Chapter 7

Exponential and Logarithm Series

Contents

Charles Hermite

The transcendence of e was proved by Charles Hermite *in* 1873 A.D. *In* 1926 A.D., D.H. Lehmer *computed the value of e to 709 decimal places by using a continued-fraction expansion.*

Newton (born 1642 A.D) *also expressed* log (1+x) *as an infinite series by expanding* $\frac{1}{(x+1)}$ *as* $(1 - x + x^2 - x^3 + ...)$. *However, it was* Nicolaus Mercator *who first published in 1668 A.D., the series* $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - .....$

The expansion of $\frac{1}{2}\log(1+x)(1-x)$ *was found by* John Wallis *in* 1695 A.D.

Exponential Series

7.1 Definition (The number *e*)

The limiting value of $\left(1+\frac{1}{n}\right)^n$ when n tends to infinity is denoted by e

i.e., e = $e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+.........\infty$ = 2.71 (Nearly)

7.2 Properties of *e*

(1) e lies between 2.7 and 2.8. *i.e.,* $2.7 < e < 2.8$ (since $\frac{1}{n!}\le\frac{1}{2^{n-1}}$ for $n\ge 2$)

(2) The value of e correct to 10 places of decimals is 2.7182818284

(3) e is an irrational (incommensurable) number

(4) e is the base of natural logarithm (Napier logarithm) *i.e.* $\ln x=\log_e x$ and $\log_{10} e$ is known as Napierian constant. $\log_{10} e=0.43429448$, $\ln x=2.303\log_{10} x$

$$\left(\text{since } \ln x=\log_{10} x.\log_e 10 \text{ and } \log_e 10=\frac{1}{\log_{10} e}=2.30258509\right)$$

7.3 Exponential Series

For $x\in R$, $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+......+\frac{x^r}{r!}+......\infty$ or $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

The above series known as exponential series and e^x is called exponential function. Exponential function is also denoted by exp. *i.e.* $\exp A=e^A$; $\therefore$ $\exp x=e^x$

7.4 Exponential Function a^x, where $a > 0$

$\because a^x=e^{\log_e a^x}=e^{x\log_e a}$

$\therefore a^x=e^{\alpha x}$ ....(i) , where $\alpha=\log_e a$. We have, $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}........+\frac{x^r}{r!}+........\infty$

Replacing x *by* αx in this series, $e^{\alpha x}=1+\frac{\alpha x}{1!}+\frac{\alpha^2x^2}{2!}+\frac{\alpha^3x^3}{3!}+.....+\frac{\alpha^rx^r}{r!}+.....\infty$

Hence from (i), $a^x = 1 + \frac{\log_e a}{1!}x + \frac{(\log_e a)^2}{2!}x^2 + ... + \frac{(\log_e a)^r x^r}{r!} + ...\infty$

7.5 Some Important Results from Exponential Series

We have the exponential series

(1) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ...\infty = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ .....(i)

(2) Replacing x by $-x$ in (i), we obtain $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + ...\infty = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$ .....(ii)

(3) Putting $x = 1$ in (i) and (ii), we get, $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\infty = \sum_{n=0}^{\infty} \frac{1}{n!}$

$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ...\infty = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

(4) From (i) and (ii), we obtain $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...\infty = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\infty = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

(5) $\frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + ...\infty = \sum_{n=0}^{\infty} \frac{1}{(2n)!}$, $\frac{e - e^{-1}}{2} = 1 + \frac{1}{3!} + \frac{1}{5!} + ...\infty = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$

Note: ❑ $e - 1 = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...\infty = \sum_{r=1}^{\infty} \frac{1}{r!}$ ❑ $e - 2 = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...\infty = \sum_{r=2}^{\infty} \frac{1}{r!}$

7.6 Some Standard results

(1) $\sum_{n=0}^{\infty} \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{1}{(n-1)!} = \sum_{n=0}^{\infty} \frac{1}{(n-k)!} = e$

(2) $\sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\infty = e - 1$

(3) $\sum_{n=2}^{\infty} \frac{1}{n!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...\infty = e - 2$

(4) $\sum_{n=0}^{\infty} \frac{1}{(n+1)!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\infty = e - 1$

(5) $\sum_{n=0}^{\infty} \frac{1}{(n+2)!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...\infty = e - 2$

(6) $\sum_{n=1}^{\infty} \frac{1}{(n+1)!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...\infty = e - 2$

(7) $\sum_{n=0}^{\infty} \frac{1}{(2n)!} = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + ...\infty = \frac{e + e^{-1}}{2} = \sum_{n=1}^{\infty} \frac{1}{(2n-2)!}$

(8) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)!} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + ...\infty = \frac{e - e^{-1}}{2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$

(9) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ...\infty$

$\therefore T_{n+1} =$ General term in the expansion of $e^x = \frac{x^n}{n!}$ and coefficient of x^n in $e^x = \frac{1}{n!}$

(10) $e^{-x}=1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+........+(-1)^n\frac{x^n}{n!}+........\infty$

$\therefore T_{n+1}=$ General term in the expansion of $e^{-x}=(-1)^n\frac{x^n}{n!}$ and coefficient of x^n in $e^{-x}=\frac{(-1)^n}{n!}$

(11) $e^{ax}=1+\frac{(ax)}{1!}+\frac{(ax)^2}{2!}+\frac{(ax)^3}{3!}+.........+\frac{(ax)^n}{n!}+.........\infty$

$\therefore T_{n+1}=$General term in the expansion of $e^{ax}=\frac{(ax)^n}{n!}$ and coefficient of x^n in $e^{ax}=\frac{a^n}{n!}$

(12) $\sum_{n=0}^{\infty}\frac{n}{n!}=e=\sum_{n=1}^{\infty}\frac{n}{n!}$

(13) $\sum_{n=0}^{\infty}\frac{n^2}{n!}=2e=\sum_{n=1}^{\infty}\frac{n^2}{n!}$

(14) $\sum_{n=0}^{\infty}\frac{n^3}{n!}=5e=\sum_{n=1}^{\infty}\frac{n^3}{n!}$

(15) $\sum_{n=0}^{\infty}\frac{n^4}{n!}=15e=\sum_{n=1}^{\infty}\frac{n^4}{n!}$

Example: 1 $\frac{2}{1!}+\frac{4}{3!}+\frac{6}{5!}+\frac{8}{7!}+.........\infty=$ **[JMI CET 2000]**

(a) $1/e$ (b) e (c) $2e$ (d) $3e$

Solution: (b) $\frac{2}{1!}+\frac{4}{3!}+\frac{6}{5!}+\frac{8}{7!}+.........\infty=\frac{(1+1)}{1!}+\frac{(1+3)}{3!}+\frac{(1+5)}{5!}+\frac{(1+7)}{7!}+........\infty$

$=\left(\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+........\infty\right)+\left(1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+........\infty\right)=\frac{e-e^{-1}}{2}+\frac{e+e^{-1}}{2}=e$

Example: 2 $\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+.........\infty=$ **[MNR 1979; MP PET 1995, 2002]**

(a) e (b) $2e$ (c) e^2 (d) $1/e$

Solution: (d) Here $T_n=\frac{(2n+1)-1}{(2n+1)!}=\frac{1}{(2n)!}-\frac{1}{(2n+1)!}\Rightarrow S=\sum_{n=1}^{\infty}T_n=\left(\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+.......\infty\right)-\left(\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+.......\infty\right)$

$\Rightarrow S=\left(\frac{e+e^{-1}}{2}-1\right)-\left(\frac{e-e^{-1}}{2}-1\right)\Rightarrow e^{-1}=\frac{1}{e}$

Example: 3 $1+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+.........\infty=$ **[MNR 1976; MP PET 1997]**

(a) $2e$ (b) $3e$ (c) $4e$ (d) $5e$

Solution: (d) $S=\frac{1^3}{1!}+\frac{2^3}{2!}+\frac{3^3}{3!}+........+\frac{n^3}{n!}+........$

Here $T_n=\frac{n^3}{n!}\Rightarrow S_n=\sum_{n=1}^{\infty}\frac{n^3}{n!}=5e$

Example: 4 The coefficient of x^n in the expansion of $\frac{e^{7x}+e^x}{e^{3x}}$ is

(a) $\frac{4^{n-1}+(-2)^n}{n!}$ (b) $\frac{4^{n-1}+2^n}{n!}$ (c) $\frac{4^{n-1}+(-2)^{n-1}}{n!}$ (d) $\frac{4^n+(-2)^n}{n!}$

Solution: (d) We have $\frac{e^{7x}+e^x}{e^{3x}}=e^{4x}+e^{-2x}=\sum_{n=0}^{\infty}\frac{(4x)^n}{n!}+\sum_{n=0}^{\infty}\frac{(-2x)^n}{n!}$ $\therefore$ coefficient of x^n in $\frac{e^{7x}+e^x}{e^{3x}}=\frac{4^n+(-2)^n}{n!}$

Example: 5 $1+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+........\infty=$ **[Roorkee 1999; MP PET 2003]**

(a) e (b) $3e$ (c) $e/2$ (d) $3e/2$

Solution: (d) $T_n = \frac{\sum n}{n!} = \frac{n(n+1)}{2.n!} = \frac{1}{2}\left[\frac{(n+1)}{(n-1)!}\right] = \frac{1}{2}\left[\frac{n-1}{(n-1)!} + \frac{2}{(n-1)!}\right] = \frac{1}{2}\left[\frac{1}{(n-2)!} + \frac{2}{(n-1)!}\right]$

$$Sn = \sum_{n=1}^{\infty} T_n = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(n-2)!} + \sum_{n=1}^{\infty}\frac{1}{(n-1)!} = \frac{e}{2} + e = \frac{3e}{2}$$

Logarithmic Series

7.7 Logarithmic Series

An expansion for $\log_e(1+x)$ as a series of powers of x which is valid only when, $|x| < 1$,

Expansion of $\log_e(1+x)$; if $|x| < 1$, then $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$

7.8 Some Important Results from the Logarithmic Series

(1) Replacing x by $-x$ in the logarithmic series, we get

$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ....\infty$ or $-\log_e(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ...\infty$

(2) (i) $\log_e(1+x) + \log_e(1-x) = \log_e(1-x^2) = -2\left\{\frac{x^2}{2} + \frac{x^4}{4} + ...\infty\right\}, (-1 < x < 1)$

(ii) $\log_e(1+x) - \log_e(1-x) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + ...\infty\right]$ or $\log_e\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + ...\infty\right]$

(3) The series expansion of $\log_e(1+x)$ may fail to be valid if $|x|$ is not less than 1. It can be proved that the logarithmic series is valid for x=1. Putting x=1 in the logarithmic series.

We get, $\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ...\infty = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + ...\infty$

(4) When $x = -1$, the logarithmic series does not have a sum. This is in conformity with the fact that log(1-1) is not a finite quantity.

7.9 Difference between the Exponential and Logarithmic Series

(1) In the exponential series $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\infty$ all the terms carry positive signs whereas in the logarithmic series $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$ the terms are alternatively positive and negative.

(2) In the exponential series the denominator of the terms involve factorial of natural numbers. But in the logarithmic series the terms do not contain factorials.

(3) The exponential series is valid for all the values of x. The logarithmic series is valid when $|x| < 1$.

Example: 6 $0.5 - \frac{(0.5)^2}{2} + \frac{(0.5)^3}{3} - \frac{(0.5)^4}{4} + ...$ **[MP PET 1995]**

(a) $\log_e\left(\frac{3}{2}\right)$ (b) $\log_{10}\left(\frac{1}{2}\right)$ (c) $\log_e(n!)$ (d) $\log_e\left(\frac{1}{2}\right)$

Solution: (a) We know that, $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} .... \infty = \log_e(1+x)$

Putting x = 0.5, we get, $0.5 - \frac{(0.5)^2}{2} + \frac{(0.5)^3}{3} - \frac{(0.5)^4}{4} + ... \infty = \log_e(1+0.5) = \log_e\left(\frac{3}{2}\right)$

Example: 7 $\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + ... \infty =$ **[Roorkee 1992; MP PET 1999; AIEEE 2003]**

(a) $\log_e\left(\frac{4}{e}\right)$ (b) $\log_e \frac{e}{4}$ (c) $\log_e 4$ (d) $\log_e 2$

Solution: (a) We know that, $\log_e 2 = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + ... \infty$ ....(i)

Also $\log_e 2 = 1 - \left(\frac{1}{2.3}\right) - \left(\frac{1}{4.5}\right) - \left(\frac{1}{6.7}\right) - .... \infty$ ....(ii)

By adding (i) and (ii), we get, $2\log_e 2 = 1 + \left(\frac{1}{1.2} - \frac{1}{2.3}\right) + \left(\frac{1}{3.4} - \frac{1}{4.5}\right) + ...$

$\Rightarrow 2\log_e 2 - 1 = \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + ... \Rightarrow \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + ... \infty = \log_e 4 - \log_e e = \log_e\left(\frac{4}{e}\right)$

Example: 8 The coefficient of x^n in the expansion of $\log_e(1+3x+2x^2)$ is

(a) $(-1)^n\left[\frac{2^n+1}{n}\right]$ (b) $\frac{(-1)^{n+1}}{n}[2^n+1]$ (c) $\frac{2^n+1}{n}$ (d) None of these

Solution: (b) We have, $\log_e(1+3x+2x^2) = \log_e(1+x) + \log_e(1+2x)$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2x)^n}{n} = \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n} + \frac{2^n}{n}\right)x^n = \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1+2^n}{n}\right)x^n$$

So coefficient of $x^n = (-1)^{n-1}\left(\frac{2^n+1}{n}\right) = \frac{(-1)^{n+1}(2^n+1)}{n}$ $\left[\because (-1)^n = (-1)^{n+2} = ...\right]$

Example: 9 The equation $x^{\log_x(2+x)^2} = 25$ holds for **[MP PET 1992]**

(a) $x = 6$ (b) $x = -3$ (c) $x = 3$ (d) $x = 7$

Solution: (c) Given equation $x^{\log_x(2+x)^2} = 25 \Rightarrow (2+x)^2 = 25$ hold for $x = 3$

Example: 10 If $y = -\left(x^3 + \frac{x^6}{2} + \frac{x^9}{3} + ...\right)$, then $x =$ **[MNR 1975]**

(a) $\frac{1+e^y}{3}$ (b) $\frac{1-e^y}{3}$ (c) $(1-e^y)^{\frac{1}{3}}$ (d) $(1-e^y)^3$

Solution: (c) $y = -\left[x^3 + \frac{(x^3)^2}{2} + \frac{(x^3)^3}{3} + ...\right] = \log_e(1-x^3) \Rightarrow e^y = 1 - x^3 \Rightarrow x = (1-e^y)^{\frac{1}{3}}$