities (Fig. 5.39) with small holes of equal diameters d = 1.0 cm and perfectly reflecting outer surfaces. The

Fig. 5.39.

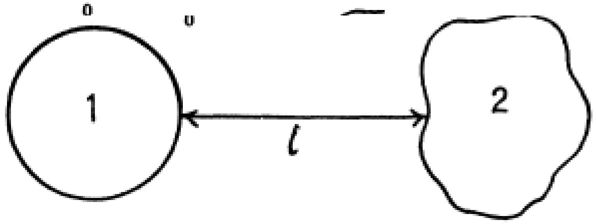
distance between the holes is l = 10 cm. A constant temperature T_1 = 1700 K is maintained in cavity 1. Calculate the steady-state temperature inside cavity 2. Instruction. Take into account that a black body radiation obeys the cosine emission law.

Ans. Taking account of cosine low of emission we write for the energy radiated per second by the hole in cavity # 1 as

$$dI(\Omega) = A\cos\theta\, d\Omega$$

where A is an constant, dΩ is an element of solid angle around some direction defined by the symbol Ω . Integrating over the whole forward hemisphere we get

$$I = A\int_0^{\pi/2} \cos\theta\, 2\pi \sin\theta\, d\theta = \pi A$$



We find A by equating this to the quantity $\sigma T_1^4 \cdot \frac{\pi d^2}{4}$ σ is stefan- Boltzman constant and d is the diameter of th hole.

Then $A = \frac{1}{4}\sigma d^2 T_1^4$

Now energy reaching 2 from 1 is $(\cos\theta \approx 1)$

$$\frac{1}{4}\sigma d^2 T_1^4 \cdot \Delta\Omega$$

where $\Delta\Omega = \frac{(\pi d^2/4)}{l^2}$ is the solid angle subtended by the hole of 2 at 1. {We are

assuming

$d << l$ so $\Delta\Omega$ = area of hole / (distance $)^2$ }.

This must equal $\sigma T_2^4 \pi d^2/4$

which is the energy emitted by 2. Thus equating

$$\frac{1}{4}\sigma d^2 T_1^4 \frac{\pi d^2}{4 l^2} = \sigma T_2^4 \frac{\pi d^2}{4}$$

or $T_2 = T_1 \sqrt{\frac{d}{2l}}$

Substituting we get T_2 = 0.380 k K = 380 K.

Q.253. A cavity of volume V = 1.0 1 is filled with thermal radiation at a temperature T = 1000 K. Find:
(a) the heat capacity C_v;
(b) the entropy S of that radiation.

Ans. (a) The total internal eneigy of the cavity is

$$U = \frac{4\sigma}{c} T^4 V$$

Hence $C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{16\sigma}{C} T^3 V$

$$= \frac{16 \times 5{\cdot}67 \times 10^8}{3 \times 10^8} \times 10^9 \times 10^{-3} \text{ Joule/ °K}$$

$$= \frac{1{\cdot}6 \times 5{\cdot}67}{3} n\,\text{J/K} = 3{\cdot}024\, n\,\text{J/K}$$

(b) From first law

$$TdS = dU + p\,dV$$
$$= V\,dU + U\,dV + \frac{u}{3} dV \quad \left(p = \frac{U}{3}\right)$$

$$= V\,dU + \frac{4U}{3} dV$$

$$= \frac{16\sigma}{C} V T^3 \, dT + \frac{16\sigma}{3C} T^4 \, dV$$

so $dS = \frac{16\sigma}{C} V T^2 \, dT + \frac{16\sigma}{3C} T^3 \, dV$

$$= d\left(\frac{16\sigma}{3C} V T^3\right)$$

Hence $S = \frac{16\sigma}{3C} V T^3 = \frac{1}{3} C_V = 1{\cdot}008 \, n \, J/K.$

Q.254. Assuming the spectral distribution of thermal radiation energy to obey Wien's formula $u(\omega, T) = A\omega^3 \exp(-a\omega/T)$, where a = 7.64 ps•K , find for a temperature T = 2000 K the most probable
(a) radiation frequency;
(b) radiation wavelength.

Ans. We are given

$$u(\omega, T) = A\,\omega^3 \exp(-a\,\omega/T)$$

(a) then $\frac{du}{d\omega} = \left(\frac{3}{\omega} - \frac{a}{T}\right) u = 0$

so $\omega_{pr} = \frac{3T}{a} = \frac{6000}{7{\cdot}64} \times 10^{12} \, s^{-1}$

(b) We determine the spectral distribution in wavelength.

$$-\tilde{u}(\lambda, T) \, d\lambda = u(\omega, T) \, d\omega$$

But $\omega = \frac{2\pi c}{\lambda}$ or $\lambda = \frac{2\pi c}{\omega} = \frac{C'}{\omega}$

so $d\lambda = -\frac{C'}{\omega^2} d\omega, \; d\omega = -\frac{C'}{\lambda^2} d\lambda$

(we have put a minus sign before dλ to subsume just this fact dλ is -ve where dω is +ve.)

$$\tilde{u}(\lambda, T) = \frac{C'}{\lambda^2} u\left(\frac{C'}{\lambda}, T\right) = \frac{C'^4 A}{\lambda^5} \exp\left(-\frac{a C'}{\lambda T}\right)$$

This is maximum when

$$\frac{\partial \tilde{u}}{\partial \lambda} = 6 = \tilde{u}\left[\frac{-5}{\lambda} + \frac{a C'}{\lambda^2 T}\right]$$

or

$$\lambda_{pr} = \frac{a C'}{5 T} = \frac{2 \pi c a}{5 T} = 1{\cdot}44 \, \mu\mathrm{m}$$

Q.255. Using Planck's formula, derive the approximate expressions for the space spectral density u_ω of radiation
(a) in the range where hω << kT (Rayleigh-Jeans formula);
(b) in the range where Nω >> kT (Wien's formula).

Ans. From Planek's formula

$$u_\omega = \frac{\hbar \omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar \omega / kT} - 1}$$

(a) In a range $\hbar \omega << kT$ (long wavelength or high temperature).

$$u_\omega \rightarrow \frac{\hbar \omega^3}{\pi^2 c^3} \frac{1}{\frac{\hbar \omega}{kT}}$$

$$= \frac{\omega^2}{\pi^2 c^3} kT, \text{ using } e^x = 1 + x$$

for small x.

(b) In the range $\hbar \omega >> T$ (high frequency or low temperature) :

$$\frac{\hbar \omega}{kT} >> 1 \quad \text{so } e^{\frac{\hbar \omega}{kT}} >> 1$$

And

$$u_\omega = \frac{\hbar \omega^3}{\pi^2 c^3} e^{-\hbar \omega / kT}$$

Q.256. Transform Planck's formula for space spectral density u_ω. of radiation from the variable ω to the variables v (linear frequency) and λ (wavelength).

Ans. We write

$$u_\omega \, d\omega = \tilde{u}_\nu \, d\nu \quad \text{where} \quad \omega = 2\pi\nu$$

Then

$$\tilde{u}_\nu = \frac{2\pi\hbar(2\pi\nu)^3}{\pi^2 c^3} \frac{1}{e^{2\pi\hbar\nu/kT} - 1} = \frac{16\pi^2\hbar\nu^3}{c^3} \frac{1}{e^{2\pi\hbar\nu/kT} - 1}$$

Also $-\tilde{u}(\lambda, T)\, d\lambda = u_\omega \, d\omega \quad \text{where} \quad \lambda = \frac{2\pi c}{\omega},$

$$d\omega = -\frac{2\pi c}{\lambda^2} d\lambda$$

$$\tilde{u}(\lambda, T) = \frac{2\pi c}{\lambda^2} u\left(\frac{2\pi c}{\lambda}, T\right)$$

$$= \frac{2\pi c}{\lambda^2}\left(\frac{2\pi c}{\lambda}\right)^3 \frac{\hbar}{\pi^2 c^3} \frac{1}{e^{2\pi\hbar c/\lambda kT} - 1} = \frac{16\pi^2 c\hbar}{\lambda^5} \frac{1}{e^{2\pi\hbar c/\lambda kT} - 1}$$

Q.257. Using Planck's formula, find the power radiated by a unit area of a black body within a narrow wavelength interval Δλ = = 1.0 nm close to the maximum of spectral radiation density at a temperature T = 3000 K of the body.

Ans. We write the required power in terms of the radiosity by considering only the energy radiated in the given range. Then from the previous problem

$$\Delta P = \frac{c}{4}\tilde{u}(\lambda_m, T)\,\Delta\lambda$$

$$= \frac{4\pi^2 c^2\hbar}{\lambda_m^5} \frac{\Delta\lambda}{e^{2\pi c\hbar/k\lambda_m T} - 1}$$

But $\lambda_m T = b$

so $$\Delta P = \frac{4\pi^2 c^2\hbar}{\lambda_m^5} \frac{\Delta\lambda}{e^{2\pi c\hbar/kb} - 1} \Delta\lambda$$

Using the data

$$\frac{2\pi c\hbar}{kb} = \frac{2\pi \times 3 \times 10^8 \times 1{\cdot}05 \times 0^{-34}}{1{\cdot}38 \times 10^{-23} \times 2{\cdot}9 \times 10^{-3}} = 4.9643$$

$$\frac{1}{e^{2\pi ch/kb}-1} = 7{\cdot}03 \times 10^{-3}$$

and $\Delta P = 0{\cdot}312 \ \mathrm{W/cm^2}$

Q.258. Fig. 5.40 shows the plot of the function y (x) representing a fraction of the total power of thermal radiation falling within

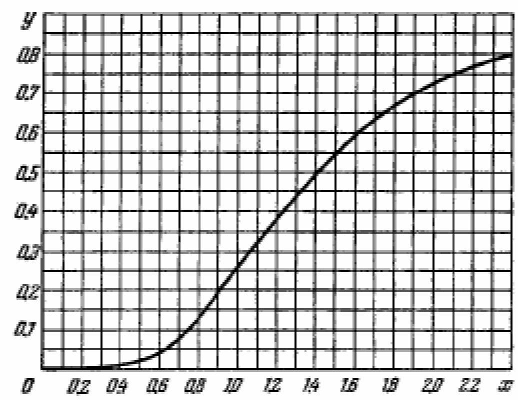


Fig. 5.40.

the spectral interval from 0 to x. Here $x = \lambda/\lambda_m$ (λ_m **is the wavelength corresponding to the maximum of spectral radiation density). Using this plot, find:**
(a) the wavelength which divides the radiation spectrum into two equal (in terms of energy) parts at the temperature 3700 K;
(b) the fraction of the total radiation power falling within the visible range of the spectrum (0.40-0.76 Rm) at the temperature 5000 K;
(c) how many times the power radiated at wavelengths exceeding 0.76 Jim will increase if the temperature rises from 3000 to 5000 K.

Ans. (a) From the curve of the function y(x) we see that y = 0-5 when x = 1.41

Thus $\lambda = 1{\cdot}41 \times \frac{0{\cdot}29}{3700} \ \mathrm{cm} = 1{\cdot}105 \ \mu\mathrm{m}.$

(b) At 5000 K

$$\lambda = \frac{0{\cdot}29}{{\cdot}5} \times 10^{-6} \ \mathrm{m} = 0{\cdot}58 \ \mu\mathrm{m}$$

So the visible range (0.40 to 0.70) ^im corresponds to a range (0.69 to 1.31) of x
From the curve
y (0.69) = 0.07
y(1-31) = 0.44

so the fraction is 0.37

(c) The value of x corresponding to 0.76 are

$$x_1 = 0{\cdot}76 \Big/ \frac{0{\cdot}29}{0{\cdot}3} = 0{\cdot}786 \text{ at } 3000\,\text{K}$$

$$x_2 = 0{\cdot}76 \Big/ \frac{0{\cdot}29}{0{\cdot}5} = 1{\cdot}31 \text{ at } 5000\,\text{K}$$

The requisite fraction is then

$$\left(\frac{P_2}{P_1}\right) = \underbrace{\left(\frac{T_2}{T_1}\right)^4}_{\substack{\text{ratio of}\\ \text{total power}}} \times \underbrace{\frac{1-y_2}{1-y_1}}_{\substack{\text{ratio of the}\\ \text{fraction of}\\ \text{required wavelengths}\\ \text{in the radiated power}}}$$

$$= \left(\frac{5}{3}\right)^4 \times \frac{1-0{\cdot}44}{1-0{\cdot}12} = 4{\cdot}91$$

Q.259. Making use of Planck's formula, derive the expressions determining the number of photons per 1 cm^3 **of a cavity at a temperature T in the spectral intervals** $(\omega,\ \omega + d\omega)$ and $(\lambda,\ \lambda + d\lambda)$.

Ans. We use the formula $\varepsilon = \hbar\omega$
Then the number of photons in the spectral interval (ω, ω + dω) is

$$n(\omega)\,d\omega = \frac{u(\omega, T)\,d\omega}{\hbar\omega} = \frac{\omega^2}{\pi^2 c^3}\,\frac{1}{e^{\hbar\omega/kT}-1}\,d\omega$$

using $n(\omega)\,d\omega = -\tilde{n}(\lambda)\,d\lambda$

we get $d\lambda\,\tilde{n}(\lambda) = n\left(\frac{2\pi c}{\lambda}\right)\frac{2\pi c}{\lambda^2}\,d\lambda$

$$= \frac{(2\pi c)^3}{\pi^2 c^3 \lambda^4}\,\frac{1}{e^{2\pi\hbar c/k\lambda T}-1}\,d\lambda$$

$$= \frac{8\pi}{\lambda^4} \frac{d\lambda}{e^{2\pi\hbar c/\lambda kT} - 1}$$

Q.260. An isotropic point source emits light with wavelength λ = 589 nm. The radiation power of the source is P = 10 W. Find:
(a) the mean density of the flow of photons at a distance r = = 2.0 m from the source;
(b) the distance between the source and the point at which the mean concentration of photons is equal to n = 100 cm^{-3}.

Ans. (a) The mean density of the flow of photons at a distance r is

$$<j> = \frac{P}{4\pi r^2} \Big/ \frac{2\pi\hbar c}{\lambda} = \frac{P\lambda}{8\pi^2\hbar c r^2} \text{ m}^{-2}\text{s}^{-2}$$

$$= \frac{10 \times \cdot 589 \times 10^{-6}}{8\pi^2 \times 1{\cdot}054 \times 10^{-34} \times 10^8 \times 4} \text{m}^{-2}\text{s}^{-1}$$

$$= \frac{10 \times \cdot 589}{8\pi^2 \times 1{\cdot}054 \times 12} \times 10^{16} \text{ cm}^{-2}\text{s}^{-1}$$

$$= 5{\cdot}9 \times 10^{13} \text{ cm}^{-2}\text{s}^{-1}$$

(b) If n(r) is the mean concentration (number per unit volume) of photons at a distance r form the source, then, since all photons are moving outwards with a velocity c, there is an outward flux of cm which is balanced by the flux from the source. In steady state, the two are equal and so

$$n(r) = \frac{<j>}{c} = \frac{P\lambda}{8\pi^2\hbar c^2 r^2} = n$$

so $r = \frac{1}{2\pi c}\sqrt{\frac{P\lambda}{2\hbar n}}$

$$= \frac{1}{6\pi \times 10^8}\sqrt{\frac{10 \times \cdot 589 \times 10^{-6}}{2 \times 1{\cdot}054 \times 10^{-34} \times 10^2 \times 10^6}}$$

$$= \frac{10^2}{6\pi}\sqrt{\frac{5{\cdot}89}{2{\cdot}108}} = 8{\cdot}87 \text{ metre}$$

Thermal Radiation. Quantum Nature of Light (Part - 2)

Q.261. From the standpoint of the corpuscular theory demonstrate that the momentum transferred by a beam of parallel light rays per unit time does not depend on its spectral composition but depends only on the energy flux Φ_e

Ans. The statement made in the question is not always correct. However it is correct in certain cases, for example, when light is incident on a perfect reflector or perfect absorber.

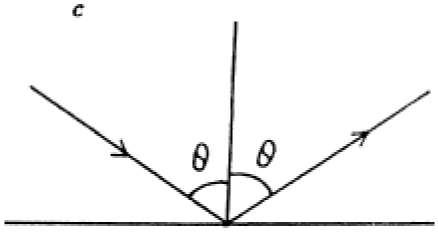
Consider the former case. If light is incident at an angle θ and reflected at the angle θ, then momentum transfered by each photon is

$$2\frac{h\nu}{c}\cos\theta$$

If there are n (v) dv photons in frequency interval (v, v + dv), then total momentum transfered is

$$\int_0^\infty 2n(\nu)\frac{h\nu}{c}\cos\theta\, d\nu$$

$$= \frac{2\Phi_e}{c}\cos\theta$$



Q.262. A laser emits a light pulse of duration ζ = 0.13 ms and energy E = 10 J. Find the mean pressure exerted by such a light pulse when it is focussed into a spot of diameter d = 10 pm on a surface perpendicular to the beam and possessing a reflection coefficient ρ = 0.50.

Ans. The mean pressure <p> is related to the force Fexerted by the beam by

$$<p> \times \frac{\pi d^2}{4} = F$$

The force F equals momentum transfered per second. This is (assuming that photons, not reflected, are absorbed)

$$2\rho\frac{E}{c\tau}+(1-\rho)\frac{E}{c\tau}=(1+\rho)\frac{E}{c\tau}.$$

The first term is the momentum transfered on reflection (see problem (261)); the second on absorption.

$$<p>=\frac{4(1+\rho)E}{\pi d^2 c\tau}$$

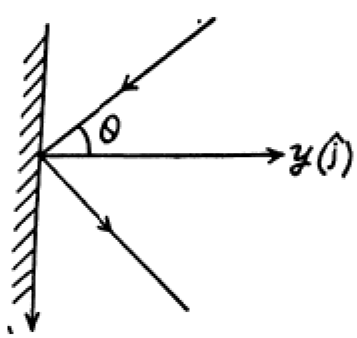
Substituting the values we get

$<p>=48{\cdot}3$ atmosphere.

Q.263. A short light pulse of energy E = 7.5 J falls in the form of a narrow and almost parallel beam on a mirror plate whose reflection coefficient is ρ = 0.60. The angle of incidence is 30°. In terms of the corpuscular theory find the momentum transferred to the plate.

Ans. The momentum transfered to the plate is

$$=\underbrace{\frac{E}{c}(1-\rho)\{\sin\theta\,\hat{i}-\cos\theta\,\hat{j}\}}_{\text{(momentum transfered on absorption)}}+\underbrace{\frac{E}{c}\rho\{-2\cos\theta\,\hat{j}\}}_{\text{(momentum transtered on reflection)}}$$



$$=\frac{E}{c}(1-\rho)\sin\theta\,\hat{j}-\frac{E}{c}(1+P)\cos\theta\,\hat{j}\quad(\hat{i})$$

Its magnitude is

$$\frac{E}{c}\sqrt{(1-\rho)^2\sin^2\theta+(1+\rho)^2\cos^2\theta} = \frac{E}{c}\sqrt{1+\rho^2+2\rho\cos 2\theta}$$

Substitution gives 35 n N.s as the answer.

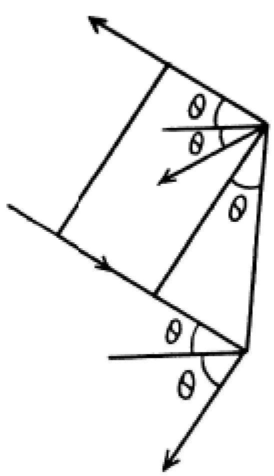
Q.264. A plane light wave of intensity I = 0.20 W/cm² falls on a plane mirror surface with reflection coefficient ρ = 0.8. The angle of incidence is 45°. In terms of the corpuscular theory find the magnitude of the normal pressure exerted by light on that surface.

Ans. Suppose the mirror has a surface area A.
The incident bean then has a cross section of A cos θ and the incident energy is $IA\cos\theta$;
Then the momentum transfered per second (= Force) is from the last problem

$$-\frac{IA\cos\theta}{c}(1+\rho)\cos\theta\,\hat{j}+\frac{IA\cos\theta}{c}(1-\rho)\sin\theta\,\hat{i}$$

The normal pressure is then $p = \frac{I}{c}(1+\rho)\cos^2\theta$

($\hat{j}$ is the unit vector $\perp^r$ to the plane mirror.)

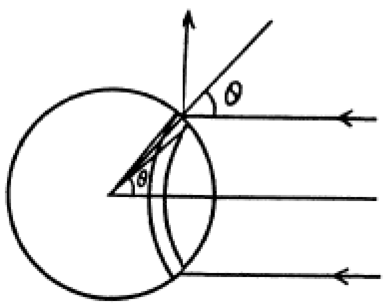


Putting in the values

$$p = \frac{0{\cdot}20\times10^4}{3\times10^8}\times1{\cdot}8\times\frac{1}{2} = 0{\cdot}6\,n\,\mathrm{N\,cm^{-2}}$$

Q.265. A plane light wave of intensity I = 0.70 W/cm[2] illuminates a sphere with ideal mirror surface. The radius of the sphere is R = 5.0 cm. From the standpoint of the corpuscular theory find the force that light exerts on the sphere.

Ans. We consider a strip defined by the angular range $(\theta, \theta + d\theta)$ From the previous problem the normal pressure exerted on this strip is



$$\frac{2I}{c}\cos^2\theta$$

This pressure gives rise to a force whose resultant, by symmetry is in the direction of the incident light. Thus

$$F = \frac{2I}{c}\int_0^{\pi/2}\cos^2\theta \cdot \cos\theta \cdot 2\pi R^2 \sin\theta\, d\theta = \pi R^2 \frac{I}{c}$$

Putting in the values

$$F = \pi \times 25 \times 10^{-4}\frac{0{\cdot}70 \times 10^4}{3 \times 10^8}\mathrm{N} = 0{\cdot}183\,\mu\mathrm{N}$$

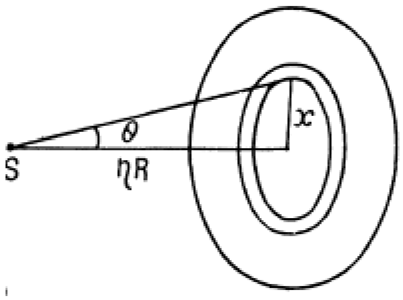
Q.266. An isotropic point source of radiation power P is located on the axis of an ideal mirror plate. The distance between the source and the plate exceeds the radius of the plate n-fold. In terms of the corpuscular theory find the force that light exerts on the plate.

Ans. Consider a ring of radius x on the plate. The normal pressure on this ring is, by problem (264),

$$\frac{2}{c}\frac{P}{4\pi(x^2+\eta^2R^2)}\cdot\cos^2\theta$$

$$= \frac{P}{2\pi c} \frac{\eta^2 R^2}{(x^2 + \eta^2 R^2)^2}$$

The total force is then



$$\int_0^R \frac{P}{2\pi c} \frac{\eta^2 R^2}{(x^2 + \eta^2 R^2)^2} 2\pi x \, dx$$

$$= \frac{P\eta^2 R^2}{2c} \int_{\eta^2 R^2}^{R^2(1+\eta^2)} \frac{dy}{y^2}$$

$$= \frac{P\eta^2 R^2}{2c} \left[\frac{1}{\eta^2 R^2} - \frac{1}{R^2(1+\eta^2)} \right] = \frac{P}{2c(1+\eta^2)}$$

Q.267. In a reference frame K a photon of frequency ω falls normally on a mirror approaching it with relativistic velocity V. Find the momentum imparted to the mirror during the reflection of the photon (a) in the reference frame fixed to the mirror; (b) in the frame K.

Ans. (a) In the reference frame fixed to the mirror, the frequency of the photon is, by the Doppler shift formula

$$\bar{\omega} = \omega \sqrt{\frac{1+\beta}{1-\beta}} \left(= \omega \frac{\sqrt{1-\beta^2}}{1-\beta} \right).$$

(see Eqn. (5.6b))

In this frame momentum imparted to the mirror of is

$$\frac{2\hbar\bar{\omega}}{c} = \frac{2\hbar\omega}{c} \sqrt{\frac{1+\beta}{1-\beta}}$$

(b) In the K frame, the incident particle carries a momentum of $\hbar\omega$ / c and returns with momentum

$$\frac{\hbar\omega}{c}\frac{1+\beta}{1-\beta}$$

(see problem 229). The momentum imparted to the mirror, then, has the magnitude

$$\frac{\hbar\omega}{c}\left[\frac{1+\beta}{1-\beta}+1\right]=\frac{2\hbar\omega}{c}\frac{1}{1-\beta}$$

Here $\beta=\frac{V}{c}$.

Q.268. A small ideal mirror of mass m = 10 mg is suspended by a weightless thread of length l = 10 cm. Find the angle through which the thread will be deflected when a short laser pulse with energy E = 13 J is shot in the horizontal direction at right angles to the mirror. Where does the mirror get its kinetic energy?

Ans. When light falls on a small mirror and is reflected by it, the mirror recoils. The energy of recoil is obtained from the incident beam photon and the frequency of reflected photons is less than the frequency of the incident photons. This shift of frequency can however be neglected in calculating quantities related to recoil (to a first approximation.) Thus, the momentum acquired by the mirror as a result of the laser pulse is

$$\left|\vec{p_f}-\vec{p_i}\right|=\frac{2E}{c}$$

Or assuming $\vec{p_i}=0$, we get

$$\left|\vec{p_f}\right|=\frac{2E}{c}$$

Hence the kinetic energy of the mirror is

$$\frac{p_f^2}{2m}=\frac{2E^2}{mc^2}$$

Suppose the mirror is deflected by an angle θ. Then by conservation of energy final P.E.

$$=mgl(1-\cos\theta)=\text{Initial K. E.}=\frac{2E^2}{mc^2}$$

or $$m g l 2 \sin^2 \frac{\theta}{2} = \frac{2E^2}{mc^2}$$

or $$\sin \frac{\theta}{2} = \left(\frac{E}{mc}\right)\frac{1}{\sqrt{g l}}$$

Using the data. $$\sin \frac{\theta}{2} = \frac{13}{10^{-5} \times 3 \times 10^8 \sqrt{9{\cdot}8 \times {\cdot}1}} = 4{\cdot}377 \times 10^{-3}$$

This gives $\theta = 0.502$ degrees .

Q.269. A photon of frequency ω_o is emitted from the surface of a star whose mass is M and radius R. Find the gravitational shift of frequency $\Delta\omega/\omega o$ of the photon at a very great distance from the star.

Ans. We shall only consider stars which are not too compact so that the gravitational field at their surface is weak:

$$\frac{\gamma M}{c^2 R} << 1$$

We shall also clarify the problem by making clear the meaning of the (slightly changed) notation.
Suppose the photon is emitted by some atom whose total relativistic energies (including the rest mass) are E_1 & E_2 with $E_1 < E_2$. These energies are defined in the absence of gravitational field and we have
$$\omega_0 = \frac{E_2 - E_1}{\hbar}$$
as the frequency at infinity of the photon that is emitted in $2 \rightarrow 1$ transition. On the surface of the star, the energies have the values

$$E'_2 = E_2 - \frac{E_2}{c^2} \cdot \frac{\gamma M}{R} = E_2\left(1 - \frac{\gamma M}{c^2 R}\right)$$

$$E'_1 = E_1\left(1 - \frac{\gamma M}{c^2 R}\right)$$

Thus, from $\hbar\omega = E'_2 - E'_2$ we get

$$\omega = \omega_0\left(1 - \frac{\gamma M}{c^2 R}\right)$$

Here ω is the frequency of the photon emitted in the transition 2 → 1 when the atom is on the surface of the star. In shows that the frequency of spectral lines emitted by atoms on the surface of some star is less than the frequency of lines emitted by atoms here on earth (where the gravitational effect is quite small).

Finally $\frac{\Delta \omega}{\omega_0} = -\frac{\gamma M}{c^2 R}$.

The answer given in /the book is incorrect in general though it agrees with the above

result for $\frac{\gamma M}{c^2 R} << 1$.

Q.270. A voltage applied to an X-ray tube being increased η = = 1.5 times, the short-wave limit of an X-ray continuous spectrum shifts by Δλ = 26 pm. Find the initial voltage applied to the tube.

Ans. The general formula is

$$\frac{2\pi\hbar c}{\lambda} = eV$$

Thus $\lambda = \frac{2\pi\hbar c}{eV}$

Now $\Delta\lambda = \frac{2\pi\hbar c}{eV}\left(1 - \frac{1}{\eta}\right)$

Hence $V = \frac{2\pi\hbar c}{e\Delta\lambda}\left(\frac{\eta - 1}{\eta}\right) = 15{\cdot}9\text{ kV}$

Q.271. A narrow X-ray beam falls on a NaC1 single crystal. The least angle of incidence at which the mirror reflection from the system of crystallographic planes is still observed is equal to α = = 4.1°. The interplanar distance is d = 0.28 nm. How high is the voltage applied to the X-ray tube?

Ans. We have as in the above problem

$$\frac{2\pi\hbar c}{\lambda} = eV$$

On the other hand, from Bragg's law $2d\sin\alpha = k\lambda = \lambda$

since k = 1 when α takes its smallest value.

Thus $V = \frac{\pi \hbar c}{e d \sin \alpha} = 30{\cdot}974\, k\, V \approx 31\, \text{kV}.$

Q.272. Find the wavelength of the short-wave limit of an X-ray continuous spectrum if electrons approach the anticathode of the tube with velocity v = 0.85 c, where c is the velocity of light.

Ans. The wavelength of X - rays is the least when all the K .E. of the electrons approaching the anticathode is converted into the eneigy of X - rays.
But the K.E. of electron is

$$T_m = m c^2 \left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right]$$

(me^2 = rest mass eneigy of electrons = 0.511 MeV)

Thus $\frac{2 \pi \hbar c}{\lambda} = T_m$

or $$\lambda = \frac{2 \pi \hbar c}{T_m} = \frac{2 \pi \hbar}{m c} \left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right]^{-1}$$

$$= \frac{2 \pi \hbar}{m c (\gamma - 1)}, \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = 2{\cdot}70 \text{ pm}.$$

Q.273. Find the photoelectric threshold for zinc and the maximum velocity of photoelectrons liberated from its surface by electromagnetic radiation with wavelength 250 nm.

Ans. The work function of zinc is

$$A = 3{\cdot}74\, romane = 3{\cdot}74 \times 1{\cdot}602 \times 10^{-19} \text{ Joule}$$

The threshold wavelength for photoelectric effect is given by

$$\frac{2 \pi \hbar c}{\lambda_0} = A$$

Or

$$\lambda_0 = \frac{2\pi\hbar c}{A} = 331{\cdot}6 \text{ nm}$$

The maximum velocity of photoelectrons liberated by light of wavelength λ is given by

$$\frac{1}{2} m v_{max}^2 = 2\pi\hbar c\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)$$

So $$v_{max} = \sqrt{\frac{4\pi\hbar c}{m}\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)} = 6{\cdot}55 \times 10^5 \text{ m/s}$$

Q.274. Illuminating the surface of a certain metal alternately with light of wavelengths λ_1 = 0.35 tim and λ_2 = 0.54 lam, it was found that the corresponding maximum velocities of photoelectrons differ by a factor η = 2.0. Find the work function of that metal.

Ans. From the last equation of the previous problem, we find

$$\eta = \frac{(v_1)_{max}}{(v_2)_{max}} = \sqrt{\frac{\frac{1}{\lambda_1} - \frac{1}{\lambda_0}}{\frac{1}{\lambda_2} - \frac{1}{\lambda_0}}}$$

Thus $$\eta^2\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_0}\right) = \frac{1}{\lambda_1} - \frac{1}{\lambda_0}$$

or $$\frac{1}{\lambda_0}(\eta^2 - 1) = \frac{\eta^2}{\lambda_2} - \frac{1}{\lambda_1}$$

and $$\frac{1}{\lambda_0} = \left(\frac{\eta^2}{\lambda_2} - \frac{1}{\lambda_1}\right) \Big/ (\eta^2 - 1)$$

So $$A = \frac{2\pi\hbar c}{\lambda_0} = \frac{2\pi\hbar c}{\lambda_2}\,\frac{\eta^2 - \frac{\lambda_2}{\lambda_1}}{\eta^2 - 1} = 1{\cdot}88 \text{ eV}$$

Q.275. Up to what maximum potential will a copper ball, remote from all other bodies, be charged when irradiated by electromagnetic radiation of wavelength λ = 140 nm?

Ans. When light of sufficiently short wavelength falls on the ball, photoelectrons are ejected and the copper ball gains positive change. The charged ball tends to resist further emission of electrons by attracting them. When the copper ball has enough chaige even the most energetic electrons are unable to leave it we can calculate this final maximum potential of the copper ball. It is obviously equal in magnitude (in volt) to the maximum K.E of electrons (in electron volts) initially emitted. Hence

$$\varphi_{max} = \frac{2\pi\hbar c}{\lambda e} - A_{cu}$$

= 8.86 - 4.47 = 4.39 volts

(Acu is the work function of copper.)

Thermal Radiation. Quantum Nature of Light (Part - 3)

Q.276. Find the maximum kinetic energy of photoelectrons liberated from the surface of lithium by electromagnetic radiation whose electric component varies with time as E = a (1 + cos ωt) cos ω_0t, where a is a constant, ω = $6.0.10^{14}$ s^{-1} and ω_o = $3.60.10^{15}$ s^{-1} .

Ans. We write

$$E = a(1+\cos\omega t)\cos\omega_0 t$$

$$= a\cos\omega_0 t + \frac{a}{2}\left[\cos(\omega_0-\omega)t + \cos(\omega_0+\omega)t\right]$$

It is obvious that light has three frequencies and the maximum K.E. of photo electrons ejected is

$$\hbar(\omega+\omega_0) - A_{Li}$$

where $A_{Li} = 2{\cdot}39\ \text{eV}$. Substituting we get 0.37 eV.

Q.277. Electromagnetic radiation of wavelength λ = 0.30 μm falls on a photocell operating in the saturation mode. The corresponding spectral sensitivity of the photocell is J = 4.8 mA/W. Find the yield of photoelectrons, i.e. the number of photoelectrons produced by each incident photon.

Ans. Suppose N photons fall on the photocell per sec. Then the power incident is $N\frac{2\pi\hbar c}{\lambda}$

This will give rise to a photocurrent of $N\frac{2\pi\hbar c}{\lambda}\cdot J$

which means that $N\frac{2\pi\hbar c}{e\lambda}\cdot J$

electrons have been emitted. Thus the number of photoelectrons produced by each photon is

$$w = \frac{2\pi\hbar c J}{e\lambda} = 0{\cdot}0198 \approx 0{\cdot}02$$

Q.278. There is a vacuum photocell whose one electrode is made of cesium and the other of copper. Find the maximum velocity of photoelectrons approaching the

copper electrode when the cesium electrode is subjected to electromagnetic radiation of wavelength 0.22 p.m and the electrodes are shorted outside the cell.

Ans. A simple application of Einstein's equation

$$\frac{1}{2} m v_{\max}^2 = \hbar\nu - \hbar\nu_0 = \frac{2\pi\hbar c}{\lambda} - A_{cs}$$

gives incorrect result in this case because the photoelectrons emitted by the Cesium electrode are retarded by the small electric Geld that exists between the cesium electrode and the Copper electrode even in the absence of external emf. This small electric Geld is caused by the contact potential difference whose magnitude equals the difference of work functions

$$\frac{1}{e}(A_{cu} - A_{cs}) \text{ volts .}$$

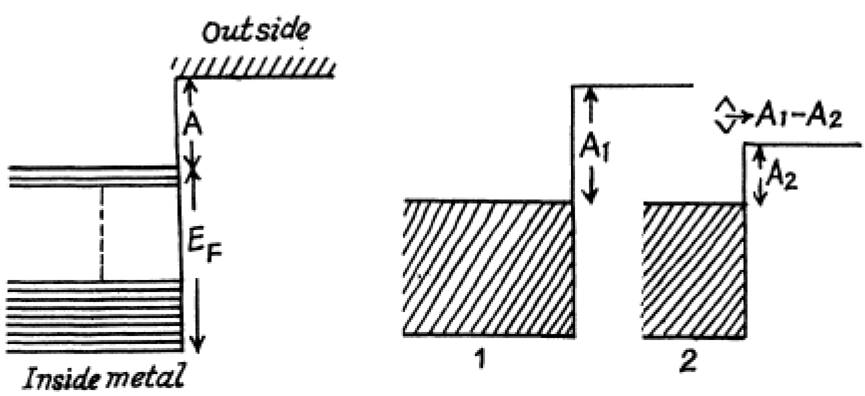
Its physical origin is explained below.
The maximum velocity of the photoelectrons reaching the copper electrode is then

$$\frac{1}{2} m v_m^2 = \frac{1}{2} m v_0^2 - (A_{cu} - A_{cs}) = \frac{2\pi\hbar c}{\lambda} - A_{cu}$$

Here v_0 is the maximum velocity of the photoelectrons immediately after emission. Putting the values we get, on using $A_{cu} = 4{\cdot}47\,\text{eV},\ \lambda = 0{\cdot}22\,\mu\text{m}$,

$$v_m = 6{\cdot}41 \times 10^5\,\text{m/s}$$

The origin of contact potential difference is the following. Inside the metals free electrons can be thought of as a Fermi gas which occupy energy levels upto a maximum called the Fermi energy E_P . The work function A measures the depth of the Fermi level.



When two metals 1 & 2 are in contact, electrons flow from one to the other till their Fermi levels are the same. This requires the appearance of contact potential difference of $A_1 - A_2$ between the two metals externally.

Q.279. A photoelectric current emerging in the circuit of a vacuum photocell when its zinc electrode is subjected to electromagnetic radiation of wavelength 262 nm is cancelled if an external decelerating voltage 1.5 V is applied. Find the magnitude and polarity of the outer contact potential difference of the given photocell.

Ans. The maximum K.E. of the photoelectrons emitted by the Zn cathode is

$$E_{max} = \frac{2\pi\hbar c}{\lambda} - A_{zn}$$

On calculating this comes out to be 0.993 eV ≈ 1.0 eV Since an external decelerating voltage of 1.5 V is required to cancel this current, we infer that a contact potential difference of 1.5 - 1.0 = 0.5 V exists in the circuit whose polarity is opposite of the decelerating voltage.

Q.280. Compose the expression for a quantity whose dimension is length, using velocity of light c, mass of a particle m, and Planck's constant $\hbar$. What is that quantity?

Ans. The unit of $\hbar$. is Joule-sec. Since mc^2 is the rest mass energy, $\frac{\hbar}{mc^2}$ as dimension of time and multiplying by c we get a quantity

$$\bar{\lambda}_c = \frac{\hbar}{mc}$$

whose dimension is length. This quantity is called reduced Compton wavelength.

(The name Compton wavelength is traditionally reserved for m $\frac{2\pi\hbar}{mc}$)

Q.281. Using the conservation laws, demonstrate that a free electron cannot absorb a photon completely.

Ans. We consider the collision in the rest frame of the initial electron. Then the reaction is

$$\gamma + e\,(\text{rest}) \longrightarrow e\,(\text{moving})$$

Energy momentum conservation gives

$$\hbar\omega + m_0 c^2 = m_0 c^2 \Big/ \sqrt{1-\beta^2}$$

$$\frac{\hbar\omega}{c} = \frac{m_0 c \beta}{\sqrt{1-\beta^2}}$$

where ω is the angular frequency of the photon.

Eliminating hω we get

$$m_0 c^2 = m_0 c^2 \frac{1-\beta}{\sqrt{1-\beta^2}} = m_0 c^2 \sqrt{\frac{1-\beta}{1+\beta}}$$

This gives $\beta = 0$ which implies $\hbar\omega = 0$

But a zero energy photon means no photon.

282. Explain the following features of Compton scattering of light by matter:
(a) the increase in wavelength Δλ, is independent of the nature of the scattering substance;
(b) the intensity of the displaced component of scattered light grows with the increasing angle of scattering and with the diminishing atomic number of the substance;
(c) the presence of a non-displaced component in the scattered radiation.

Ans. (a) Compton scattering is the scattering of light by free electrons. (The free electrons are the electrons whose binding is much smaller than the typical energy transfer to the electrons). For this reason the increase in wavelength Δλ is independent of the nature of the scattered substance.

(b) This is because the effective number of free electrons increases in both cases. With increasing angle of scattering, the energy transfered to electrons increases. With diminishing atomic number of the substance the binding energy of the electrons decreases.

(c) The presence of a non-displaced component in the scattered radiation is due to scattering from strong bound (inner) electrons as well as nuclei. For scattering by these the atom essentially recoils as a whole and there is very little energy transfer.

Q.283. A narrow monochromatic X-ray beam falls on a scattering substance. The wavelengths of radiation scattered at angles θ_1 = 60° and θ_2 = 120° differ by a factor η = 2.0. Assuming the free electrons to be responsible for the scattering, find the incident radiation wavelength.

Ans. Let λ_0 - wavelength of the incident radiation.
Then wavelength of the radiation scattered at $\boldsymbol{\theta_1}$ = 60°

$$= \lambda_1 = \lambda_0 + 2\pi \lambda\!\!\!^{-}_c (1 - \cos\theta_1) \quad \text{where } \lambda\!\!\!^{-}_c = \frac{\hbar}{mc}.$$

and similarly

$$\lambda_2 = \lambda_0 + 2\pi \lambda\!\!\!^{-}_c (1 - \cos\theta_2)$$

From the data $\theta_1 = 60°, \ \theta_2 = 120°$

$$\lambda_2 = \eta \lambda_1$$

Thus

$$(\eta - 1)\lambda_0 = 2\pi \lambda\!\!\!^{-}_c \left[1 - \cos\theta_2 - \eta(1 - \cos\theta_1)\right]$$

$$= 2\pi \lambda\!\!\!^{-}_c \left[1 - \eta + \eta\cos\theta_1 - \cos\theta_2\right]$$

Hence

$$\lambda_0 = 2\pi \lambda\!\!\!^{-}_c \left[\frac{\eta\cos\theta_1 - \cos\theta_2}{\eta - 1} - 1\right]$$

$$= 4\pi \lambda\!\!\!^{-}_c \left[\frac{\sin^2\theta_2/2 - \eta\sin^2\theta_1/2}{\eta - 1}\right] = 1{\cdot}21 \text{ pm}.$$

The expression λ_0 given in the book contains misprints.

Q.284. A photon with energy $\hbar\omega$ = 1.00 MeV is scattered by a stationary free electron. Find the kinetic energy of a Compton electron if the photon's wavelength changed by η = 25% due to scattering.

Ans. The wave lengths of the photon has increased by a fraction η so its final wavelength is

$$\lambda_f = (2 + \eta)\lambda_i$$

and its energy is $\frac{\hbar\omega}{1+\eta}$

The K.E. of the Compton electron is the energy lost by the photon and is

$$T = \hbar\omega\left(1 - \frac{1}{1+\eta}\right) = \hbar\omega\frac{\eta}{1+\eta}$$

Q.285. A photon of wavelength λ = 6.0 pm is scattered at right angles by a stationary free electron. Find:
(a) the frequency of the scattered photon;
(b) the kinetic energy of the Compton electron.

Ans. (a) From the Compton formula

$$\lambda' = 2\pi\lambda\!\!\!^{-}_c(1 - \cos 90) + \lambda$$

Thus $\omega' = \frac{2\pi c}{\lambda'} = \frac{2\pi c}{\lambda + 2\pi\lambda\!\!\!^{-}_c}$ where $2\pi\lambda\!\!\!^{-}_c = \frac{h}{mc}$.

Substituting the values, we get $\omega' = 2{\cdot}24 \times 10^{20}$ rad/sec

(b) The kinetic energy of the scattered electron (in the frame in which the initial electron was stationary) is simply

$$T = \hbar\omega - \hbar\omega'$$

$$= \frac{2\pi\hbar c}{\lambda} - \frac{2\pi\hbar c}{\lambda + 2\pi\lambda\!\!\!^{-}_c}$$

$$= \frac{4\pi^2\hbar c\lambda\!\!\!^{-}_c}{\lambda(\lambda + 2\pi\lambda_c)} = \frac{2\pi\hbar c/\lambda}{1 + \lambda/2\pi\lambda\!\!\!^{-}_c} = 59{\cdot}5 \text{ kV}$$

Q.286. A photon with energy how $\hbar\omega$ = 250 keV is scattered at an angle θ = 120° by a stationary free electron. Find the energy of the scattered photon.

Ans. The wave length of the incident photon is

$$\lambda_0 = \frac{2\pi c}{\omega}$$

Then the wavelength of the final photon is

$$\frac{2\pi c}{\omega} + 2\pi \lambda\!\!\!^{-}_c (1 - \cos\theta)$$

and the energy of the final photon is

$$\hbar\omega' = \frac{2\pi\hbar c}{\frac{2\pi c}{\omega} + 2\pi\lambda_c(1-\cos\theta)} = \frac{h\omega}{1 + \frac{h\omega}{mc^2}(1-\cos\theta)}$$

$$= \frac{h\omega}{1 + 2\left(\frac{h\omega}{mc^2}\right)\sin^2(\theta/2)} = 144{\cdot}2\ \text{kV}$$

Q.287. A photon with momentum p = 1.02 MeV/c, where c is the velocity of light, is scattered by a stationary free electron, changing in the process its momentum to the value p' = 0.255 MeV/c. At what angle is the photon scattered?

Ans. We use the equation

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar}{p}.$$

Then from Compton formula

$$\frac{2\pi\hbar}{p'} = \frac{2\pi\hbar}{p} + 2\pi\frac{\hbar}{mc}(1-\cos\theta)$$

so $\frac{1}{p'} = \frac{1}{p} + \frac{1}{mc}\cdot 2\sin^2\theta/2$

Hence $\sin^2\frac{\theta}{2} = \frac{mc}{2}\left(\frac{1}{p'} - \frac{1}{p}\right)$

$$= \frac{mc(p-p')}{2pp'}$$

or $\sin\frac{\theta}{2} = \sqrt{\frac{mc(p-p')}{2pp'}}$,

Substituting from the data

$$\sin\frac{\theta}{2} = \sqrt{\frac{mc^2(cp - cp')}{2cp\cdot cp'}} = \sqrt{\frac{0{\cdot}511\,(1{\cdot}02 - 0{\cdot}255)}{2 \times 1{\cdot}02 \times 0{\cdot}255}}$$

This gives **θ** = 120.2 degrees.

Q.288. A photon is scattered at an angle θ = 120° by a stationary free electron. As a result, the electron acquires a kinetic energy T = 0.45 MeV. Find the energy that the photon had prior to scattering.

Ans. From the Compton formula

$$\lambda = \lambda_0 + \frac{2\pi\hbar}{mc}(1 - \cos\theta)$$

From conservation of eneigy

$$\frac{2\pi\hbar c}{\lambda_0} = \frac{2\pi\hbar c}{\lambda} + T = \frac{2\pi\hbar c}{\lambda_0 + \frac{2\pi\hbar}{mc}(1-\cos\theta)} + T$$

or
$$\frac{4\pi\hbar}{mc}\sin^2\frac{\theta}{2} = \frac{T}{2\pi\hbar c}\lambda_0\left(\lambda_0 + \frac{4\pi\hbar}{mc}\sin^2\frac{\theta}{2}\right)$$

or introducing $\hbar\omega_0 = 2\pi\hbar c/\lambda_0$

$$\frac{2\sin^2\theta/2}{mc^2} = \frac{T}{\hbar\omega_0}\left(\frac{1}{\hbar\omega_0} + \frac{2}{mc^2}\sin^2\frac{\theta}{2}\right)$$

Hence
$$\left(\frac{1}{\hbar\omega_0}\right)^2 + 2\frac{1}{\hbar\omega_0}\frac{\sin^2\frac{\theta}{2}}{mc^2} - \frac{2\sin^2\frac{\theta}{2}}{mc^2T} = 0$$

$$\left(\frac{1}{\hbar\omega_0} + \frac{\sin^2\frac{\theta}{2}}{mc^2}\right)^2 = \frac{2\sin^2\frac{\theta}{2}}{mc^2T} + \left(\frac{\sin^2\frac{\theta}{2}}{mc^2}\right)^2$$

$$\frac{1}{\hbar\omega_0} = \frac{\sin^2\frac{\theta}{2}}{mc^2}\left[\sqrt{1 + \frac{2mc^2}{T\sin^2\theta/2}} - 1\right.$$

$$\hbar\omega_0 = \frac{mc^2/\sin^2\theta/2}{\sqrt{1+\frac{2mc^2}{T\sin^2\frac{\theta}{2}}}-1}$$

or

$$= \frac{T}{2}\left[\sqrt{1+\frac{2mc^2}{T\sin^2\theta/2}}+1\right]$$

Substituting we get $\hbar\omega_0 = 0{\cdot}677\ \text{MeV}$

Q.289. Find the wavelength of X-ray radiation if the maximum kinetic energy of Compton electrons is T_{max} = 0.19 MeV.

Ans. We see from the previous problem that the electron gains the maximum K.E. when the photon is scattered backwards θ = 180°. Then

$$\omega_0 = \frac{mc^2/\hbar}{\sqrt{1+\frac{2mc^2}{T_{max}}}-1}$$

Hence

$$\lambda_0 = \frac{2\pi c}{\omega_0} = \frac{2\pi\hbar}{mc}\left[\sqrt{1+\frac{2mc^2}{T_{max}}}-1\right]$$

Substituting the values we get **Δλ** = 3.695 pm .

Q.290. A photon with energy $\hbar\omega$ = 0.15 MeV is scattered by a stationary free electron changing its wavelength by Δλ = 3.0 pm. Find the angle at which the Compton electron moves.

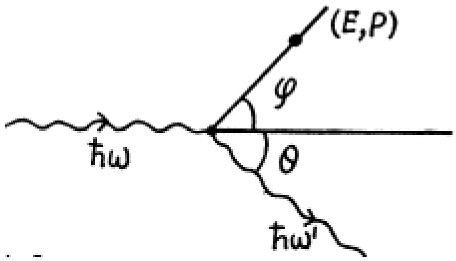
Ans. Refer to the diagram. Energy momentum conservation gives

$$\frac{\hbar\omega'}{c} - \frac{\hbar\omega'}{c}\cos\theta = p\cos\varphi$$

$$\frac{h\omega'}{c}\sin\theta = p\sin\varphi$$

$$h\omega + mc^2 = h\omega' + E$$

Where $E^2 = c^2 p^2 + m^2 c^4$. we see



$$\tan \varphi = \frac{\omega' \sin \theta}{\omega - \omega' \cos \theta} = \frac{\frac{1}{\lambda'} \sin \theta}{\frac{1}{\lambda} - \frac{1}{\lambda'} \cos \theta}$$

$$= \frac{\lambda \sin \theta}{\lambda' - \lambda \cos \theta} = \frac{\sin \theta}{\frac{\Delta \lambda}{\lambda} + 2 \sin^2 \frac{\theta}{2}}$$

where $\Delta \lambda = \lambda' - \lambda = 2 \pi \lambda_c (1 - \cos \theta) = 4 \pi \lambda_c \sin^2 \frac{\theta}{2}$

Hence $\tan \varphi = \dfrac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\frac{\Delta \lambda}{\lambda} + \frac{\Delta \lambda}{2 \pi \lambda_c}}$

But $\sin \theta = 2 \sqrt{\dfrac{\Delta \lambda}{4 \pi \lambda_c}} \sqrt{1 - \dfrac{\Delta \lambda}{4 \pi \lambda_c}} = \dfrac{\Delta \lambda}{2 \pi \lambda_c} \sqrt{\dfrac{4 \pi \lambda_c}{\Delta \lambda} - 1}$

Thus $\tan \varphi = \dfrac{\sqrt{\frac{4 \pi \hbar}{m c \Delta \lambda} - 1}}{1 + \frac{2 \pi \hbar}{m c \lambda}} = \dfrac{\sqrt{\frac{4 \pi \hbar}{m c \Delta \lambda} - 1}}{1 + \frac{\hbar \omega}{m c^2}} = 31 \cdot 3°$

Q.291. A photon with energy exceeding η = 2.0 times the rest energy of an electron experienced a head-on collision with a stationary free electron. Find the curvature radius of the trajectory of the Compton electron in a magnetic field B = 0.12 T. The Compton electron is assumed to move at right angles to the direction of the field.

Ans. By head on collision we understand that the electron moves on in the direction of the incident photon after the collision and the photon is scaltered backwards. Then, let us write

$$\hbar \omega = \eta m c^2$$
$$\hbar \omega' = \sigma m c^2$$

$(E, p) = (\varepsilon m c^2, \mu m c)$ of the electron.

Then by energy momentum conservation (cancelling factors of mc[2] and me)

$$1 + \eta = \sigma + \varepsilon$$

$$\eta = \mu - \sigma$$
$$\varepsilon^2 = 1 + \mu^2$$

So eliminating σ & ε $1 + \eta = -\eta + \mu + \sqrt{\mu^2 + 1}$

or $(1 + 2\eta - \mu) = \sqrt{\mu^2 + 1}$

Squaring $(1 + 2\eta)^2 - 2\mu(1 + 2\eta) = 1$

$$4\eta + 4\eta^2 = 2\mu(1 + 2\eta)$$

or $\mu = \dfrac{2\eta(1 + \eta)}{1 + 2\eta}$

Thus the momentum of the Compton electron is

$$p = \mu m c = \frac{2\eta(1 + \eta) m c}{1 + 2\eta}.$$

Now in a m agentic field p = B e p

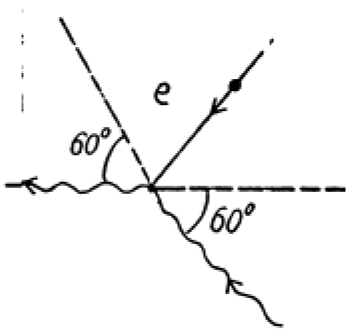
Thus $\rho = 2\eta(1 + \eta) \Big/ (1 + 2\eta) \dfrac{m c}{B e}.$

Substituting the values ρ = 3.412 cm.

Q.292. Having collided with a relativistic electron, a photon is deflected through an angle θ = 60° while the electron stops. Find the Compton displacement of the wavelength of the scattered photon.

Ans. This is the inverse of usual Compton scattering.
When we write down the energy-momentum conservation equation for this process we find that they are the same for the inverse process as they arc for the usual process.

If follows that the formula for Compton shift is applicable \ except that the energy (frequency) of the photon is ^ increased on scattering and the wavelength is shifted downward. With this understanding, we write



$$\Delta \lambda = 2\pi \frac{\hbar}{mc}(1 - \cos\theta)$$

$$= 4\pi\left(\frac{\hbar}{mc}\right)\sin^2\frac{\theta}{2} = 1{\cdot}21 \text{ pm}$$