10. Matrix Analysis.

10.1 Introduction:-

$$x + y - z = 3$$
$$-2x + y + z = 5$$
$$-x - 2y + 3z = 7$$

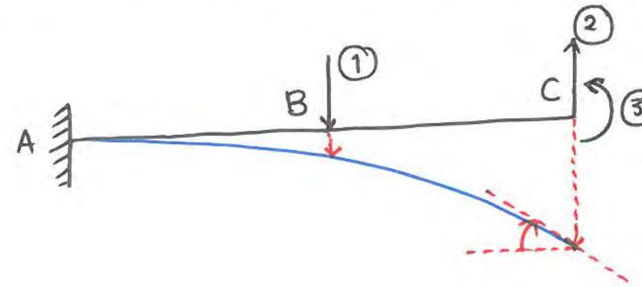
In matrix form :-

$$\begin{bmatrix} 1 & 1 & -1 \\ -2 & 1 & 1 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

Simultaneous equations are solved using matrix. In structural analysis also, simultaneous equations are formulated corresponding to static or kinematic indeterminacy. It means, matrix can be used to analyze structure also.

10.2 Co-ordinate System:-

Co-ordinate system of matrix is the direction corresponding to assumed redundant force in force method and corresponding to KI in displacement method. It has no relation with co-ordinate system of mathematics.

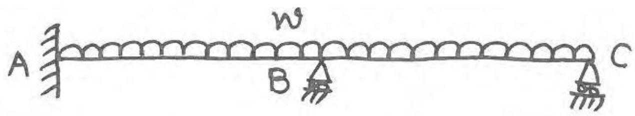


Deflection at B = +ve becoz along co-ordinate 1

Deflection at C = −ve becoz opp co-ordinate 2

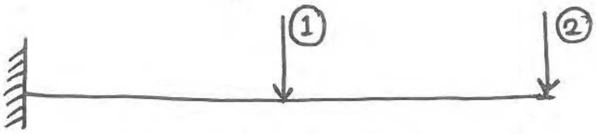
Rotation at C = −ve becoz opp co-ordinate 3.

10.3 Flexibility Matrix Method of Analysis:-

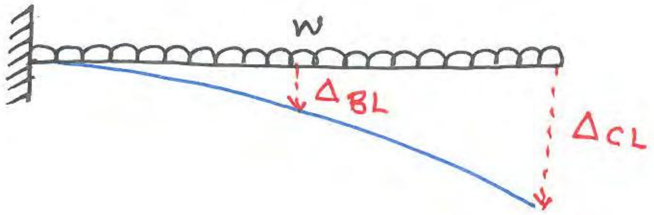


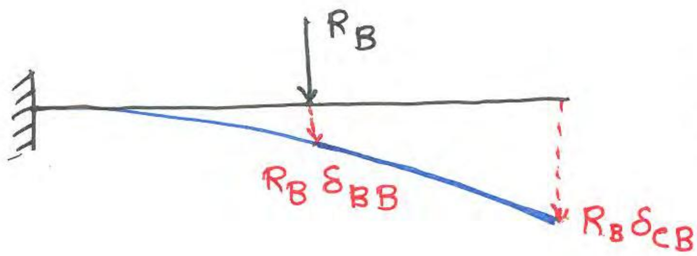
DSI = 2 so R_B & R_C are taken as redundants.

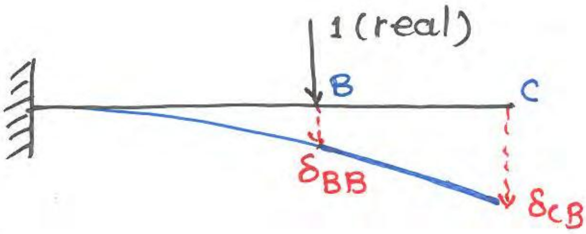
Co-ordinate system.

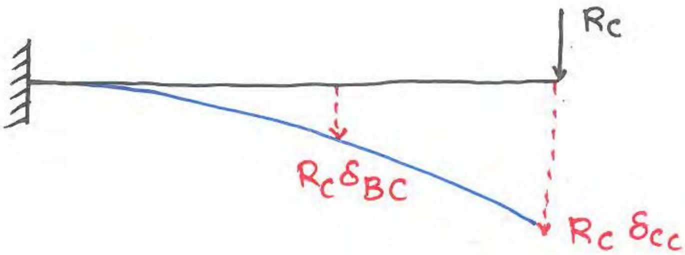


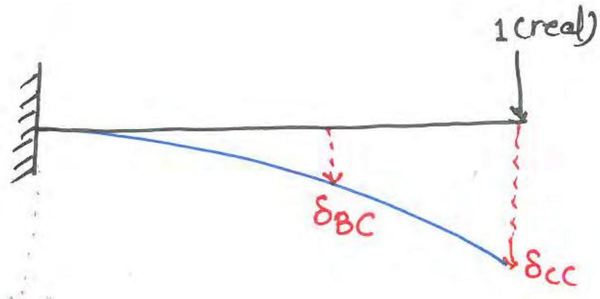
Using method of consistent deformation.











Compatibility at B:-

$$\Delta_{BL} + R_B \delta_{BB} + R_C \delta_{BC} = \Delta_B$$

Compatibility at C:-

$$\Delta_{CL} + R_B \cdot \delta_{CB} + R_C \delta_{CC} = \Delta_C$$

In matrix form:-

$$\begin{bmatrix} \Delta_{BL} \\ \Delta_{CL} \end{bmatrix} + \begin{bmatrix} \delta_{BB} & \delta_{BC} \\ \delta_{CB} & \delta_{CC} \end{bmatrix} \begin{bmatrix} R_B \\ R_C \end{bmatrix} = \begin{bmatrix} \Delta_B \\ \Delta_C \end{bmatrix}$$

putting $B \rightarrow 1$ and $C \rightarrow 2$

$$\begin{bmatrix} \Delta_{1L} \\ \Delta_{2L} \end{bmatrix} + \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}$$

$$\Rightarrow [\Delta_L] + [F][R] = [\Delta]$$

$$\Rightarrow \quad [R] = [F]^{-1} \left\{ [\Delta] - [\Delta_L] \right\}$$

- **Flexibility:-**

Displacement produced by unit force/moment is called flexibility.

δ_{ij} = Displacement at ith co-ordinate due to unit force/moment at jth co-ordinate.

Since all terms of matrix $[F]$ represents flexibility so it is called flexibility matrix.

10.3.1. Properties of Flexibility Matrix:-

1) It is always a square matrix with size equal to number of redundants.
2) Always symmetric matrix because of Maxwell's Reciprocal theorem.

$$\delta_{ij} = \delta_{ji}$$

3) Diagonal terms are always +ve value.

10.3.2 Procedure to contruct Flexibility Matrix:-

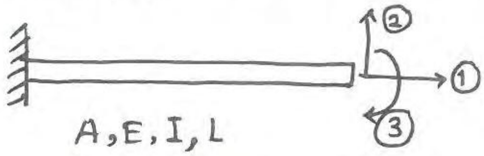
Step I:- Apply unit load at co-ordinate 1.

Step II:- Calculate displacements at all co-ordinates.

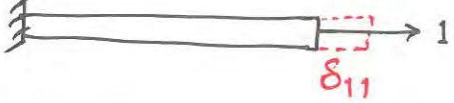
Step III:- Arrange all calculated displacements of step II in 1st column of flexibility matrix.

Step IV:- Repeat above three steps for other columns of matrix.

Ex. Construct Flexibility Matrix.



For 1^{st} column :-

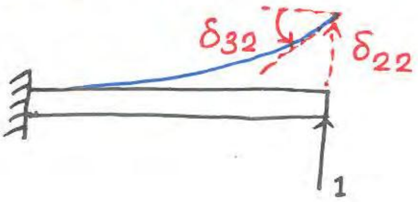


$\delta_{11} = \frac{PL}{AE} = \frac{L}{AE}$ (+ve becoz alon co-ordinate ①)

$\delta_{21} = 0$

$\delta_{31} = 0$

For 2^{nd} column :-

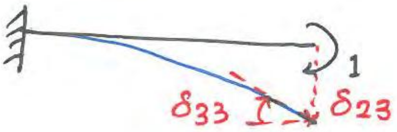


$\delta_{12} = 0$

$\delta_{22} = \frac{PL^3}{3EI} = \frac{L^3}{3EI}$ (+ve beoz along ②)

$\delta_{32} = -\frac{PL^2}{2EI} = -\frac{L^2}{2EI}$ (-ve bcoz opp ③)

For 3^{rd} column :-



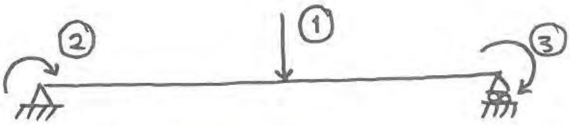
$\delta_{13} = 0$

$\delta_{23} = -\frac{ML^2}{2EI} = -\frac{L^2}{2EI}$ (-ve bcoz opp ②)

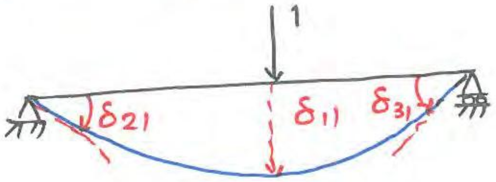
$\delta_{33} = \frac{ML}{EI} = \frac{L}{EI}$ (+ve bcoz along ③)

Flexibility Matrix :-

$$[F]_{3\times3} = \begin{bmatrix} \frac{L}{AE} & 0 & 0 \\ 0 & \frac{L^3}{3EI} & -\frac{L^2}{2EI} \\ 0 & -\frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix}$$



For 1st column:-

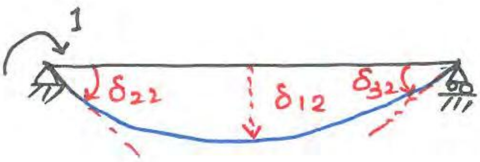


$$\delta_{11} = \frac{PL^3}{48EI} = \frac{L^3}{48EI} \quad \left(\begin{array}{c}+ve\ bcoz \\ along\ (1)\end{array}\right)$$

$$\delta_{21} = \frac{PL^2}{16EI} = \frac{L^2}{16EI} \quad \left(\begin{array}{c}+ve\ bcoz \\ along\ (2)\end{array}\right)$$

$$\delta_{31} = -\frac{PL^2}{16EI} = \frac{-L^2}{16EI} \quad \left(\begin{array}{c}-ve\ bcoz \\ opp\ (3)\end{array}\right)$$

For 2nd column:-



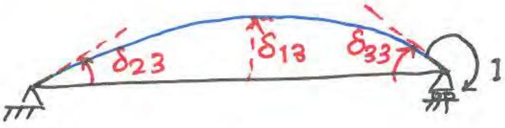
δ_{12} = No standard formula
so using Maxwell reciprocal theorem

$$\delta_{12} = \delta_{21} = \frac{L^2}{16EJ} \quad \left(+ve\ bcoz\ along\ (1)\right)$$

$$\delta_{22} = \frac{ML}{3EI} = \frac{L}{EI} \quad \left(\begin{array}{c}+ve\ bcoz \\ along\ (2)\end{array}\right)$$

$$\delta_{32} = -\frac{ML}{6EI} = -\frac{L}{6EI} \quad \left(\begin{array}{c}-ve\ bcoz \\ opp\ (3)\end{array}\right)$$

For 3rd column:-



δ_{13} = No standard formula
So using Maxwell Reciprocal theorem.

$$\delta_{13} = \delta_{31} = -\frac{L^2}{16EI} \quad (-ve\ bcoz\ opp\ (1))$$

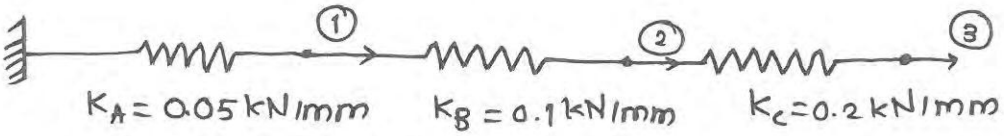
$$\delta_{23} = -\frac{ML}{6EI} = -\frac{L}{6EI} \quad (-ve\ bcoz\ opp\ (2))$$

$$\delta_{33} = \frac{ML}{3EI} = \frac{L}{3EI} \quad (+ve\ bcoz\ along\ (3))$$

Flexibility Matrix :-

$$[F]_{3\times3} = \begin{bmatrix} \frac{L^3}{48EI} & \frac{L^2}{16EI} & -\frac{L^2}{16EI} \\ \frac{L^2}{16EI} & \frac{L}{EI} & -\frac{L}{6EI} \\ -\frac{L^2}{16EI} & -\frac{L}{6EI} & \frac{L}{3EI} \end{bmatrix}$$

Ex.



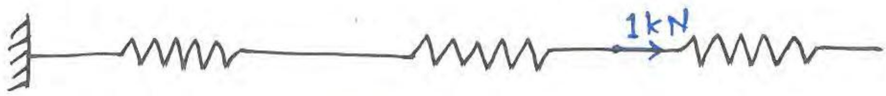
For 1^{st} column :-

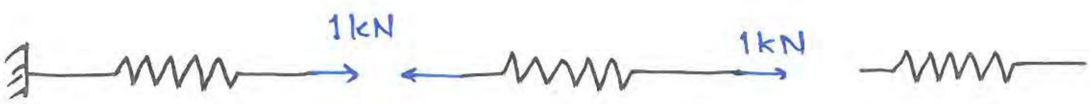
δ_{11} = Elongation of spring A

$= \frac{1}{K_A} = \frac{1}{0.05} = 20mm$

δ_{21} = same as movement of 1 = 20 mm

δ_{31} = same as movement of 2 = 20 mm





δ_{12} = Elongation of spring A

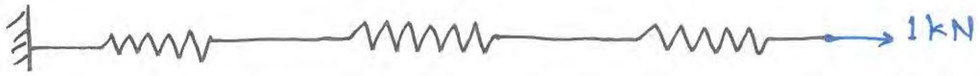
$= \frac{1}{K_A} = \frac{1}{0.05} = 20\,mm$

δ_{22} = Elongation of spring A+B

$= \frac{1}{K_A} + \frac{1}{K_B} = \frac{1}{0.05} + \frac{1}{0.1} = 30\,mm$

δ_{32} = same as movement of 2 = 30mm

For 3rd column :-



δ_{13} = Elongation of spring A

$$= \frac{1}{K_A} = \frac{1}{0.05} = 20 \text{ mm}$$

δ_{23} = Elongation of spring A+B

$$= \frac{1}{K_A} + \frac{1}{K_B} = \frac{1}{0.05} + \frac{1}{0.1} = 30 \text{ mm}$$

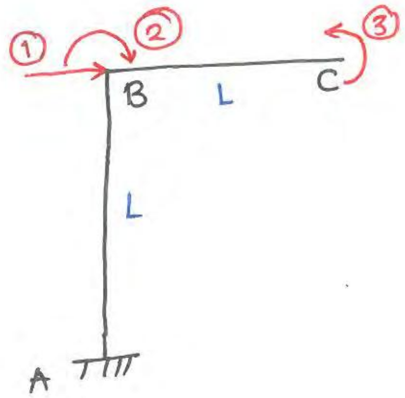
δ_{33} = Elongation of spring A+B+C

$$= \frac{1}{K_A} + \frac{1}{K_B} + \frac{1}{K_C} = \frac{1}{0.05} + \frac{1}{0.1} + \frac{1}{0.2} = 35 \text{ mm}$$

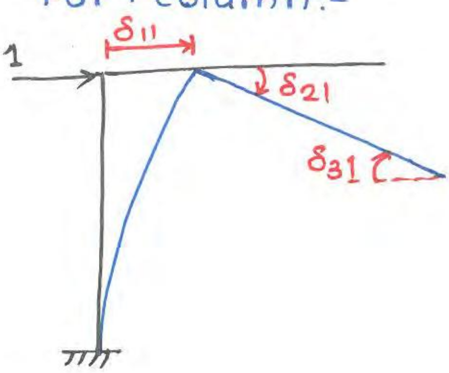
Flexibility Matrix

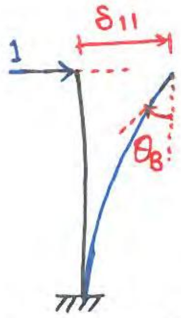
$$[F]_{3\times3} = \begin{bmatrix} 20 & 20 & 20 \\ 20 & 30 & 20 \\ 20 & 20 & 35 \end{bmatrix}$$

Ex.



For 1^{st} column:-



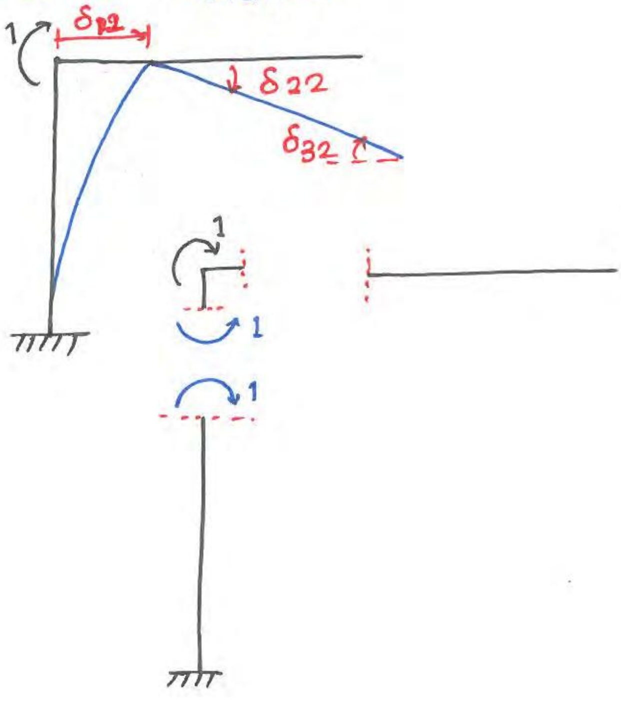


$$\delta_{11} = \frac{PL_{AB}^3}{3EI} = \frac{L^3}{3EI} \quad \left(\text{+ve bcoz along ①}\right)$$

$$\delta_{21} = \theta_B = \frac{PL_{AB}^2}{2EI} = \frac{L^2}{2EI} \quad \left(\text{+ve bcoz along ②}\right)$$

$$\delta_{31} = -\theta_B = -\frac{PL_{AB}^2}{2EI} = -\frac{L^2}{2EI} \quad \left(\text{-ve bcoz opp ③}\right)$$

For 2^{nd} column:-



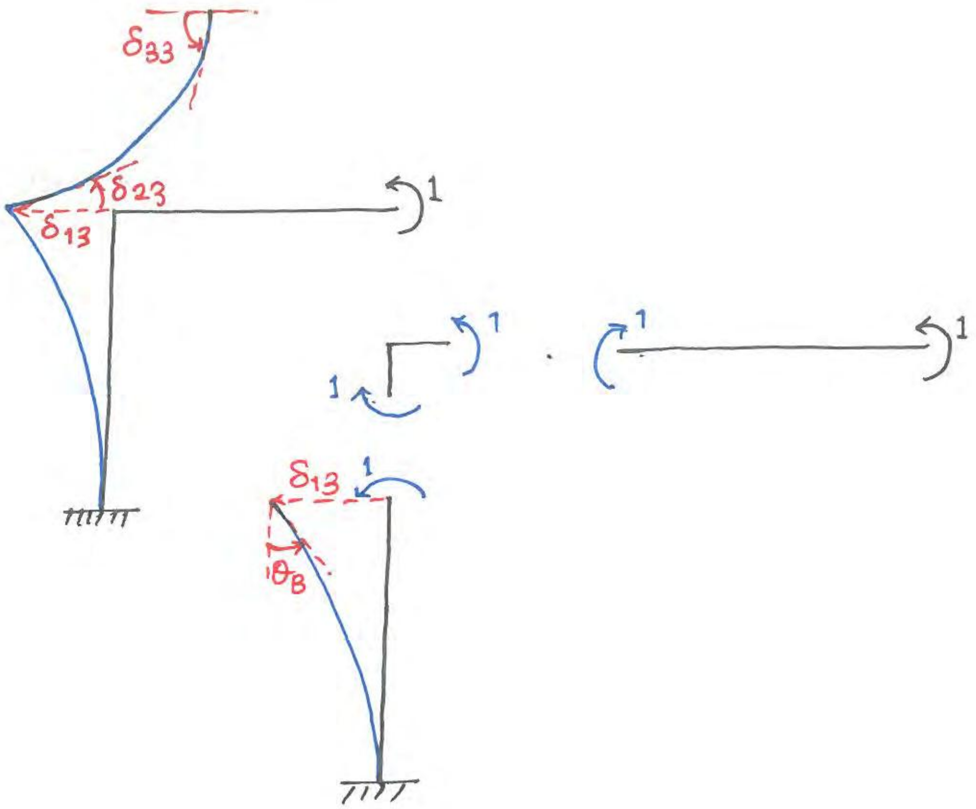
$$\delta_{12} = \frac{ML_{AB}^2}{2EI}$$

$$= \frac{L^2}{2EI} \quad \left(\text{+ve bcoz along ①}\right)$$

$$\delta_{22} = \theta_B = \frac{L_{AB}}{EI} = \frac{L}{EI} \quad \left(\text{+ve bcoz along ②}\right)$$

$$\delta_{32} = -\theta_B = -\frac{L_{AB}}{EI} = -\frac{L}{EI}$$

$$\left(\text{-ve bcoz opp ③}\right)$$



$$\delta_{13} = \frac{-ML_{AB}^2}{2EI} = -\frac{L^2}{2EI} \quad (\text{-ve bcoz opp } ①)$$

$$\delta_{23} = -\theta_B = -\frac{ML_{AB}}{EI} = -\frac{L}{EI} \quad (\text{-ve bcoz opp } ②)$$

$\delta_{33} =$ θ_B + Rotation at C due to bending of BC

$$= \frac{ML_{AB}}{EI} + \frac{ML_{BC}}{EI}$$

$$\delta_{33} = \frac{2L}{EI} \quad (\text{+ve bcoz along } ③)$$

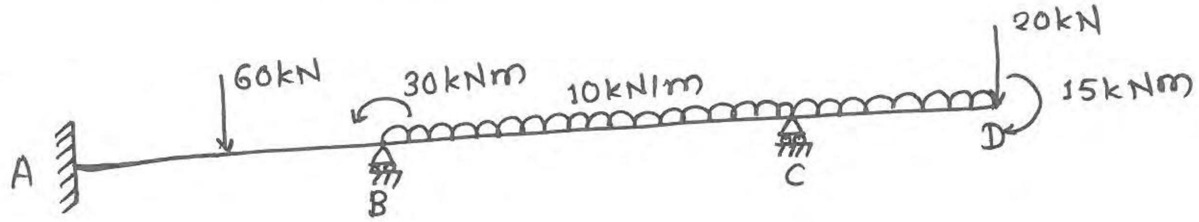
10.4 Stiffness Matrix Method of Analysis:-

Similar to flexibility matrix method, matrix equation for stiffness matrix method can be written as follows:-

$$\begin{bmatrix} \Delta \\ \theta \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} F_{Final} \\ M_{Final} \end{bmatrix} - \begin{bmatrix} F_{Fixed\ End} \\ M_{Fixed\ End} \end{bmatrix} \right\}$$

$$[\Delta] = [k]^{-1} \left\{ [F_{Final}] - [F_{Fixed\ End}] \right\}$$

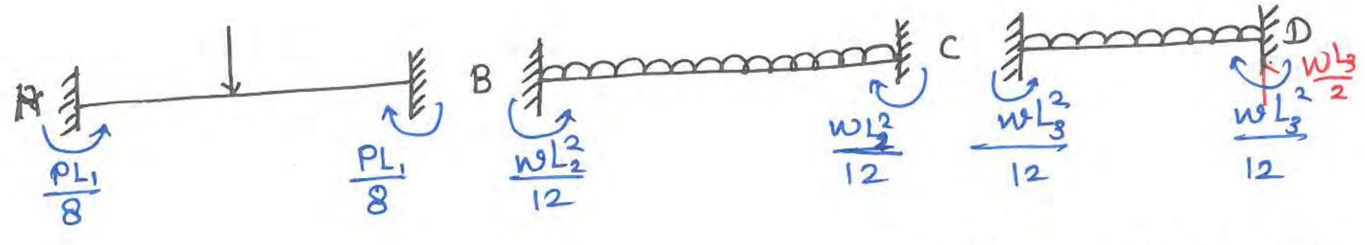
Ex.



$KI = 4 \quad (\theta_B, \theta_C, \theta_D, \Delta_{YD})$

Co-ordinate system

$$[\Delta] = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \Delta_4 \end{bmatrix} \qquad [F_{Final}] = \begin{bmatrix} -30 \\ 0 \\ 15 \\ -20 \end{bmatrix}$$



$M_{FE1} = \frac{PL_1}{8} - \frac{wL_2^2}{12}$ (+ve if along ①)

$M_{FE3} = \frac{wL_3^2}{12}$ (+ve along ③)

$M_{FE2} = -\frac{wL_2^2}{12} + \frac{wL_3^2}{12}$ (+ve if along ②)

$F_{FE4} = \frac{wL_3}{2}$ (+ve bcoz along ④)

K_{ij} = Force/Moment required/developed at ith co-ordinate due to unit displacement (deflection/rotation) at jth co-ordinate.

$i = j \longrightarrow$ required

$i \neq j \longrightarrow$ Developed.

10.4.1 Properties of Stiffness Matrix:-

Same as flexibility matrix.

10.4.2 Procedure to Construct Stiffness Matrix:-

Step I: Lock all co-ordinates.

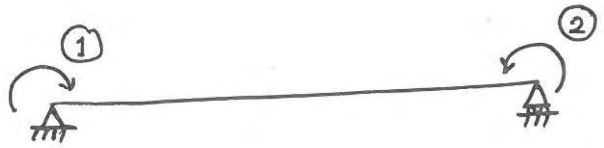
Step II: Give unit displacement in the direction of co-ordinate ① only.

Step III:- Calculate Force/Moment required at co-ordinate ① and developed at other co-ordinates.

Step IV:- Arrange the values calculated in step III in 1st column of stiffness matrix.

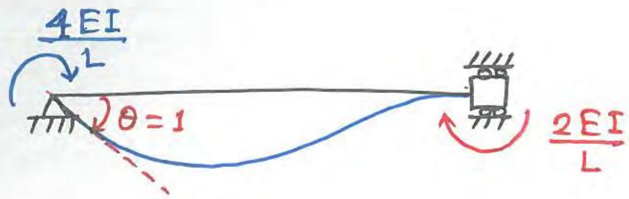
Step V:- Repeat step II to step IV for other columns of matrix

Ex. Construct a stiffness matrix



Locked Structure

For 1^{st} column:-



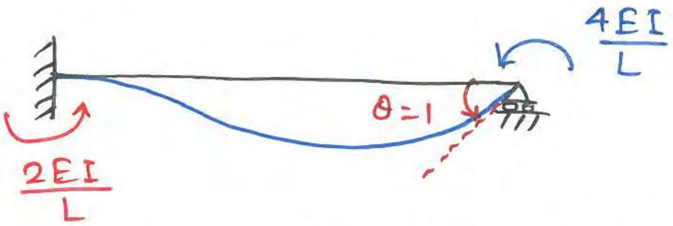
k_{11} = Moment required at co-ordinate ① due to unit rotation at co-ordinate ①

$= \frac{4EI}{L}$ (+ve bcoz along ①)

k_{21} = Moment developed at co-ordinate ② due to unit rotation at co-ordinate ①.

$= -\frac{2EI}{L}$ (-ve bcoz opp ②)

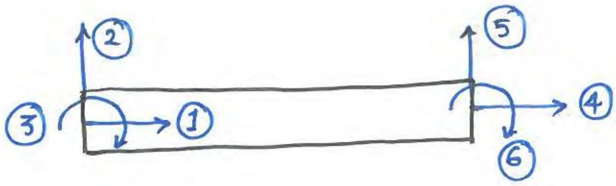
For column 2 :-



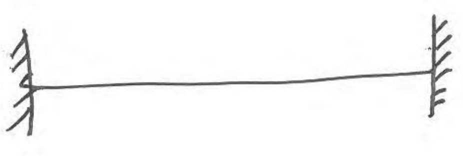
$k_{12} = -\frac{2EI}{L}$ (-ve bcoz opp ①)

$k_{22} = \frac{4EI}{L}$ (+ve bcoz along ②)

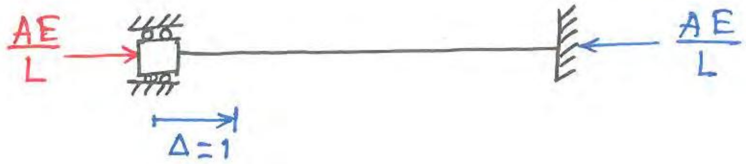
Ex. Construct a stiffness matrix of beam element.



Locked structure



For 1st column:-



$k_{11} = \frac{AE}{L}$ (+ve bcoz along ①)

$k_{21} = 0$

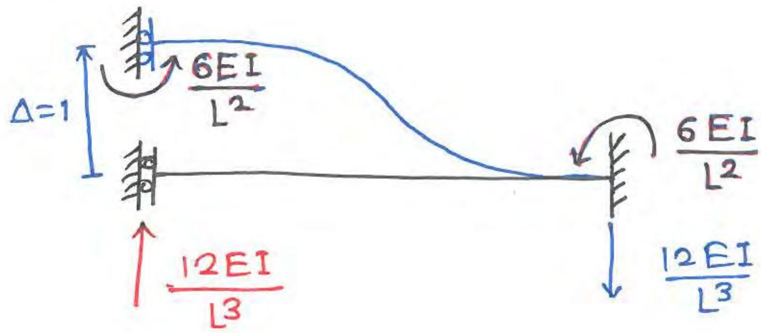
$k_{31} = 0$

$k_{41} = -\frac{AE}{L}$ (-ve bcoz opp ④)

$k_{51} = 0$

$k_{61} = 0$

For 2nd column:-



$k_{12} = 0$

$k_{22} = \frac{12EI}{L^3}$ (+ve bcoz along ②)

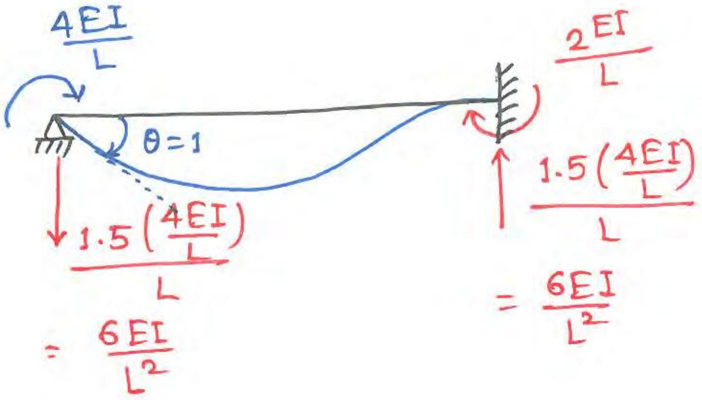
$k_{32} = -\frac{6EI}{L^2}$ (-ve bcoz opp. ③)

$k_{42} = 0$

$k_{52} = \frac{-12EI}{L^3}$ (-ve bcoz opp ⑤)

$k_{62} = -\frac{6EI}{L^2}$ (-ve bcoz opp ⑥)

For 3rd column:-



$k_{13} = 0$

$k_{23} = -\dfrac{6EI}{L^2}$ (-ve bcoz opp ②)

$k_{33} = \dfrac{4EI}{L}$ (+ve bcoz along ③)

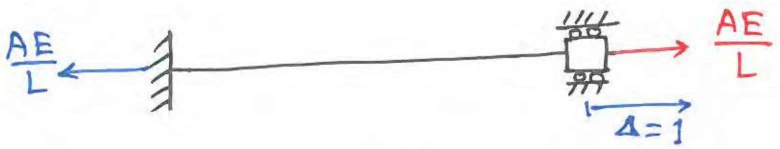
$k_{43} = 0$

$k_{53} = \dfrac{6EI}{L^2}$ (+ve bcoz alon ⑤)

$k_{63} = \dfrac{2EI}{L}$ (+ve bcoz along ⑥)

For 4th column:-

k_1



$k_{14} = -\dfrac{AE}{L}$ (-ve bcoz opp ①)

$k_{24} = 0$

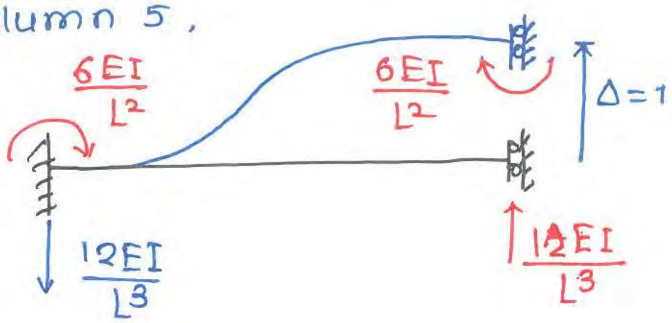
$k_{34} = 0$

$k_{44} = \dfrac{AE}{L}$ (+ve bcoz along ④)

$k_{54} = 0$

$k_{64} = 0$

For column 5,



$k_{15} = 0$

$k_{25} = -\frac{12EI}{L^3}$ (-ve bcoz opp 2)

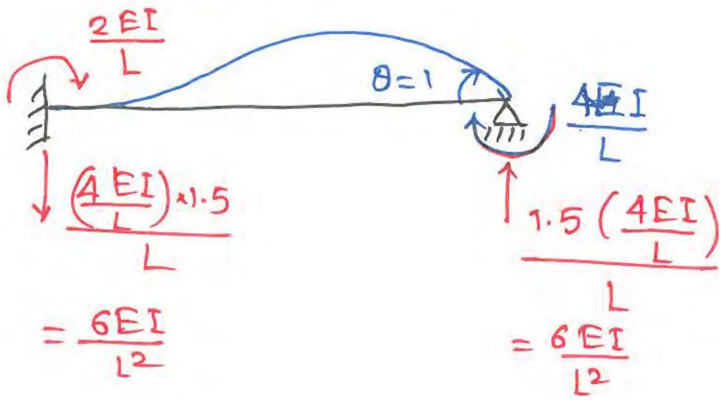
$k_{35} = \frac{6EI}{L^2}$ (+ve bcoz along 3)

$k_{45} = 0$

$k_{55} = \frac{12EI}{L^3}$ (+ve bcoz along 5)

$k_{65} = \frac{6EI}{L^2}$ (+ve bcoz along 6)

For 6th column:-



$k_{16} = 0$

$k_{26} = \frac{-6EI}{L^2}$ (-ve bcoz opp 2)

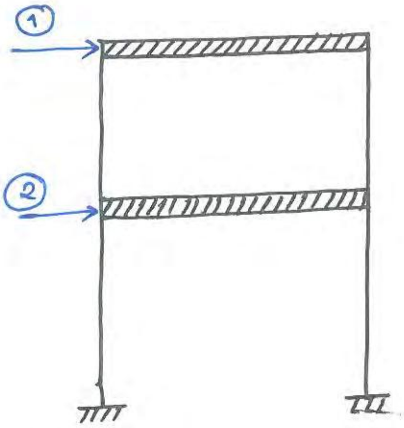
$k_{36} = \frac{2EI}{L}$ (+ve bcoz along 3)

$k_{46} = 0$

$k_{56} = \frac{6EI}{L^2}$ (+ve bcoz along 5)

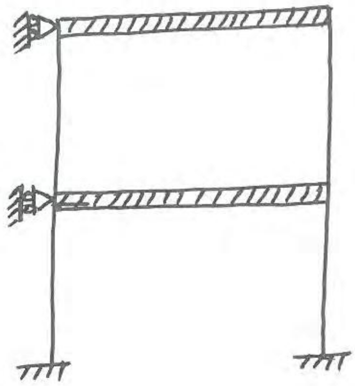
$k_{66} = \frac{4EI}{L}$ (+ve bcoz along 6)

Ex

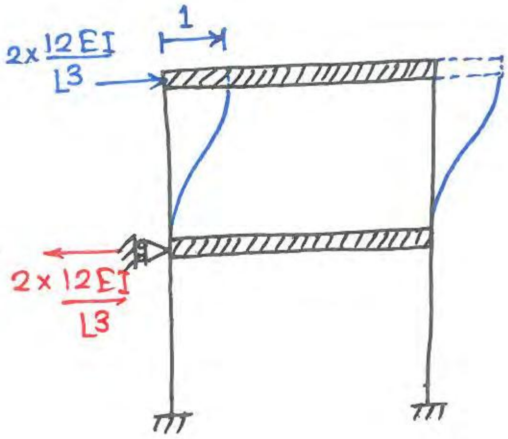


⇒

Locked Structure:-



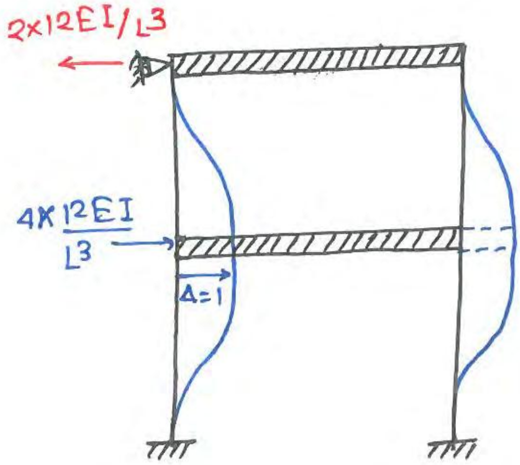
For 1st column:-



$k_{11} = \frac{24EI}{L^3}$ (+ve bcoz along ①)

$k_{21} = -\frac{24EI}{L^3}$ (-ve bcoz opp ②)

For 2nd column:-



$k_{21} = -\frac{24EI}{L^3}$ (-ve becoz opp ①)

$k_{22} = \frac{48EI}{L^3}$ (+ve bcoz along ②)

$$[k]_{2\times 2} = \begin{bmatrix} \frac{24EI}{L^3} & \frac{-24EI}{L^3} \\ -\frac{24EI}{L^3} & \frac{48EI}{L^3} \end{bmatrix}$$