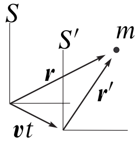
3.2 Frames of reference

Galilean transformations

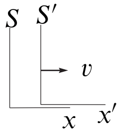
Time and position[a]	$\boldsymbol{r}=\boldsymbol{r}'+\boldsymbol{v}t$	(3.1)	$\boldsymbol{r},\boldsymbol{r}'$	position in frames S and S'
	$t=t'$	(3.2)	$\boldsymbol{v}$	velocity of S' in S
			t,t'	time in S and S'
Velocity	$\boldsymbol{u}=\boldsymbol{u}'+\boldsymbol{v}$	(3.3)	$\boldsymbol{u},\boldsymbol{u}'$	velocity in frames S and S'
Momentum	$\boldsymbol{p}=\boldsymbol{p}'+m\boldsymbol{v}$	(3.4)	$\boldsymbol{p},\boldsymbol{p}'$	particle momentum in frames S and S'
			m	particle mass
Angular momentum	$\boldsymbol{J}=\boldsymbol{J}'+m\boldsymbol{r}'\times\boldsymbol{v}+\boldsymbol{v}\times\boldsymbol{p}'t$	(3.5)	$\boldsymbol{J},\boldsymbol{J}'$	angular momentum in frames S and S'
Kinetic energy	$T=T'+m\boldsymbol{u}'\cdot\boldsymbol{v}+\frac{1}{2}mv^2$	(3.6)	T,T'	kinetic energy in frames S and S'



[a]Frames coincide at $t=0$.

Lorentz (spacetime) transformations[a]

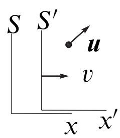
Lorentz factor	$\gamma=\left(1-\frac{v^2}{c^2}\right)^{-1/2}$	(3.7)	γ	Lorentz factor
			v	velocity of S' in S
			c	speed of light
Time and position	$x=\gamma(x'+vt'); \quad x'=\gamma(x-vt)$	(3.8)	x,x'	x-position in frames S and S' (similarly for y and z)
	$y=y'; \quad y'=y$	(3.9)		
	$z=z'; \quad z'=z$	(3.10)		
	$t=\gamma\left(t'+\frac{v}{c^2}x'\right); \quad t'=\gamma\left(t-\frac{v}{c^2}x\right)$	(3.11)	t,t'	time in frames S and S'
Differential four-vector[b]	$\mathrm{d}\boldsymbol{X}=(c\,\mathrm{d}t,-\mathrm{d}x,-\mathrm{d}y,-\mathrm{d}z)$	(3.12)	$\boldsymbol{X}$	spacetime four-vector



[a]For frames S and S' coincident at $t=0$ in relative motion along x. See page 141 for the transformations of electromagnetic quantities.
[b]Covariant components, using the (1,−1,−1,−1) signature.

Velocity transformations[a]

Velocity			γ	Lorentz factor $=[1-(v/c)^2]^{-1/2}$
	$u_x=\frac{u_x'+v}{1+u_x'v/c^2}; \quad u_x'=\frac{u_x-v}{1-u_xv/c^2}$	(3.13)	v	velocity of S' in S
	$u_y=\frac{u_y'}{\gamma(1+u_x'v/c^2)}; \quad u_y'=\frac{u_y}{\gamma(1-u_xv/c^2)}$	(3.14)	c	speed of light
	$u_z=\frac{u_z'}{\gamma(1+u_x'v/c^2)}; \quad u_z'=\frac{u_z}{\gamma(1-u_xv/c^2)}$	(3.15)	u_i,u_i'	particle velocity components in frames S and S'



[a]For frames S and S' coincident at $t=0$ in relative motion along x.

Momentum and energy transformations[a]

Momentum and energy

$$p_x=\gamma(p'_x+vE'/c^2); \qquad p'_x=\gamma(p_x-vE/c^2) \qquad (3.16)$$

$$p_y=p'_y; \qquad p'_y=p_y \qquad (3.17)$$

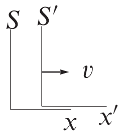
$$p_z=p'_z; \qquad p'_z=p_z \qquad (3.18)$$

$$E=\gamma(E'+vp'_x); \qquad E'=\gamma(E-vp_x) \qquad (3.19)$$

$$E^2-p^2c^2=E'^2-p'^2c^2=m_0^2c^4 \qquad (3.20)$$

Four-vector[b] $\qquad \boldsymbol{P}=(E/c,-p_x,-p_y,-p_z) \qquad (3.21)$

γ	Lorentz factor $=[1-(v/c)^2]^{-1/2}$
v	velocity of S' in S
c	speed of light
p_x,p'_x	x components of momentum in S and S' (sim. for y and z)
E,E'	energy in S and S'
m_0	(rest) mass
p	total momentum in S
$\boldsymbol{P}$	momentum four-vector



[a]For frames S and S' coincident at $t=0$ in relative motion along x.
[b]Covariant components, using the (1,−1,−1,−1) signature.

Propagation of light[a]

Doppler effect

$$\frac{\nu'}{\nu}=\gamma\left(1+\frac{v}{c}\cos\alpha\right) \qquad (3.22)$$

Aberration[b]

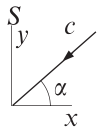
$$\cos\theta=\frac{\cos\theta'+v/c}{1+(v/c)\cos\theta'} \qquad (3.23)$$

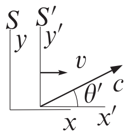
$$\cos\theta'=\frac{\cos\theta-v/c}{1-(v/c)\cos\theta} \qquad (3.24)$$

Relativistic beaming[c]

$$P(\theta)=\frac{\sin\theta}{2\gamma^2[1-(v/c)\cos\theta]^2} \qquad (3.25)$$

ν	frequency received in S
ν'	frequency emitted in S'
α	arrival angle in S
γ	Lorentz factor $=[1-(v/c)^2]^{-1/2}$
v	velocity of S' in S
c	speed of light
θ,θ'	emission angle of light in S and S'
$P(\theta)$	angular distribution of photons in S





[a]For frames S and S' coincident at $t=0$ in relative motion along x.
[b]Light travelling in the opposite sense has a propagation angle of $\pi+\theta$ radians.
[c]Angular distribution of photons from a source, isotropic and stationary in S'. $\int_0^\pi P(\theta)\,\mathrm{d}\theta=1$.

Four-vectors[a]

Covariant and contravariant components

$$x_0=x^0 \qquad x_1=-x^1$$

$$x_2=-x^2 \qquad x_3=-x^3 \qquad (3.26)$$

Scalar product

$$x^iy_i=x^0y_0+x^1y_1+x^2y_2+x^3y_3 \qquad (3.27)$$

Lorentz transformations

$$x^0=\gamma[x'^0+(v/c)x'^1]; \qquad x'^0=\gamma[x^0-(v/c)x^1] \qquad (3.28)$$

$$x^1=\gamma[x'^1+(v/c)x'^0]; \qquad x'^1=\gamma[x^1-(v/c)x^0] \qquad (3.29)$$

$$x^2=x'^2; \qquad x'^3=x^3 \qquad (3.30)$$

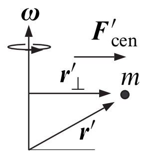
x_i	covariant vector components
x^i	contravariant components
x^i,x'^i	four-vector components in frames S and S'
γ	Lorentz factor $=[1-(v/c)^2]^{-1/2}$
v	velocity of S' in S
c	speed of light

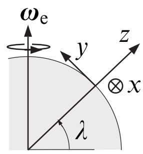
[a]For frames S and S', coincident at $t=0$ in relative motion along the (1) direction. Note that the (1,−1,−1,−1) signature used here is common in special relativity, whereas (−1,1,1,1) is often used in connection with general relativity (page 67).

Rotating frames

Vector transformation	$\left[\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right]_S = \left[\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right]_{S'} + \boldsymbol{\omega}\times\boldsymbol{A}$	(3.31)
Acceleration	$\dot{\boldsymbol{v}} = \dot{\boldsymbol{v}}' + 2\boldsymbol{\omega}\times\boldsymbol{v}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}')$	(3.32)
Coriolis force	$\boldsymbol{F}'_{\mathrm{cor}} = -2m\boldsymbol{\omega}\times\boldsymbol{v}'$	(3.33)
Centrifugal force	$\boldsymbol{F}'_{\mathrm{cen}} = -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}')$	(3.34)
	$= +m\omega^2\boldsymbol{r}'_{\perp}$	(3.35)
Motion relative to Earth	$m\ddot{x} = F_x + 2m\omega_{\mathrm{e}}(\dot{y}\sin\lambda - \dot{z}\cos\lambda)$	(3.36)
	$m\ddot{y} = F_y - 2m\omega_{\mathrm{e}}\dot{x}\sin\lambda$	(3.37)
	$m\ddot{z} = F_z - mg + 2m\omega_{\mathrm{e}}\dot{x}\cos\lambda$	(3.38)
Foucault's pendulum[a]	$\Omega_{\mathrm{f}} = -\omega_{\mathrm{e}}\sin\lambda$	(3.39)

$\boldsymbol{A}$	any vector
S	stationary frame
S'	rotating frame
$\boldsymbol{\omega}$	angular velocity of S' in S
$\dot{\boldsymbol{v}}, \dot{\boldsymbol{v}}'$	accelerations in S and S'
$\boldsymbol{v}'$	velocity in S'
$\boldsymbol{r}'$	position in S'
$\boldsymbol{F}'_{\mathrm{cor}}$	coriolis force
m	particle mass
$\boldsymbol{F}'_{\mathrm{cen}}$	centrifugal force
$\boldsymbol{r}'_{\perp}$	perpendicular to particle from rotation axis
F_i	nongravitational force
λ	latitude
z	local vertical axis
y	northerly axis
x	easterly axis
Ω_{f}	pendulum's rate of turn
ω_{e}	Earth's spin rate





[a]The sign is such as to make the rotation clockwise in the northern hemisphere.