Chapter 6

Application of Derivatives

Tangents at the Origin

B. Tangents at the Origin

If a curve passing through the origin be given by a rational integral algebraic equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation.

e.g., if the equation of a curve be $x^2 - 4y^2 + x^4 + 3x^3y + 3x^2 y^2 + y^4 = 0$, the tangents at the origin are given by $x^2 - 4y^2 = 0$ or $x + 2y$ and $x - 2y = 0$.

In the curve $x^2 + y^2 + ax + by = 0$, $ax + by = 0$, is the equation of the tangent at the origin; and in the curve $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$, $x^2 - y^2 = 0$ is the equation of a pair of tangents at the origin.

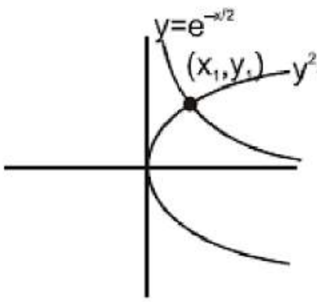
If the equation of a curve be $x^2 + y^2 + x^3 + 3 x^2 y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ i.e. $x + y = 0$ and $x - y = 0$

C. Angle of intersection

Angle of intersection between two curves is defined as the angle between the two tangents drawn to the two curves at their point of intersection . If the angle between two curves is 90° then they are called **ORTHOGONAL** curves.

Ex.10 Find the angle between curves $y^2 = 4x$ and $y = e^{-x/2}$

Sol.



Let the curves intersect at point (x_1,y_1)

for $y^2 = 4x\ \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \frac{2}{y_1}$

and for $y = e^{-x/2}\ \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = -\frac{1}{2}e^{-x_1/2}$

$= -\frac{y_1}{2} \Rightarrow m_1, m_2 = -1$ Hence $\theta=90°$

Note : here that we have not actually found the intersection point but geometrically we can see that the curves intersect.

Ex.11 Show that the curves $y = 2\sin^2 x$ and $y = \cos 2x$ intersect at $\pi/6$. What is their angle of intersection ?

Sol. Given curves are $y = 2\sin^2 x$ ...(1)

and $y = \cos 2x$ ...(2)

Solving (1) and (2), we get $2\sin^2 x = \cos 2x$

$\Rightarrow 1 \cos 2x = \cos 2x \Rightarrow \cos 2x = 1/2 \Rightarrow \cos \pi/3 \Rightarrow 2x = \pm \pi/3$

$x = \pm\pi/6$ are the points of intersection

From (1), $dy/dx = 4 \sin x \cos x = 2 \sin 2x = m_1$ (say)

From (2) $dy/dx = -2 \sin 2x = m_2$ (say)

If angle of intersection is θ, then $\tan\theta$ $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{4\sin 2x}{1 - 4\sin^2 2x}\right|$

$$\therefore \quad (\tan\theta)_{x = \pm\pi/6} = \left|\frac{4 \times \pm\frac{\sqrt{3}}{2}}{1 - 4 \times \frac{3}{4}}\right| = \left|\frac{\pm 2\sqrt{3}}{-2}\right| = \sqrt{3}$$

$$\therefore \quad \theta = \frac{\pi}{3}$$

Ex.12 Show that the angle between the tangents at any point P and the line joining P to the origin `O' is the same at all points of the curve ln $(x^2 + y^2) = c \tan^{-1} (y/x)$ where c is constant.

Sol. Let the point P(x, y) on the curve ln $(x^2 + y^2) = c \tan^{-1} (y/x)$ where c is constant.

Differentiating both sides w.r.t. x, we get

$$\frac{2x+2yy'}{(x^2+y^2)} = \frac{c(xy'-y)}{(x^2+y^2)} \Rightarrow \quad y' = \frac{2x+cy}{cx-2y} = m_1 \text{ (say)}$$

Slope of OP $= y/x = m_2$ (say)

Let the angle between the tangents at P and OP be θ

$$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right| = \left|\frac{\frac{2x+cy}{cx-2y} - \frac{y}{x}}{1+\frac{2xy+cy^2}{cx^2-2xv}}\right| = \frac{2}{c}.$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{2}{c}\right)$$ which is independent of x and y..

Ex.13 Show tht the curves $\frac{x^2}{a^2+k_1} + \frac{y^2}{b^2+k_1} = 1$ and $\frac{x^2}{a^2+k_2} + \frac{y^2}{b^2+k_2} = 1$ intersect orthogonally.

Sol.

Given $\frac{x^2}{a^2+k_1} + \frac{y^2}{b^2+k_1} = 1 \quad \ldots(1)$

and $\frac{x^2}{a^2+k_2} + \frac{y^2}{b^2+k_2} = 1 \qquad \ldots(2)$

Subtracting (2) from (1), we get $x^2\left(\frac{1}{a^2+k_1} - \frac{1}{a^2+k_2}\right) + y^2\left(\frac{1}{b^2+k_1} - \frac{1}{b^2+k_2}\right) = 0$

$$\Rightarrow x^2\left(\frac{k_2-k_1}{(a^2+k_1)(a^2+k_2)}\right) + y^2\left(\frac{k_2-k_1}{(b^2+k_1)(b^2+k_2)}\right) = 0$$

$$\therefore \frac{x^2}{y^2} = -\frac{(a^2+k_1)(a^2+k_2)}{(b^2+k_1)(b^2+k_1)} \quad ...(3)$$

Now from (1), $\frac{2x}{(a^2+k_1)} + \frac{2y}{(b^2+k_1)}\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\frac{x(b^2+k_1)}{y(a^2+k_1)} = m_1 \text{ (say)}$$

Similarly from (2), $\frac{dy}{dx} = -\frac{x(b^2+k_2)}{y(a^2+k_2)} = m_2$ (say)

$$\Rightarrow m_1 m_2 = \frac{x^2(b^2+k_1)(b^2+k_2)}{y^2(a^2+k_1)(a^2+k_2)} = -1 \text{ [From (3)]}$$

Hence given curves intersect orthogonally.

Ex.14 Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.

Sol.

Equation of the given curves are $xy = 4$ ....(i) and $x^2 + y^2 = 8$ ....(ii)

from (i), $1.y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$ ....(iii), from (ii),

$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$ ....(iv)

Putting the value fo y from (i) i (ii), we get $x^2 + 16/x^2 = 8$ or or $x^4 + 16 = 8x^2$

or $x^4\ 8x^2 + 16 = 0$ or $(x^2\ 4)^2 = 0$ or $x^2\ 4 = 0$ or $x^2 = 4$

from (i) ; when $x = 2, y = 2$ and when $x = 2, y = -2$

Hence points of intersection of the two curves are (2, 2) and (2, 2).

Slope of the tangent to the curve (i) at point (2, 2) $\Rightarrow m_1 = -2/2 = -1$ ...(from iii)

Slope of tangent to the curve (ii) at point (2, 2) $\Rightarrow m_2 = -2/2 = -1$....(from iv)

Since $m_3 = m_4$, hence the two curves touch each other at (-2, -2). Thus curves (i) and (ii) touch each other.

Slope of tangent to curve (i), $m_3 = -\left(\frac{-2}{-2}\right) = -1$

Slope of tangent to curve (ii), $m_4 = -\left(\frac{-2}{-2}\right) = -1$

Since $m_3 = m_4$, hence the two curves touch each other at (2, 2). Thus curves (i) and (ii) touch each other.

Ex.15 The gradient of the common tangent to the two curves $y = x^2 - 5x + 6$ & $y = x^2 + x + 1$ is

(A) - 1/3

(B) - 2/3

(C) - 1

(D) - 3

Sol. $y = ax + b$ on solving with both curves and putting $D = 0$ gives

$a^2 + 10a + 4b + 1 = 0$ and $a^2 - 2a + 4b - 3 = 0 \Rightarrow a = -1/3$ & $b = 5/9$

$\Rightarrow 3x + 9y = 5$; point of contact $(7/3, -2/9)$ & $(-2/3, 7/9)$

D. Length of Tangent

(a) Length of the tangent (PT) = $\frac{y_1\sqrt{1+[f'(x_1)]^2}}{f'(x_1)}$

(b) Subtangent (MT) = $\frac{y_1}{f'(x_1)}$

(c) Length of Normal (PN) = $y_1\sqrt{1+[f'(x_1)]^2}$

(d) Subnormal (MN) = $y_1 f'(x_1)$

Ex.16 What should be the value of n in the equation of curve $y = a^{1-n}. x^n$, so that the sub-normal may be of constant length ?

Sol. Given curve is $y = a^{1-n} . x^n$

Taking logarithm of both sides, we get, $\ell n\ y = (1 - n)\ \ell n\ a + n\ \ell n\ x$

Differentiating both sides w.r.tx, we get $\frac{1}{y}.\frac{dy}{dx}=0+\frac{n}{x}$ or $\frac{dy}{dx}=\frac{ny}{x}$ ...(1)

Lengths of sub-normal = y dy/dx = y .ny/x ....{from 1}

$$= \frac{ny^2}{x} = n.\frac{(a^{1-n}x^n)^2}{x}$$

$$(\because y = a^{1-n} . x^n) \quad = n . a^{2-2n} . x^{2n-1}$$

Since lengths of sub-normal is to be constant, so x should not appear in its value i.e., 2n 1 = 0. n =1/2

Ex.17 If the relation between sub-normal SN and sub-tangent ST at any point S on the curve

$by^2 = (x + a)^3$ is p(SN) = q(ST)^2; then p/q is

(A) 8b/27

(B) b

(C) 1

(D) none of these

Sol.

$$b \times 2y\frac{dy}{dx} = 3(x+a)^2 \quad \Rightarrow \quad \frac{dy}{dx}=\frac{3}{2}\frac{(x+a)^2}{by}$$

$$\Rightarrow \quad \frac{p}{q}=\frac{(S_T)^2}{S_N}=\left|\frac{y_0}{(f'(x_0))^3}\right|$$

Let a point by (x_0, y_0) lying on the curve by = $(x_0 + a)^3$ ....(i)

$$\frac{p}{q}=\left|\frac{y_0}{\left(\frac{3}{2}\frac{(x_0+a)^2}{by_0}\right)^3}\right|=\left|\frac{8y_0^4\times b^3}{27(x_0+a)^6}\right|=\frac{8}{27}b$$

(from equation (i))

Ex.18 For the curve y = $a \ \ell n \ (x^2 - a^2)$ show that sum of lengths of tangent & subtangent at any point is proportional to coordinates of point of tangency.

Sol.

Let point of tangency be (x_1, y_1) $\Rightarrow$ $m = \left.\frac{dy}{dx}\right|_{x_1} = \frac{2ax_1}{x_1^2 - a^2}$

tangent + subtangent = $y_1 \sqrt{1 + \frac{1}{m^2}} + \frac{y_1}{m} = y_1 \sqrt{1 + \frac{(x_1^2 - a^2)^2}{4a^2x_1^2}} + \frac{y_1(x_1^2 - a^2)}{2ax_1}$

$$= y_1 \frac{\sqrt{x_1^4 + a^4 + 2a^2x_1^2}}{2ax_1} + \frac{y_1(x_1^2 - a^2)}{2ax_1}$$

$= \frac{y_1(x_1^2 + a^2)}{2ax_1} + \frac{y_1(x_1^2 - a^2)}{2ax_1} = \frac{y_1(x_1^2)}{2ax_1} = \frac{x_1y_1}{2a}$ Hence proved.

Ex.19 Show that the segment of the tangent to the curve y $= \frac{a}{2}\ell n\left(\frac{a + \sqrt{a^2 - x^2}}{a - \sqrt{a^2 - x^2}}\right) - \sqrt{(a^2 - x^2)}$ contained between the y-axis and point of tangency has a constant length.

Sol.

Let $x = a \sin \phi$ then $y = \frac{a}{2}\ell n\left(\frac{a + a\cos\phi}{a - a\cos\phi}\right) - a\cos\phi$

$\frac{dx}{d\phi} = a\cos\phi$ and $\frac{dy}{d\phi} = \frac{a}{\sin\phi} + \sin\phi = \frac{a\cos^2\phi}{\sin\phi}$

Hence $\frac{dy}{dx} = \frac{\left(\frac{dy}{d\phi}\right)}{\left(\frac{dx}{d\phi}\right)} = -\cot\phi$

Equation of tangent at 'ø' y a ln cot ø/2 + a cos ø = $= \frac{-\cos\phi}{\sin\phi}(x - a\sin\phi)$

⇒y sin ø a sinø ln cot ø/2 + a sin ø cos ø - x cosø + a sin ø cos ø

⇒x cos ø+ y sin ø = a sin ø ln cot ø/2

Point on y-axis   P ≡ (0, a ln cot ø/2) and point of tangency

Q≡ (a sinø a ln cotø/2 a cos ø)

$$PQ = \sqrt{(a^2 \sin^2 \phi + a^2 \cos^2 \phi)} = \sqrt{a^2} = a = \text{constant.}$$

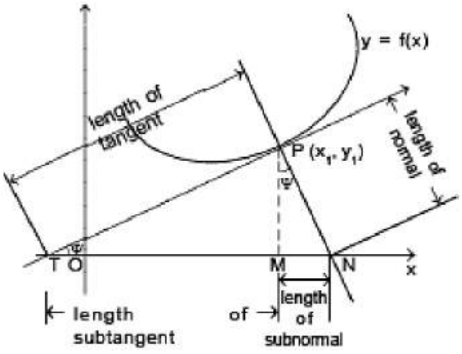
E. Solving Equations

Ex.20 For what values of c does the equation *l*n x = cx² have exactly one solution ?

Sol.

Let's start by graphing y = In x and $y = cx^2$ for various values of c. We know that for c * 0, $y = cx^2$ is a parabola that opens upward if c > 0 and downward if c < 0. Figure 1 shows the parabolas $y = cx^2$ for several positive values of c. Most of them don't intersect y = In x at all and one intersect twice. We have the feeling that there must be a value of c (somewhere between 0.1 and 0.3) for which the curves intersect exactly once, as in Figure 2.

To find that particular value of c, we let 'a' be the x-coordinate of the single point of intersection. In other words, In a = ca^2, so 'a' is the unique solution of the given equation. We see from Figure 2 that the curves just touch, so they have a common tangent line when x = a. That means the curves y = In x and $y=cx^2$ have the same slope when x = a. Therefore 1/a = 2ca



Solving the equation In a = ca^2 and 1/a = 2ca we get $ln\ a = ca^2 = c.\frac{1}{2c}=\frac{1}{2}$

Thus, $a = e^{1/2}$ and $c = \frac{ln\ a}{a^2}=\frac{ln\ e^{1/2}}{e}=\frac{1}{2e}$

For negative values of c we have the situation illustrated in Figure 3: All parabolas $y = cx^2$ with negative values of c intersect $y = \text{In } x$ exactly once. And let's not forget about $c = 0$: The curve $y = 0\ x^2 = 0$ just he x-axis, which intersects $y = \text{In } x$ exactly once.

To summarize, the required values of c are $c=1/(2e)$ and $c<0$

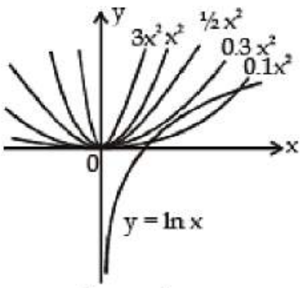


Figure 1

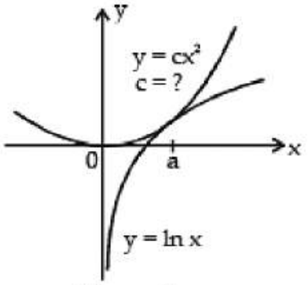


Figure 2

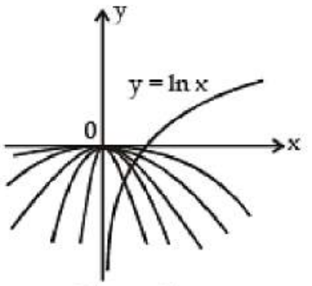
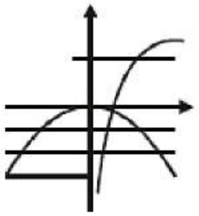


Figure 3

Ex.21 The set of values of p for which the equation $px^2 = ln\,x$ possess a single root is

Sol.



for $p \le 0$, there is obvious one solution ; for $p > 0$ one root

$\Rightarrow$ the curves touch each .

$2\ px_1 = 1/x_1 \Rightarrow\ x_1^2 = 1/2p$;

Also $px_1^2 = \ln x_1 \Rightarrow\ p\ (1/2p) = \ln x_1 \Rightarrow\ x_1 = e^{1/2}$

$\Rightarrow\ 2\ p = 1/e\ \Rightarrow\ p = 1/2e$. Hence $p \in (- \propto, 0]\ U\ \{1/2e\}$

F. Shortest distance

Shortest distance between two non-intersecting curves always along the common normal (wherever defined)

Ex.22 Find the shortest distance between the line $y = x - 2$ and the parabola $y = x^2 + 3x + 2$.

Sol. Let $P(x_1, y_1)$ be a point closest to the line $y = x - 2$ then $\left.\frac{dy}{dx}\right|_{(x_1,y_1)}$ = slope of line

$\Rightarrow 2x_1 + 3 = 1 \Rightarrow x_1 = -1 \Rightarrow y_1 = 0$ Hence point $(-1, 0)$ is the closest and its perpendicular distance from the

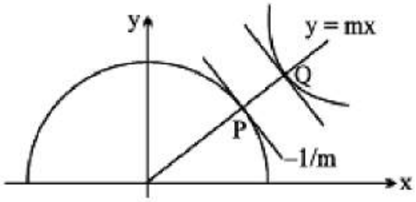
line $y = x - 2$ will give the shortest distance $\Rightarrow p = \frac{3}{\sqrt{2}}$.

Ex.23 Let P be a point on the curve C_1: $y = \sqrt{2 - x^2}$ and Q be a point on the curve C_2: $xy = 9$, both P and Q lie in the first quadrant. If 'd' denotes the minimum value between P and Q, find the value of d^2.

Sol. Note that C_1 is a semicircle and C_2 is a rectangular hyperbola.

PQ will be minimum if the normal at P on the semicircle is also a normal at Q on $xy = 9$

Let the normal at P be $y = mx$ ....(1) $(m > 0)$ solving it with $xy = 9$



$$mx^2 = 9 \Rightarrow x = \frac{3}{\sqrt{m}};\ y = \frac{9\sqrt{m}}{3} \quad \therefore \quad Q \equiv \left(\frac{3}{\sqrt{m}}, 3\sqrt{m}\right)$$

differentiating $xy = 9$

$$x\frac{dy}{dx} + y = 0 \Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

$$\left.\frac{dy}{dx}\right|_Q = -\frac{3\sqrt{m}\cdot\sqrt{m}}{3} = -m$$

$\therefore$ tangent at P and Q must be parallel

$$\therefore \quad -m = -\frac{1}{m} \quad \Rightarrow \quad m^2 = 1 \quad \Rightarrow \quad m = 1$$

$\therefore$ normal at P and Q is $y = x$

solving P(1, 1) and Q(3, 3)

$(PQ)^2 = d^2 = 4 + 4 = 8$

G. Rate Measurement

Ex.24 A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall ?

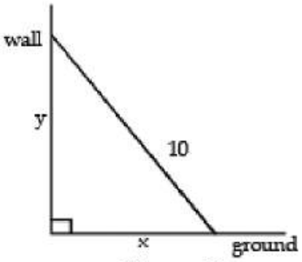
Sol. We first draw a diagram and label it as in Figure 1. Let x feet be the distance from the bottom of the ladder to the wall and y feet the distance from the top of the ladder to the ground. Note that x and y are both function of t (time). We are given that $dx/dt = 1$ ft/s and we are asked to find dy/dt when $x = 6$ ft (see Figure 2). In this problem, the relationship between x and y is given by the Pythagorean Theorem : $x^2 + y^2 = 100$

Differentiating each side with respect to t using the Chain Rule, we have $2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} = 0$

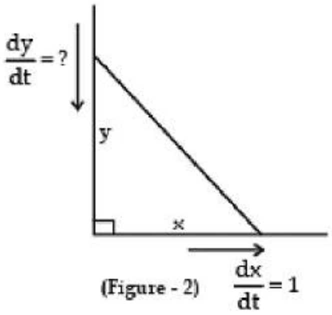
and solving this equation for the desired rate, we obtain $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$

When $x = 6$, the Pythagorean Theorem gives $y = 8$ and so, substituting these values and $dx/dt = 1$,

we have $\frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4}$ ft/s



(Figure - 1)



(Figure - 2)

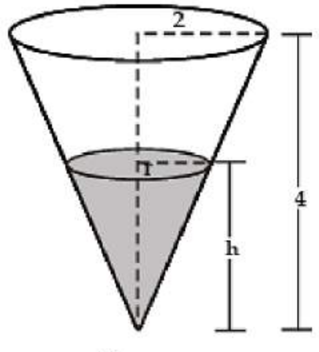
The fact that dy/dt is negative means thatthe distance from the top of the ladder to the ground is decreasing at a rate of 3/4 ft/s In other words, the top of the ladder is sliding down the wall at a rate of 3/4 ft/s .

Ex.25 A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

Sol.

We first sketch the cone and label it as in Figure. Let V, r, and h be the volume of the water, the radius of the surface, and the height at time t, where t is measured in minutes.

We are given that $dV/dt = 2m^3/min$ and we are asked to find dh/dt when h is 3 m. The quantities V and h are related by the equation $V = 1/3pr^2h$But it is the very useful to express V as a function of h alone.



Figure

In order to eliminate r, we use the similar triangles in Figure to write $\frac{r}{h} = \frac{2}{4}$ $r = \frac{h}{2}$ and the expression for V becomes $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$

$$\frac{dV}{dt}=\frac{\pi}{4}h^2\frac{dh}{dt} \qquad \text{so} \qquad \frac{dh}{dt}=\frac{4}{\pi h^2}\frac{dV}{dt}$$

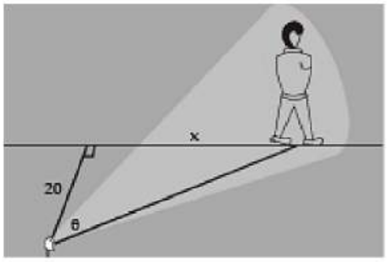
Substituting h = 3 m and dV/dt = 2m³/min, we have $\frac{dh}{dt}=\frac{4}{\pi(3)^2}.2=\frac{8}{9\pi}$

The water level is rising at a rate of 8/(9p) ≈ 0.28 m/min.

Ex.26 A man walks along a straight path at the speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight ?

Sol. We draw Figure and let x be the distance from the man to the point on the path closest to the searchlight. We let θ be the distance from the man to the point on the path closest to the searchlight and the perpendicular to the path.

We are given that dx/dt = 4 ft/s and are asked to find dq/dt when x = 15. The equation that relates



Differentiating each side with respect to t, we get dx/dt = 20sec² θ dθ/dt

$$\text{so} \qquad \frac{d\theta}{dt}=\frac{1}{20}\cos^2\theta\frac{dx}{dt}=\frac{1}{20}\cos^2\theta(4)=\frac{1}{5}\cos^2\theta$$

when x = 15, the length of the beam is 25, so cosθ = 4/5

$$\text{and } \frac{d\theta}{dt}=\frac{1}{5}\left(\frac{4}{5}\right)^2=\frac{16}{125}=0.128$$

The searchlight is rotating at a rate of 0.128 rad/s.

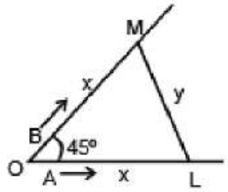
Ex.27 Two men A and B start with velocities v at the same time from the junction of two roads inclined at 45º to each other. If they travel by different roads, find the rate at which they are being separated.

Sol.

Let L and M be the positions of men A and B at any time t,

Let OL = x and LM = y. Then OM = x

given, dx/dt = v; to find dy/dt from ΔLOM,



$$\cos 45° = \frac{OL^2 + OM^2 - LM^2}{2.OL.OM} \text{ or,}$$

$$\frac{1}{\sqrt{2}} = \frac{x^2 + x^2 - y^2}{2.x.x} = \frac{2x^2 - y^2}{2x^2}$$

or, $\sqrt{2x^2} = 2x^2 - y^2$

or $(2 - \sqrt{2})\, x^2 = y^2$

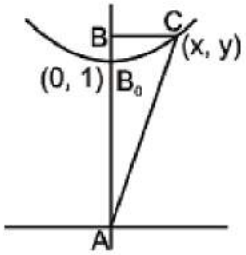
differentiating w. r. t. we get $\frac{dy}{dt} = \sqrt{2\sqrt{2}}\,\frac{dx}{dt} = \sqrt{2-\sqrt{2}}\ v$

$$\left[\because\ \frac{dx}{dt} = v\right]$$

∴ they are being separated from each other at the rate $\sqrt{2-\sqrt{2}}$ v.

Ex.28 A variable triangle ABC in the xy plane has its orthocentre at vertex 'B', a fixed vertex 'A' at the origin & the third vertex 'C' restricted to lie on the parabola y = $1 + \frac{7x^2}{36}$. The point B starts at the point (0, 1) at time t = 0 & moves upward along the y axis at a constant velocity of 2 cm/sec. How fast is the area of the triangle increasing when t = 7/2 sec ?

Sol.



$A = \frac{xy}{2} = \frac{x}{2}\left(1 + \frac{7x^2}{36}\right)$; $\frac{dA}{dt} = \left(\frac{1}{2} + \frac{7}{24}x^2\right)\frac{dx}{dt}$

at $t = \frac{7}{2}$; $y = 2 \times \frac{7}{2} = 7 \Rightarrow AB = 8$

when $y = 8$ then $x = 6 \Rightarrow \frac{dA}{dt} = \left(\frac{1}{2} + \frac{7}{24} 3.6\right)\frac{dx}{dt} = 11 . \frac{dx}{dt}$

Also $\frac{dy}{dt} = 2 = \frac{14x}{36}\ \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{36}{7x} = \frac{6}{7}$

$\Rightarrow \frac{dA}{dt} = 11 . \frac{6}{7} = \frac{66}{7}$

Ex.29 Find the approximate value of $(1.999)^6$.

Sol.

Let $f(x) = x^6$. Now, $f(x + \delta x)\ f(x) = f'(x) . \delta x = 6x^5\ \delta x$

We may write, $1.999 = 2\ 0.001$

Taking $x = 2$ and $\delta x = 0.001$, we have $f(1.999)\ f(2) = 6(2)^5 \times 0.001$

$\Rightarrow f(1.999) = f(2)\ 6 \times 32 \times 0.001 = 64\ 64 \times 0.003 = 64 \times 0.997 = 63.808$ (approx).

Tangent and Normal

Definition

- The tangent line to the graph of f at the point P(a, f(a)) is
  1. The line on P with slope f'(a) if f'(a) exists ;
  2. The line x = a if $\lim_{x \to a}\left|\frac{f(x) - f(a)}{x - a}\right| = \infty$.

- In neither (1) nor (2) holds, then the graph of does not have a tangent line at the point P(a, f(a)).
- In case f'(a) exists, then y - f(a) = f'(a) (x - a) is an equation of the tangent line to the graph of f at the point P(a, f(a)).
- The normal line N to the graph of a function f at the point P(a, f(a)) is defined to be the line through P perpendicular to the tangent line.
- It follows that if f'(a) $\neq$ 0 the slope of N is -1/f'(a) and $y - f(a) = -\frac{1}{f'(a)}(x - a)$ is an equation of N.
- If f'(a) = 0, then N is the vertical line x = a; and if the tangent line is vertical, then N is the horizontal line y = f(a).

Note :

1. The point P (x_1, y_1) will satisfy the equation of the curve & the equation of tangent & normal line.

2. If the tangent at any point P on the curve is parallel to the axis of x then dy/dx = 0 at the point P.

3. If the tangent at any point on the curve is parallel to the axis of y, then dy/dx = ∞ or dx/dy = 0.

4. If the tangent at any point on the curve is equally inclined to both the axes there dy/dx = $\pm$ 1.

5. For equation of tangent at (x_1, y_1), substitute xx_1 for x^2, yy_1 for y^2, $\frac{x + x_1}{2}$ for x, $\frac{y + y_1}{2}$ for y and $\frac{xy_1 + x_1y}{2}$ for xy and keep the constant as such. This method is applicable only for second degree curves, i.e., $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

6. Method to find normal at (x_1, y_1) of second degree conics $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ The equation of normal at (x_1, y_1) is $\frac{x - x_1}{ax_1 + hy_1 + g} = \frac{y - y_1}{hx_1 + by_1 + f}$

Tangents at the Origin

- If a curve passing through the origin be given by a rational integral algebraic equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation.

e.g., if the equation of a curve be $x^2 - 4y^2 + x^4 + 3x^3y + 3x^2 y^2 + y^4 = 0$, the tangents at the origin are given by $x^2 - 4y^2 = 0$ or $x + 2y$ and $x - 2y = 0$.

- In the curve $x^2 + y^2 + ax + by = 0$, $ax + by = 0$, is the equation of the tangent at the origin; and in the curve $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$, $x^2 - y^2 = 0$ is the equation of a pair of tangents at the origin.
- If the equation of a curve be $x^2 + y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ i.e. $x + y = 0$ and $x - y = 0$

Angle of intersection

Angle of intersection between two curves is defined as the angle between the two tangents drawn to the two curves at their point of intersection . If the angle between two curves is 90° then they are called **ORTHOGONAL** curves.

Example 1. Find the equation of tangent to the ellipse $3x^2+y^2+x+2y = 0$ which are perpendicular to the line $4x - 2y = 1$.

Sol. Since, tangent is the perpendicular to the line $4x - 2y = 1$,

(slope of tangent) × (slope of normal) = 1

$$\Rightarrow \frac{dy}{dx} \times 2 = -1 \qquad \Rightarrow \frac{dy}{dx} = -\frac{1}{2} \text{....(i)}$$

The given equation $3x^2 + y^2 + x + 2y = 0$ ....(iii)

$$\Rightarrow \qquad 6x + 2y\frac{dy}{dx} + 1 + 2\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = -\frac{(6x+1)}{2(y+1)}$$

Let (x_1, y_1) be the point of contact of the tangent and the curve

From (i) and (iii), we get

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -\frac{(6x_1+1)}{2(y_1+1)} = -\frac{1}{2} \text{ i.e.,} \quad y_1 = 6x_1\text{....(iv)}$$

Substituting this in (ii) [since the points lies on the curve] we get,

$3x_1^2 + 36x_1^2 + x_1 + 12x_1 = 0$

i.e., $13x_1 (3x^1 + 1) = 0 \Rightarrow x_1 = 0, 1/3$

Using (iv), $x_1 = 0 \Rightarrow y_1 = 0$ and $x_1 = 1/3 \Rightarrow y_1 = 2$

Hence, the points where tangent has slope 1/2 are P(0, 0) and Q(1/3, 2).

Equation of tangents at P, Q are $y = -1/2x$ i.e. $x+2y = 0$

and $y + 2 = -\frac{1}{2}\left(x + \frac{1}{3}\right)$ i.e., $3x + 6y + 13 = 0$ respectively..

Example 2. Find the equation of normal to the curve $x + y = x^y$, where it cuts x-axis.

Sol.

Given curve is $x + y = x^y$ .....(i)

at xaxis $y = 0$, $x + 0 = x^0 \Rightarrow x = 1$

Point is A(1, 0)

Now to differentiation $x + y = x^y$ taking log of both sides

$\Rightarrow \log(x + y) = y \log x$

$$\frac{1}{x+y}\left\{1+\frac{dy}{dx}\right\} = y \cdot \frac{1}{x} + (\log x)\ \frac{dy}{dx}$$

Putting $x = 1$, $y = 0$ $\left\{1+\frac{dy}{dx}\right\} = 0$ $\Rightarrow$

$$\left(\frac{dy}{dx}\right)_{(1,0)} = -1$$

slope of normal = 1

Equation of normal is, $\frac{y-0}{x-1} = 1$

$\Rightarrow y = x\ 1$

Example 3. At what points on the curve $y = \frac{2}{3}x^3 + \frac{1}{2}x^2$, the tangents make equal angles with co-ordintae axes ?

Sol.

Given curve is $y = \frac{2}{3}x^3 + \frac{1}{2}x^2$, ...(1)

Differentiating both sides w.r.t.x, then $dy/dx = 2x^2 + 2$

$$\therefore \quad \frac{dy}{dx} = \pm 1 \quad \text{or} \quad 2x^2 + x = \pm 1$$

or $2x^2 + 2x\ x\ 1 = 0$ or $(2x\ 1)\ (x + 1)$

$\therefore x = \frac{1}{2}, -1$ (If $2x^2 + x + 1 = 0$ then x is imaginary)

From (1), for $x = 1/2$, $y = \frac{2}{3}.\frac{1}{8} + \frac{1}{2}.\frac{1}{4} = \frac{5}{24}$

and for $x = -1$, $y = -\frac{2}{3} + \frac{1}{2} = -\frac{1}{6}$

Hence points are $\left(\frac{1}{2}, \frac{5}{24}\right)$ and $\left(-1, \frac{1}{6}\right)$.

Example 4. Show that the curve $x = 1 - 3t^2$, $y = t - 3t^3$ is symmetrical about x-axis and has no real point for $x > 1$. If the tangent at the point t is inclined at an angle φ to OX. Prove that $3t = \tan\varphi + \sec\varphi$. If the tangent at P(-2, 2) meets the curve again at Q, prove that the tangents at P and Q are at right angles.

Sol. Given curve is $x = 1 - 3t^2$ ...(1)

& $y = t - 3t^3$ ...(2)

From (1) and (2), $y = tx$ or $x = 1 - 3\left(\frac{y}{x}\right)^2 \Rightarrow x^3 = x^2 - 3y^2$

Since all powers of y are even, so curve is symmetrical about x-axis.

For $x > 1 \Rightarrow 1\ 3t^2 > 1 \Rightarrow 3t^2 > 0$ Impossible

From (1) and (2),

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{1-9t^2}{-6t} = \tan\phi$$

...given (3)

$$\sec^2\phi = 1 + \tan^2\phi = 1 + \left(\frac{1-9t^2}{6t}\right)^2 = \left(\frac{1+9t^2}{6t}\right)^2$$

$$\therefore \quad \sec\phi = \frac{1-9t^2}{6t}$$

Adding (3) and (4) we get, $\tan ø + \sec ø = 3t$

$P(2, 2)$

$1\ 3t^2 = 2$ and $2 = t\ 3t^3$ then we get $t = 1$

$$\left.\frac{dy}{dx}\right|_{t=-1} = \frac{1-9(-1)^2}{-6(-1)} = -\frac{4}{3}$$

Equation of tangent at (2, 2) is $Y - 2 = -\frac{4}{3}(X + 2)$

$$\Rightarrow \quad t - 3t^3 - 2 = -\frac{4}{3}(1 - 3t^2 + 2)$$
$$\Rightarrow \quad 9t^3 + 12t^2 - 3t - 6 = 0$$

$$\Rightarrow \quad 3t - 9t^3 - 6 = -12 + 12t^2$$
$$\Rightarrow \quad (t + 1)^2 (3t - 2) = 0$$

Therefore the tangent at t = 1 meets the curve again at $t = \frac{2}{3}, Q\left(-\frac{1}{3}, -\frac{2}{9}\right)$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{t=2/3} = \frac{1-9\left(\frac{4}{9}\right)}{-6\left(\frac{2}{3}\right)} = \frac{3}{4}$$

Hence $\left.\frac{dy}{dx}\right|_{t=-1} \times \left.\frac{dy}{dx}\right|_{t=-2/3} = -1$

Hence the tangents at P and Q are at right angles.

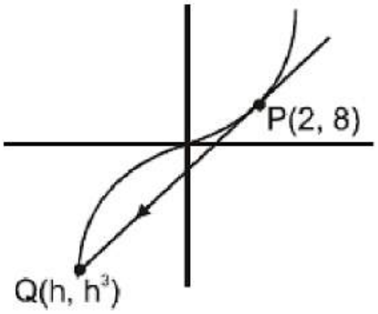
Example 5. Tangent at P(2, 8) on the curve $y = x^3$ meets the curve again at Q. Find coordinates of Q.

Sol. Equation of tangent at (2, 8) is $y = 12x - 16$

Solving this with $y = x^3$ we get $x^3 - 12x + 16 = 0$

this cubic must give all points of intersection of line and curve $y = x^3$

i.e., point P and Q.



But, since line is tangent at P so $x = 2$ will be a repeated root of equation $x^3 - 12x + 16 = 0$ and another root will be $x = h$.

Using theory of equations sum of roots $\Rightarrow 2 + 2 + h = 0 \;\; \Rightarrow h = -4$

Hence coordinates of Q are (-4, -64)

Example 6. If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle φ with the axis of x, show that its equation is $y \cos\varphi - x \sin\varphi = a \cos 2\varphi$.

Sol. Given curve is $x^{2/3} + y^{2/3} = a^{2/3}$ ...(1)

Differentiating both sides w.r.t.x, we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$$

Slope of normal $= -dx = \dfrac{dx}{dy} = \dfrac{x^{1/3}}{y^{1/3}} = \tan\phi$ (given) $\therefore$

$x = y \tan^3 \phi \quad ...(2)$

From (1) and (2), y$^{2/3}$ (1 + tan2Ø) = a$^{2/3}$ Þ y$^{2/3}$ = a$^{2/3}$ cos2Ø

y = a cos^3 Ø and x = a sin^3 Ø

Therefore equation of normal is y a cos^3 Ø = tan Ø (x a sin^3 Ø)

y cos Ø a cos^4 Ø = x sin Ø a sin^4 Ø

y cosØ x sin Ø = a (cos^4 Ø sin^4 Ø)

= a (cos^2 Ø + sin^2 Ø) (cos^2 Ø- sin^2 Ø) = a . 1 . cos 2Ø

Hence y cos Ø x sin Ø = a cos 2Ø

Example 7. (a) Find y' if $x^3 + y^3 = 6xy$.

(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point (3, 3).

(c) At what points on the curve is the tangent line horizontal ?

Sol. (a) Differentiating both sides $x^3 + y^3 = 6xy$ with respect to x, regarding y as a function of x, and using the Chain Rule on the y^3 term and the Product Rule on the 6xy term, we get

$3x^2 + 3y^2y' = 6y + 6xy'$ or $x^2 + y^2y' = 2y + 2xy'$

We now solve for y' :$y^2y'\ 2xy' = 2y\ x^2$

$(y^2\ 2x)y' = 2y\ x^2$, $\quad y' = \dfrac{2y - x^2}{y^2 - 2x}$

(b) When x = y = 3

$$y' = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = -1$$

So, tangent to the folium of Descartes is $y 3 = 1(x 3)$ or $x + y = 6$

(c) The tangent line is horizontal if $y' = 0$. Using the expression for y' from part (a), we see that $y' = 0$ when $2y x^2 = 0$. Substituting $y = 1/2x^2$, n the equation of the curve, we get $x^3 + \left(\frac{1}{2}x^2\right)^3 = 6x\left(\frac{1}{2}x^2\right)$

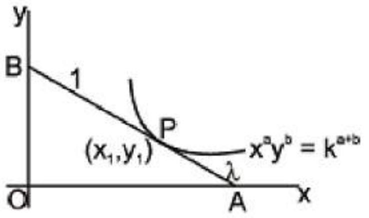
which simplifies to $x^6 = 16x^3$. so either $x = 0$ or $x^3 = 16$. If $x = 16^{1/3} = 2^{4/3}$, then $y = 1/2(2^{8/3}) = 2^{5/3}$. Thus, the tangent is horizontal at $(0, 0)$ and at $(2^{4/3}, 2^{5/3})$.

Example 8. In the curve $x^a y^b = k^{a+b}$, (a b > 0) prove that the portion of the tangent intercepted between the coordinate axes is divided at its point of contact into segments which are in constant ratio.

Sol. Let $P(x_1, y_1)$ be the point of contact of the tangent.

Here, $x^a y^b = k^{a+b}$

$\therefore$ a log x + b log y = (a + b) log k.



Differentiating, $\frac{a}{x} + \frac{b}{y}\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = -\frac{ay}{bx}$;

$$\therefore \left(\frac{dy}{dx}\right)_{x_1\text{-}y_1} = \frac{-ay_1}{bx_1}$$

Solving with $y = 0$,

$$-y_1 = \frac{-ay_1}{bx_1}(x - x_1) \quad \text{or} \quad bx_1 = a(x - x_1);$$

$$\therefore \qquad x = \frac{(a+b)x_1}{a}$$

Solving (1) with $x = 0$, $y - y_1 = \frac{-ay_1}{bx_1}(-x_1)$

or $\quad y = y_1 + \frac{ay_1}{b} = \frac{(a+b)y_1}{b}$

$$A = \left(\frac{a+b}{a}x_1, 0\right) \text{ and } B = \left(0, \frac{a+b}{b}y_1\right)$$

Let P divide AB in the ratio $\lambda : 1$. Then

$$P = \left(\frac{\lambda.0 + 1.\frac{a+b}{b}x_1}{\lambda+1}, \frac{\lambda.\frac{a+b}{b}y_1 + 1.0}{\lambda+1}\right)$$

$$= \left(\frac{a+b}{a(\lambda+1)}x_1, \frac{\lambda(a+b)}{b(\lambda+1)}y_1\right)$$

$$x_1 = \frac{a+b}{a(\lambda+1)}x_1 \text{ and } \frac{\lambda(a+b)}{b(\lambda+1)}y_1$$

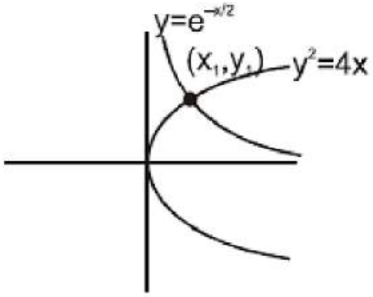
$\Rightarrow \quad a(\lambda + 1) = a + b$ and $b(\lambda + 1) = \lambda(a + b)$

$\Rightarrow \quad \lambda = \frac{b}{a}$ and $b = \lambda a$, i.e., $\lambda = \frac{b}{a}$

P divides AB in the constant ratio b : a.

Example 9. Find the angle between curves $y^2 = 4x$ and $y = e^{-x/2}$

Sol.



Let the curves intersect at point (x_1,y_1)

for $y^2 = 4x$ $\left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \frac{2}{y_1}$

and for$y = e^{-x/2}$ $\left.\frac{dy}{dx}\right|_{(x_1,y_1)} = -\frac{1}{2}e^{-x_1/2}$

$= -\frac{y_1}{2} \Rightarrow m_1, m_2 = -1$ Hence $\theta = 90°$

Note: here that we have not actually found the intersection point but geometrically we can see that the curves intersect.

Example 10. Show that the curves $y = 2\sin^2 x$ and $y = \cos 2x$ intersect at $\pi/6$. What is their angle of intersection ?

Sol. Given curves are $y = 2\sin^2 x$ ...(1)

and $y = \cos 2x$ ...(2)

Solving (1) and (2), we get $2\sin^2 x = \cos 2x$

$\Rightarrow 1 \cos 2x = \cos 2x \Rightarrow \cos 2x = 1/2 \Rightarrow \cos \pi/3 \Rightarrow 2x = \pm \pi/3$

$x = \pm\pi/6$ are the points of intersection

From (1), $dy/dx = 4\sin x \cos x = 2\sin 2x = m_1$ (say)

From (2) $dy/dx = -2\sin 2x = m_2$ (say)

If angle of intersection is θ, then $\tan\theta$ $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{4\sin 2x}{1 - 4\sin^2 2x}\right|$

$\therefore$ $(\tan\theta)_{x = \pm\pi/6} = \left|\frac{4 \times \pm\frac{\sqrt{3}}{2}}{1 - 4 \times \frac{3}{4}}\right| = \left|\frac{\pm 2\sqrt{3}}{-2}\right| = \sqrt{3}$

$\therefore$ $\theta = \frac{\pi}{3}$

Example 11. Show that the angle between the tangents at any point P and the line joining P to the origin `O' is the same at all points of the curve ln $(x^2 + y^2) = c \tan^{-1} (y/x)$ where c is constant.

Sol. Let the point P(x, y) on the curve ln $(x^2 + y^2) = c \tan^{-1} (y/x)$ where c is constant.

Differentiating both sides w.r.t. x, we get

$$\frac{2x+2yy'}{(x^2+y^2)} = \frac{c(xy'-y)}{(x^2+y^2)} \Rightarrow \quad y' = \frac{2x+cy}{cx-2y} = m_1 \text{ (say)}$$

Slope of OP $= y/x = m_2$ (say)

Let the angle between the tangents at P and OP be θ

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right| = \left|\frac{\frac{2x+cy}{cx-2y} - \frac{y}{x}}{1+\frac{2xy+cy^2}{cx^2-2xv}}\right| = \frac{2}{c}.$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{2}{c}\right)$$ which is independent of x and y.

Example 12. Show tht the curves $\frac{x^2}{a^2+k_1} + \frac{y^2}{b^2+k_1} = 1$ and $\frac{x^2}{a^2+k_2} + \frac{y^2}{b^2+k_2} = 1$ intersect orthogonally.

Sol.

Given $\frac{x^2}{a^2+k_1} + \frac{y^2}{b^2+k_1} = 1 \quad ...(1)$

and $\frac{x^2}{a^2+k_2} + \frac{y^2}{b^2+k_2} = 1 \qquad ...(2)$

Subtracting (2) from (1), we get $x^2\left(\frac{1}{a^2+k_1} - \frac{1}{a^2+k_2}\right) + y^2\left(\frac{1}{b^2+k_1} - \frac{1}{b^2+k_2}\right) = 0$

$$\Rightarrow x^2\left(\frac{k_2-k_1}{(a^2+k_1)(a^2+k_2)}\right) + y^2\left(\frac{k_2-k_1}{(b^2+k_1)(b^2+k_2)}\right) = 0$$

$$\therefore \frac{x^2}{y^2} = -\frac{(a^2+k_1)(a^2+k_2)}{(b^2+k_1)(b^2+k_1)} \quad ...(3)$$

Now from (1), $\frac{2x}{(a^2+k_1)} + \frac{2y}{(b^2+k_1)}\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\frac{x(b^2+k_1)}{y(a^2+k_1)} = m_1 \text{ (say)}$$

Similarly from (2), $\frac{dy}{dx} = -\frac{x(b^2+k_2)}{y(a^2+k_2)} = m_2$ (say)

$$\Rightarrow m_1\, m_2 = \frac{x^2(b^2+k_1)(b^2+k_2)}{y^2(a^2+k_1)(a^2+k_2)} = -1 \text{ [From (3)]}$$

Hence given curves intersect orthogonally.

Example 13. Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.

Sol.

Equation of the given curves are $xy = 4$ ....(i) and $x^2 + y^2 = 8$ ....(ii)

from (i), $1.y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$ ....(iii), from (ii),

$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$ ....(iv)

Putting the value fo y from (i) i (ii), we get $x^2 + 16/x^2 = 8$ or or $x^4 + 16 = 8x^2$

or $x^4\ 8x^2 + 16 = 0$ or $(x^2\ 4)^2 = 0$ or $x^2\ 4 = 0$ or $x^2 = 4$

from (i) ; when $x = 2, y = 2$ and when $x = 2, y = -2$

Hence points of intersection of the two curves are (2, 2) and (2, 2).

Slope of the tangent to the curve (i) at point (2, 2) $\Rightarrow m_1 = -2/2 = -1$ ...(from iii)

Slope of tangent to the curve (ii) at point (2, 2) $\Rightarrow m_2 = -2/2 = -1$....(from iv)

Since $m_3 = m_4$, hence the two curves touch each other at (-2, -2). Thus curves (i) and (ii) touch each other.

Slope of tangent to curve (i), $m_3 = -\left(\frac{-2}{-2}\right) = -1$

Slope of tangent to curve (ii), $m_4 = -\left(\frac{-2}{-2}\right) = -1$

Since $m_3 = m_4$, hence the two curves touch each other at (2, 2). Thus curves (i) and (ii) touch each other.

Example 14. The gradient of the common tangent to the two curves $y = x^2 - 5x + 6$ & $y = x^2 + x + 1$ is

(A) - 1/3

(B) - 2/3

(C) - 1

(D) - 3

Sol. $y = ax + b$ on solving with both curves and putting $D = 0$ gives

$a^2 + 10a + 4b + 1 = 0$ and $a^2 - 2a + 4b - 3 = 0 \Rightarrow a = -1/3$ & $b = 5/9$

$\Rightarrow 3x + 9y = 5$; point of contact (7/3, - 2/9) & (- 2/3, 7/9)

Length of Tangent

(a) Length of the tangent (PT) = $\frac{y_1\sqrt{1 + [f'(x_1)]^2}}{f'(x_1)}$

(b) Subtangent (MT) = $\frac{y_1}{f'(x_1)}$

(c) Length of Normal (PN) = $y_1\sqrt{1 + [f'(x_1)]^2}$

(d) Subnormal (MN) = $y_1 f'(x_1)$

Example 15. What should be the value of n in the equation of curve $y = a^{1-n} . x^n$, so that the sub-normal may be of constant length ?

Sol. Given curve is $y = a^{1-n} . x^n$

Taking logarithm of both sides, we get, $\ell n\, y = (1 - n)\, \ell n\, a + n\, \ell n\, x$

Differentiating both sides w.r.tx, we get $\frac{1}{y}.\frac{dy}{dx}=0+\frac{n}{x}$ or $\frac{dy}{dx}=\frac{ny}{x}$ ...(1)

Lengths of sub-normal $= y\ dy/dx = y\ .ny/x$ ....{from 1}

$$= \frac{ny^2}{x} = n.\frac{(a^{1-n}x^n)^2}{x}$$

$$(\because\ y = a^{1-n} . x^n) \quad = n\ .\ a^{2-2n} . x^{2n-1}$$

Since lengths of sub-normal is to be constant, so x should not appear in its value i.e., $2n\ 1 = 0$. $n = 1/2$

Example 16. If the relation between sub-normal SN and sub-tangent ST at any point S on the curve

$by^2 = (x + a)^3$ is $p(SN) = q(ST)^2$; then p/q is

(A) 8b/27

(B) b

(C) 1

(D) none of these

Sol.

$$b \times 2y\ \frac{dy}{dx} = 3(x + a)^2 \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{3}{2}\frac{(x+a)^2}{by}$$

$$\Rightarrow \qquad \frac{p}{q} = \frac{(S_T)^2}{S_N} = \left|\frac{y_0}{(f'(x_0))^3}\right|$$

Let a point by (x_0, y_0) lying on the curve by $= (x_0 + a)^3$ ....(i)

$$\frac{p}{q} = \left|\frac{y_0}{\left(\frac{3}{2}\frac{(x_0+a)^2}{by_0}\right)^3}\right| = \left|\frac{8y_0^4 \times b^3}{27(x_0+a)^6}\right| = \frac{8}{27}b$$

(from equation (i))

Example 17. For the curve y = $a\ \ell n\ (x^2 - a^2)$ **show that sum of lengths of tangent & subtangent at any point is proportional to coordinates of point of tangency.**

Sol.

Let point of tangency be (x_1, y_1) ⇒ $m = \left.\frac{dy}{dx}\right|_{x_1} = \frac{2ax_1}{x_1^2 - a^2}$

tangent + subtangent = $y_1\sqrt{1+\frac{1}{m^2}} + \frac{y_1}{m} = y_1\sqrt{1+\frac{(x_1^2-a^2)^2}{4a^2x_1^2}} + \frac{y_1(x_1^2-a^2)}{2ax_1}$

$$= y_1\ \frac{\sqrt{x_1^4 + a^4 + 2a^2x_1^2}}{2ax_1} + \frac{y_1(x_1^2-a^2)}{2ax_1}$$

$= \frac{y_1(x_1^2+a^2)}{2ax_1} + \frac{y_1(x_1^2-a^2)}{2ax_1} = \frac{y_1(x_1^2)}{2ax_1} = \frac{x_1y_1}{2a}$ Hence proved.

Example 18. Show that the segment of the tangent to the curve y $= \frac{a}{2}\ell n\left(\frac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}}\right) - \sqrt{(a^2-x^2)}$ **contained between the y-axis and point of tangency has a constant length.**

Sol.

Let $x = a \sin \phi$ then $y = \frac{a}{2}\ell n\left(\frac{a + a\cos\phi}{a - a\cos\phi}\right) - a\cos\phi$

$\frac{dx}{d\phi} = a\cos\phi$ and $\frac{dy}{d\phi} = \frac{a}{\sin\phi} + \sin\phi = \frac{a\cos^2\phi}{\sin\phi}$

Hence $\frac{dy}{dx} = \frac{\left(\frac{dy}{d\phi}\right)}{\left(\frac{dx}{d\phi}\right)} = -\cot\phi$

Equation of tangent at 'ø' y a ln cot ø/2 + a cos ø = $= \frac{-\cos\phi}{\sin\phi}(x - a\sin\phi)$

⇒y sin ø a sinø ln cot ø/2 + a sin ø cos ø - x cosø + a sin ø cos ø

$\Rightarrow x \cos ø + y \sin ø = a \sin ø \ln \cot ø/2$

Point on y-axis $P \equiv (0, a \ln \cot ø/2)$ and point of tangency

$Q \equiv (a \sin ø \; a \ln \cot ø/2 \; a \cos ø)$

$PQ = \sqrt{(a^2 \sin^2 \phi + a^2 \cos^2 \phi)} = \sqrt{a^2} = a =$ constant.

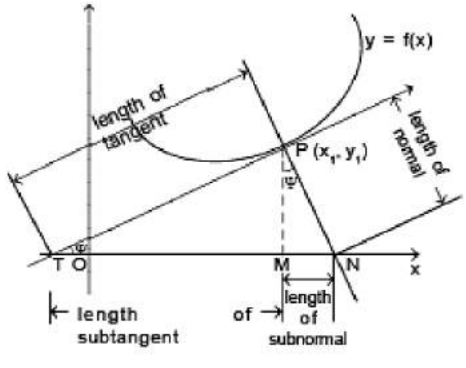
Solving Equations

Example 19. For what values of c does the equation $\ell n\, x = cx^2$ have exactly one solution ?

Sol.

Let's start by graphing y = In x and $y = cx^2$ for various values of c. We know that for c * 0, $y = cx^2$ is a parabola that opens upward if $c > 0$ and downward if $c < 0$. Figure 1 shows the parabolas $y = cx^2$ for several positive values of c. Most of them don't intersect y = In x at all and one intersect twice. We have the feeling that there must be a value of c (somewhere between 0.1 and 0.3) for which the curves intersect exactly once, as in Figure 2.

To find that particular value of c, we let 'a' be the x-coordinate of the single point of intersection. In other words, In $a = ca^2$, so 'a' is the unique solution of the given equation. We see from Figure 2 that the curves just touch, so they have a common tangent line when x = a. That means the curves y = In x and $y=cx^2$ have the same slope when x = a. Therefore $1/a = 2ca$



Solving the equation In $a = ca^2$ and $1/a = 2ca$

we get $ln\ a = ca^2 = c.\frac{1}{2c} = \frac{1}{2}$

Thus, $a = e^{1/2}$ and $c = \frac{\ln a}{a^2} = \frac{\ln e^{1/2}}{e} = \frac{1}{2e}$

For negative values of c we have the situation illustrated in Figure 3: All parabolas $y = cx^2$ with negative values of c intersect y = In x exactly once. And let's not forget about $c = 0$: The curve $y = 0\ x^2 = 0$ just he x-axis, which intersects y = In x exactly once.

To summarize, the required values of c are $c=1/(2e)$ and $c<0$

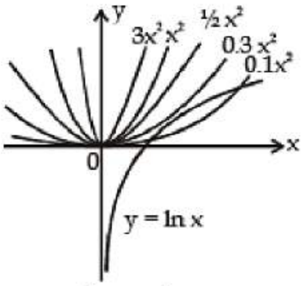


Figure 1

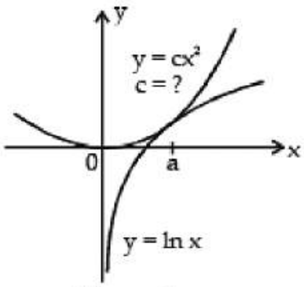


Figure 2

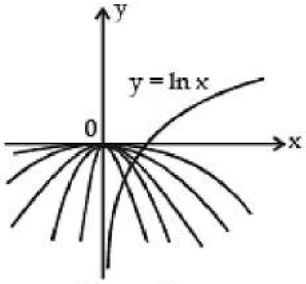
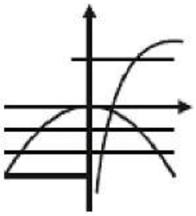


Figure 3

Example 20. The set of values of p for which the equation $px^2 = ln\,x$ possess a single root is

Sol.



for $p \le 0$, there is obvious one solution ; for $p > 0$ one root

$\Rightarrow$ the curves touch each .

$2\ px_1 = 1/x_1 \Rightarrow\ x_1^2 = 1/2p$;

Also $px_1^2 = \ln x_1 \Rightarrow\ p\ (1/2p) = \ln x_1 \Rightarrow\ x_1 = e^{1/2}$

$\Rightarrow\ 2\ p = 1/e\ \Rightarrow\ p = 1/2e$. Hence $p \in (-\propto, 0]\ U\ \{1/2e\}$

Shortest distance

Shortest distance between two non-intersecting curves always along the common normal (wherever defined)

Example 21. Find the shortest distance between the line $y = x - 2$ and the parabola $y = x^2 + 3x + 2$.

Sol. Let $P(x_1, y_1)$ be a point closest to the line $y = x - 2$ then $\left.\frac{dy}{dx}\right|_{(x_1,y_1)}$ = slope of line

$\Rightarrow 2x_1 + 3 = 1 \Rightarrow x_1 = -1 \Rightarrow y_1 = 0$ Hence point (-1, 0) is the closest and its perpendicular distance from the

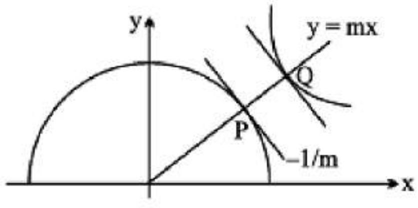
line $y = x - 2$ will give the shortest distance $\Rightarrow p = \frac{3}{\sqrt{2}}$.

Example 22. Let P be a point on the curve C_1: $y = \sqrt{2 - x^2}$ and Q be a point on the curve C_2: $xy = 9$, both P and Q lie in the first quadrant. If 'd' denotes the minimum value between P and Q, find the value of d^2.

Sol. Note that C_1 is a semicircle and C_2 is a rectangular hyperbola.

PQ will be minimum if the normal at P on the semicircle is also a normal at Q on $xy = 9$

Let the normal at P be $y = mx$ ....(1) $(m > 0)$ solving it with $xy = 9$



$$mx^2 = 9 \Rightarrow x = \frac{3}{\sqrt{m}};\ y = \frac{9\sqrt{m}}{3} \quad \therefore \quad Q \equiv \left(\frac{3}{\sqrt{m}}, 3\sqrt{m}\right)$$

differentiating $xy = 9$

$x\frac{dy}{dx} + y = 0 \Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$

$\left.\frac{dy}{dx}\right|_Q = -\frac{3\sqrt{m}\cdot\sqrt{m}}{3} = -m$

$\therefore$ tangent at P and Q must be parallel

$\therefore \quad -m = -\frac{1}{m} \quad \Rightarrow \quad m^2 = 1 \quad \Rightarrow \quad m = 1$

$\therefore$ normal at P and Q is $y = x$

solving $P(1, 1)$ and $Q(3, 3)$

$(PQ)^2 = d^2 = 4 + 4 = 8$

Rate Measurement

Example 23. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall ?

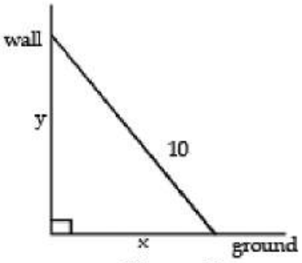
Sol. We first draw a diagram and label it as in Figure 1. Let x feet be the distance from the bottom of the ladder to the wall and y feet the distance from the top of the ladder to the ground. Note that x and y are both function of t (time). We are given that $dx/dt = 1$ ft/s and we are asked to find dy/dt when $x = 6$ ft (see Figure 2). In this problem, the relationship between x and y is given by the Pythagorean Theorem : $x^2 + y^2 = 100$

Differentiating each side with respect to t using the Chain Rule, we have $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

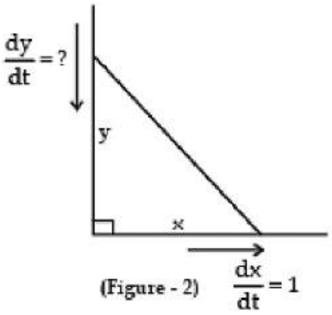
and solving this equation for the desired rate, we obtain $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$

When $x = 6$, the Pythagorean Theorem gives $y = 8$ and so, substituting these values and $dx/dt = 1$,

we have $\frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4}$ ft/s



(Figure - 1)



(Figure - 2)

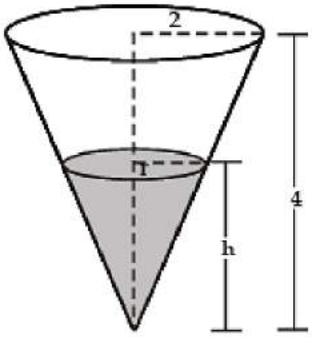
The fact that dy/dt is negative means thatthe distance from the top of the ladder to the ground is decreasing at a rate of 3/4 ft/s In other words, the top of the ladder is sliding down the wall at a rate of 3/4 ft/s .

Example 24. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

Sol.

We first sketch the cone and label it as in Figure. Let V, r, and h be the volume of the water, the radius of the surface, and the height at time t, where t is measured in minutes.

We are given that $dV/dt = 2m^3/min$ and we are asked to find dh/dt when h is 3 m. The quantities V and h are related by the equation $V = 1/3pr^2h$But it is the very useful to express V as a function of h alone.



Figure

In order to eliminate r, we use the similar triangles in Figure to write $\frac{r}{h} = \frac{2}{4}$ $r = \frac{h}{2}$ and the expression for V becomes $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$

$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$ so $\frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt}$

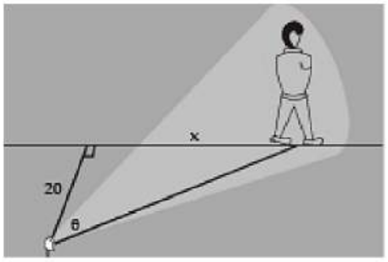
Substituting h = 3 m and dV/dt = $2m^3$/min, we have $\frac{dh}{dt} = \frac{4}{\pi(3)^2} .2 = \frac{8}{9\pi}$

The water level is rising at a rate of 8/(9p) ≈ 0.28 m/min.

Example 25. A man walks along a straight path at the speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight ?

Sol. We draw Figure and let x be the distance from the man to the point on the path closest to the searchlight. We let θ be the distance from the man to the point on the path closest to the searchlight and the perpendicular to the path.

We are given that dx/dt = 4 ft/s and are asked to find dq/dt when x = 15. The equation that relates



Differentiating each side with respect to t, we get dx/dt = $20\sec^2 \theta$ dθ/dt

so $\frac{d\theta}{dt} = \frac{1}{20} \cos^2 \theta \frac{dx}{dt} = \frac{1}{20} \cos^2 \theta (4) = \frac{1}{5} \cos^2 \theta$

when x = 15, the length of the beam is 25, so cosθ = 4/5

and $\frac{d\theta}{dt} = \frac{1}{5}\left(\frac{4}{5}\right)^2 = \frac{16}{125} = 0.128$

The searchlight is rotating at a rate of 0.128 rad/s.

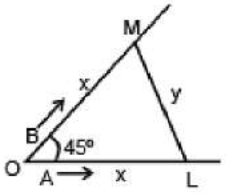
Example 26. Two men A and B start with velocities v at the same time from the junction of two roads inclined at 45º to each other. If they travel by different roads, find the rate at which they are being separated.

Sol.

Let L and M be the positions of men A and B at any time t,

Let OL = x and LM = y. Then OM = x

given, dx/dt = v; to find dy/dt from ΔLOM,



$$\cos 45^{\circ} = \frac{OL^2 + OM^2 - LM^2}{2.OL.OM} \text{ or,}$$

$$\frac{1}{\sqrt{2}} = \frac{x^2 + x^2 - y^2}{2.x.x} = \frac{2x^2 - y^2}{2x^2}$$

or, $\sqrt{2x^2} = 2x^2 - y^2$

or $(2 - \sqrt{2})\, x^2 = y^2$

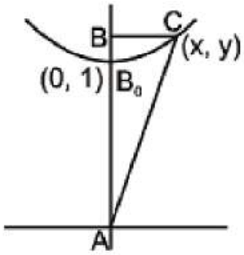
differentiating w. r. t. we get $\frac{dy}{dt} = \sqrt{2\sqrt{2}}\,\frac{dx}{dt} = \sqrt{2-\sqrt{2}}\ v$

$\left[\because\ \frac{dx}{dt} = v\right]$

∴ they are being separated from each other at the rate $\sqrt{2-\sqrt{2}}$ v.

Example 27. A variable triangle ABC in the xy plane has its orthocentre at vertex 'B', a fixed vertex 'A' at the origin & the third vertex 'C' restricted to lie on the parabola y = $1 + \frac{7x^2}{36}$. The point B starts at the point (0, 1) at time t = 0 & moves upward along the y axis at a constant velocity of 2 cm/sec. How fast is the area of the triangle increasing when t = 7/2 sec ?

Sol.



$$A = \frac{xy}{2} = \frac{x}{2}\left(1 + \frac{7x^2}{36}\right) \;;\; \frac{dA}{dt} = \left(\frac{1}{2} + \frac{7}{24}x^2\right)\frac{dx}{dt}$$

$$\text{at } t = \frac{7}{2} \;;\; y = 2 \times \frac{7}{2} = 7 \Rightarrow AB = 8$$

$$\text{when } y = 8 \text{ then } x = 6 \Rightarrow \frac{dA}{dt} = \left(\frac{1}{2} + \frac{7}{24}3.6\right)\frac{dx}{dt} = 11 . \frac{dx}{dt}$$

$$\text{Also } \frac{dy}{dt} = 2 = \frac{14x}{36}\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{36}{7x} = \frac{6}{7}$$

$$\Rightarrow \frac{dA}{dt} = 11 . \frac{6}{7} = \frac{66}{7}$$

Example 28. Find the approximate value of $(1.999)^6$.

Sol.

Let $f(x) = x^6$. Now, $f(x + \delta x)\ f(x) = f'(x) . \delta x = 6x^5\ \delta x$

We may write, $1.999 = 2\ 0.001$

Taking $x = 2$ and $\delta x = 0.001$, we have $f(1.999)\ f(2) = 6(2)^5 \times 0.001$

$\Rightarrow f(1.999) = f(2)\ 6 \times 32 \times 0.001 = 64\ 64 \times 0.003 = 6$

Maxima and Minima of a Function

MAXIMA AND MINIMA

A. Classification of Maxima & Minima

A function f(x) is said to have a local maximum at x = a if f(a) is greater than every other value assumed by f(x) in the immediate neighbourhood of x = a . Symbolically

$$\left.\begin{matrix} f(a) > f(a+h) \\ f(a) > f(a-h) \end{matrix}\right] \Rightarrow x = a \text{ gives local maxima for a sufficiently small positive h .}$$

Similarly, a function f(x) is said to have a local minimum value at x = b if f(b) is least than every other value assumed by f(x) in the immediate neighbourhood at x = b . Symbolically if

$$\left.\begin{matrix} f(b) < f(b+h) \\ f(b) < f(b-h) \end{matrix}\right] \Rightarrow x = b \text{ gives local minima for a sufficiently small positive h .}$$

Note that :

(i) The local maximum & minimum values of a function are also known as relative maxima or relative minima as these are the greatest & least values of the function relative to some neighbourhood of the point in question .

(ii) The term 'extremum' or 'turning point' is used both for local maximum or minimum values.

(iii) A local maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.

(iv) A function can have several local maximum & minimum values & a local minimum value may even be greater than a local maximum value.

B. Fermat's Theorem

If f(x) has a local maximum or minimum at x = c and if f ' (c) exists, then f ' (c) = 0 .

(i) The set of values of x for which f ' (x) = 0 are often called as stationary points. The rate of change of function is zero at a stationary point.

(ii) In case f ' (c) does not exist, f(c) may be a maximum or a minimum.

C. The First Derivative Test

Suppose that c is a critical number of a continuous function f.

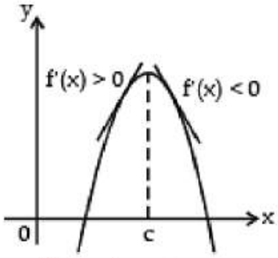
(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

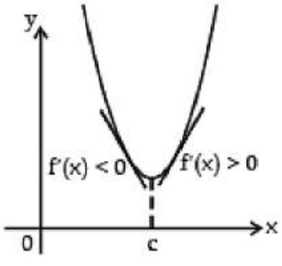
(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

In part (a), since the sign of $f'(x)$ change from positive to negative at c, f is increasing to the left of c and decreasing to the right of c. It follows that f has local maximum at c.

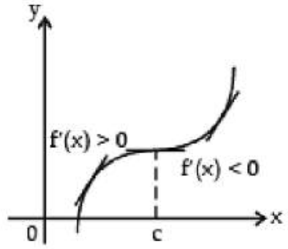
It is easy to remember the First Derivative Test by visualizing the following diagrams.



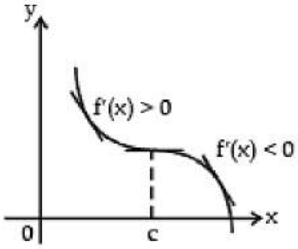
(a) Local maximum



(b) Local minimum



:) No maximum or minimum



(d) No maximum or minimum

Sufficient conditions for an extremum

If x_0 is a critical point of the function f(x) and the inequalities $f'(x_0 - h) > 0$, $f'(x_0 + h) < 0$ are satisfied for an arbitrary, sufficiently small $h > 0$, then the function f(x) possesses a maximum at the point x_0; now if $f'(x_0 + h) < 0$, $f'(x_0 + h) > 0$, then the function f(x) possesses a minimum at the point x_0. If the signs of $f'(x_0 + h)$ and $f'(x_0 + h)$ are the same, then the function f(x) does not possess an extremum at the point x_0.

Ex.1 Test the function $y = (x - 2)^{2/3}(2x + 1)$ for extremum.

Sol. We find $y' = \frac{10}{3}\cdot\frac{x-1}{\sqrt[3]{x-2}}$. The critical points are x = 1 (the derivative is zero) and x = 2 (the derivative does not exist). The inequalities $y'(1 - h) > 0$, $y'(1 + h) < 0$, $y'(2 - h) < 0$, $y'(2 + h) > 0$ hold at a sufficiently small $h > 0$. Consequently, at the point $x = 1$ the function possesses a maximum $y_{max} = 3$ and at the point $x = 2$ it possesses a minimum $y_{min} = 0$.

Ex.2 Let $f(x) = \begin{cases} |x-2| + a^2 - 9a - 9, & \text{if } x < 2 \\ 2x - 3, & \text{if } x \geq 2 \end{cases}$. Then find the value of `a' for which f(x) has local minimum at x=2.

Sol. We have $f(x) = \begin{cases} |x-2| + a^2 - 9a - 9, & \text{if } x < 2 \\ 2x - 3, & \text{if } x \geq 2 \end{cases}$.

f(x) has local minima at $x = 2$. Since, $f(x) = 2x - 3$ for $x \geq 2$ (is strictly increasing)

$\lim_{x \to 2^-} f(x) \geq f(2)$ or $\lim_{h \to 0} f(2-h) \geq f(2)$ $\{\because f(2) = 2 \times 2 - 3 = 1\}$

$\lim_{h \to 0} \{ |2 - h - 2| + a^2 - 9a - 9| \geq 1$

$a^2\ 9a\ 10 \geq 0 \Rightarrow (a + 1)\ (a\ 10) \geq 0 \Rightarrow a \leq 1$ or $a \geq 10$

Ex.3 Find the local maximum and minimum values of the function $g(x) = x + 2\sin x \quad 0 \leq x \leq 2\pi$

Sol. To find the critical number of g, we differentiate : $g'(x) = 1 + 2\cos x$

So $g'(x) = 0$ when $\cos x = -\frac{1}{2}$. The solutions of this equation are $2\pi/3$ and $4\pi/3$.

Because g is differentiable everywhere, the only critical numbers are $2\pi/3$ and $4\pi/3$ and so we analyze g in the following table.

Interval	$g'(x) = 1 + 2\cos x$	g
$0 < x < 2\pi/3$	+	increasing on $(0, 2\pi/3)$
$2\pi/3 < x < 4\pi/3$	−	decreasing on$(2\pi/3, 4\pi/3)$
$4\pi/3 < x < 2\pi$	+	increasing on$(4\pi/3, 2\pi)$

Because $g'(x)$ changes from positive to negative at $2\pi/3$, the First Derivative Test tells us that there is a local maximum at $2\pi/3$ and the local maximum value is

$$g(2\pi/3) = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3} = \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} + \sqrt{3} \approx 3.83$$

Likewise, g'(x) changes from negative to positive at $4\pi/3$ and so

$$g(4\pi/3) = \frac{4\pi}{3} + 2\sin\frac{4\pi}{3} = \frac{4\pi}{3} + 2\left(-\frac{\sqrt{3}}{2}\right) = \frac{4\pi}{3} - \sqrt{3} \approx 2.46$$

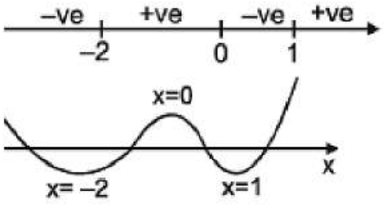
Ex.4 Find the values of a for which all roots of the equation $3x^4 + 4x^3 - 12x^2 + a = 0$ are real and distinct.

Sol. Consider the function $f(x) = 3x^4 + 4x^3 - 12x^2 + a$.

Then $f'(x) = 12(x^3 + x^2 - 2x) = 12x\,(x - 1)\,(x + 2)$.

From the sign scheme for f'(x), we can see that the shape of the curve will be as shown alongside.

For four real and distinct roots, the two minima must lie below the X-axis and the maxima must lie above the x-axis.



Thus, we have $f(2) < 0$ i.e. $48\ 32\ 48 + a < 0$ ...(i) i.e. $a < 32$

and $f(1) < 0$ i.e. $3 + 4\ 12 + a < 0$ ...

(ii) i.e. $a < 5$

and $f(0) > 0$ i.e. $a > 0$ ...(iii)

Taking intersection of inequalities (1), (2) and (3) we have $a \in (0, 5)$.

Ex.5 If $f(x) = x^3 + 3(a - 7)x^2 + 3(a^2 - 9)x - 1$. If f(x) attains maxima at some positive value of x, then find the possible values of a.

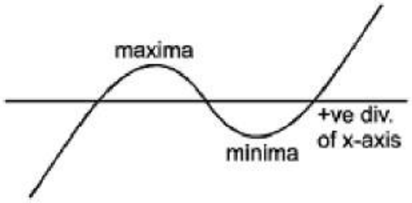
Sol. We have $f(x) = x^3 + 3(a - 7)x^2 + 3(a^2 - 9)x - 1$ and $f'(x) = 3x^2 + 6(a - 7)x + 3(a^2 - 9)$

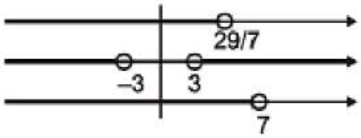
which shows that there are two critical points (real or imaginary). According to the given condition, there is one real critical point (maxima), then the other critical point must also be real (minima).

Also, we have $f(-\infty) = -\infty$ and $f(\infty) = \infty$

From the above facts, the graph of the curve $y = f(x)$ can be drawn as shown alongside. Thus, if maxima occurs at some +ve value of x, then the minima must also occur at some +ve value of x(see fig.).

Thus, the roots of equation $f'(x) = 0$ are +ve and distinct, which is possible if discriminant > 0





i.e. $(a\ 7)^2 > a^2\ 9$ i.e. $14\,a + 58 > 0$

i.e. $a < 29/7$ ...(1)

and product of the roots > 0

i.e. $a^2\ 9 > 0$

i.e. $a < 3$ or $a > 3$ ...(2)

and sum of the roots > 0

i.e. $a\ 7 < 0$ i.e. $a < 7$ ...(3)

Drawing the number line for inequalities (1), (2), (3) and taking intersection, gives

$a \in (-\infty, -3) \cup \left(3, \frac{29}{7}\right)$.

Ex.6 For what real values of a and b are all the extrema of the function; f(x) = a²x³ - 0.5 a x² - 2x - b, is positive and the minimum is at the point $x_0 = 1/3$.

Sol. For extrema, $f'(x) = 0 \Rightarrow$

$3a^2x^2 - ax - 2 = 0$ at $x = \frac{1}{3}$ (as at $x = \frac{1}{3}$ function is minimum)

$$3a^2\left(\frac{1}{3}\right)^2 - a\left(\frac{1}{3}\right) - 2 = 0$$

$$\Rightarrow \quad \frac{a^2}{3} - \frac{a}{3} - 2 = 0 \text{ or}$$

$a^2 \ a \ 6 = 0 \Rightarrow a = 2, 3$

So their arises two cases as :

Case I : at a = 3, if function attains minimum and is positive,

$$\therefore 9\left(\frac{1}{3}\right)^3 - (0.5)\ (3)\left(\frac{1}{3}\right)^2 - 2\left(\frac{1}{3}\right) - b > 0$$

$$\left\{\text{since minimum at } x = \frac{1}{3} \text{ when } a = 3 \Rightarrow f\left(\frac{1}{3}\right) > 0 \text{ when } a = 3\right\}$$

$$\Rightarrow \quad b < \frac{1}{3} - \frac{1.5}{9} - \frac{2}{3}$$

$$\text{or} \quad b < -\frac{1}{2}$$

Case II : at a = -2, if function attains minimum and is positive,

$$\therefore (-2)^2\left(\frac{1}{3}\right)^3 - (0.5)(-2)\left(\frac{1}{3}\right)^2 - 2\left(\frac{1}{3}\right) - b > 0$$

$$\left\{\text{since minimum at } x = \frac{1}{3} \text{ when } a = -2 \Rightarrow f\left(\frac{1}{3}\right) > 0 \text{ when } a = -2\right\}$$

$\Rightarrow \quad b < \frac{4}{27}+\frac{1}{9}-\frac{2}{3} \qquad \text{or} \quad b < -\frac{11}{27}$

$\therefore$ when $a = 3 \Rightarrow b < -\frac{1}{2}$ and when $a = -2 \Rightarrow b < -\frac{11}{27}$.

Ex.7 For what values of `a' the point of local minima of $f(x) = x^3 - 3ax^2 + 3(a^2 - 1)x + 1$ is less than 4 and point of local maxima is greater than -2.

Sol. $f'(x) = 3(x^2 - 2ax + a^2 - 1)$

Clearly roots of the equation $f'(x) = 0$ must be distinct and lie in the interval (-2, 4)

$D > 0 \Rightarrow a \in R$ ...(1)

$f'(2) > 2 \Rightarrow a^2 + 4a + 3 > 0 \Rightarrow a < 3$ or $a > 1$ ...(2)

$f'(4) > 0 \Rightarrow a^2\ 8a + 15 > 0 \Rightarrow a > 5$ or $a < 3$ ...(3)

and $-2 < -\frac{B}{2A} < 4 \Rightarrow -2 < a < 4$

From (1), (2) and (3) $-1 < a < 3$

Alternate : $f'(x) = 3(x - (a - 1))\,(x - (a + 1))$

clearly $-2 < a + 1 < 4$ and $-2 < a - 1 < 4 \Rightarrow -1 < a < 3$

D. Extremum at End-points

A point (c, f(c)) is called an endpoint of the graph of the function f if there exists an interval (a, b) containing c such that the domain of f contains every number of the interval (a, c) and no number of the interval (c, b), or vice versa.

If (c, f(c)) is an endpoint of the graph of f such that f(c) is the maximum or minimum value of in some interval containing c, then f(c) is called an endpoint extremum of f. Note the difference between this definition and that of a relative extremum, in which it is assumed that some open interval containing c is contained in the domain of the function.

Consider $f(x) = \sqrt{4-x^2}$. Clearly, $f(-2) = 0$ and $f(2) = 0$ are endpoint extrema of f. Also, $f(0) = 2$ is a (relative) maximum value of f.

E. Second-Derivative Test For Extremum

Let c be a critical point of f in an open interval (a, b); that is, assume $a < c < b$ and $f'(c) = 0$. Assume also that the second derivative f'' exists in (a, b). Then we have the following :

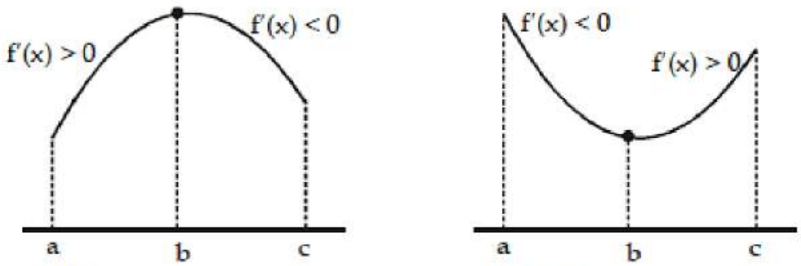
(a) If f'' is negative in (a, b), f has a relative maximum at c.

(b) If f'' is positive in (a, b), f has a relative minimum at c.

The two cases are illustrated in Figure

Proof. Consider case (a), $f'' < 0$ in (a, b). The function f' is strictly decreasing in (a, b). But $f'(c) = 0$, so f' changes its sign from positive to negative at c, as shown in Figure (a). Hence, f has a relative maximum at c. The proof in case (b) is entirely analogous.

If f'' is continuous at c, and if $f''(c) \neq 0$, there will be a neighbourhood of c in which f'' has the same sign as f''(c). Therefore, if $f'(c) = 0$, the function f has a relative maximum at c if f''(c) is negative, and a relative minimum if f''(c) is positive. This test suffices for many examples that occur in practice.



(a) Relative maximum at c (b) Relative minimum at c

Figure : An extremum occurs when the derivative changes sign.

Ex.8 Find all possible values of ' a ' for which the cubic $f(x) = x^3 + ax + 2$ is non monotonic and has exactly one real root .

Sol. Note that

(i) for 3 real and distinct roots we have or $f(x_1) . f(x_2) < 0 \Rightarrow f(x)$ is non monotonic in this case

(ii) for exactly one real root and monotonic the graph will be as shown. Here $f'(x) \neq 0$

(iii) for exactly one real root and non monotonic the graph will be as shown. Here $f(x_1) . f(x_2) > 0$

Now $f(x) = x^3 + ax + 2 \Rightarrow f'(x) = 3x^2 + a$

if $a \geq 0$, f (x) is always increasing .

Now let $a < 0$ $f'(x) = 0 \Rightarrow x = \pm\sqrt{-\frac{a}{3}} = \pm\sqrt{b}$ where $b = -\frac{a}{3} > 0$

$f''(x) = 6x$; $f'' > 0 \Rightarrow$ minima and $f'' < 0 \Rightarrow$ maxima

for exactly one real root and non monotonic (case iii)

$f''(x) = 6x$; $f''\left(\sqrt{b}\right) > 0 \quad \Rightarrow$ minima and

$f''\left(-\sqrt{b}\right) < 0 \Rightarrow$ maxima for exactly one real root and non monotonic (case iii)

$f\left(\sqrt{b}\right) . f\left(-\sqrt{b}\right) > 0$

$\Rightarrow (b^{3/2} + a\,b^{1/2} + 2)\,(-b^{3/2} - a\,b^{1/2} + 2) > 0$

or $(b^{3/2} + a\,b^{1/2})^2 - 4 < 0$

or $b^3 + a^2 b + 2ab^2 - 4 < 0$; now substituting $b = -\frac{a}{3}$
we get $a^3 + 27 > 0$. But $a < 0 \Rightarrow a \in (-3, 0)$

Note that

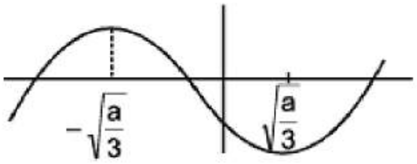
(i) for a = - 3 , $f(x) = x^3 - 3x + 2 = (x + 2)(x - 1)^2$ i.e. f (x) has two coincident roots .

(ii) for a < - 3 , f (x) has 3 real and distinct roots

(iii) for $a \geq 0$, f (x) is exactly real root and is always monotonic increasing find $(x_2 - x_1)^2$ from (1) and get V as a function of y

Ex.9 Let 'p' & 'q' be real numbers . Prove that the cubic $y = x^3 + px + q$ has three distinct real roots, if $4p^3 + 27q^2 < 0$.

Sol.



Let $f(x) = x^3 + px + q \Rightarrow f'(x) = 3x^2 + p$

If $p > 0 \Rightarrow$ no root (f (x) is monotonic)

If $p < 0 \Rightarrow x = \pm\sqrt{-\frac{p}{3}} = \pm\sqrt{\frac{a}{3}}$ ($a = -p$)

3 distinct real roots f (x) must have exactly one maxima & minima .

$f''(x) = 6x$; $f''\left(\sqrt{\frac{a}{3}}\right) > 0 \Rightarrow$ min. at $x = \sqrt{\frac{a}{3}}$

and $f''\left(-\sqrt{\frac{a}{3}}\right) < 0 \Rightarrow$ max. at $x = -\sqrt{\frac{a}{3}}$

Hence $f\left(\sqrt{\frac{a}{3}}\right) . f\left(-\sqrt{\frac{a}{3}}\right) < 0$

$$\Rightarrow \left[\left(\frac{a}{3}\right)^{3/2} + p\sqrt{\frac{a}{3}} + q\right] \times \left[-\left(\frac{a}{3}\right)^{3/2} - p\sqrt{\frac{a}{3}} + q\right] < 0$$

$$\Rightarrow \quad \left(\left(\frac{a}{3}\right)^{3/2} + \sqrt{\frac{a}{3}}\,p + q\right) . \left(\left(\frac{a}{3}\right)^{3/2} + \sqrt{\frac{a}{3}}\,p + q\right) > 0$$

$$\Rightarrow \quad \left(\left(\frac{a}{3}\right)^{3/2} + p\sqrt{\frac{a}{3}}\right)^2 - q^2 > 0$$

$$\Rightarrow \quad \left(\frac{a}{3}\right)^3 + p^2\frac{a}{3} + 2p\left(\frac{a}{3}\right)^2 - q^2 > 0$$

$$\Rightarrow \quad a^3 + 9p^2a + 6pa^2 - 27q^2 > 0$$

$$\Rightarrow \quad -p^3 - 9p^3 + 6p^3 - 27q^2 > 0$$

$$\Rightarrow \quad 4p^3 + 27q^2 < 0$$

Ex.10 Suppose f(x) is real valued polynomial function of degree 6 satisfying the following conditions ;

(a) f has minimum value at $x = 0$ and 2

(b) f has maximum value at $x = 1$

(c) for all x, $\underset{x \to 0}{\text{Limit}} \frac{1}{x} \, ln \begin{vmatrix} \frac{f(x)}{x} & 1 & 0 \\ 0 & \frac{1}{x} & 1 \\ 1 & 0 & \frac{1}{x} \end{vmatrix} = 2$ **.**

Sol.

Determine f(x). $D = 1 + \frac{f(x)}{x^3}$

$$\Rightarrow \underset{x \to 0}{\text{Limit}} \; ln\left(1 + \frac{f(x)}{x^3}\right)^{1/x} = 2$$

$\Rightarrow$ f(x) have co-efficient of x^3, x^2, x or constant term zero in order that the limit may exist.

$$= ln\, e^{\underset{x \to 0}{\text{Limit}} \frac{1}{x} \cdot \frac{f(x)}{x^3}} = \underset{x \to 0}{\text{Limit}} \frac{f(x)}{x^4} = 2$$

$$= \underset{x \to 0}{\text{Limit}} \frac{ax^6 + bx^5 + cx^4}{x^4} = 2 \Rightarrow c = 2 .$$

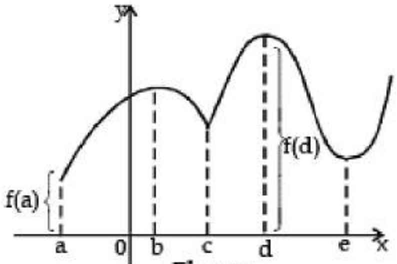
Hence $f(x) = ax^6 + bx^5 + cx^4 \Rightarrow f'(x) = x^3 (6\,ax^2 + 5\,bx + 8)$

$f'(1) = 0$ and $f'(2) = 0$ gives $6a + 5b + 8 = 0$ and $24\,a + 10\,b + 8 = 0$

$$\Rightarrow a = \frac{2}{3} \; ; \; b = -\frac{12}{5} \quad \Rightarrow f(x) = \frac{2}{3}x^6 - \frac{12}{5}x^5 + 2x^4$$

F. GLOBAL MAXIMUM AND MINIMUM

Definition A function f has an absolute maximum (or global maximum) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number f(c) is called the maximum value of f on D. Similarly, f has an absolute minimum at c if $f(c) \leq f(x)$ for all x in D and the number $f(c)$ is called the mini mum value of f on D. The maximum and minimum values of f are called the extreme values of f



Figure

Minimum value f(a), Maximum value f(d)

Figure shows the graph of a function f with absolute maximum at d and absolute minimum at a. Note that $(d, f(d))$ is the highest point on the graph and $(a, f(a))$ is the lowest point.

The Extreme Value Theorem - Application Of Derivatives, Class 12, Maths

The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value of $f(d)$ at some numbers c and d in $[a, b]$.

The Extreme Value Theorem is illustrated in Figure 1. Note that an extreme value can be attained at more than one point. Although the Extreme Value Theorem is intuitively very plausible, it is difficult to prove and so we omit the proof.

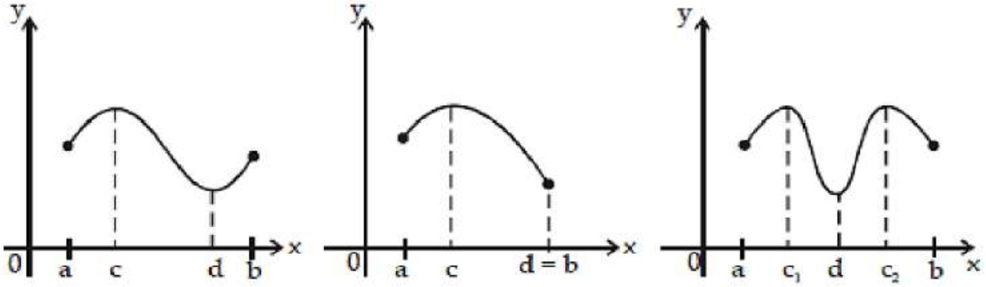


Figure 1

Conditions of Extreme Value Theorem

Figure 2, 3 show that a function need not possess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem. The function f whose graph is shown in Figure 2 is defined on the closed interval [0, 2] but has no maximum value. (Notice that the range of f is [0, 3). The function takes on values arbitrarily close to 3, but never actually attains the value 3.) This does not contradict the extreme value theorem.

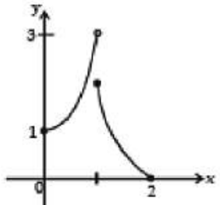


Figure 2
This function has minimum value f(2) = 0, but no maximum value.

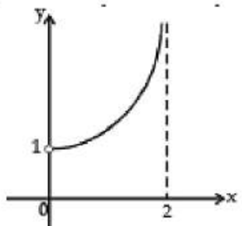


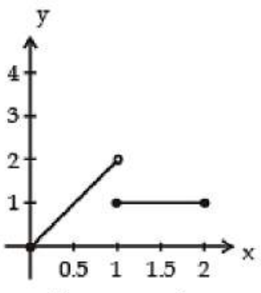
Figure 3
This continuous function g has no maximum or minimum

The function f shown in Figure 3 is continuous on the open interval (0, 2) but has neither a maximum nor a minimum value. The range of g is (1, ∝). The function takes on arbitrarily large values.] This does not contradict the Extreme Value Theorem because the interval (0, 2) is not closed.

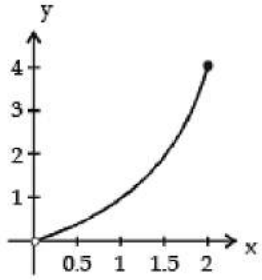
In each case, explain why the given function does not contradict the extreme value theorem.

a. $f(x) = \begin{cases} 2x & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x \le 2 \end{cases}$

b. $g(x) = x^2$ on $0 < x \le 2$



Does not have a maximum value



g does not have a minimum value
(but it does have a maximum value.)

a. The function f has no maximum. It takes on all values arbitrarily close to 2, but it never reaches the value 2. The extreme value theorem is not violated because f is not continuous on [0, 2].

b. Although the functional values of g(x) become arbitrarily small as x approaches 0, it never reaches the value 0, so g has no minimum. The function g is continuous on the interval (0, 2], but the extreme value theorem is not violated because the interval is not closed.

Procedure for Finding the Extrema of a Continuous Function

Suppose a continuous function f is differentiable at all except a finite number of values of x in its domain, the closed interval $a \le x \le b$.

1. Find all x in $a < x < b$ that satisfy the equation $f'(x) = 0$ or at which $f'(x)$ does not exist; let $x = r$, $x = s$, $x = t$, ... be such x. The numbers r, s, t, . . . are often called **critical points** of f.

2. Evaluate f at each critical point; that is, find f(r), f(s), f(t),........

3. Evaluate f(a) and f(b).

4. The largest of the numbers computed in Step 2 and Step 3 is the maximum of f(x) for a £ x £ b, and the smallest number is the minimum.

Ex.11 Let $f(x) = 2x^3 - 9x^2 + 12x + 6$. discuss the global maxima and global minima of f(x) in (1, 3).

Sol.

$f(x) = 2x^3\ \ 9x^2 + 12x + 6 \Rightarrow f'(x) = 6x^2\ \ 18x + 12 \Rightarrow f'(x) = 6\ (x\ 1)\ (x\ 2)$

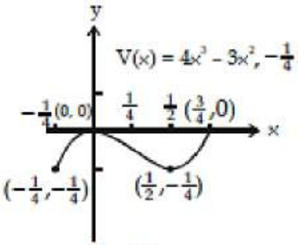
let $f'(x) = 0 \Rightarrow x = 1, 2. \therefore f(1) = 11$ and $f(2) = 10$ ...(i) let us consider the open interval (1, 3). Clearly $x = 2$ is the only point in (1, 3) and $f(2)=10$ [from (i)

Now $\lim_{x \to 1^+} f(x) = 11$ and $\lim_{x \to 3^-} f(x) = 15$

Thus, $x = 2$ is the point of global minima in (1, 3) and global maxima does not exist in (1, 3).

Ex.12 Let $w(x) = 4x^2 - 3x^2$ on $-\frac{1}{4} < x < \frac{3}{4}$. Discuss the extrema of w.

Sol. w(x) has a maximum at $x = 0$ and a minimum at $x = 1/2$, and these two values are in the given interval.



Figure

Ex.13 The greatest value of the function $f(x) = 2.3^{3x} - 3^{2x} . 4 + 2.3^{x}$ in the interval [- 1, 1] is

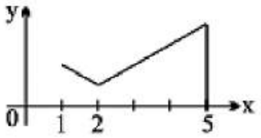
Sol. $f'(x) = 2.3^{x} . \ln 3 [3.3^{2x} - 4.3 x + 2]$

$= 3.2 . 3^{x} \ln 3 \left[\left(e^{x} - \frac{2}{3}\right)^2 + \frac{2}{3} \right] > 0$ in $[-1, 1]$

Hence f (x) is greatest when $x = 1$ & $f(1) = 24$

Ex.14 Let $f(x) = ax^2 - 4ax + b$ ($a > 0$) be defined in $1 \le x \le 5$. Suppose the average of the maximum value and the minimum value of the function is 14, and the difference between the maximum value and minimum value is 18. Find the value of $a^2 + b^2$.

Sol.



$f(x) = ax^2\ \ 4ax + b\ (a > 0) \Rightarrow f'(x) = 2ax\ 4a = 0$

at $x = 2$ also, $f'(x) = 2a(x\ 2) \Rightarrow$ for $x \in (1, 2)$ f is

Hence minimum occurs as $x = 2$

$f(2) = 4a\ 8a + b$

$f(2) = b\ 4a$

maximum will occur at f (5) and

$f(5) = 25a - 20a + b = b + 5a$
$M = b + 5a$
$m = b - 4a$
————

$M\ m = 9a = 18 \Rightarrow a = 2$ also $M+m/2 = 14 \Rightarrow M + m = 28 = 2b + a \Rightarrow b = 13$

Hence $a = 2$ and $b = 13$

$a^2 + b^2 = 4 + 169 = 173$

Ex.15 If f(x)= $(x - a)(x - b) - \left(\frac{a+b}{2}\right)$ **and f(x) = 0 has both non-negative roots, then prove that** $f(x)^3 - \frac{(a+b)^2}{2}$.

Sol. Given that $f(x) = (x - a)(x - b) - \left(\frac{a+b}{2}\right)$

Sum of the root of the equation $f(x) = 0$, will be positive $\Rightarrow (a + b) > 0$

The product of the roots of the equation will be greater than and equal to zero

$$\Rightarrow \quad ab - \left(\frac{a+b}{2}\right) \geq 0.$$

Now f(x) will be minimum, when $f'(x) = 0 \Rightarrow x = a+b/2$

$$\Rightarrow \quad (f(x))_{\min} = \left(\frac{a+b}{2}\right)^2 - \frac{(a+b)^2}{2} + ab - \left(\frac{a+b}{2}\right)$$

$$=\frac{-(a+b)^2}{4}+ab-\left(\frac{a+b}{2}\right)$$

$$\Rightarrow \qquad \frac{-(a+b)^2-4ab+4ab}{4}=\frac{-(a+b)^2}{4}$$

Ex.16 If x > 0, let f(x) = 5 x^2 + A x^{-5}, where A is a positive constant . Find the smallest A such that $f(x) \geq 24$ for all x > 0 .

Sol.

f ' (x) = 10 x - 5 A x^{-6} and f '' (x) = 10 + 30 A $x^{-7} > 0$ i.e. f ' (x) = 0 gives a minima

$$\Rightarrow x^7=\frac{A}{2} \Rightarrow x=\left(\frac{A}{2}\right)^{1/7}$$

Since A > 0 ⇒ we get only one minima and no maxima . Hence smallest value of f(x) will be at $x=\left(\frac{A}{2}\right)^{1/7}$

$$\text{i.e. } f(x)\big]_{\min}=5\,.\left(\frac{A}{2}\right)^{2/7}+A\left(\frac{A}{2}\right)^{-5/7}=24$$

$$\text{or } \ 5\left(\frac{A}{2}\right)^{2/7}+2\left(\frac{A}{2}\right)^{2/7}=24$$

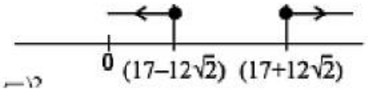
$$\Rightarrow A=2\left(\frac{24}{7}\right)^{7/2}$$

Ex.17 Find the sum of the local maximum and local minimum values of the function $f(x)=\frac{\tan 3x}{\tan^3 x}$ on interval $(0, \pi/2)$.

Sol.

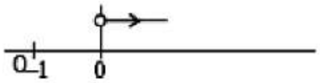
$$y=\frac{\tan 3x}{\tan^3 x}=\frac{3\tan x-\tan^3 x}{\tan^3 x(1-3\tan^2 x)}$$

$$=\frac{3-\tan^2 x}{\tan^2 x(1-3\tan^2 x)}=\frac{3-t}{t(1-3t)} \text{ where } \tan^2 x = t > 0$$



$\Rightarrow (t\ 3t^2)y = 3\ t\ ,\ 3yt^2\ \ (1 + y)t + 3 = 0$

$t > 0 \Rightarrow D \geq 0$; Sum of roots > 0 ; Product of roots $>$



$\frac{1+y}{3y} > 0$ and $\frac{1}{y} > 0$ hence $y > 0$

hence $(1 + y)^2\ \ 36y \geq 0$;

$\Rightarrow y^2\ \ 34y\ 1 \geq 0 \Rightarrow (y\ 17)^2 \geq 288 \Rightarrow (y - 17)^2 - \left(12\sqrt{2}\right)^2 \geq 0$

$\Rightarrow (y\ 17\ 12\sqrt{2}\ )(y\ 17 + 12\sqrt{2}\ ) \geq 0$

$\Rightarrow [y\ (17 + 12\sqrt{2}\ )]\ [y\ (17\ 12\sqrt{2}\ )] \geq 0$

Hence $y_{max} = 17\ 12\sqrt{2}$, $y_{min} = 17 + 12\sqrt{2} \Rightarrow y_{max} + y_{min} = 34$ which is rational

Ex.18 For a certain curve $\frac{d^2y}{dx^2}$ = 6x - 4 and y has a local maximum value 5 when x = 1 . Find the equation and the global maximum and minimum values of y, given that $0 \leq x \leq 2$.

Sol.

Integrating, $\frac{dy}{dx} = 3x^2 - 4x + A$; $\left.\frac{dy}{dx}\right]_{x=1} = 0 \Rightarrow A = 1$

Hence $\frac{dy}{dx} = 3x^2 - 4x + 1$;

Integrating again, $y = x^3 - 2x^2 + x + B$; $y]_{x=1} \Rightarrow B = 5$.

Thus $y = x^3 - 2x^2 + x + 5$.

given x = 1/3 and z = 1 f(1/3) =139/27; f(1) = 5

also f(0) = 5 ; f(2) = 7 . Hence GMV = 7 ; gmv = 5

Ex.19 Find the least and the greatest value of $f(x, y) = x^2 + y^2 - xy$ where x and y are connected by the relation $x^2 + 4y^2 = 4$.

Sol.

Here $x^2 + 4y^2 = 4$

$\Rightarrow \frac{x^2}{4} + y^2 = 1$ (which is clearly an ellipse)

$\Rightarrow$ Let $x = 2 \cos \theta, y = \sin\theta$

Hence, $f(x, y) = x^2 + y^2\ xy = 4 \cos 2\theta + \sin 2\theta\ 2 \sin\theta \cos\theta = 2 (1 + \cos 2\theta) + 1/2 (1 - \cos 2\theta) - \sin 2\theta$

$$= \left(2 - \frac{1}{2}\right) \cos 2\theta - \sin 2\theta + \frac{5}{2}$$

$$= \frac{3}{2} \cos 2\theta - \sin 2\theta + \frac{5}{2}$$

“Since we know a $\sin\theta + b \cos\theta$ lies between $-\sqrt{a^2 + b^2}$ to $\sqrt{a^2 + b^2}$”

$$-\frac{\sqrt{13}}{2} + \frac{5}{2} \le \frac{3}{2} \cos 2\theta - \sin 2\theta + \frac{5}{2} \le \frac{\sqrt{13}}{2} + \frac{5}{2}$$

Thus, greatest value of $f(x, y) = \frac{5 + \sqrt{13}}{2}$ and least value of $f(x, y) = \frac{5 - \sqrt{13}}{2}$

G. Geometrical Problems

Working Rule

1. When possible , draw a figure to illustrate the problem & label those parts that are important in the problem . Constants & variables should be clearly distinguished.

2. Write an equation for the quantity that is to be maximized or minimized . If this quantity is denoted by `y' , it must be expressed in terms of a single independent variable x . This may require some algebraic manipulations.

3. If $y = f(x)$ is a quantity to be maximum or minimum, find those values of x for which $f'(x) = 0$ or $f'(x)$ does not exist.

4. Test each value of x to determine whether it provides a maximum or minimum or neither. The usual tests are:

(a) If d^2y/dx^2 is positive when $dy/dx = 0 \Rightarrow y$ is minimum.
If d^2y/dx^2 is negative when $dy/dx = 0 \Rightarrow y$ is maximum.
If $d^2y/dx^2 = 0$ when $dy/dx = 0$, the test fails.

(b) $\text{If}\ \frac{dy}{dx}\ \text{is} \left.\begin{matrix} \text{positive} & \text{for} & x < x_0 \\ \text{zero} & \text{for} & x = x_0 \\ \text{negative} & \text{for} & x > x_0 \end{matrix}\right]$ $\Rightarrow$ a maximum occurs at $x = x_0$.

But if dy/dx changes sign from negative to zero to positive as x advances through x_0 there is a minimum. If dy/dx does not change sign, neither a maximum nor a minimum.

5. If the function $y = f(x)$ is defined for only a limited range of values $a \leq x \leq b$ then examine $x = a$ & $x = b$ for possible extreme values.

Useful Formulae Of Mensuration

1. Volume of a cuboid = lbh .
2. Surface area of a cuboid = 2 (lb + bh + hl) .
3. Volume of a prism = area of the base x height .
4. Lateral surface of a prism = perimeter of the base x height .
5. Total surface of a prism = lateral surface + 2 area of the base (Note that lateral surfaces of a prism are all rectangles) .
6. Volume of a pyramid = 1/3 area of the base x height .
7. Curved surface of a pyramid = 1/2 (perimeter of the base) x slant height .
8. (Note that slant surfaces of a pyramid are triangles) .
9. Volume of a cone $= 1/3\pi r^2 h$
10. Curved surface of a cylinder = $2\pi rh$.
11. Total surface of a cylinder = $2\pi rh + 2\pi r^2$
12. Volume of a sphere = $4/3\pi r^3$
13. Surface area of a sphere = $4\pi r^2$
14. Area of a circular sector = $1/2 r^2 \theta$, when θ is in radians.

Ex.20 A trapezium ABCD is inscribed into a semicircle of radius *l* so that the base AD of the trapezium is a diameter and the vertices B & C lie on the circumference . Find the base angle q of the trapezium ABCD which has the greatest perimeter .

Sol. Hint : P = AB + BC + CD + DA = (AB + CD) + BC + DA

$CD = AB = 2\, l \cos\theta$; $AD = 2\, l$ and

$x = AB \cos\theta \Rightarrow BC = 2\, l - 2\, x = 2\, l - 2\, AB \cos\theta = 2\, l - 4\, l \cos^2\theta$

Hence $P = 4\, l + 4\, l \cos\theta - 4\, l \cos^2\theta$

$\Rightarrow \frac{dP}{d\theta} = 0$ gives $\theta = 0$ (not possible)

or $\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

Ex.21 A bus contractor agrees to run special buses for the employees of ABC Co. Ltd . He agrees to run the buses if atleast 200 persons travel by his buses . The fare per person is to be Rs. 10/- per day if 200 travel and will be decreased for everybody by 2 praise per person over 200 that travels . How many passengers will give the contractor maximum daily revenue ?

Sol. Let number of passengers be x, which will yield maximum profit

$f(x) = x\left[10 - (x - 200)\frac{2}{100}\right]$ for $x \geq 200$

$\Rightarrow f'(x) = 0 \Rightarrow x = 350$; $f(x)]_{\max} = 2450$

Ex.22 Find the radius of the smallest circular disk large enough to cover every isosceles triangle of a given perimeter L.

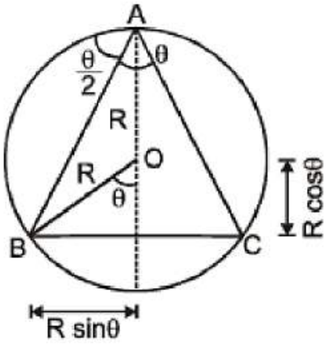
Sol.

$AB = (R + R\cos\theta)\sec\frac{\theta}{2}$

Hence $L = 2\, AB + BC = 2R\left[(1 + \cos\theta)\sec\frac{\theta}{2} + \sin\theta\right]$

$$= 2R\left[2\cos\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]$$

$$L = 4R\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\right) \qquad R = \frac{L}{4\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\right)}$$



$$\text{Let } f(\theta) = \cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\right)$$

$$\Rightarrow f'(\theta) = -\frac{1}{2}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\right) + \frac{1}{2}\cos^2\frac{\theta}{2}$$

$$f'(\theta) = \frac{1}{2}\cos\theta - \frac{1}{2}\sin\frac{\theta}{2} = 0$$

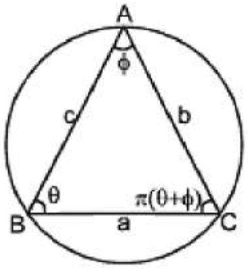
$$\Rightarrow \theta = \frac{\pi}{3} \qquad \text{But } 0 < \theta < \frac{\pi}{2}$$

$$f\left(\frac{\pi}{2}\right) = 0.207 \text{ If } \theta = \frac{\pi}{2}$$

$$\Rightarrow R = \frac{L}{4} \text{ at } \theta = 0 \text{ is the required radius}$$

Ex.23 Through a point A on the circumference of a circle of radius r, two straight lines are drawn enclosing an angle f . If the straight lines meet the circle again at B & C, find the maximum area of triangle ABC .

Sol.



$$\frac{a}{\sin \phi} = \frac{b}{\sin \theta} = \frac{c}{\sin (\theta + \phi)}$$

$$A = \frac{1}{2} \text{ bc} \sin \phi$$

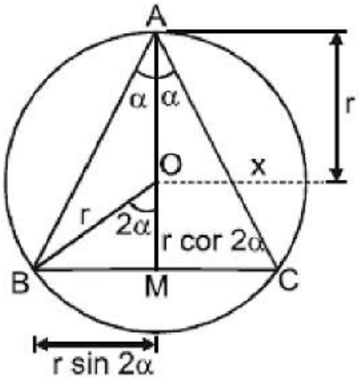
$$A = 2r^2 \sin ø \sin ø \sin (\theta + ø) = r^2 \sin ø [\cos ø - \cos (2\theta + ø)]$$

$$\frac{dA}{d\theta} = 0 \Rightarrow \theta = \frac{\pi}{2} - \frac{\phi}{2}$$

Ans. : $r^2 \sin ø (1 + \cos ø)$ sq. units

Ex.24 An isosceles triangle is inscribed in a circle of radius r . If the angle 2 α at the apex is restricted to lie between $0 \text{ and } \frac{\pi}{2}$, find the largest and the smallest value of the perimeter of the triangle . Give sufficient details of your reasoning .

Sol.



$$0 < 2\alpha < \frac{\pi}{2} \Rightarrow 0 < \alpha < \frac{\pi}{4}$$

$$P = 2x + 2r \sin 2\alpha$$

$$= \frac{2r(1+\cos 2\alpha)}{\cos\alpha} + 2r\sin 2\alpha$$

$$= 2r\left[\frac{1+\cos 2\alpha + \sin 2\alpha\cos\alpha}{\cos\alpha}\right]$$

$$P = f(\alpha) = 4r(1+\sin\alpha)\cos\alpha$$

$$f'(\alpha) = 4r[\cos 2\alpha - (1+\sin\alpha)\cos\alpha] = 4r(1-2\sin\alpha)(1+\sin\alpha)$$

$\Rightarrow \sin\alpha = \frac{1}{2}$ or $\sin\alpha = -1$ (not possible)

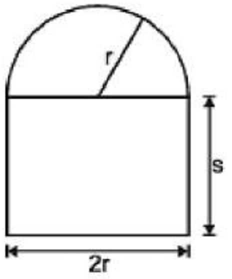
Hence $P_{max} = 4r\left(\frac{3}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}\,r$,

$P_{min} = 4r$ when $\alpha = 0$

$$P\left(\alpha = \frac{\pi}{4}\right) = 4r\left(\frac{\sqrt{2}+1}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{2}} = 2r\left(\sqrt{2}+1\right)$$

Ex.25 The plan view of a swimming pool consists of a semicircle of radius r attached to a rectangle of length '2r' and width 's' . If the surface area A of the pool is fixed, for what value of 'r' and 's' the perimeter 'P' of the pool is minimum .

Sol.



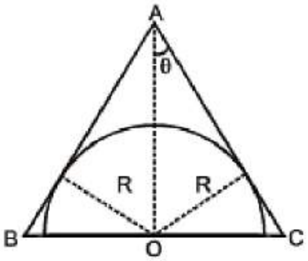
$A = \frac{\pi r^2}{2} + 2rs$ $\qquad P = 2s + 2r + \pi r$

$P = \left(\frac{\pi r}{2} + 2s\right) + 2r + \frac{\pi r}{2}$ $\qquad P = \frac{A}{r} + 2r + \frac{\pi r}{2}$

Now $\frac{dP}{dr} = 0 \Rightarrow r = \sqrt{\frac{2A}{\pi+4}} \Rightarrow s = \sqrt{\frac{2A}{\pi+4}}$

Ex.26 Find the altitude of a cone of the least volume that can be drawn around a hemisphere of radius R (the centre of the base of the cone falls on the centre of the sphere).

Sol.



Let AO = H, BO = r = radius of the base of the cone

R = radius of the hemisphere, ∠OAC = θ (θ ∈(0, π/2))

$\frac{r}{H} = \tan\theta$ and $\frac{R}{H} = \sin\theta$

$\Rightarrow\ r = \frac{R}{\sin\theta}.\tan\theta = \frac{R}{\cos\theta}$

$V(\theta) = \frac{1}{3}\pi\left(\frac{R}{\cos\theta}\right)^2\left(\frac{R}{\sin\theta}\right) = \frac{1}{3}\pi R^3.\frac{1}{\cos^2\theta}.\frac{1}{\sin\theta}$

$= \frac{1}{3}\pi R^3.\frac{1}{\sin\theta - \sin^3\theta}$

Now $V'(\theta) = \frac{\pi R^3 \cos\theta}{(\sin\theta - \sin^3\theta)^2}\left(\sin\theta + \frac{1}{\sqrt{3}}\right)\left(\sin\theta - \frac{1}{\sqrt{3}}\right)$

Clearly V(θ) has only one critical point namely $\theta = \sin^{-1} 1/\sqrt{3}$ .Using sign scheme for V'(θ)

we get, $\theta = \sin^{-1} 1/\sqrt{3}$ to be the point of maxima. Hence corresponding altitude H R/sinθ = R√3 .

Ex.27 What normal to the curve $y = x^2$ forms the shortest chord.

Sol. Let (t, t^2) be any point on the parabola $y = x^2$

Now $\frac{dy}{dx} = 2x \Rightarrow \left(\frac{dy}{dx}\right)_{(t, t^2)} = 2t$, which is slope of tangent.

So, the slope of the normal to $y = x^2$ at (t, t^2) is $(-1/2t)$

The equation of the normal to $y = x^2$ at (t, t^2) is $y\ t^2 =(-1/2t)\ (x-t)$....(1)

Suppose equation (i) meets the curve again at $B(t_1, t_1^2)$, then,

$$t_1^2 - t^2 = -\frac{1}{2t}\ (t_1 - t) \Rightarrow \quad t_1 + t = -\frac{1}{2t} \quad t_1 = -t - \frac{1}{2t} \qquad .....(ii)$$

Let L be the length of the chord AB (as normal)

$L = AB^2 = (t\ t_1)^2 + (t^2\ \ t_1^2)^2 = (t\ t_1)^2\ [1 + (t + t_1)^2]$

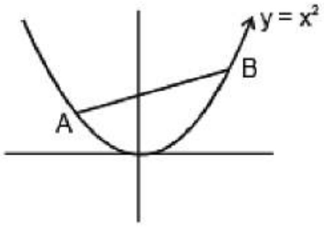
$$=\left(t+t+\frac{1}{2t}\right)^2\left[1+\left(t+t+\frac{1}{2t}\right)^2\right] \text{ (using (ii))}$$

$$=\left(2t+\frac{1}{2t}\right)^2\left(1+\frac{1}{4t^2}\right) = 4t^2\left(1+\frac{1}{4t^2}\right)^3$$

$$\Rightarrow \quad \frac{dL}{dt} = 8t\left(1+\frac{1}{4t^2}\right)^2 + 12t^2\left(1+\frac{1}{4t^2}\right)^2\left(-\frac{2}{4t^3}\right)$$

$$\Rightarrow \quad \frac{dL}{dt} = 2\left(1+\frac{1}{4t^2}\right)^2 + \left[4t\left(1+\frac{1}{4t^2}\right) - \frac{3}{t}\right]$$

$$\Rightarrow \quad \frac{dL}{dt} = 2\left(1+\frac{1}{4t^2}\right)^2\left(4t+\frac{2}{t}\right) = 4\left(1+\frac{1}{4t^2}\right)^2\left(2t-\frac{1}{t}\right)$$



For extremum let $\frac{dL}{dt} = 0$

$$\Rightarrow \quad t = \pm \frac{1}{\sqrt{2}}$$

Again, $\frac{d^2L}{dt^2} = 8\left(1+\frac{1}{4t^2}\right)\left(-\frac{1}{2t^2}\right)\left(2t-\frac{1}{t}\right)+4\left(1-\frac{1}{4t^2}\right)^2\left(2+\frac{1}{t^2}\right)$

$$\Rightarrow \quad \left(\frac{d^2L}{dt^2}\right)_{t=\pm\frac{1}{\sqrt{2}}} > 0$$

$\therefore$ minimum when $t = \pm \frac{1}{\sqrt{2}}$.

Thus, points are $A = \left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and B $((\mp\sqrt{2}, 2)$

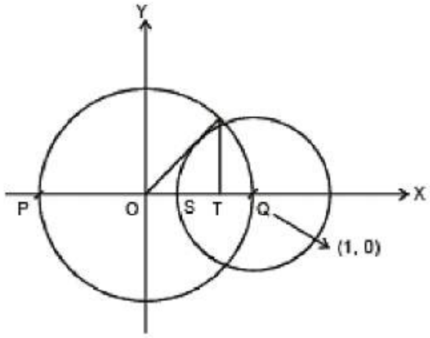
$\Rightarrow$ equation of normal AB is $\sqrt{2}\,x + 2y\ 2 = 0$ and $\sqrt{2}\,x\ 2y + 2 = 0$

Ex.28 The circle $x^2 + y^2 = 1$ cuts the x-axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis and the line segment PQ at S. Find the maximum area of the triangle QSR.

Sol. The centre of the circle $x^2 + y^2 = 1$ ....(i) is (0, 0) and radius OP=1=OQ so, co-ordinates of Q are (1, 0)

Let the radius of the variable circle be r. Hence, its equation is $(x - 1)^2 = (y)^2 = r^2$ ...(ii)

Now, $RT = \sqrt{OR^2 - OT^2} = \sqrt{1-\left(1-\frac{r^2}{2}\right)^2}$



Now, the area of Δ QSR is,

$$A = \frac{1}{2} . QS . RT \qquad \therefore \qquad A^2 = \frac{1}{4} (QS^2) . (RT^2)$$

$$A^2 = \frac{1}{4} r^2 \left(r^2 - \frac{r^4}{4}\right) = \frac{1}{16} (4r^4 - r^6) \text{ [using (ii) and (iv)]}$$

Thus, $\frac{d(A^2)}{dr} = \frac{1}{16} (16r^3 - 6r^5) = 0$ (for extremum)

$$\Rightarrow \qquad r = 2\sqrt{\frac{2}{3}}$$

Also, $\frac{d^2(A^2)}{dr^2} = \frac{1}{16} (48r^2 - 30r^4) = -\frac{16}{3} < 0$

where $r = 2\sqrt{\frac{2}{3}}$

Hence, area is maximum at $r = 2\sqrt{\frac{2}{3}}$ and $A_{max.} = \frac{4}{3\sqrt{3}}$ sq. units.

H. Maximum and Minimum for Discrete Valued Functions

Ex.29 Find the largest term in the sequence $a_n = \frac{n}{n^2 + 10}$ $(n \in N)$.

Sol.

Consider the function $f(x) = \frac{x}{x^2 + 10}$, $x > 0$.

Then $f'(x) = \frac{(x^2+10)-2x^2}{(x^2+10)^2}$

$= \frac{-(x+\sqrt{10})(x-\sqrt{10})}{(x^2+10)^2} > 0 \; \forall \; 0 < x < \sqrt{10}$

$\Rightarrow$ f(x) strictly increases in $(0, \sqrt{10})$ strictly decreases in $(10, \propto)$

$\Rightarrow$ f(x) has greatest value at $x = \sqrt{10}$

Hence, the given sequence has greatest value at $n = 3$ or $n = 4$.

Now, we have $a_3 = 3/19$ and $a_4 = 4/26$. Hence, $a_3 = 3/19$. is the largest term of the given sequence.

I. Concavity and Point of Inflection

The graph of the function $y = f(x)$ is said to be concave down on the interval (a, b) if it lies below the tangent drawn at any point of that interval (Fig. 1).

The graph of the function $y = f(x)$ is said to be concave up on the interval (a, b) if it lies above the tangent drawn at any point of that interval (Fig. 2).

The sufficient condition for the concavity of the graph of a function.

If $f''(x) < 0$ on the open interval (a, b) then the graph of the function is concave down on that interval; now if $f''(x) > 0$, then on the open interval (a, b) the graph of the function is concave up.

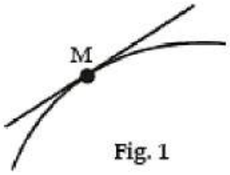


Fig. 1

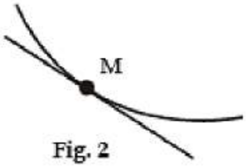


Fig. 2

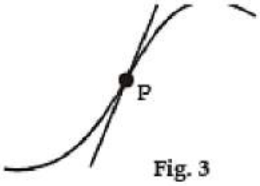


Fig. 3

The point $(x_0; f(x_0))$ of the graph of the function separating its concave down part from the concave up part is called a point of inflection (Fig. 3).

If x_0 is the abscissa of the inflection point of the graph of the function $y = f(x)$, then the second derivative is equal to zero or does not exist. The points at which $f''(x) = 0$ or $f''(x)$ does not exist are called critical points of the 2nd kind.

If x_0 is a critical point of the 2nd kind and the inequalities $f''(x_0 - h) < 0, f''(x_0 + h) > 0$ (or inequalities $f''(x_0 - h) > 0, f''(x_0 + h) < 0$) hold for an arbitrary sufficiently small $h > 0$, then the point of the curve $y = f(x)$ with the abscissa x_0 is a point of inflection.

If $f''(x_0 - h)$ and $f''(x_0 + h)$ are of the same sign, then the point of the curve $y = f(x)$ with the abscissa x_0 is not a point of inflection.

Ex.30 Find the intervals of concavity of the graph of the function $y = x^5 + 5x - 6$.

Sol. We have $y' = 5x^4 + 5$, $y'' = 20x^3$. If $x < 0$, then $y'' < 0$ and the curve is concave down ; now if $x > 0$, then $y'' > 0$ and the curve is concave up. Thus we see that the curve is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, +\infty)$.

Ex.31 Find the inflection points of the curve $y = (x - 5)^{5/3} + 2$.

Sol. We find $y' = \frac{5}{3}(x-5)^{2/3}$, $\quad y'' = \frac{10}{9\sqrt[3]{(x-5)}}$.

The second derivative does not vanish for any value of x and does not exist at $x = 5$. The value $x = 5$ is the abscissa of the inflection point since $y''(5 - h) < 0$, $y''(5 + h) > 0$. Thus, (5, 2) is the inflection point. Inflection points can also occur if $\frac{d^2y}{dx^2}$ fails to exist .

Cusp :

A point on a graph where the curve makes an abrupt change in direction is called a **cusp.** Our next example features a graph with such a point.

Find the first and second derivatives and write them in factored form.

Let $f(x) = 2x^{5/3} + 5x^{2/3}$.

$$f'(x) = 2\left(\frac{5}{3}\right) x^{2/3} + 5\left(\frac{2}{3}\right) x^{-1/3} = \frac{10}{3} x^{-1/3} (x + 1)$$

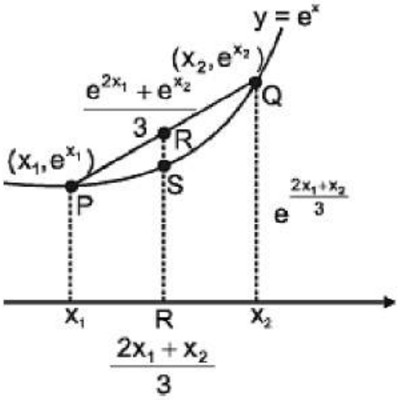
$$f''(x) = \frac{10}{3}\left(\frac{2}{3}\right)x^{-1/3} + \frac{10}{3}\left(-\frac{1}{3}\right) x^{-4/3} = \frac{10}{9} x^{-4/3} (2x - 1)$$

Note that the graph is concave down on both sides of $x = 0$ and that the slope $f'(x)$ decreases without bound to the left of $x = 0$ and increases without bound to the right. This means the graph changes direction abruptly at x = 0, and we have a cusp at the origin.

Ex.32 Prove that for any two numbers x_1 & x_2 $\frac{e^{2x_1}+e^{x_2}}{3} > e^{\frac{2x_1+x_2}{3}}$

Sol.

Assume $f(x) = e^x$ and let x_1 & x_2 be two points on the curve $y = e^x$.
Let R be another point which divides P and Q in ratio 1 : 2.



y coordinate of point R is $\frac{e^{2x_1}+e^{x_2}}{3}$ and y coordinate of point S is $e^{\frac{2x_1+x_2}{3}}$. .Since f(x) = ex is always concave up, hence point R will always be above point S

$$\Rightarrow \quad \frac{e^{2x_1}+e^{x_2}}{3} < e^{\frac{2x_1+x_2}{3}}$$

(above inequality could also be easily proved using AM and GM.)

Ex.33 If $0 < x_1 < x_2 < x_3 < \pi$ then prove that

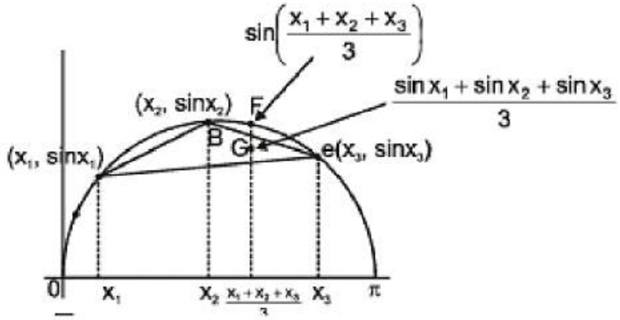
sin $\left(\frac{x_1 + x_2 + x_3}{3}\right) > \frac{\sin x_1 + \sin x_2 + \sin x_3}{3}$. Hence or otherwise prove that if A, B, C are angles of triangle then maximum value of $\sin A + \sin B + \sin C$ is $\frac{3\sqrt{3}}{2}$.

Sol.

Let point A, B, C form a triangle y coordinate of centroid G is $\frac{\sin x_1 + \sin x_2 + \sin x_3}{3}$ and y coordinate of point F is $\sin\left(\frac{x_1 + x_2 + x_3}{3}\right)$.

Hence $\sin\left(\frac{x_1 + x_2 + x_3}{3}\right) > \frac{\sin x_1 + \sin x_2 + \sin x_3}{3}$.

If $A + B + C = \pi$, then $\sin\left(\frac{A + B + C}{3}\right) > \frac{\sin A + \sin B + \sin C}{3}$



$$\Rightarrow \quad \sin\frac{\pi}{3} > \frac{\sin A + \sin B + \sin C}{3}$$

$$\Rightarrow \frac{3\sqrt{3}}{2} > \sin A + \sin B + \sin C$$

⇒ maximum value of (sin A + s in B + s in C) $= \frac{3\sqrt{3}}{2}$.

Increasing and Decreasing Functions & Monotonicity

Increasing and Decreasing Functions

Before explaining the increasing and decreasing function along with monotonicity, let us understand what functions are. A function is basically a relation between input and output such that, each input is related to exactly one output.
Functions can increase, decrease or can remain constant for intervals throughout their entire domain. Functions are continuous and differentiable in the given intervals.

An interval in Maths is defined as a continuous/connected portion on the real line. Since it is a "portion of a line", it basically is a line segment which has two endpoints. So, an interval has two endpoints. Easy to keep track, let's name our interval and the endpoints and in an interval, assume any two points viz.

x_1 and x_2 such that $x_1 < x_2$. Now, there can be a total of four different cases:

If $f(x_1) \leq f(x_2)$, the function is said to be non-decreasing in l
If $f(x_1) \geq f(x_2)$, the function is said to be non-increasing in l
If $f(x_1) > f(x_2)$, the function is said to be decreasing (strictly) in l
If $f(x_1) < f(x_2)$, the function is said to be increasing (strictly) in l

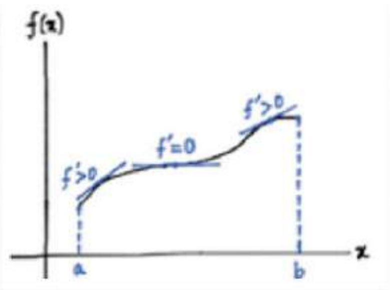
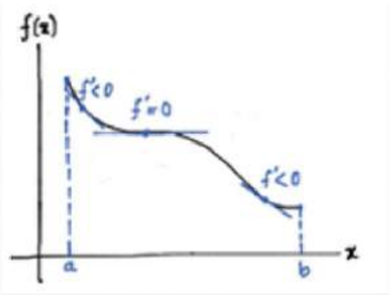
This increasing or decreasing behaviour of functions is commonly referred to as monotonicity of the function. A monotonic function is defined as any function which follows one of the four cases mentioned above. Monotonic basically has two terms in it. Mono means one and tonic means tone. Together, it means, "in one tone". When we tell that a function is non-decreasing, does it mean that it is increasing? No. It can also mean the function not varying at all! In other words, function having a constant value for some interval. Never confuse non-decreasing with increasing. That was the definition of increasing and decreasing functions. Let us now see how to know where and in which way the function is behaving.

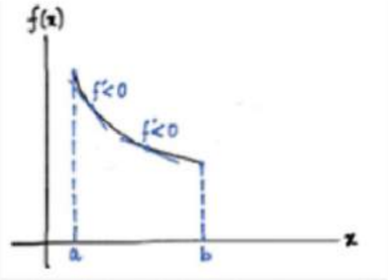
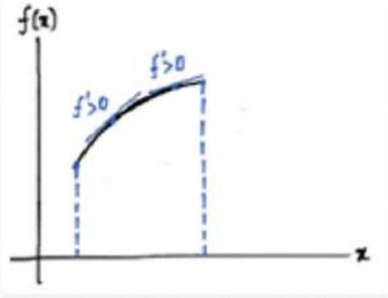
Test for increasing and decreasing functions

Let us now use derivative of a function to determine the behaviour of a function. To test the monotonicity of a function f, we first calculate it's derivative f′. There is a small catch here. Before starting the test, make sure that is continuous in the interval [a, b] and differentiable in (a, b). So, for all of the four cases discussed in

previous heading, we have tests as:
For function to be non-decreasing in l, $f'(x) \geq 0, \forall\ x \in (a, b)$
For function to be non-increasing in l, $f'(x) \leq 0, \forall\ x \in (a, b)$
For function to be decreasing/strictly decreasing in l, $f'(x) < 0, \forall\ x \in (a, b)$
For function to be increasing/strictly increasing in l,$f'(x) > 0, \forall\ x \in (a, b) <$

Let us see examples of each case.

Nature of graph of f	Cases	Behaviour	Test
	$f(x1) \leq f(x2)$	Non – decreasing	$f'(x) \geq 0,\ \forall\ x \in (a, b)$
	$f(x1) \geq f(x2)$	Non – increasing	$f'(x) \leq 0,\ \forall\ x \in (a, b)$

	$f(x1) > f(x2)$	Decreasing/ Strictly decreasing	$f'(x) < 0, \ \forall x \in (a, b)$
	$f(x1) < f(x2)$	Increasing/ Strictly Increasing	$f'(x) > 0, \ \forall x \in (a, b)$

That was increasing and decreasing functions, monotonic functions and monotonicity explained. This concept is particularly very useful for drawing graphs of functions.

A. Definitions

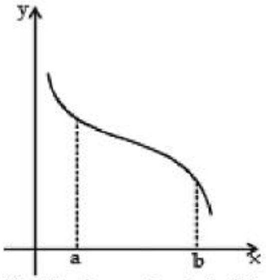
The function f(x) is called strictly increasing on the open interval (a, b) if for any two points x_1 and x_2 belonging to the indicated interval and satisfying the inequality $x_1 < x_2$ the inequality $f(x_1) < f(x_2)$ holds true.

The function f(x) is called strictly decreasing on the open interval (a, b) if for any points x_1 and x_2 belonging to the indicated interval and satisfying the inequality $x_1 < x_2$ the inequality $f(x_1) > f(x_2)$ holds true.

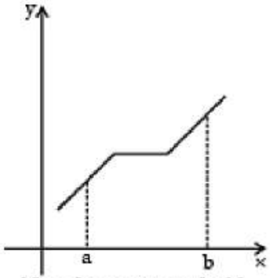
A function f is said to be non-decreasing in an interval I contained in the domain of f

If $f(x_1) \le f(x_2)$ whenever $x_1 \le x_2$ for all numbers x_1, x_2 in I.

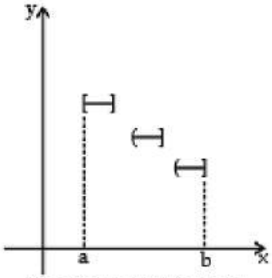
If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ for all numbers x_1, x_2 in I, then f is said to be strictly increasing in the interval I. Non-increasing and strictly decreasing functions are defined in a similar way. If f is strictly increasing in I, then the graph of f is rising as we traverse it from left to right; if f is strictly decreasing in I, the graph of f is falling in I. Some examples are show in Figure.



Strictly decreasing in [a, b]



Non-decreasing in [a, b]



Non-increasing in [a, b]

If a function f is either non-decreasing in an interval I or non-increasing in I, then f is said to be monotonic in I. Similarly, f is said to be strictly monotonic in I if f is either strictly increasing in I or strictly decreasing in I.

Basic definition test :

The function f(x) is said to be strictly increasing at a point x_0 if for a sufficiently small $h > 0$ the condition (Fig. 1) $f(x_0 - h) < f(x_0) < f(x_0 + h)$ is fulfilled.

The function f(x) is said to be strictly decreasing at a point x_0 if for a sufficiently small $h > 0$ the condition (Fig. 2) $f(x_0 - h) > f(x_0) > f(x_0 + h)$ is fulfilled.

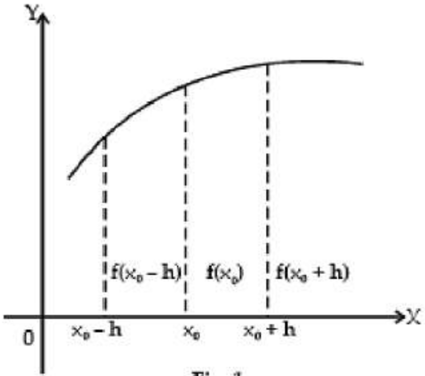


Fig. 1

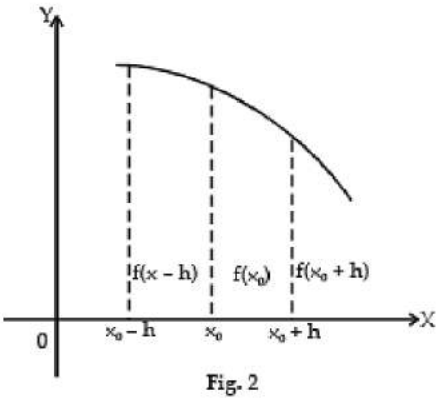


Fig. 2

A differentiable function is called increasing in an interval (a, b) if it is increasing at every point within the interval (but not necessarily at the end points). A function decreasing in an interval (a, b) is similarly defined.

Sufficiency Test :

If the derivative function $f'(x)$ in an interval (a, b) is every where positive, then the function $f(x)$ in this interval is Increasing ; If $f'(x)$ is every where negative, then $f(x)$ is Decreasing.

Note : The test (criterion) also holds true when the derivative takes on zero values in the interval (a, b) so long as $f(x)$ does not identically become zero throughout the interval (a, b) or in some interval (a', b') comprising a part of (a, b). The function $f(x)$ would be a constant on such an interval.

If $f'(a) = 0$ then examine the sign of $f'(a^+)$ and $f'(a^-)$

(a) If $f'(a^+) > 0$ and $f'(a^-) > 0$ then strictly increasing

(b) If $f'(a^+) < 0$ and $f'(a^-) < 0$ then strictly decreasing

Note : If a function is invertible it has to be either increasing or decreasing.

If a function is continuous in the intervals in which it rises and falls may be separated by points at which its derivative is zero or it fails to exist.

B. Critical Point

A critical point of a function f is a number c in the domain of f such the either $f'(c) = 0$ or $f'(c)$ does not exist.

Ex.1 Find the critical points of $f(x) = x^{3/5} (4 - x)$.

Sol.

$$f'(x) = \frac{3}{5}x^{-2/5} (4 - x) + x^{3/5} (-1)$$

$$= \frac{3(4-x)}{5x^{2/5}} - x^{3/5} = \frac{3(4-x)-5x}{5x^{2/5}} = \frac{12-8x}{5x^{2/5}}$$

Therefore, $f'(x) = 0$ if $12\ 8x = 0$, that is, $x = 3/2$. and $f'(x)$ does not exist when $x = 0$.

Thus, the critical points are $3/2$ and 0.

Ex.2 Find the critical numbers for the function $f(x) = \frac{e^x}{x-2}$

Sol.

$$f'(x) = \frac{(x-2)e^x - e^x(1)}{(x-2)^2} = \frac{e^x(x-3)}{(x-2)^2}$$

The derivative is not defined at x = 2, but f is not defined at 2 either, so x = 2 is not a critical number. The actual critical numbers are found by solving f'(x) = 0 :

$$\frac{e^x(x-3)}{(x-2)^2} = 0$$

x = 3 This is the only critical number since $e^x > 0$.

Ex.3 Find all possible values of the parameter ' b ' for which the function,

$f(x) = \sin 2x - 8(b+2)\cos x - (4b^2 + 16b + 6)x$

is monotonic decreasing throughout the number line and has no critical points.

Sol. $f'(x) = 2\cos 2x + 8(b+2)\sin x - (4b^2 + 16b + 6)$

$= 2(1 - 2\sin^2 x) + 8(b+2)\sin x - (4b^2 + 16b + 6)$

$= -4[\sin^2 x - 2(b+2)\sin x + (b^2 + 4b + 1)]$

for monotonic decreasing and no critical points $f'(x) > 0 \ \forall x \in R$

Now, $D = 4(b+2)^2 - 4(b^2 + 4b + 1) = 4[3] = 12$ which is always positive .

Now let $\sin x = y$; $y \varepsilon [-1, 1]$

$g(y) = y^2 - 2(b+2)y + (b^2 + 4b + 1)$

we have to find those values of ' b ' for which $g(y) > 0$ for all $y \varepsilon (-1, 1)$

Conditions are $g(-1) > 0 \ \& \ -\frac{b}{2a} < -1$ or $g(1) > 0$ and $-\frac{b}{2a} > 1$

First condition gives $1 + 2(b+2) + b^2 + 4b + 1 > 0$

$b^2 + 6b + 6 > 0$ .....(1) & $\frac{2(b+2)}{2} < -1$ or $b < -3$ .....(2)

Similarly second condition gives $b > \sqrt{3} - 1$

(1) & (2) $\Rightarrow$ $b < -\left(3+\sqrt{3}\right)$

Hence $b \in \left(-\infty, -\left(3+\sqrt{3}\right)\right) \cup \left(\sqrt{3}-1, \infty\right)$

Ex.4 If $f(x) = \frac{x^2}{2 - 2\cos x}$; $g(x) = \frac{x^2}{6x - 6\sin x}$ **where $0 < x < 1$, then**

Sol. Put $x = \pi/6$ & $\pi/3$ and observe the behavior of $f(x)$ & $g(x)$. Alternatively

$$f'(x) = \frac{1}{2}\left[\frac{(1-\cos x)\,2x - x^2 \sin x}{(1-\cos x)^2}\right]$$

consider $= 2\,(1 - \cos x) - x \sin x$

$$= 4\sin^2 \quad - 2x \sin\frac{x}{2} \;\; \cos\frac{x}{2}$$

$$= 2x \sin\frac{x}{2} \;\; \cos\frac{x}{2}\left[\frac{\tan\frac{x}{2}}{\frac{x}{2}} - 1\right] \;\Rightarrow\; \text{f is increasing.}$$

$$g'(x) = \frac{1}{6}\left[\frac{(x-\sin x)\,2x - x^2(1-\cos x)}{(x-\sin x)^2}\right]$$

consider $x - 2\sin x + x\cos x$

$= 2x\cos^2 x - 4\sin x\cos x$

$$= 2x\cos^2 x \quad \left[1 - \frac{\tan\frac{x}{2}}{\frac{x}{2}}\right] \;\Rightarrow\; \text{g is decreasing.}$$

Ex.5 Find possible values of a such that $f(x) = e^{2x} - (a+1)\,e^x + 2x$ is strictly increasing for $x \in R$.

Sol. $f(x) = e^{2x} - (a+1)\,e^x + 2x$

$f'(x) = 2e^{2x} - (a+1)\,e^x + 2$

Now, $f(x) = e^{2x} - (a+1)\,e^x + 2x \geq 0$ for $x \in R$

$\Rightarrow\ 2\left(e^x + \frac{1}{e^x}\right) - (a+1) \geq 0\ \ \forall\ \ x \in R$

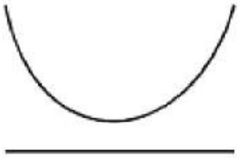
$(a+1) \leq 2\left(e^x + \frac{1}{e^x}\right)\ \ \forall x \in R \Rightarrow\ a + 1 \leq 4$

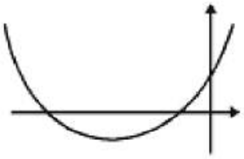
$\Rightarrow\ a \leq 3\ \ \left(\because\ \ e^x + \frac{1}{e^x}\text{has minimum value 2}\right)$

Aliter : $2e^{2x}\ \ (a + 1)\ e^x + 2 \geq 0$ for $x \in R$

putting $e^x = t$; $t \in (0,\propto)$

$2t^2\ \ (a + 1)\ t + 2 \geq\ 0$ for $t \in (0, \propto)$





Hence either

(i) $D \leq 3\ \Rightarrow\ (a + 1)^2\ \ 4 \leq 0 \Rightarrow\ (a + 5)\ (a\ 3) \leq 0 \Rightarrow\ a \in [5, 3]$

(ii) both roots are negative

$D \geq 0$ & $-b/2a < 0$ & $f(0) \geq 0$

$\Rightarrow\qquad a \in (-\infty, -5] \cup [3, \infty)$

$\Rightarrow\qquad a \in (-\infty, -5] \cup [3, \infty)$

$\Rightarrow\qquad a \in (-\infty, -5]$

Taking union of (i) and (ii), we get $a \in (\propto, 3]$.

Ex. 6 Prove that the function $f(x) = \frac{\ell n\, x}{x}$ is strictly decreasing in (e, ∞). Hence, Prove that $303^{202} < 202^{303}$.

Sol.

We have $\mathbf{f(x)} = \frac{\ell n\, x}{x}$,$x > 0$,

Then $f'(x) = \frac{1-\ell n\, x}{x^2} < 0 \ \forall\ x > e$

⇒ f(x) strictly decreases in (e,∝) Thus, we have f(303) < f(202)

i.e. $\frac{\ell n\,(303)}{303} < \frac{\ell n\,(202)}{202}$

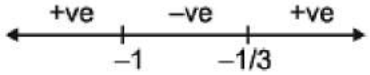
i.e . 202 ln (303) < 303 ln (202)

⇒ $303^{202} < 202^{303}$ which is the desired result.

Ex.7 Let $f(x) = x^3 + 2x^2 + x + 5$. Show that f(x) has only one real root a such that [α] = -3.

Sol. We have $f(x) = x^3 + 2x^2 + x + 5, x \in R$ and $f'(x) = 3x^2 + 4x + 1 = (x + 1)(3x + 1), x \in R$

Drawing the number line for f'(x), we have f(x) strictly increases in (∝, 1) strictly decreases in (1, 1/3) strictly increases in (1/3, ∝)

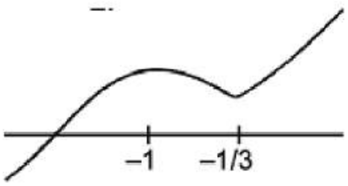


Also, we have f(1) = 1 + 2 1 + 5 = 5 and $f\left(\frac{-1}{3}\right) = \frac{-1}{27} + \frac{2}{9} - \frac{1}{3} + 5 = 5 - \frac{4}{27} = 4.85$

The graph of f(x) (see fig.) shows that f(x) cuts the X-axis only once.

Now, we havef(3) = 27 + 12 3 + 5 = 13 and f(2) = 8 + 8 2 + 5 = 3.

Which are of opposite signs. This proves that the curve cuts the X-axis somewhere between 2 and 3.



$\Rightarrow$ f(x) = 0 has a root α lying between 2 and 3. Hence $[\alpha] = 3$

Ex.8 Find the number of real roots of the equation $\sum_{i=1}^{n} \frac{a_i^2}{x - b_i}$ = c where $b_1 < b_2 < \ldots.. < b_n$.

Sol.

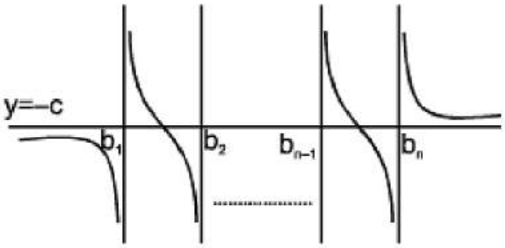
Consider the function f(x) = $\sum_{i-1}^{n} \frac{a_i^2}{x - b_i} - c = \frac{a_1^2}{x - b_1} + \frac{a_2^2}{x - b_2} + \ldots. + \frac{a_n^2}{x - b_n} - c$

and $\quad f'(x) = \left[\frac{a_1^2}{(x - b_1)^2} + \frac{a_2^2}{(x - b_2)^2} + \ldots. + \frac{a_n^2}{(x - b_n)^2}\right]$

$< 0 \ \forall \ x \in R \sim \{b_1, b_2, \ldots\ldots., b_n\}$

= f(x) strictly decreases in (α, b_1) U (b_1, b_2) U ... U $(b_{n\ 1}, b_n)$

Now, we have



$f(-\infty) = -c = f(\infty)$
$f(b_1^-) = -\infty$ and $f(b_1^+) = \infty$
$f(b_2^-) = -\infty$ and $f(b_2^+) = \infty$
....
....
$f(b_n^-) = -\infty$ and $f(b_n^+) = \infty$

The plot of the curve $y = f(x)$ is shown alongside.

Ex.9 If $f : R \to R$ and f is a polynomial with $f(x) = 0$ has real and distinct roots, show that the equation, $[f'(x)]^2 - f(x) . f''(x) = 0$ cannot have real roots.

Sol. Let $f(x) = c(x - x_1)(x - x_2) ...... (x - x_n)$

Again Let $h(x) = \frac{f'(x)}{f(x)} = \left(\frac{1}{x-x_1} + \frac{1}{x-x_2} + ... + \frac{1}{x-x_n}\right)$

$$h'(x) = \frac{f(x) . f''(x) - [f'(x)]^2}{f^2(x)}$$

$$= -\left(\frac{1}{(x-x_1)^2} + \frac{1}{(x-x_2)^2} + ... + \frac{1}{(x-x_n)^2}\right)$$

$\Rightarrow h'(x) < 0 \Rightarrow f(x) . f''(x) - [f'(x)]^2 < 0$

Alternatively : a function $f(x)$ satisfying the equation $[f'(x)]^2 - f(x) . f''(x) = 0$ is $f(x) = c . e^{c_1 x}$ which can't have any root.

C. Intervals of Monotonocity

Ex.10 Find the intervals of monotonocity of the following functions :

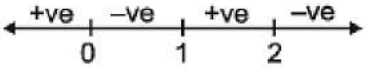
(a) $f(x) = \frac{|x-1|}{x^2}$

(b) $f(x) = 2x^2 - \ell n|x|$

(c) $f(x) = \frac{x^3}{x^4 + 27}$

Sol.

(a) We have $f(x) = \frac{1-x}{x^2}, x < 1 \; ; \; f(x) = \frac{x-1}{x^2}, x \geq 1$



and $f'(x) = \frac{-2}{x^3} + \frac{1}{x^2} = \frac{x-2}{x^3}, x < 1$;

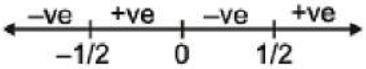
$f'(x) = \frac{2-x}{x^3}, x > 1$

Now, from the sign scheme for f'(x), we have ⇒ f(x) strictly increases i (-∞, 0)

strictly decreases in (0, 1) ; strictly increases in (1, 2) ; strictly decreases in (2, ∞)

Ans. : Increases in (-∞, 0), (1, 2); Decreases in (0, 1), (2, ∞)

(b) We have $f(x)=2x^2 \ln|x|$ and $f'(x) = 4x\ 1/x = \frac{4\left(x+\frac{1}{x}\right)\left(x-\frac{1}{2}\right)}{x}$



Now, from the sign scheme for f'(x), we have ⇒ f(x) strictly decreases in (-∞, -1/2)

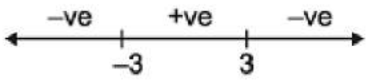
strictly increases in (-1/2, 0) ; strictly decreases in (0, 1/2) ; strictly increases in (-1/2, ∞)

Ans. : Increases in $\left(-\frac{1}{2}, 0\right), \left(\frac{1}{2}, \infty\right)$; Decreases in $\left(-\infty, -\frac{1}{2}\right), \left(0, \frac{1}{2}\right)$

(c) We have $f(x) = \frac{x^3}{x^4+27}$ and

$$f'(x) = \frac{(x^4+27)(3x^2) - x^3(4x^3)}{(x^4+27)^2}$$

$$= \frac{-x^2(x^4-81)}{(x^4+27)^2} = \frac{x^2(x^2+9)(x+3)(x-3)}{(x^4+27)^2}$$



$\Rightarrow$ f(x) strictly decreases in $(-\infty, -3)$; strictly increases in $(-3, 3)$; strictly decreases in $(3, \infty)$.

Ans : Increases in $(-3, 3)$; Decreases in $(-\infty, -3), (3, \infty)$

Ex.11 A function f (x) is given by the equation, $x^2 f'(x) + 2x f(x) - x + 1 = 0$ ($x \neq 0$). If $f(1) = 0$, then find the intervals of monotonocity of f.

Sol.

$$\frac{d}{dx}[x^2 y] = x - 1 \Rightarrow x^2 y = \int (x-1)\, dx$$

wherey = f (x)

This gives $y = \frac{1}{2} - \frac{1}{x} + \frac{1}{2x^2}$

Find $\frac{dy}{dx}$ and solve

$\frac{dy}{dx} > 0$ & $\frac{dy}{dx} < 0$

Ans. : I in $(-\propto, 0)$ U $(1, \propto)$; D in $(0, 1)$

D. Operations on Monotonous Functions

I. (a) Negative : If f is an increasing function then its negative i.e. h = f is a decreasing function.

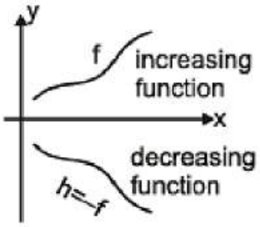
By derivative $h'(x) = f'(x), f'(x) > 0 \therefore h'(x) < 0$

$\Rightarrow$ h is a decreasing function

In short (an increasing function) = a decreasing function i.e. I = D Similarly D = I

(b) Reciprocal : Reciprocal of an increasing function is a decreasing function

By Graph



In short $\frac{1}{\text{an increasing function}}$ = a decreasing function

i.e. (i) $\frac{1}{1} = D$ & (ii) $\frac{1}{D} = I$

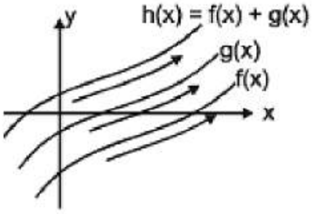
II.(a) Sum : If f is an increasing function and g is also an increasing function their h = f + g is an increasing function.

By derivative

$h'(x) = f'(x) + g'(x)$ f & g are increasing function, ⇒ $f'(x)$ & $g'(x)$ are positive ⇒ $f'(x) + g'(x)$ is positive ⇒ $f(x) + g(x)$ increases

In short, An increasing function + An increasing function = An increasing function

By Graph



i.e. (i) I + I = I (ii) I + D = can't say (iii) D + D = D

(b) Difference : Monotonocity of the difference of two function can be predicted using I(a) and II(a)

I - I = I + (-I) = I + D = can't say

I - D = I + (-D) = I + I = increasing

D - I = D + (-I) = D + D = decreasing

D - D = D + (-D) = D + I = can't say

III. (a) Product : Consider h = f × g

Case I : Both the function involved in the product i.e. f & g are positive

If f & g both are increasing function then h = f × g is also an increasing function.

In short I × I = I, I × D = can't say, D × D = D.

Case II : If any of the function takes negative values then we can predict the monotonocity by using I(a) & case I of III(a). If a function f is increasing & takes negative values & another function g is decreasing & takes positive values.

then h(x) = f(x) × g(x) = (-f(x)) × g(x) = $-\left[\underbrace{\phi(x)}_{\text{Decrreasing}} \times \underbrace{g(x)}_{\text{Decrreasing}}\right]$ = increasing

(b) Division : Monotonocity of division of two functions can be predicted by using I(b) & III(a).

$\frac{I}{D} = I \times \frac{I}{D}$ = I × I = I (assuming that both the functions I & D take positive values).

IV. Composition :

(I) I (I)= I

(II) I(D) = D

(III) D (I) = D

(IV) D(D) = I

Let h(x) = D(D(x)) x increases ⇒ D(x) decreases ⇒ D(D(x)) increases

E. Inequalities

General Approach to prove Inequalities :

To prove f(x) ≥ g(x) for x ≥ a, we Assume h(x) = f(x) g(x)

Find $h'(x) = f'(x)\ g'(x)$

If $h'(x) \geq 0$ Apply increasing function h on $x \geq a$ to get $h(x) \geq h(a)$.

If $h(a) \geq 0$ then $h(x) \geq 0$ for $x \geq a$ i.e. then given inequality is true.

If $h'(x) \leq 0$ Apply decreasing function h on $x \geq a$ to get $h(x) \leq h(a)$.

If $h(a) \leq 0$ then $h(x) \leq 0$ for $x \geq a$ i.e. the given inequality is false

Note : If the sign of $h'(x)$ is not obvious then to determine its sign assume $g(x) = h'(x)$ & apply the above procedure on $g(x)$.

Ex.12 Prove that, $2x \sec x + x > 3 \tan x$ for $0 < x < \pi/2$.

Sol. $f(x) = 2x \sec x + x - 3 \tan x$

$f'(x) = 2 \sec x + 2x \sec x \tan x + 1 - 3 \sec^2 x = \sec^2 x\ [2 \cos x + 2x \sin x + \cos^2 x - 3]$

Consider $g(x) = 2 \cos x + 2x \sin x + \cos^2 x - 3$

$g'(x) = -2 \sin x + 2x \cos x + 2 \sin x - 2 \sin x \cos x = 2 \cos x\ (x - \sin x) > 0$ for $x\ \varepsilon\ (0, \pi/2)$

Ex.13 Prove that $\tan x > x + \frac{x^3}{3}$ for all $x\ \varepsilon \left(0, \frac{\pi}{2}\right)$.

Sol.

Let $f(x) \tan x - x - \frac{x^3}{3}$ ...(1)

Clearly, $f(x)$ is defined at all $x \in (0, \pi/2)$.

Now, $f'(x) = \sec^2 x\ 1\ x\ x^2$ ...(2)

$f''(x) = 2 \sec^2 x \cdot \tan x\ 2x$...(3)

$f'''(x) = 2 \sec^4 x + 4 \sec^2 x. \tan^2 x\ 2 = 2\ (1 + \tan^2 x)^2 + 4 \sec^2 x \cdot \tan^2 x\ 2$

$= 2\tan^4 x + 4 \tan^2 x + 4 \sec^2 x \cdot \tan^2 x > 0$ for all $x \in (0, \pi/2)$

$\Rightarrow f'''(x) > 0$ in the interval $(0, p/2) \Rightarrow f''(x)$ is monotonic increasing in $(0, \pi/2)$

$f''(x) > f''(0)$ when $x \in (0, \pi/2)$.

But from (3), $f''(0) = 0$. Thus, $f''(x) > 0$ for all $x \in (0, \pi/2)$

$\therefore f'(x)$ is monotonic increasing in $(0, \pi/2)$

$\therefore f'(x) > f'(0)$ when $x \in (0, \pi/2)$

But from (2), $f'(0) = 1\ 1\ 0 = 0$.

Thus, $f'(x) > 0$ for all $x \in (0, p/2)$

$\therefore f(x)$ is monotonic increasing in $(0, \pi/2)$

$\therefore f(x) > f(0)$ when $x \in (0, \pi/2)$

But from (1), $f(0) = 0$.

Thus, $f(x) > 0$ for all $x \in (0, \pi/2)$

$$\therefore \tan x - x = \frac{x^3}{3} > 0 \text{ for all } x \in (0, \pi/2)$$

or $$\tan x > x + \frac{x^3}{3} \text{ for all } x \in (0, \pi/2)$$

Ex.14 Show that $1 + x \log (x + \sqrt{x^2+1}) \geq \sqrt{1+x^2}$ for all $x \geq 0$.

Sol.

Let $f(x) = 1 + x \log (x + \sqrt{x^2+1}) - \sqrt{1+x^2}$

or $$f'(x) = \frac{x}{[x+\sqrt{x^2+1}]}.\left[1+\frac{1.(2x)}{2\sqrt{x^2+1}}\right] + \log(x+\sqrt{x^2+1}).1 - \frac{1(2x)}{2\sqrt{1+x^2}}$$

or $$f'(x) = \frac{x}{(x+\sqrt{x^2+1})}\left[\frac{\sqrt{x^2+1}+x}{2\sqrt{x^2+1}}\right] + \log(x+\sqrt{x^2+1}) - \frac{x}{\sqrt{1+x^2}}$$

$$f'(x) = \frac{x}{\sqrt{x^2+1}} + \log(x + \sqrt{x^2+1}) - \frac{x}{\sqrt{1+x^2}}$$

$$f'(x) = \ = \log\ (x + \sqrt{x^2+1})$$

so, $\log\ (x + \sqrt{x^2+1}) \geq 0$

Since, f(x) is increasing for, $x \geq 0 \Rightarrow f(x) \geq f(0)$

$$\Rightarrow\ 1 + x \log\ (x + \sqrt{x^2+1}) - \sqrt{1+x^2} \geq 1 + 0 - \sqrt{1}$$

$$\Rightarrow\ 1 + x \log\ (x + \sqrt{x^2+1} \geq \sqrt{1+x^2}\ ,\ \text{ for } x \geq 0.$$

Ex.15 Examine which is greater sin x tan x or x^2. Hence evaluate $\lim\limits_{x\to 0}\left[\frac{\sin x \tan x}{x^2}\right]$, where $x \in \left(0, \frac{\pi}{2}\right)$

Sol. Let $f(x) = \sin x \,.\, \tan x - x^2$

$f'(x) = \cos x \,.\, \tan x + \sin x \,.\, \sec^2 x - 2x = \sin x + \sin x \sec^2 x - 2x$

$\Rightarrow f''(x) = \cos x + \cos x \sec^2 x + 2 \sec^2 x \sin x \tan x - 2$

$\Rightarrow f''(x) = (\cos x + \sec x - 2) + 2 \sec^2 x \sin x \tan x$

Now $\cos x + \sec x\ 2 = (\sqrt{\cos x} - \sqrt{\sec x})^2$ and $2 \sec^2 x \tan x \,.\, \sin x > 0$

because $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow f''(x) > 0 \Rightarrow f'(x)$ is M.I.

Hence $f'(x) > f'(0)$

$\Rightarrow f'(x) > 0 \Rightarrow f(x)$ is M.I. $\Rightarrow f(x) > 0 \Rightarrow \sin x \tan x\ x^2 > 0$

Hence $\sin x \tan x\ x^2$

$$\Rightarrow \quad \frac{\sin x \tan x}{x^2} > 1$$

$$\Rightarrow \qquad \lim_{x\to 0}\left[\frac{\sin x \tan x}{x^2}\right] = 1$$

Ex.16 Prove : $1 + \cot x \le \cot \frac{x}{2} \ \forall\, x \in (0, \pi)$

Sol. Consider the function f(x) = cot (x/2) 1 cot x, x ∈ (0, p)

Then $f'(x) = \frac{-1}{2}\ \csc^2\left(\frac{x}{2}\right) + \csc^2 x$

$$= \frac{1}{\sin^2 x} - \frac{1}{2\sin^2(x/2)} = \frac{1}{2\sin^2(x/2)}\left[\frac{1}{2\cos^2(x/2)} - 1\right]$$

$$= \frac{-\cos x}{4\sin^2(x/2)\cos^2(x/2)} = \frac{-\cos x}{\sin^2 x} < 0 \ \forall\, x \in (0, \pi/2)$$

⇒ f(x) strictly decreases in (0, π/2)   strictly increases in (π/2, π)

⇒ f(x) has least value at x = π/2 ⇒ f(x) ≥ f(π/2) = 0

i.e. $\cot\left(\frac{x}{2}\right) \ge 1 + \cot x$ which proves the desired result.

Ex.17 Prove that $\ell n\left(1+\frac{1}{x}\right) > \frac{1}{1-x^2}$, x>0. **Hence, show that the function f(x)=** $\left(1+\frac{1}{x}\right)^2$ **strictly increases in (0, ∞).**

Sol.

Consider the fucntion g(x) = $\ell n\left(1+\frac{1}{x}\right) > \frac{1}{1-x} \ \forall\, x > 0.$

Then $g'(x) = \frac{-1/x^2}{1+\frac{1}{x}} + \frac{1}{(1+x)^2}$

$= \frac{-1}{x(1+x)} + \frac{1}{(1+x)^2} = \frac{-1}{x(1+x)} < 0 \ \forall \ x > 0$

$\Rightarrow$ g(x) strictly decreases in (0, ∝) $\Rightarrow$ $> \lim_{x\to\infty} g(x) = 0$

i.e. $\ell n\left(1+\frac{1}{x}\right) > \frac{1}{x+1}$ ...(1)

which gives the desired result

Now, we have f(x) $= \left(1+\frac{1}{x}\right)^x$, x > 0 and

$f'(x) = \left(1+\frac{1}{x}\right)^x \ell n\left(1+\frac{1}{x}\right) + x\left(1+\frac{1}{x}\right)^{x-1}\left(\frac{-1}{x^2}\right)$

$= \left(1+\frac{1}{x}\right)^x \left[\ell n\left(1+\frac{1}{x}\right) - \frac{1}{1+x}\right] > 0 \ \forall \ x > 0$ [using result (1)] $\Rightarrow$ f(x) strictly increases in (0, ∝)

Ex.18 Prove that $\sin x \tan x > x^2 \ \forall \ x \in \left(0, \frac{\pi}{2}\right)$.

Sol.

Let f(x) = sin x tan x x^2 $\Rightarrow$ f'(x) = sin x $\sec^2 x$ + sin x 2x

$\Rightarrow$ f''(x) = 2 sin x sec2 x tan x + cos x 2 + sec x

= 2 sin x tan x sec2 x + (cos x + sec x 2) $> 0 \ \forall \ x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow$ f'(x) is an increasing function.

$\Rightarrow$ f'(x) > f'(0) $\Rightarrow$ sin x sec2 x + sin x 2x > 0

$\Rightarrow$ f(x) is an increasing function $\Rightarrow$ f(x) > f(0) sin x tan x x^2 > 0 $\Rightarrow$ sin x tan x > x^2

Ex.19 Prove that $\sin 1 > \cos(\sin 1)$. Also show that the equation $\sin(\cos(\sin x)) = \cos(\sin(\cos x))$ has only one solution in $\left[0, \frac{\pi}{2}\right]$.

Sol. $\sin 1 > \cos(\sin 1)$ if $\cos\left(\frac{\pi}{2} - 1\right) > \cos(\sin 1)$

$\Rightarrow$ if $\frac{\pi}{2} - 1 < \sin 1 \Rightarrow$ if $\sin 1 > \left(\frac{\pi - 2}{2}\right)$ ...(1)

and $\sin 1 > \sin \frac{\pi}{4} > \frac{1}{\sqrt{2}}$.

Hence (1) is true $\Rightarrow \sin 1 > \cos(\sin 1)$.

Now let $f(x) = \sin(\cos(\sin x)) \cos(\sin(\cos x))$

$\Rightarrow \quad f'(x) < 0 \ \forall \ x \in \left[0, \frac{\pi}{2}\right]$

$\Rightarrow \quad f(x)$ is decreasing in $\left[0, \frac{\pi}{2}\right]$

and $f(0) = \sin 1 \cos(\sin 1) > 0$

$f\left(\frac{\pi}{2}\right) = \sin(\cos(1)) - 1 > 0$

Since $f(0)$ is positive and $f(x) = 0$ has one solution in $\left[0, \frac{\pi}{2}\right]$.

Ex.20 Using calculus establish the inequality, $(x^b + y^b)^{1/b} < (x^a + y^a)^{1/a}$, where $x > 0$, $y > 0$ and $b > a > 0$.

Sol. $(x_b + y_b)^{1/b} < (x_a + y_a)^{1/a}$ $\Rightarrow \left(\left(\frac{x}{y}\right)^b + 1\right)^{1/b} < \left(\left(\frac{x}{y}\right)^a + 1\right)^{1/a}$

or T P T $(t_b + 1)^{a/b} < t_a + 1$

Let $f(t) = (t_b + 1)^{a/b} - t_a - 1$

$$\Rightarrow \quad f'(t) = \frac{a}{b}\left(t^b+1\right)^{\frac{a}{b}-1} \ . \ b t^{b-1} - a t^{a-1}$$

$$\Rightarrow \quad f'(t) = a t^{a-1}\left[t^{b-a}\left(t^b+1\right)^{\frac{a}{b}-1} - 1\right]$$

$$= a t^{a-1}\left[\left(1+\frac{1}{t^b}\right)^{\frac{a}{b}-1} - 1\right]$$

Now since $1 + \frac{1}{t^b} > 1$ & $\frac{a}{b} - 1 < 0$

therefore $\left(1+\frac{1}{t^b}\right)^{\frac{a}{b}-1} < 1$

Hence $f'(t) < 0$ i.e. $f(t)$ is decreasing function

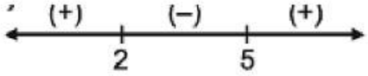
So $f(t) < f(0)$ but $f(0) = 0 \Rightarrow (t_b + 1)^{a/b} < t^a + 1$ Hence proved

Ex.21 Prove that the function $f(x) = 2x^3 + 21x^2 - 60x + 41$ is strictly positive in the interval $(-\infty, 1)$.

Sol.

$f(x) = 2x^3 + 21x^2 \ 60x + 41$

$f'(x) = 6x^2 + 42x \ 60 = 6(x^2 \ 7x + 10) = 6(x \ 5)\ (x \ 2)$



$x \in (2, 5) \Rightarrow f'(x) > 0$, i.e., $f(x)$ is m.i.

and x Ï $(2, 5) \Rightarrow f'(x) < 0$ i.e., $f(x)$ is m.d. $\therefore x \in (\propto, 1) \Rightarrow f(x)$ is m.d.

When $x \in (\propto, 1)$, $x < 1$; so, $f(x) > f(1)$.

But $f(1) = 2 + 21 \ 60 + 41 = 0$.

$\therefore x \in (\propto, 1) \Rightarrow f(x) > f(1) = 0$

$\therefore f(x)$ is strictly positive in the interval $(\propto, 1)$.

F. Rolle's Theorem

Let f be a function that satisfies the following three hypotheses :

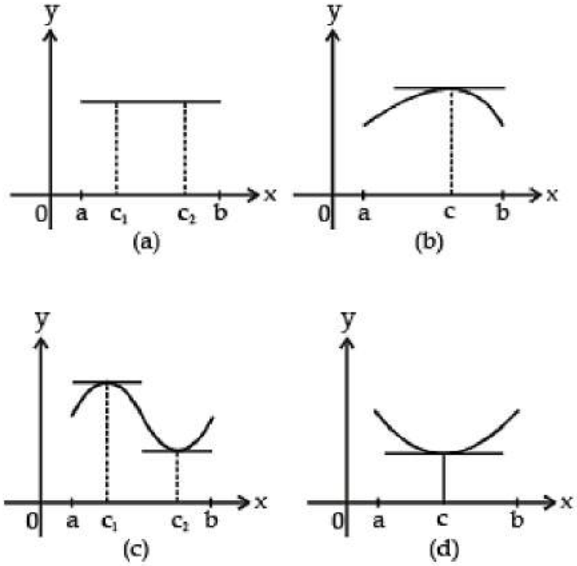
1. f is continuous on the closed interval [a, b].

2. f is differentiable on the open interval (a, b).

3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$

Before given the proof let's take a look at the graphs of some typical functions that satisfy the three hypotheses. Figure 1 shows the graph of four such functions. In each case it appears that there is atleast one point (c, f(c)) on the graph where the tangent is horizontal and therefore $f'(c) = 0$. Thus, Rolle's Theorem is plausible.



(a) (b) (c) (d)

Proof : There are three cases :

Case I : $f(x) = k$, a constant. Then $f'(x) = 0$, so the number c can taken to be any number in (a, b).

Case II : $f(x) > f(a)$ for some x in (a, b) [as in Figure 1(b) or (c)]

By the Extreme Value Theorem (which we can apply by hypothesis 1), f has a maximum value somewhere in [a, b]. Since $f(a) = f(b)$, it must attain this maximum value at a number c in the open interval (a, b). Then f has a local maximum at c and, by hypothesis 2, f is differentiable at c. Therefore, $f'(c) = 0$ by Fermat's Theorem.

Case III : $f(x) < f(a)$ for some x in (a, b) [as in Figure 1(c) or (d)]

By the Extreme Value Theorem, f has minimum value in [a, b] and, since $f(a) = f(b)$, it attains this minimum value at a number c in (a, b). Again $f'(c) = 0$ by Fermat's Theorem.

Ex.22 Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

Sol. First we use the Intermediate Value Theorem to show that a root exists. Let $f(x) = x^3 + x - 1$. Then $f(0) = -1 < 0$ and $f(1) = 1 > 0$. Since f is a polynomial, it is continuous, so the Intermediate Value Theorem states that there is a number c between 0 and 1 such that $f(c) = 0$. Thus, the given equation has a root.

To show that the equation has no other real root, we use Rolle's Theorem and argue by contradiction. Suppose that it had two roots a and b. Then $f(a) = 0 = f(b)$ and, since f is a polynomial, it is differentiable on (a, b) and continuous on [a, b]. Thus, by Rolle's Theorem, there is a number c between a and b such that $f'(c) = 0$.
But $f'(x) = 3x^2 + 1 \geq 1$ for all x

(since $x^2 \geq 0$) so $f'(x)$ can never be 0. This gives a contradiction. Therefore, the equation can't have two real roots.

Ex.23 Let f (x) & g (x) be differentiable for $0 \leq x \leq 1$, such that $f(0) = 2$, $g(0) = 0$, $f(1) = 6$. Let there exist a real number c in [0, 1] such that $f'(c) = 2g'(c)$, then the value of g (1)

Sol. Consider $\varphi(x) = f(x) - 2g(x)$ defined on [0, 1] since f (x) and g(x) are differentiable for $0 \leq x \leq 1$, therefore f (x) is differentiable on (0, 1) and continuous on [0, 1]

$\phi(0) = \phi(0) - 2g(0) = 2 - 0 = 2\ \phi(1) = f(1) - 2g(1) = 6 - 2g(1)$

Now $f'(x) = f'(x) - 2g'(x) \Rightarrow f'(c) = f'(c) - 2g'(x) = 0$ (given)

$\Rightarrow f(x)$ satisfies Rolle's theorem on [0, 1] $\therefore \phi(0) = \phi(1) \Rightarrow 2 - 6 - 2g(1) \Rightarrow g(1) = 2$

Our main use of Rolle's Theorem is in proving the following important theorem, which was first stated by another French mathematician, Joseph-Louis Lagrange.

Ex.24 If f(x) is continuous in [a, b] and differentiable in (a, b), prove that there is atleast one c ε(a, b), such that $\frac{f'(c)}{3c^2} = \frac{f(b)-f(a)}{b^3-a^3}$.

Sol. Let us consider a function, $h(x) = f(x) - f(a) + A(x^3 - a^3)$

Where A is obtained from the relation $h(b) = 0$.

So that, $0 = h(b) = f(b) - f(a) + A(b^3 - a^3)$ ...(i)

also, $h(a) = 0$

Since, **(1)** h(x) is continuous in [a, b] **(2)** h(x) is differentiable in (a, b) and **(3)** $h(a) = 0 = h(b)$

hence, all the three condition of Rolle's theorem. Then there must exists a'c' ε (a, b) such that $f'(c) = 0$.

$\Rightarrow f'(c) + A(3c^2) = 0$ or

$$f'(c) = 3c^2 \frac{f(b)-f(a)}{b^3-a^3} \{\text{using (i)}\} \quad \Rightarrow \quad \frac{f'(c)}{3c^2} = \frac{f(b)-f(a)}{b^3-a^3}$$

G. The Mean Value Theorem

Let f be a function that satisfies the following hypotheses :

1. f is continuous on the closed interval [a, b].

2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

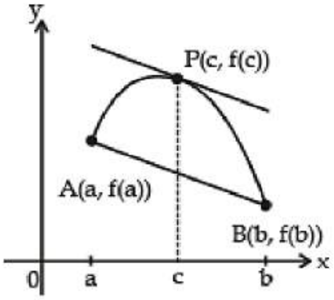
(I) $f'(c) = \frac{f(b)-f(a)}{b-a}$ or, equivalently,

(II) $f(b) - f(a) = f'(c)(b - a)$

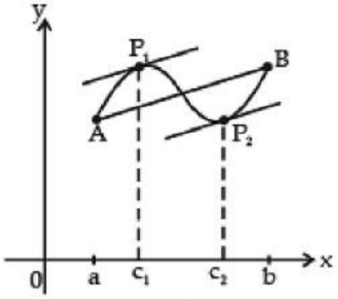
Before proving this theorem, we can see that it is reasonable by interpreting it geometrically. Figures (a) and (b) show that points A(a, f(a)) and B(b(b, f(b)) on the graphs of two differentiable functions. The slope of the secant line AB is

(III) $m_{AB} = \dfrac{f(b) - f(a)}{b - a}$

which is the same expression as on the roght side od eq. 1. Since f'(c) is the slope of the tangent line at the point (c, f(c)), the Mean Value Theorem, in the form given by Equation 1, says that there is at least one point P(c, f(c)) on the graph where the slope of the tangent line is the same as the slope of the secant line AB. In other words, there is a point P where the tangent line is parallel to the secant line AB.

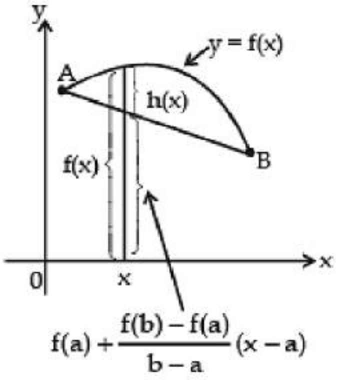


(a)



(b)

Proof We apply Rolle's Theorem to a new function h defined as the difference between f and the function whose graph is the secant line AB. Using Equation 3, we see that the equation of the line AB can be written as



$$y - f(x) = \frac{f(b)-f(a)}{b-a}(x-a)$$

or as $y = f(a) + \frac{f(b)-f(a)}{b-a}\ (x-a)$

(IV) $h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}\ (x-a)$

First we must verify that h satisfies the three hypotheses of Rolle's Theorem.

1. The function h is continuous on [a, b] because it is the sum of f and a first-degree polynomial, both of which are continuous.

2. The function h is differentiable on (a, b) because both f and the first-degree polynomial are differentiable. In fact we can compute h' directly from Equation 4 :

$h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$ (Note that f(a) and [f(b) - f(a)]/(b - a) are constants.)

3.

$h(a) = f(a)\ f(a) - \frac{f(b)-f(a)}{b-a}\ (a-a) = 0$

$h(b) = f(b)\ f(a) - \frac{f(b)-f(a)}{b-a}\ (b-a)$

Since h satisfies the hypotheses of Rolle's Theorem, that theorem says there is a number c in (a, b) such that h'(c) = 0. Therefore

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

and so $f'(c) = \frac{f(b) - f(a)}{b - a}$

Ex.25 To illustrate the Mean Value Theorem with a specific function, let's consider $f(x) = x^3 - x$, a= 0, b = 2. since f is a polynomial, it is continuous and differentiable for x, so it is certainly continuous on [0, 2] and differentiable on (0, 2) such that $f(2) - f(0) = f'(c)(2 - 0)$

Sol.

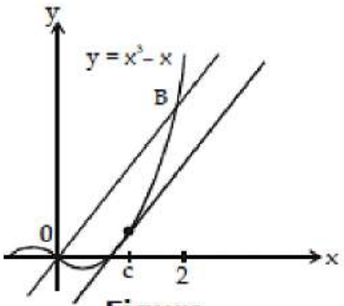
Now $f(2) = 6$, $f(0) = 0$, and $f'(x)\ 3x^2 - 1$, so this equation becomes

$$6 = (3c^2 - 1)2 = 6c^2 - 2$$

which gives $c^2 = \frac{4}{3}$, tha is , $c = \pm 2/\sqrt{3}$.

But c must lie in (0, 2), so $c = 2/\sqrt{3}$.

The tangent line at this value of c is parallel to the secant line OB.



Figure

Ex.26 If $f'(x) = \frac{1}{1+x^2}$ for all x and $f(0) = 0$, show that $0.4 < f(2) < 2$

Sol.

Given $f'(x) = \frac{1}{1+x^2}$ for all x ...(1)

$f'(x) > 0$ for all x $[\therefore 1 + x^2 > 0]$

Also given $f(0) = 0$ ...(2)

From (1), it follows that f(x) is differentiable at all x, therefore f(x) is also continuous at all x

$\therefore$ by Lagrange's mean value theorem in [0, 2]

$$\frac{f(2)-f(0)}{2-0} = f'(c) = \frac{1}{1+c^2}, \text{ where } 0 < c < 2$$

$$\text{or, } \quad \frac{f(2)-0}{2-0} = \frac{1}{1+c^2}$$

$$\text{or} \quad f(2) = \frac{2}{1+c^2} \quad \ldots(3)$$

Now $0 < c < 2$

$$\therefore \frac{2}{1+c^2} < \frac{2}{1+0^2} \qquad \text{or } \frac{2}{1+c^2} < 2 \quad \ldots(4)$$

$$\text{and} \quad \frac{2}{1+c^2} > \frac{2}{1+2^2} = \frac{2}{5} = 0.4$$

$$\text{or, } \quad \frac{2}{1+c^2} > 0.4 \quad \ldots(5)$$

From (3), (4) and (5) it follows that $0.4 < f(2) < 2$.

H. Curve Sketching

The following checklist is intended as a guide to sketching a curve y = f(x). Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.) But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

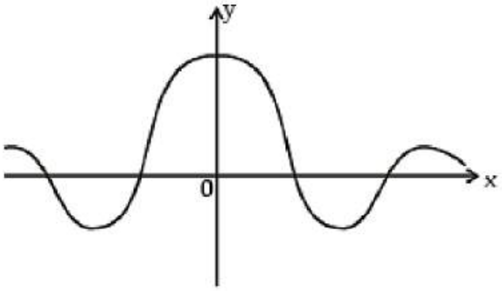
I. Domain It's often useful to start by determining the domain D of f, that is, the set of values of x for which f(x) is defined.

II. Intercepts The y-intercept is f(0) and this tells us where the curve intersect the y-axis. To find the x-intercepts, we set y = 0 and solve for x. (You can omit this step if the equation is difficult to solve.)

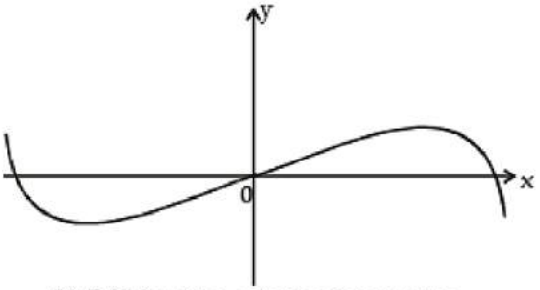
III. Symmetry

(a) If f(-x) = f(x) for all x in D, that is, the equation of the curve is unchanged when x is replaced by -x, then f is an even function and the curve is symmetric about the y-axis. This means that out work is cut in half. If we know what the curve looks like for $x \geq 0$, then we need only reflect about the y-axis to obtain the complete curve [see Figure (a)]. Here are some examples : $y = x^2$, $y = x^4$, $y = |x|$, and y = cos x.

(b) If f(-x) = -f(x) for all x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geq 0$, [Rotate 180º about the origin; see Figure (b).] Some simple examples of odd functions are y = x, $y = x^3$, $y = x^5$, and y = sin x.



(a) Even function : reflectional symmetry



(b) Odd function rotational symmetry

Figure

(c) If f(x + p) = f(x) for all x in D, where p is positive constant, then f is called a **periodic function** and the smallest such number p is called the **period.** For instance, y = sin x has period 2 π and y = tan x has period π . If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph (see Figure).

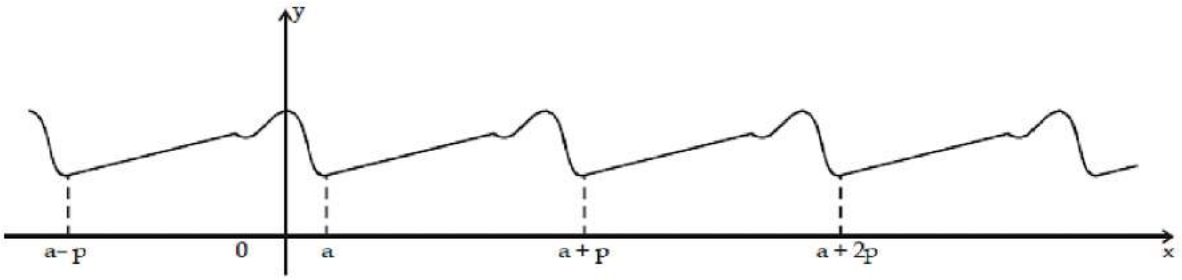


Figure Periodic Function : translational symmetry

IV. Asypmtotes

(a) Horizontal Asymptotes. If either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to\infty} f(x) = L$, then the line y = L is a horizontal asymptote of the curve y = f(x). If it turns out that $\lim_{x\to\infty} f(x) = \infty$ (or -∞), then we do not have an asymptote to the right, but that is still useful information for sketching the curve.

(b) Vertical Asymptotes. The line x = a is a vertical asymptote if at least one of the following statements is true :

$\lim_{x\to a^+} f(x) = \infty$ $\lim_{x\to a^-} f(x) = \infty$

$\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to a^-} f(x) = -\infty$

(For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.) Furthermore, in sketching the curve it is very useful to know exactly which of the statements in (ii) is true. If f(a) is not defined but a is an endpoint of the domain of f, then you should compute $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$, whether or not this limit is infinite,

V. Interval of Increase / Decrease Use the I/D Test. Compute f'(x) and find the intervals on which f'(x) is positive (f is increasing) and the intervals on which f'(x) is negative (f is decreasing).

VI. Local Maximum and Minimum Value Find the critical numbers of f [the number c where f'(c) = 0 or f'(c) does not exist]. Then use the First Derivative Test. If f' changes from positive to negative at a critical number c, then f(c) is a local maximum. If f' changes from negative to positive at c, then f(c) is a local minimum. Although it is usually preferable to use the First Derivative Test, you can use the

Second Derivative Test if c is a critical number such that $f''(c) \neq 0$. Then $f''(c) > 0$ implies that f(c) is a local minimum, whereas $f''(c) < 0$ implies that f(c) is a local maximum.

VII. Concavity and Points of Inflection Compute f'(x) and use the Concavity Test. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Inflection points occur where the direction of concavity changes.

VIII. Sketch the Curve Using the information in items A - G, draw the graph. Sketch the asymptotes as dashed lines, Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes. If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

Ex.27 Use the guidelines to sketch the curve y = $\frac{2x^2}{x^2-1}$.

Sol.

I. The domain is $\{x|x^2 \ 1 \neq 0\} = \{x \mid x \neq \pm 1\} = (\propto, 1) \cup (1, 1) \cup (1. \propto)$

II. The x-and y-intercepts are both 0.

III. Since f(-x) = f(x), the function f is even. The curve is symmetric about the y-axis.

IV. $\lim_{x\to\pm\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\pm\infty} \frac{2}{1-1/x^2} = 2.$ Therefore, the line y = 2 is a horizontal asymptote.

Since the Denominator is 0 when $x = \pm 1$, we compute the following limits :

$$\lim_{x\to 1^+} \frac{2x^2}{x^2-1} = \infty, \quad \lim_{x\to 1^-} \frac{2x^2}{x^2-1} = \infty,$$

$$\lim_{x\to -1^+} \frac{2x^2}{x^2-1} = -\infty, \quad \lim_{x\to -1^-} \frac{2x^2}{x^2-1} = -\infty$$

Therefore, the lines $x = 1$ and $x = -1$ are vertical asymptotes. This information about limits and asymptotes enables us to draw the preliminary sketch in Figure, showing the parts of the curve near the aymptotes.

V. $f'(x) = \dfrac{4x(x^2-1)-2x^2.2x}{(x^2-1)^2} = \dfrac{-4x}{(x^2-1)^2}$

Since $f'(x) > 0$ when $x < 0$ $(x \neq -1)$ and $f'(x) < 0$ when $x > 0$ when $x > 0$ $(x \neq 1)$, f is increasing on $(-\propto, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \propto)$.

VI. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = 0$ is local maximum by the First Derivative Test.

VII. $f''(x) = \dfrac{-4(x^2-1)^2 - 4x.2(x^2-1)2x}{(x^2-1)^4} = \dfrac{12x^2+4}{(x^2-1)^3}$

Since $12x^2 + 4 > 0$ for all x, we have

$f''(x) > 0 \Leftrightarrow x^2\ 1 > 0 \Leftrightarrow |x| > 1$

VIII. Using the information in V VI, we finish the sketch in Figure.

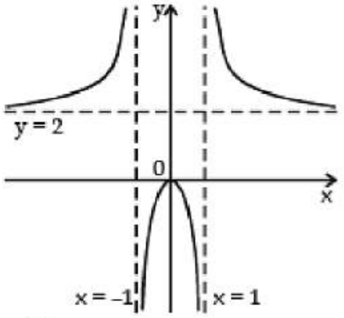


Figure
Finished sketch of $y = \dfrac{2x^2}{x^2-1}$

Ex.28 Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x+1}}$.

Sol.

I. Domain $= \{x|x + 1 > 0\} = \{x|x > 1\} = (1, \propto)$

II. The x- and y-intercepts are both 0.
III. Symmetry : None

IV. Since $\lim_{x \to \infty} \frac{x^2}{\sqrt{x+1}} = \infty$ there is no horizontal asymptote.

Since $\sqrt{x+1} \to 0$ as $x \to 1^+$ and f(x) is always positive, we have $\lim_{x \to -1^+} \frac{x^2}{\sqrt{x+1}} = \infty$ and so the line x = 1 is a vertical asymptote.

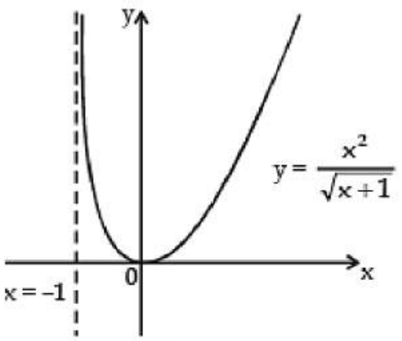
V. $f'(x) = \frac{2x\sqrt{x+1} - x^2.1/(2\sqrt{x+1})}{x+1} = \frac{x(3x+4)}{2(x+1)^{3/2}}$

We see that f'(x) = 0 when x = 0 (notice that - 4/3 is not in the domain of f), so the only critical number is 0. Since f'(x) < 0 when -1 < x < 0 and f'(x) > 0 when x> 0, f is decreasing on (-1, 0) and increasing on (0, ∝)

VI. Since f'(0) = 0and f' changes from negative to positive at 0, f(0) = 0 is a local (and absolute) minimum by the first derivative.

VII. $f''(x) = \frac{2(x+1)^{3/2}(6x+4) - (3x^2+4x)3(x+1)^{1/2}}{4(x+1)^3} = \frac{3x^2+8x+8}{4(x+1)^{5/2}}$

Note that the denominator is always positive. The numerator is the quadratic $3x^2$ + 8x + 8, which is always positive because its discriminant is b^2 4ac = 32, which is negative, and the coefficient of x^2 is positive. Thus, f''(x) > 0 for all x in the domain of f, which means that f is concave upward on (-1, to) and there is no point of inflection.



Figure

VIII. The curve is sketched in Figure.

Ex.29 Sketch the graph of $f(x) = xe^x$

Sol. I. The domain is R.

II. The x- and y-intercepts are both 0.

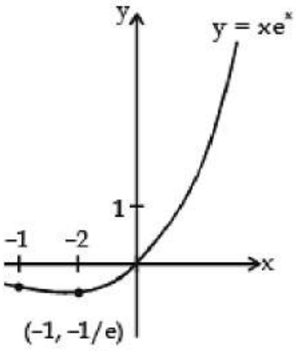
III. Symmetry : None

IV. Because both x and e^x become large as $x \to \propto$, we have $\lim\limits_{x\to\infty} xe^x = \infty$. As $x \to -\infty$, however, $e^x \to 0$ and so we have an indeterminate product that requires the use of L'Hospital's Rule :

$\lim\limits_{x\to-\infty} xe^x = \lim\limits_{x\to-\infty} \frac{x}{e^{-x}} = \lim\limits_{x\to-\infty} \frac{1}{-e^{-x}} = \lim\limits_{x\to-\infty} (-e^x) = 0$. Thus, the x-axis is horizontal asymptote.

V $f'(x) = xe^x + e^x = (x + 1)e^x$. Since e^x is always positive, we see that $f'(x) > 0$ when $x + 1 > 0$, and $f'(X) < 0$ when $x + 1 < 0$. So f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.

VI. Because $f'(1) = 0$ and f changes from negative to positive at $x = 1$, $f(1) = e^1$ is a local (and absolute) minimum.



Figure

VII. $f''(x) = (x + 1)ex + ex = (x + 2)e^x$ Since $f''(x) > 0$ if $x > 2$ and $f''(x) < 0$ if $x < 2$, f is concave upward on $(2, \propto)$ and concave downward on $(\propto, 2)$. The inflection points in $(2, 2e^2)$

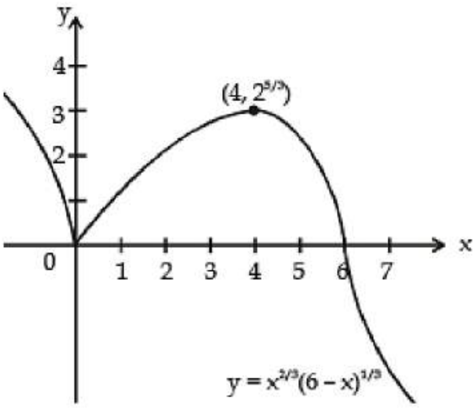
VIII. We use this information to sketch the curve in Figure.

Ex.30 Sketch the graph of the function $f(x) = x^{2/3}(6 - x)^{1/3}$.

Sol. You can use the differentiation rules to check that the first two derivatives are

Interval	$4-x$	$x^{1/3}$	$(6-x)^{2/3}$	$f'(x)$	f
$x<0$	+	−	+	−	decreasing on$(-\infty,0)$
$0<x<4$	+	+	+	+	increasing on$(0,4)$
$4<x<6$	−	+	+	−	decreasing on $(4,6)$
$x>6$	−	+	+	−	decreasing on $(6,\infty)$

$$f'(x) = \frac{4-x}{x^{1/3}(6-x)^{2/3}},\ \ f''(x) = \frac{-8}{x^{4/3}(6-x)^{5/3}}$$



Since $f'(x) = 0$ when $x = 4$ and $f'(x)$ does not exist when $x = 0$ or $x = 6$, the critical numbers are 0, 4 and 6.To find the local extreme values we use the First Derivative Test. Since f' changes from negative to positive at 0, $f(0) = 0$ is a local minimum. Since f' changes from positive to negative at 4, $f(4) = 2^{5/3}$ is a local maximum. The sign of f' does not change at 6, so there is no minimum or maximum there Looking at the expression for $f''(x)$ and noting that $x^{4/3} > 0$ for all x, we have $f''(x) < 0$ for $x < 0$ and for $0 < x < 6$ and $f''(x) > 0$ for $x > 6$. So f is concave downward on (-x, 0) and (0, 6) and concave upward on (6, x), and the only inflection point is (6, 0). The graph is sketched in Figure. Note that the curve has vertical tangents at (0, 0) and (6, 0) because $|f'(x)| \to \propto$ as $x \to 0$ and as $x \to 6$

Ex.31 Plot the following curves :

(a) $y = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + 6$

(b) $y = \frac{x}{\ell n\, x}$

(c) $y = x\ \ell n\ x$

(d) $y = \dfrac{\ell n\, x}{x}$

(e) $y = \dfrac{x+1}{(x-1)(x-7)}$

(f) $2^{|x|}\,|y| + 2^{|x|-1} = 1.$

Sol.

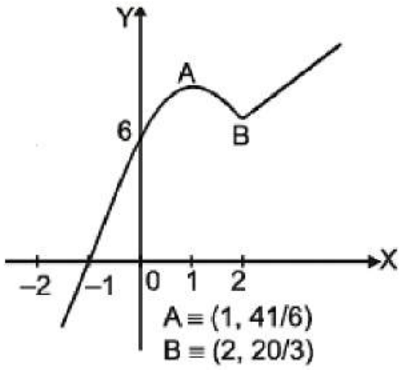
(a) We have $y = \dfrac{x^3}{3} - \dfrac{3x^2}{2} + 2x + 6$ whose domain is $x \in R$, and

$y' = x^2 \ 3x + 2 = (x \ 1)\ (x \ 2) > 0$ for $x \in (\propto, 1)\ U\ (2, \propto)$ < 0 for $x \in (1, 2)$

$\Rightarrow$ y strictly increases in $(\propto, 1)$ strictly decreases in $(1, 2)$; strictly increases in $(2, \propto)$

Now we have $y(1) = \dfrac{1}{3} - \dfrac{3}{2} + 2 + 6 = \dfrac{41}{6}$, $y(2) = \dfrac{8}{3} - 6 + 4 + 6 = \dfrac{20}{3}$

$y\ (\propto)\ \propto, y(\propto) = \propto$



The curve cuts the Y-axis at (0, 6).

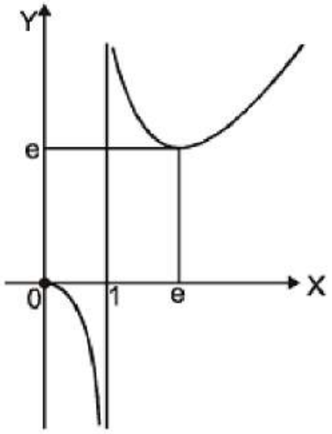
The cure cuts the ve X-axis somewhere between 1and 2, since

$y(-1) = \dfrac{-1}{3} - \dfrac{3}{2} - 2 + 6 > 0$ and $y\ (-2) = \dfrac{-8}{3} - \dfrac{12}{2} - 4 + 6 < 0.$ The plot of the curve is shown along side.

(b) We have y = x/ln x Whose domain is $x \in (0, \propto) \sim \{1\}$, and

$$y' = \frac{\ell n\, x - 1}{\ell n^2\, x} < 0 \;\forall\; x \in (0, 1) \cup (1, e)$$
$$> 0 \;\forall\; x \in (e, \infty)$$

⇒ y strictly decreases in (0, 1) U (1, e) ; strictly increases in (e, ∝)



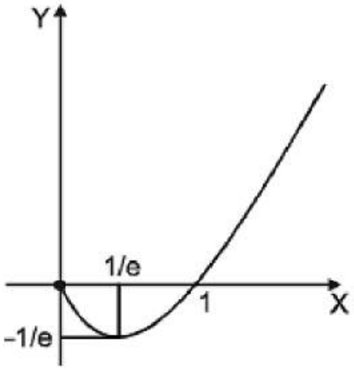
Now, we have $\lim\limits_{x\to 0^+} \frac{x}{\ell n\, x} = 0$, y (e) = e

$$\lim_{x\to\infty} \frac{x}{\ell n\, x}\left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{1}{1/x} = \infty$$

The plot of the curve is shown alongside.

(c) We have y = x ln x whose domain is $x \in (0, \propto)$, and ⇒ y strictly decreases in (0, e^1) ; strictly increases in (e1, ∝).

Now, we have $\lim\limits_{x\to 0^+} x\,\ell n\, x = \lim\limits_{x\to 0^+} \frac{\ell n\, x}{1/x}\left(\frac{\infty}{\infty}\right) = \lim\limits_{x\to 0^+} \frac{1/x}{1/x^2} = 0$



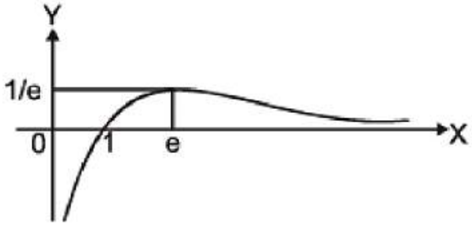
$$\lim_{x\to\infty} x\,\ell n\, x = \infty, y(e^{-1}) = \frac{-1}{e}$$

The curve cuts the X-axis at (1, 0). The plot of the curve is shown above.

(d) We have y = lnx/x whose domain is $x \in (0, \propto)$, and

$$y' = \frac{1-\ell n\, x}{x^2} > 0 \ \forall\ x \in (0, e)$$

$$< 0 \ \forall\ x \in (e, \infty)$$



⇒ y strictly increases in (0, e); strictly decreases in $(e, \propto)$.

Now, we have $\lim_{x\to\infty} \frac{\ell n\, x}{x}\left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{1/x}{1}$

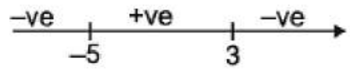
$$= 0,\ \lim_{x\to\infty} \frac{\ell n\, x}{x} - \infty, y(e) = \frac{1}{e}$$

The curve cuts the X-axis at (1, 0). The plot of the curve is shown above.

(e) We have $y = \frac{x+1}{(x-1)(x-7)}$ whose domain is $x \in R \sim \{1.7\}$, and

$$y' = \frac{(x^2 - 8x + 7) - (2x - 8)(x + 1}{(x - 1)^2 (x - 7)^2}$$

$$= \frac{-(x^2 + 2x - 15)}{(x - 1)^2 (x - 7)^2} = \frac{-(x + 5)(x - 3)}{(x - 1)^2 (x - 7)^2}$$

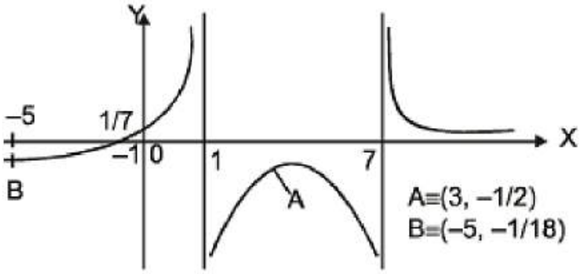


Now, we have $y(3) = \frac{4}{(2)(-4)} = \frac{-1}{2}$, $y(-5) = \frac{-4}{(-6)(-12)} = \frac{-1}{18}$

$$\lim_{x \to 1^-} \frac{x + 1}{(x - 1)(x - 7)} = \infty, \ \lim_{x \to 1^+} \frac{x + 1}{(x - 1)(x - 7)} = -\infty$$

$$\lim_{x \to 7^-} \frac{x + 1}{(x - 1)(x - 7)} = -\infty, \ \lim_{x \to 7^+} \frac{x + 1}{(x - 1)(x - 7)} = \infty$$

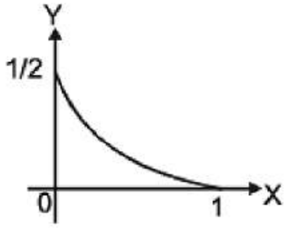
$$\lim_{x \to \infty} \frac{x + 1}{(x - 1)(x - 7)} = 0, \ \lim_{x \to \infty} \frac{x + 1}{(x - 1)(x - 7)} = 0$$



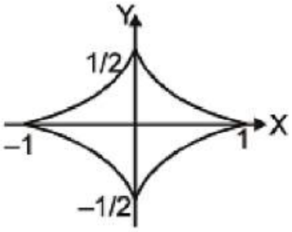
The curve cuts the Y-axis at (0, 1/7). The curve cuts the X-axis at (1, 0). The plot of the curve is shown below.

(f) We have $2^{|x|} |y| + 2^{|x| - 1} = 1$ i.e.

$$|y| = 2^{-|x|} - \frac{1}{2}$$



The curve is symmetrical about the X-axis as well as the Y-axis. In the first quadrant the equation of the curve reduces to $y = 2^x - 1/2$



whose plot is shown above.
The complete curve is drawn by taking the mirror image of the above shown curve in the X-axis and the Y-axis as shown alongside

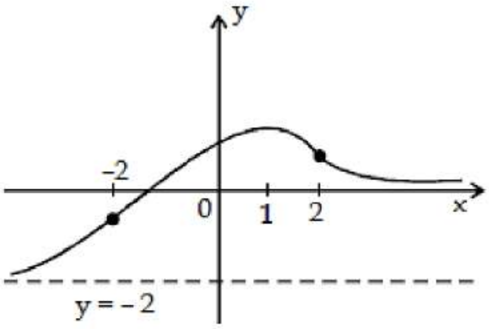
Ex.32 Sketch a possible graph of a function f that satisfies the following conditions :

(i) $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$n $(1, \infty)$

(ii) $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$

(iii) $\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = 0$

Sol. Condition (i) tells us that f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. Condition (ii) says that f is concave upward on $(-\infty, -2)$ and $(2, \infty)$, and concave downward on $(-2, 2)$. From condition (iii) we know that the graph of f has two horizontal asymptotes: $y = -2$ and $y = 0$.



Figure

We first draw the horizontal asymptote y = -2 as a dashed line (see Figure). We then draw the graph of f approaching this asymptote at the far left, increasing to its maximum point at x = 1 and decreasing toward the x-axis at the far right. We also make sure that the graph has inflection points when x = -2 and 2. Notice that we made the curve bend upward for x < -2 and x > 2. and bend downward when x is between -2 and 2.

Application of Derivatives Formulas

Things To Remember :

(i) The value of the derivative at P (x_1, y_1) gives the slope of the tangent to the curve at P. Symbolically $f'(x_1) = \left.\frac{dy}{dx}\right]_{x_1y_1}$ = Slope of tangent at P $(x_1\ y_1)$ = m (say).

(ii) Equation of tangent at (x_1, y_1) is ;

$$y - y_1 = \left.\frac{dy}{dx}\right]_{x_1y_1} (x - x_1).$$

(iii) Equation of normal at (x_1, y_1) is ;

$$y - y_1 = -\frac{1}{\left.\frac{dy}{dx}\right]_{x_1y_1}} (x - x_1).$$

Note :

1.The point P (x_1, y_1) will satisfy the equation of the curve & the eqation of tangent & normal line.

2. If the tangent at any point P on the curve is // to the axis of x then dy/dx = 0 at the point P.

3. If the tangent at any point on the curve is parallel to the axis of y, then dy/dx = ∞

or dx/dy = 0.

4. If the tangent at any point on the curve is equally inclined to both the axes then dy/dx = ± 1.

5. If the tangent at any point makes equal intercept on the coordinate axes then dy/dx = 1.

6. Tangent to a curve at the point P (x_1, y_1) can be drawn even through dy/dx at P does not exist. e.g. x = 0 is a tangent to $y = x^{2/3}$ at (0, 0).

7. If a curve passing through the origin be given by a rational integral algebraic equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation. e.g. If the equation of a curve be $x^2\ y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2\ y^2 = 0$ i.e. x + y = 0 and x − y = 0.

(iv) Angle of intersection between two curves is defined as the angle between the 2 tangents drawn to the 2 curves at their point of intersection. If the angle between two curves is 90° every where then they are called **ORTHOGONAL** curves.

(v) (a) Length of the tangent (PT) = $\dfrac{y_1\sqrt{1+[f'(x_1)]^2}}{f'(x_1)}$

(b) Length of Subtangent (MT) = $\dfrac{y_1}{f'(x_1)}$

(c) Length of Normal (PN) = $y_1\sqrt{1+[f'(x_1)]^2}$

(d) Length of Subnormal (MN) = $y_1 f'(x_1)$

(vi) Differentials :

The differential of a function is equal to its derivative multiplied by the differential of the independent variable. Thus if, y = tan x then $dy = \sec^2 x\, dx$.

In general $dy = f'(x)\, dx$.

Note that : d (c) = 0 where 'c' is a constant.
d (u + v − w) = du + dv − dw d (u v) = u d v + v d u

Note :

1. For the independent variable 'x' , increment Δ x and differential d x are equal but this is not the case with the dependent variable 'y' i.e. $\Delta y \neq dy$.

2. The relation $d\,y = f'(x)\,d\,x$ can be written as $\frac{dy}{dx} = f'(x)$; (x) ; thus the quotient of the differentials of 'y' and 'x' is equal to the derivative of 'y' w.r.t. 'x'.

Monotonocity

(Significance of the sign of the first order derivative)

Definitions :

1. A function f (x) is called an Increasing Function at a point x = a if in a sufficiently small neighbourhood

around $x = a$ we have $\left.\begin{array}{l} f(a+h) > f(a) \text{ and} \\ f(a-h) < f(a) \end{array}\right\}$ increasing;

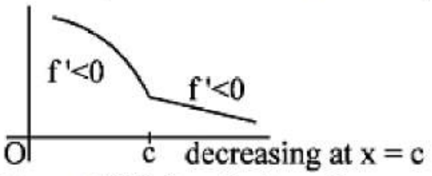
Similarly decreasing if $\left.\begin{array}{l} f(a+h) < f(a) \text{ and} \\ f(a-h) > f(a) \end{array}\right\}$ decreasing.

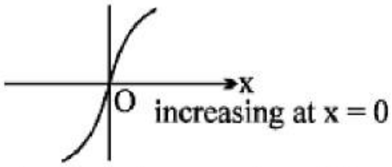
2. A differentiable function is called increasing in an interval (a, b) if it is increasing at every point within the interval (but not necessarily at the end points). A function decreasing in an interval (a, b) is similarly defined.

3. A function which in a given interval is increasing or decreasing is called "Monotonic" in that interval.

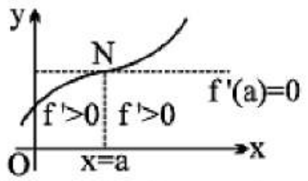
4. Tests for increasing and decreasing of a function at a point :

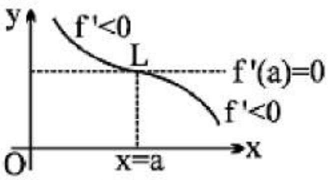
If the derivative $f'(x)$ is positive at a point $x = a$, then the function f (x) at this point is increasing. If it is negative, then the function is decreasing. Even if $f'(a)$ is not defined, f can still be increasing or decreasing.





Note : If $f'(a) = 0$, then for $x = a$ the function may be still increasing or it may be decreasing as shown. It has to be identified by a seperate rule. e.g. $f(x) = x^3$ is increasing at every point. Note that, $dy/dx = 3x^2$.





5. Tests for Increasing & Decreasing of a function in an interval :

Sufficiency Test :

If the derivative function $f'(x)$ in an interval (a , b) is every where positive, then the function f (x) in this interval is Increasing ; If f $'$(x) is every where negative, then f (x) is Decreasing.

General Note :

(1) If a function is invertible it has to be either increasing or decreasing.
(2) If a function is continuous the intervals in which it rises and falls may be separated by points at which its derivative fails to exist.
(3) If f is increasing in [a, b] and is continuous then f (b) is the greatest and f (c) is the least value of f in [a, b]. Similarly if f is decreasing in [a, b] then f (a) is the greatest value and f (b) is the least value.

6. (a) ROLLE'S THEOREM :

Let f(x) be a function of x subject to the following conditions :
(i) f(x) is a continuous function of x in the closed interval of $a \le x \le b$.
(ii) f $'$ (x) exists for every point in the open interval $a < x < b$.
(iii) f (a) = f (b). Then there exists at least one point x = c such that $a<c<b$ where f $'$ (c) = 0.

Note that if f is not continuous in closed [a, b] then it may lead to the adjacent graph where all the 3 conditions of Rolles will be valid but the assertion will not be true in (a, b).

(b) LMVT THEOREM :

Let f(x) be a function of x subject to the following conditions :
(i) f(x) is a continuous function of x in the closed interval of $a \le x \le b$.
(ii) f $'$ (x) exists for every point in the open interval $a < x < b$.
(iii) $f(a) \neq f(b)$.
Then there exists at least one point x = c such that $a < c < b$ where f $'$

(c) $= \frac{f(b) - f(a)}{b - a}$

Geometrically, the slope of the secant line joining the curve at x = a & x = b is equal to the slope of the tangent line drawn to the curve at x = c. Note the following :

- Rolles theorem is a special case of LMVT since $f(a) = f(b) \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a} = 0.$

Note :

Now [f (b) f (a)] is the change in the function f as x changes from a to b so that [f (b) f (a)] / (b a) is the average rate of change of the function over the interval [a, b]. Also f $'$(c) is the actual rate of change of the function for x = c. Thus, the theorem states that the average rate of change of a function over an interval is also the actual rate of change of the function at some point of the interval. In particular, for instance, the average velocity of a particle over an interval of time is equal to the velocity at some instant belonging to the interval. This interpretation of the theorem justifies the name "Mean Value" for the theorem.

(c) Application Of Rolles Theorem For Isolating The Real Roots Of An Equation F (X)=0

Suppose a & b are two real numbers such that ;

(i) f(x) & its first derivative $f'(x)$ are continuous for $a \le x \le b$.

(ii) f(a) & f(b) have opposite signs.

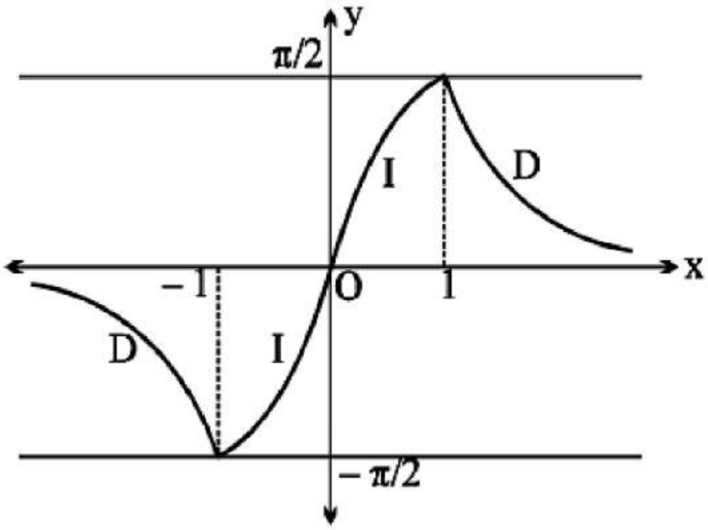
(iii) $f'(x)$ is different from zero for all values of x between a & b.

Then there is one & only one real root of the equation $f(x) = 0$ between a & b.

Maxima - Minima

Functions Of A Single Variable

How Maxima & Minima Are Classified



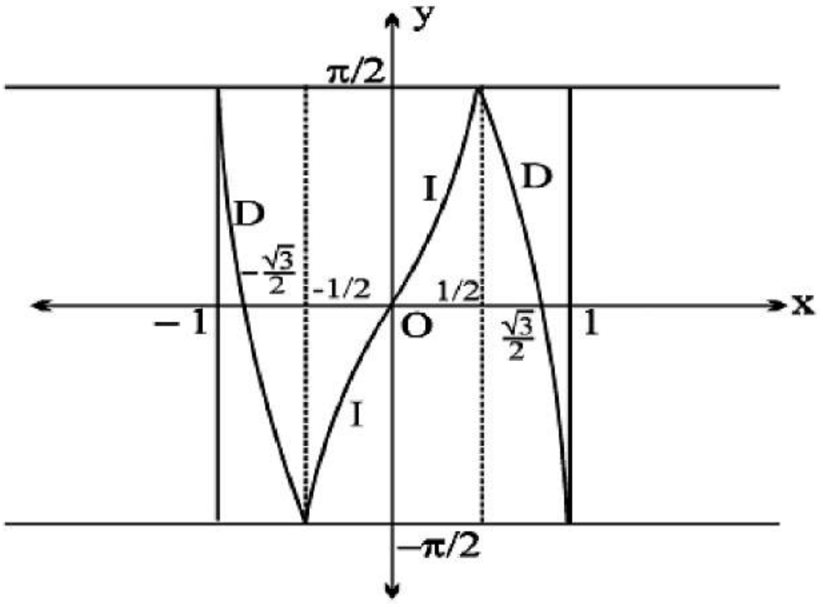
1. A function f(x) is said to have a maximum at x = a if f(a) is greater than every other value assumed by f(x) in the immediate neighbourhood of x = a.

Symbolically $\left.\begin{matrix} f(a) > f(a+h) \\ f(a) > f(a-h) \end{matrix}\right] \Rightarrow x = a$ gives maxima for a sufficiently small positive h.

Similarly, a function f(x) is said to have a minimum value at x = b if f(b) is least than every other value assumed by f(x) in the immediate neighbourhood at x = b.

Symbolically if $\left.\begin{matrix} f(b) < f(b+h) \\ f(b) < f(b-h) \end{matrix}\right] \Rightarrow x$ = b gives minima for a sufficiently small positive h.

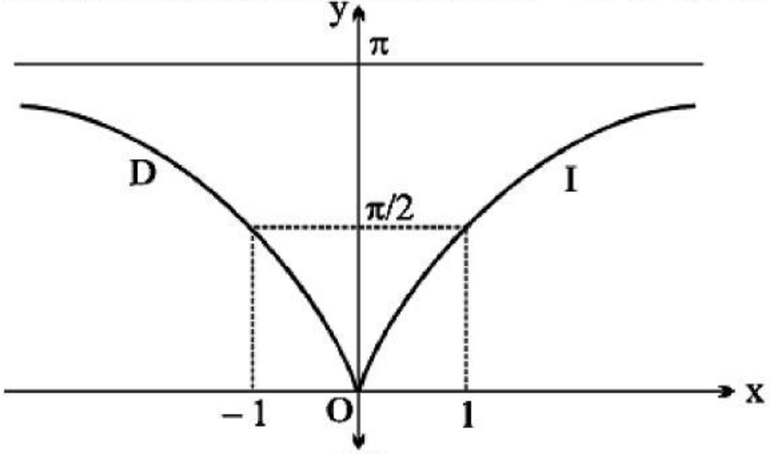
Note that :

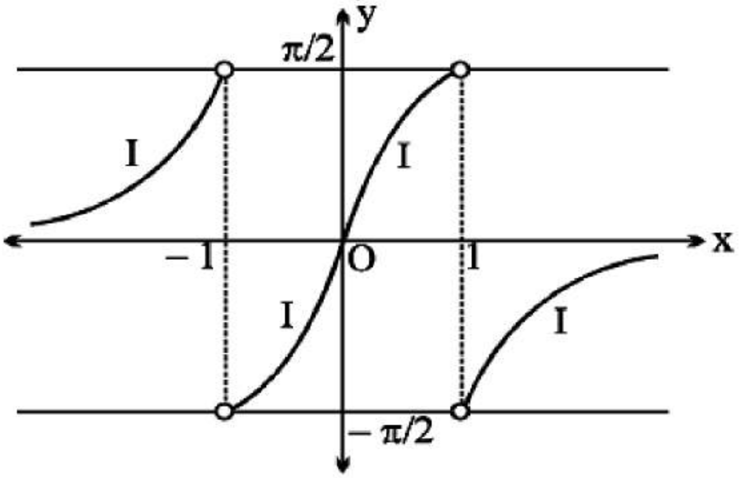


(i) the maximum & minimum values of a function are also known as local/relative maxima or local/relative minima as these are the greatest & least values of the function relative to some neighbourhood of the point in question.
(ii) the t erm 'extremum' or (extremal) or 'turning value' is used both for maximum or a minimum value.
(iii) a maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.
(iv) a function can have several maximum & minimum values & a minimum value may even be greater than a maximum value.
(v) maximum & minimum values of a continuous function occur alternately & between two consecutive maximum values there is a minimum value & vice versa.

2. A Necessary Condition For Maximum & Minimum :

If f(x) is a maximum or minimum at x = c & if $f'(c)$ exists then $f'(c) = 0$.





Note :

(i) The set of values of x for which $f'(x) = 0$ are often called as stationary points or critical points. The rate of change of function is zero at a stationary point.
(ii) In case $f'(c)$ does not exist f(c) may be a maximum or a minimum & in this case left hand and right hand derivatives are of opposite signs.
(iii) The greatest (global maxima) and the least (global minima) values of a function f in an interval [a, b] are f(a) or f(b) or are given by the values of x for which $f'(x) = 0$.

(iv) Critical points are those where $\frac{dy}{dx} = 0$, if it exists ,or it fails to exist either by virtue of a vertical tangent or by virtue of a geometrical sharp corner but not because of discontinuity of function.

3. Sufficient Condition For Extreme Values :

$$\left.\begin{matrix} f'(c-h) > 0 \\ f'(c+h) < 0 \end{matrix}\right] \Rightarrow x = c$$

is a point of local maxima, where $f'(c) = 0$.

Similarly $\left.\begin{matrix} f'(c-h) < 0 \\ f'(c+h) > 0 \end{matrix}\right] \Rightarrow x = c$ is a point of local minima, where $f'(c) = 0$.

Note : If $f'(x)$ does not change sign i.e. has the same sign in a certain complete neighbourhood of c, then f(x) is either strictly increasing or decreasing throughout this neighbourhood implying that f(c) is not an extreme value of f.

4. Use Of Second Order Derivative In Ascertaining The Maxima Or Minima:

(a) f(c) is a minimum value of the function f, if $f'(c) = 0$ & $f''(c) > 0$.

(b) f(c) is a maximum value of the function f, $f'(c) = 0$ & $f''(c) < 0$.

Note : if $f''(c) = 0$ then the test fails. Revert back to the first order derivative check for ascertaning the maxima or minima.

5. SUMMARY-WORKING RULE :

FIRST :

When possible , draw a figure to illustrate the problem & label those parts that are important in the problem. Constants & variables should be clearly distinguished.

SECOND:

Write an equation for the quantity that is to be maximised or minimised. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. his may require some algebraic manipulations.

THIRD :

If $y = f(x)$ is a quantity to be maximum or minimum, find those values of x for which $dy/dx = f'(x) = 0$.

FOURTH :

Test each values of x for which $f'(x) = 0$ to determine whether it provides a maximum or minimum or neither. The usual tests are :

(a) If d^2y/dx^2 is positive when $dy/dx = 0 \Rightarrow y$ is minimum.

If d^2y/dx^2 is negative when $dy/dx = 0 \Rightarrow y$ is maximum.

If $d^2y/dx^2 = 0$ when $dy/dx = 0$, the test fails.

(b) $$\text{If} \frac{dy}{dx} \text{ is } \left.\begin{matrix} \text{positive} & \text{for} & x < x_0 \\ \text{zero} & \text{for} & x = x_0 \\ \text{negative} & \text{for} & x > x_0 \end{matrix}\right] \Rightarrow \text{a maximum occurs at } x = x_0.$$

But if dy/dx changes sign from negative to zero to positive as x advances through x_o there is a minimum. If dy/dx does not change sign, neither a maximum nor a minimum. Such points are called **INFLECTION POINTS.**

FIFTH :

If the function $y = f(x)$ is defined for only a limited range of values $a \leq x \leq b$ then examine $x = a$ & $x = b$ for possible extreme values.

SIXTH :

If the derivative fails to exist at some point, examine this point as possible maximum or minimum.

Important Note :

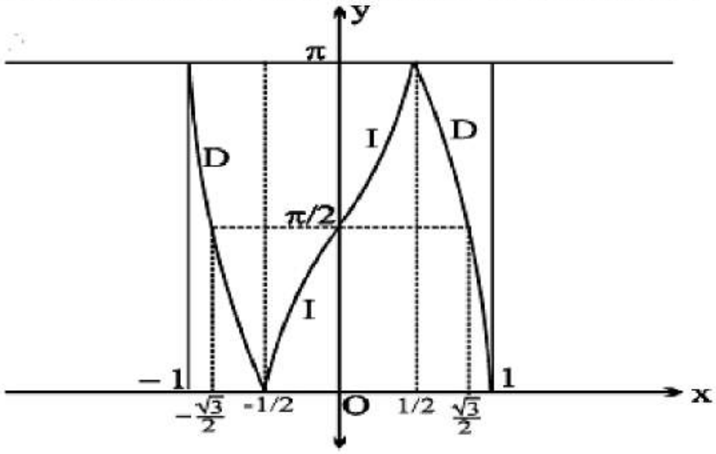
Given a fixed point $A(x_1, y_1)$ and a moving point $P(x, f(x))$ on the curve $y = f(x)$. Then AP will be maximum or minimum if it is normal to the curve at P.

If the sum of two positive numbers x and y is constant than their product is maximum if they are equal, i.e. $x + y = c$, $x > 0$, $y > 0$, then

$$xy = \frac{1}{4} [(x + y)^2 - (x - y)^2]$$

If the product of two positive numbers is constant then their sum is least if they are equal. i.e. $(x + y)^2 = (x\,y)^2 + 4xy$

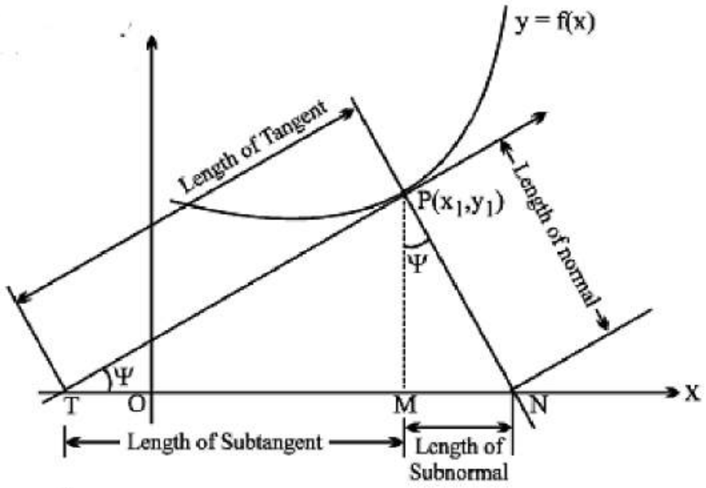
6. Useful Formulae Of Mensuration To Remember :



- Volume of a cuboid = lbh.
- Surface area of a cuboid = 2 (lb + bh + hl).
- Volume of a prism = area of the base x height.
- Lateral surface of a prism = perimeter of the base x height.
- Total surface of a prism = lateral surface + 2 area of the base
  (Note that lateral surfaces of a prism are all rectangles).
- Volume of a pyramid = 1/3 area of the base x height.
- Curved surface of a pyramid = 1/2 (perimeter of the base) x slant height.
  (Note that slant surfaces of a pyramid are triangles).
- Volume of a cone = $\frac{1}{3}\pi r^2 h$.
- Curved surface of a cylinder = $2\pi rh$.
- Total surface of a cylinder = $2\pi rh + 2\pi r^2$.
- Volume of a sphere = $\frac{4}{3}\pi r^3$.
- Surface area of a sphere = $4\pi r^2$.
- Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.

7. Significance Of The Sign Of 2nd Order Derivative And Points Of Inflection :

The sign of the 2nd order derivative determines the concavity of the curve. Such points such as C & E on the graph where the concavity of the curve changes are called the points of inflection. From the graph we find that if:



(i) $\frac{d^2y}{dx^2} > 0$ ⇒ concave upwards

(ii) $\frac{d^2y}{dx^2} < 0 \Rightarrow$ concave downwards.

At the point of inflection we find that $\frac{d^2y}{dx^2} = 0$ & $\frac{d^2y}{dx^2}$ changes sign.

Inflection points can also occur if $\frac{d^2y}{dx^2}$ fails to exist. For example, consider the graph of the function defined as,

$$f(x) = [\begin{matrix} x^{3/5} & \text{for} \quad x \in (-\infty\,,\,1) \\ 2 - x^2 & \text{for} \quad x \in (1\,,\,\infty) \end{matrix}$$

Note that the graph exhibits two critical points one is a point of local maximum & the other a point of inflection.