06

Continuity and Differentiability

6.01 Introduction

Graphically, a function is continuous in the given interval, if its graph can be drawn at this point without raising the pencil (or pen), otherwise it is discontinuous in that interval. But, only graphical understanding of the concept of continuity is not sufficient. So we must have an analytical approach to analyse the continuity of a function. We shall understand this approach with the help of limits.

6.02 Cauchy's definition of continuity

Let $f(x)$ be a function, then it is continuous at a point a in its domain, if for a small positive number $\in$, there exists a positive number δ such that

$$|f(x)-f(a)|<\in \text{ when } |x-a|<\delta$$

In other words, function $f(x)$ is called a continuous function at a point a in its domain if for every $\in>0$, for every point in interval $(a-\delta, a+\delta)$ the numerical difference of $f(x)$ and $f(a)$ may be lesser than $\in$.

6.03 Alternate definition of continuity

Let $f(x)$ is a real function on a subset of the real numbers and let a be a point in the domain of f, then f is continuous if and only if $\lim_{x\to a} f(x)$ exists and is equal to $f(a)$, i.e.

$$\lim_{x\to a} f(x) = f(a)$$

$$\Leftrightarrow \qquad \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$$

or $$f(a+0) = f(a-0) = f(a)$$

i.e., Right hand limit of $f(x)$ at a = Left hand limit of $f(x)$ at a = Value of function at a

6.04 Continuity at a point from left and right

Any function $f(x)$ at a point a of its domain.

(i) is continuous from left, if

$$\lim_{x\to a^-} f(x) = f(a)$$

or $$f(a-0) = f(a)$$

(ii) is continuous from right, if

$$\lim_{x\to a^+} f(x) = f(a)$$

or $$f(a+0) = f(a)$$

6.05 Continuous function in an open interval

A function $f(x)$ is called continuous in open interval (a, b) if it is continuous at every point in the interval.

6.06 Continuous function in a closed interval

Function $f(x)$ is called continuous in closed interval $[a, b]$ if it is

(i) Continuous from right at point a

(ii) Continuous from left at point b

(iii) Continuous in open interval (a, b)

6.07 Continuous function

If a function is continuous at every point of its domain then it is called a continuous function. Some examples of continuous function are

(i)	Identity function	$f(x) = x,$
(ii)	Constant function	$f(x) = c,$ where c is a constant
(iii)	Polynomial function	$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n,$
(iv)	Trigonometric function	$f(x) = \sin x, \cos x$
(v)	Exponential function	$f(x) = a^x, a > 0$
(vi)	Logarithmic function	$f(x) = \log_e x$
(vii)	Absolute valued function	$f(x) = \lvert x\rvert, x + \lvert x\rvert, x - \lvert x\rvert$

6.08 Discontinuous function

A function is discontinuous in its domain D if it is not continuous at atleast one point in the domain. Some examples of discontinuous function are

(i) $f(x) = [x]$, greatest integer function

(ii) $f(x) = x - [x]$, discontinuous at every integer.

(iii) $f(x) = \tan x, \sec x$, discontinuous at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

(iv) $f(x) = \cot x, \operatorname{cosec} x$, discontinuous at $x = 0, \pm\pi, \pm 2\pi, \ldots$

(v) $f(x) = \sin\frac{1}{x}, \cos\frac{1}{x}$ discontinuous at $x = 0$

(vi) $f(x) = e^{1/x}$ discontinuous at $x = 0$

(vii) $f(x) = \frac{1}{x}$, discontinuous at $x = 0$

6.09 Properties of continuous function

(i) If $f(x)$ and $g(x)$ are two continuous functions in domain D then $f(x) \pm g(x), f(x) \cdot g(x), cf(x)$ will be continuous in D However $\frac{f(x)}{g(x)}$ will be continuous for all points in D where $g(x) \neq 0, \quad \forall \; x \in D$.

(ii) If $f(x)$ and $g(x)$ are two continuous functions in their respective domains then their composite function $(g \circ f)(x)$ will be continuous.

Illustrative Examples

Example 1. Examine the continuity of function $f(x)=\begin{cases}\dfrac{x-|x|}{x}, & x\neq 0\\ 1 & , x=0\end{cases}$ at $x = 0$.

Solution : We know that $|x|=\begin{cases}-x, & \text{if } x<0\\ x, & \text{if } x\geq 0\end{cases}$

then the given function may be defined as $f(x)=\begin{cases}0\ , & x>0\\ 2\ , & x<0\\ 1\ , & x=0\end{cases}$

continuity at $x = 0$

From definition of function

$$f(0) = 1$$

$$\therefore \quad f(0-0)=\lim_{h\to 0} f(0-h)=2$$

$$f(0+0)=\lim_{h\to 0} f(0+h)=0$$

$$\therefore \quad f(0)\neq f(0-0)\neq f(0+0)$$

hence $f(x)$ is not continuous at $x = 0$

Example 2. Examine the continuity of $f(x)=|x|+|x-1|$ at $x = 0$ and $x = 1$

Solution : $f(x)$ may be written as $f(x)=\begin{cases}1-2x, & \text{if } x\leq 0\\ 1, & \text{if } 0<x<1\\ 2x-1, & \text{if } x\geq 1\end{cases}$

Continuity at $x = 0$

Here $f(0)=1-2(0)=1$

$$f(0-0)=\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(1-2x)$$

$$=\lim_{h\to 0}\{1-2(0-h)\}=1$$

$$f(0+0)=\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}1=1$$

so $f(0-0)=f(0+0)=f(0)$

hence function $f(x)$ is continuous at $x = 0$

Continuity at $x = 1$

From deinition of function

$$f(1)=2(1)-1=1$$

$$f(1-0)=\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} 1=1$$

$$f(1+0)=\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(2x-1)$$

$$=\lim_{x\to 1^+}[2(1+h)-1]=1$$

so $$f(1-0)=f(1+0)=f(1)$$

$f(x)$ is continuous at $x = 1$.

Example 3. Show that the following function $f(x)$ is not continuous at $x = 0$.

$$f(x)=\begin{cases}\dfrac{e^{1/x}}{1+e^{1/x}} & ;\ x\neq 0\\ 0 & ;\ x=0\end{cases}$$

Solution : From definition of function $f(0)=0$

Right hand limit at $x=0$ $$f(0+0)=\lim_{h\to 0} f(0+h)$$

$$=\lim_{h\to 0}\frac{e^{1/(0+h)}}{1+e^{1/(0+h)}}$$

$$=\lim_{h\to 0}\frac{1}{e^{-1/h}+1}=\frac{1}{0+1}=1$$

Left hand limit at $x=0$ $$f(0-0)=\lim_{h\to 0} f(0-h)$$

$$=\lim_{h\to 0}\frac{e^{1/(0-h)}}{1+e^{1/(0-h)}}$$

$$=\lim_{h\to 0}\frac{e^{-1/h}}{1+e^{-1/h}}=\frac{0}{1+0}=0$$

so $$f(0-0)\neq f(0+0)$$

hence $f(x)$ is not continuous at $x = 0$

Example 4. Examine the continuity of function $f(x)$ at $x = 2$.

$$f(x)=\begin{cases}x^2 & ;\quad x<1\\ x & ;\quad 1\leq x<2\\ \dfrac{x^3}{4} & ;\quad x\geq 2\end{cases}$$

Solution : From definition of function $f(2)=\dfrac{2^3}{4}=2$

Right hand limit at $x=2$ $$f(2+0)=\lim_{h\to 0} f(2+h)$$

$$= \lim_{h\to 0} \frac{(2+h)^3}{4}$$

$$= \frac{(2+0)^3}{4} = 2$$

Left hand limit at $x = 2$ $\qquad f(2-0) = \lim_{h\to 0} f(2-h)$

$$= \lim_{h\to 0}(2-h) = 2$$

so $f(2-0) = f(2+0) = f(2) = 2$

Hence $f(x)$ is continuous at $x = 2$.

Example 5. If the following function is continuous at $x = 0$, find the vlaue of c.

$$f(x) = \begin{cases} \dfrac{1-\cos(cx)}{x\sin x} & ; \quad x \neq 0 \\ \dfrac{1}{2} & ; \quad x = 0 \end{cases}$$

Solution : From definition of function $f(0) = \frac{1}{2}$ (1)

at $x = 0$ finding limit of $f(x)$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1-\cos(cx)}{x\sin x}$$

$$= \lim_{x\to 0} \frac{2\sin^2(cx/2)}{x\sin x}$$

$$= \lim_{x\to 0} \frac{(c^2/2)\left(\dfrac{\sin(cx/2)}{cx/2}\right)^2}{(\sin x/x)}$$

$$= \frac{(c^2/2)\cdot 1^2}{1} = \frac{c^2}{2} \qquad (2)$$

$\because$ $\quad f(x)$ is continuous at $x = 0$, so

$$\lim_{x\to 0} f(x) = f(0)$$

From (1) and (2)

$\Rightarrow \quad \dfrac{c^2}{2} = \dfrac{1}{2} \quad \Rightarrow \quad c^2 = 1 \quad \Rightarrow \quad c = \pm 1$

Example 6. Find the values of a and b if the given function is continuous in [4, 6]

$$f(x)=\begin{cases} 3 & ; \quad x\le 4 \\ ax+b & ; \quad 4<x<6 \\ 7 & ; \quad x\ge 6 \end{cases}$$

Solution : Given that function is continuous is [4, 6]

Right hand limit of $f(x)$ at $x = 4$

$$f(4+0)=\lim_{h\to 0} f(4+h)$$

$$=\lim_{h\to 0}\{a(4+h)+b\}$$

$$=4a+b \qquad (1)$$

and $$f(4)=3 \qquad (2)$$

Left hand limit of $f(x)$ at $x = 6$

$$f(6-0)=\lim_{h\to 0} f(6-h)$$

$$=\lim_{h\to 0}\{a(6-h)+b\}$$

$$=6a+b \qquad (3)$$

and $$f(6)=7 \qquad (4)$$

$\because$ function $f(x)$ is continuous at left extreme point at $x = 4$ of [4, 6], so $f(4+0)=f(4)$

$\Rightarrow$ $$4a+b=3 \qquad (5)$$

Similarly $f(x)$, is continuous at right extreme point at $x = 6$ of [4, 6], so $f(6-0)=f(6)$

$\Rightarrow$ $$6a+b=7 \qquad (6)$$

solving equations (5) and (6)

$$a=2,\ b=-5$$

which are required values of a and b.

Example 7. Find the condition for m, for which the funcion $f(x)$ is continous at $x = 0$.

$$f(x)=\begin{cases} x^m\sin(1/x) & ; \quad x\ne 0 \\ 0 & ; \quad x=0 \end{cases}$$

Solution : From definition of function $\qquad f(0)=0$

$$f(0-0)=\lim_{h\to 0} f(0-h)$$

$$=\lim_{h\to 0}(0-h)^m\sin\left(1/(0-h)\right)$$

$$=(-1)^{m+1}\lim_{h\to 0} h^m\sin\left(1/h\right)$$

$$=(-1)^{m+1}(0)^m\times(\text{a finite number between } -1 \text{ and } 1)$$

$$=0,\ \text{if } m>0$$

Similarly $f(0+0)=0,\ \text{if}\ \ m>0$

So $f(0-0)=f(0+0)=f(0)=0,\ \text{if}\ \ m>0$

have $f(x)$ is continuous at $x = 0$, only when $m > 0$

Example 8. Examine the continuity of function $f(x)$ at $x = 0$.

$$f(x)=\begin{cases}(\sin x)/x+\cos x & ; \quad x\neq 0\\ 2 & ; \quad x=0\end{cases}$$

Solution : From definition of function

$$f(0)=2$$

$$f(0-0)=\lim_{h\to 0} f(0-h)$$

$$=\lim_{h\to 0}\left\{\frac{\sin(-h)}{(-h)}+\cos(-h)\right\}$$

$$=\lim_{h\to 0}\left\{\frac{\sin h}{h}+\cos h\right\}=1+1=2$$

and $$f(0+0)=\lim_{h\to 0} f(0+h)$$

$$=\lim_{h\to 0}\left\{\frac{\sin h}{h}+\cos h\right\}=\{1+1\}=2$$

so $$f(0-0)=f(0+0)=f(0)=2$$

Hence $f(x)$ is continous at $x = 0$.

Exercise 6.1

1. Examine the continuity of following functions

(a) $f(x)=\begin{cases}x\{1+(1/3)\sin(\log x^2)\} & ; \quad x\neq 0\\ 0 & ; \quad x=0\end{cases}$

at $x = 0$.

(b) $f(x)=\begin{cases}\dfrac{e^{1/x}}{x} & ; \quad x\neq 0\\ 0 & ; \quad x=0\end{cases}$

at $x = 0$

(c) $f(x)=\begin{cases}1+x & ; \quad x\leq 3\\ 7-x & ; \quad x>3\end{cases}$

at $x = 3$

(d) $f(x)=\begin{cases}\sin x & ; \quad \dfrac{-\pi}{2}<x\leq 0\\ \tan x & ; \quad 0<x<\dfrac{\pi}{2}\end{cases}$

at $x = 0$

(e) $f(x) = \begin{cases} \cos(1/x) & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$

at $x = 0$

(f) $f(x) = \begin{cases} \dfrac{1}{(x-a)}.\operatorname{cosec}(x-a) & ; \quad x \neq a \\ 0 & ; \quad x = a \end{cases}$

at $x = a$

(g) $f(x) = \begin{cases} \dfrac{x^2}{a} - a, x < a & ; \quad x < a \\ 0 & ; \quad x = a \\ a - \dfrac{a^3}{x^2} & ; \quad x > a \end{cases}$

at $x = a$

2. Examine the continuity of $f(x) = x - [x]$ at $x = 3$.

3. Find the vlaue of k is the following function is continuous at $x = 2$

$$f(x) = \begin{cases} \dfrac{x^3 + x^2 - 16x + 20}{(x-2)^2} & ; \quad x \neq 2 \\ k & ; \quad x = 2 \end{cases}$$

4. Examine the continuity of following function in [–1, 2]

$$f(x) = \begin{cases} -x^2 & ; \quad -1 \leq x < 0 \\ 4x - 3 & ; \quad 0 < x \leq 1 \\ 5x^2 - 4x & ; \quad 1 < x \leq 2 \end{cases}$$

6.10 Differentiability

In previous class we had defined the derivative of a real value function and first principle of derivatives. Here we shall study a method of find derivative with special limit method. if equation of curved is $y = f(x)$ then this function is differentiable at $x = a$ if a tangent to the curve can be drawn through this point. If curve has a break or changes its direction then $f(x)$ is not differentiable at $x = a$. Mathematically we will study differentiability as follows:

1. A function $f:(a,b)\to R$ is differentiable at $c\in(a,b)$ if $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists. This limit of $f(x)$ at point c is called derivative of f and is expressed as $f'(c)$.
2. Function f is differentiable at c is for every $\in>0$, $\exists\ \delta>0$ so that

$$\left|\frac{f(x)-f(c)}{x-c}-f'(c)\right|<\in \text{ where } |x-c|<\delta$$

i.e.

$$\Rightarrow f'(c)-\in\ <\ \frac{f(x)-f(c)}{x-c}\ <\ f'(c)+\in$$

6.11 Left hand derivative of a function

A function $f(x)$ is said to be differentiable from left hand side at a point c in its domain if $\lim_{h\to 0}\frac{f(c-h)-f(c)}{-h}, h>0$ exists and is finite.

The value of this limit is represented by $LDf(c)$ or $Lf'(c)$ or $f'(c-0)$ and it is called the left hand derivative of f at c.

6.12 Right hand derivative of a function

A function $f(x)$ is said to be differentiable from right hand side at a point c in its domain if $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}, h>0$ exists and is finite.

The value of this limit is represented by $RDf(c)$ or $Rf'(c)$ or $f'(c+0)$ and it is called the right hand derivative of f at c.

6.13 Differentiable function

A function f is differentiable at a point c in its domain if both left hand derivative and right hand derivative are finite and equal.

i.e. $f'(c-0)=f'(c+0)$

$$\lim_{h\to 0}\frac{f(c-h)-f(c)}{-h}=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

Note: In the following cases $f(x)$ is not differentiable at a point c if

(i) $f'(c-0)\neq f'(c+0)$

(ii) $f'(c-0)$ and $f'(c+0)$ either or both infinite

(iii) $f'(c-0)$ and $f'(c+0)$ either or both do not exists.

6.14 Differentiability in an interval

1. Function f is differentiable in open interval (a, b) if $f(x)$ is differentiable at every point of interval.
2. Function f is differentiable is closed interval $[a, b]$ if
   (i) $f'(c)$ exists when $c\in(a,b)$
   (ii) Right hand derivative of $f(x)$ exists at poitn a
   (iii) Left hand derivative of $f(x)$ exists at point b

6.15 Some important results

(i) If a function f is diferentiable at a point c, then it is also continuous at that point but the converse of above statement needs not to be true. It is clear that if a function is not cotinuous then surely it will not be differentiable.

Note:

(i) While examining differentiability of any function, Firstly its continuity should be examined.

(ii) Every polynomial, exponential and constant functions are alway differentiable in R

(iii) Logarithmic and trigonometric functions are differentiable in their domains.

(iv) Composite functions, sum, difference, product and quotient (when denominator is not zero) of two differentiable functions are always differntiable.

Illustrative Examples

Example 9. If the following function is continuous at $x = 0$ then examine its differentiablity at $x = 0$

$$f(x) = \begin{cases} x^2\left(\dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right) & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$$

Solution : Left hand derivative of $f(x)$ at $x = 0$

$$f'(0-0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{(-h)^2\left(\dfrac{e^{-1/h} - e^{-(-1/h)}}{e^{-1/h} + e^{-(-1/h)}}\right) - 0}{-h}$$

$$= \lim_{h\to 0} -h\left(\frac{e^{-2/h} - 1}{e^{-2/h} + 1}\right)$$

$$= 0 \times \left(\frac{0-1}{0+1}\right) = 0$$

and Right hand derivative of $f(x)$ at $x = 0$

$$f'(0+0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{(h)^2\left(\dfrac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}}\right) - 0}{h}$$

$$= \lim_{h\to 0} h\left(\frac{1 - e^{-2/h}}{1 + e^{-2/h}}\right)$$

$$= 0 \times \left(\frac{1-0}{1+0}\right) = 0$$

so $f'(0-0) = f'(0+0)$

hence function $f(x)$ is differentiable at $x = 0$

Example 10. If the following function is continuous everywhere then examine its differentiability at $x = 0$

$$f(x) = \begin{cases} x\left(1+\frac{1}{3}\sin(\log x^2)\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Solution : Right hand derivative of $f(x)$ at $x = 0$.

$$f'(0+0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h\left(1+1/3.\sin(\log h^2)\right)-0}{h}$$

$$= \lim_{h\to 0}\{1+1/3.\sin(\log h^2)\}$$

This limit does not exist because $\lim_{h\to 0}\sin(\log h^2), -1$, is between -1 and 1 hence $\lim_{h\to 0}\{1+1/3.\sin(\log h^2)\}$, $2/3$ and 4 / 3. Hence $f(x)$ is not differentiable at $x = 0$.

Example 11. For what values of m is the following function differentiable at $x = 0$ and $f'(x)$ is continuous

$$f(x) = \begin{cases} x^m \sin\frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Solution : differentiability at $x = 0$ Let hand derivative of $f(x)$ at $x = 0$

$$f'(0-0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{(-h)^m \sin\frac{1}{(-h)} - 0}{-h}$$

$$= \lim_{h\to 0} (-1)^m h^{m-1} \sin\frac{1}{h} \qquad (1)$$

right hand derivative of $f(x)$ at $x = 0$

$$f'(0+0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^m \sin\frac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0} h^{m-1} \sin\frac{1}{h} \qquad (2)$$

If $f(x)$ is differentiable at $x = 0$ then $f'(0-0) = f'(0+0)$, which is possible only when $m-1>0$ or $m>1$ hence the given functionis differentiable at $x = 0$ if $m > 1$.

Test of continuity at $x = 0$

$$f'(x) = mx^{m-1}\sin(1/x) - x^{m-2}\cos(1/x) \neq 0$$

$$f'(0) = 0$$

$f'(x)$ is continuous at $x = 0$ if $m > 2$

Hence required condition is $m > 2$.

Example 12. If the function $f(x) = |x-1| + 2|x-2| + 3|x-3|, \;\forall\; x \in R$ is continuous at points $x = 1,\; 2,\; 3$ then examine the differentiability of function at these points.

Solution : We can write the function as

$$f(x) = \begin{cases} 14-6x, & \text{if} \quad x \le 1 \\ 12-4x, & \text{if} \quad 1 < x \le 2 \\ 4, & \text{if} \quad 2 < x \le 3 \\ 6x-14, & \text{if} \quad x > 3 \end{cases}$$

differentiability at $x = 1$

Let hand derivative of $f(x)$ at $x = 1$

$$f'(1-0) = \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{\{14-6(1-h)\} - \{14-6(1)\}}{-h}$$

$$= \lim_{h\to 0} \frac{(6h)}{-h} = -6 \qquad (1)$$

Right hand derivative of $f(x)$ at $x = 1$

$$f'(1+0) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{\{12-4(1+h)\} - \{14-6(1)\}}{h}$$

$$= \lim_{h \to 0} \frac{-4h}{h} = -4 \qquad (2)$$

From (1) and (2)

$$f'(1-0) \neq f'(1+0)$$

Hence function $f(x)$ is not differentiable at $x = 1$, similarly we can prove that $f(x)$ is not differentiable at $x = 2$ and $x = 3$ also.

Example 13. Test the differentiability of following function at $x = 0$.

$$f(x) = \begin{cases} e^{-1/x^2} \cdot \sin(1/x), & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0 \end{cases}$$

Solution : Left hand derivative of $f(x)$ at $x = 0$

$$f'(0-0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{e^{-1/(-h)^2} \cdot \sin(1/(-h)) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{\sin(1/h)}{he^{1/h^2}} \qquad (1)$$

Now,

$$= \lim_{h \to 0} \frac{\sin(1/h)}{h\left[1 + \frac{1}{h^2} + \frac{1}{\underline{|2}\, h^4} + \dots\right]}$$

$$= \text{(a finite number between } -1 \text{ and } 1\text{)} \;/\; \lim_{h \to 0} \left\{h + \frac{1}{h} + \frac{1}{\underline{|2}} \cdot \frac{1}{h^3} + \dots\right\} = 0 \quad (2)$$

Right hand limit of $f(x)$ at $x = 0$

$$f'(0+0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{e^{1/h^2} \cdot \sin(1/h) - 0}{h}$$

$$= \lim_{h \to 0} \frac{\sin 1/h}{he^{-1/h^2}}$$

$$= 0 \text{ (as above)}$$

so $\quad f'(0-0) = f'(0+0) = 0$

Hence $f(x)$ is differentiable at $x = 0$.

Example 14. Is function $f(x) = |x-2|$, differentiable at $x = 2$?

Solution : Left hand derivative of $f(x)$ at $x = 2$

$$f'(2-0)=\lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$$

$$=\lim_{h\to 0}\frac{|2-h-2|-0}{-h}=\lim_{h\to 0}\frac{|-h|}{-h}$$

$$=\lim_{h\to 0}\frac{h}{-h}=\lim_{h\to 0}(-1)=-1 \qquad (1)$$

Right hand derivative of $f(x)$ at $x=2$

$$f'(2+0)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{|2+h-2|-0}{h}=\lim_{h\to 0}\frac{|h|}{h}$$

$$=\lim_{h\to 0}\frac{h}{h}=\lim_{h\to 0}(1)=1 \qquad (2)$$

From (1) and (2)

Hence $f(x)$ is not differentiable at $x=2$.

Exercise 6.2

1. Prove that following functions are differentiable for every value of x.
   (i) Identity function $f(x)=x$
   (ii) Constant function $f(x)=c$, where c is constant
   (iii) $f(x)=e^x$
   (iv) $f(x)=\sin x$.
2. Prove that function $f(x)=|x|$ is not differentiable at $x=0$.
3. Examine the differentiability of the function $f(x)=|x-1|+|x|$, at $x=0$ and 1.
4. Examine the differentiability of the function $f(x)=|x-1|+|x-2|$, in [0, 2].
5. Examine the differentiability of $f(x)=\begin{cases} x\tan^{-1}x & ; \quad x\neq 0 \\ 0 & ; \quad x=0 \end{cases}$
6. Examine the differentiability of $f(x)=\begin{cases} \dfrac{1-\cos x}{2} & ; \quad x\leq 0 \\ \dfrac{x-2x^2}{2} & ; \quad x>0 \end{cases}$
7. Prove that the following function $f(x)$.

$$f(x)=\begin{cases} x^m\cos(1/x) & ; \quad x\neq 0 \\ 0 & ; \quad x=0 \end{cases}$$

(i) is continuous at $x = 0$, if $m > 0$

(ii) is differentiable at $x = 0$ if $m > 1$

8. Examine the differentiability of following function at $x = 0$

$$f(x) = \begin{cases} \dfrac{1}{1+e^{1/x^2}} & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$$

9. Examine the differentiability of following function at $x = 0$.

$$f(x) = \begin{cases} \dfrac{|x|}{x} & ; \quad x \neq 0 \\ 1 & ; \quad x = 0 \end{cases}$$

10. Examine the Differentiability of following function at $x = \pi/2$

$$f(x) = \begin{cases} 1+\sin x & ; \quad 0 < x < \pi/2 \\ 2+\left(x-\pi/2\right)^2 & ; \quad x \geq \pi/2 \end{cases}$$

11. Find the values of m and n if

$$f(x) = \begin{cases} x^2+3x+m, & \text{when} \quad x \leq 1 \\ nx+2, & \text{when} \quad x > 1 \end{cases}$$

if differentiable at every point

Miscellaneous Exercise 6

1. If $f(x) = \dfrac{x^2-9}{x-3}$ is continuous at $x = 3$ then the value of $f(3)$ will be

(a) 6 (b) 3 (c) 1 (d) 0.

2. If $f(x) = \begin{cases} \dfrac{\sin 3x}{x} & ; \quad x \neq 0 \\ m & ; \quad x = 0 \end{cases}$, $x = 0$ is continuous at $x = 0$ then the value of m

(a) 3 (b) 1 / 3 (c) 1 (d) 0.

3. If $f(x) = \begin{cases} \dfrac{\log(1+mx)-\log(1-nx)}{x} & ; \quad x \neq 0 \\ k & ; \quad x = 0 \end{cases}$, is continuous at $x = 0$, the value of k will be

(a) o (b) $m+n$ (c) $m-n$ (d) $m \cdot n$.

4. If $f(x) = \begin{cases} x+\lambda & ; \quad x < 3 \\ 4 & ; \quad x = 3 \\ 3x-5 & ; \quad x > 3 \end{cases}$, is continuous at $x = 3$, then the vlaue of λ is

(a) 4 (b) 3 (c) 2 (d) 1.

5. If $f(x) = \cot x,$ is not continuous at $x = \frac{n\pi}{2}$ when

(a) $n \in Z$ (b) $n \in N$ (c) $n/2 \in Z$ (d) only $n = 0$.

6. The set of those points on $f(x) = x\,|\,x\,|$, where the function is differentiable

(a) $(0, \infty)$ (b) $(-\infty, \infty)$ (c) $(-\infty, 0)$ (d) $(-\infty, 0) \cup (0, \infty)$

7. Which of the following function is not differentiable at $x = 0$

(a) $x\,|\,x\,|$ (b) $\tan x$ (c) e^{-x} (d) $x + |\,x\,|$

8. The value of left hand derivative of $f(x)$ at $x = 2$ is; $f(x) = \begin{cases} 1+x, & \text{when } x \le 2 \\ 5-x, & \text{when } x > 2 \end{cases}$

(a) -1 (b) 1 (c) -2 (d) 2.

9. Function $f(x) = [x]$ is not differentiable at

(a) every integer (b) every rational number

(c) origin (d) everywhere

10. The vlaue of right hand derivative of $f(x)$ at $x = 0$ is; $f(x) = \begin{cases} \frac{\sin x^2}{x}, & \text{when } x \ne 0 \\ 0, & \text{when } x = 0 \end{cases}$,

(a) -1 (b) 1 (c) 0 (d) Infinite

11. Examine the continuity of following function $f(x) = |\sin x| + |\cos x| + |\,x\,|, \ \forall \ x \in R$

12. Find the value of m, when the following function is continous at $x = 0$

$$f(x) = \begin{cases} \frac{\sin(m+1)x + \sin x}{x} & ; \quad x < 0 \\ 1/2 & ; \quad x = 0 \\ \frac{x^{3/2}+1}{2} & ; \quad x > 0 \end{cases}$$

13. Find the values of m and n when the following function is continuous

$$f(x) = \begin{cases} x^2 + mx + n & ; \quad 0 \le x < 2 \\ 4x - 1 & ; \quad 2 \le x \le 4 \\ mx^2 + 17n & ; \quad 4 < x \le 6 \end{cases}$$

14. Examine the continuity of the function $f(x) = \begin{cases} \frac{\tan x}{\sin x} & ; \quad x \ne 0 \\ 1 & ; \quad x = 0 \end{cases}$, at $x = 0$

15. Examine the continuity of following function at $x = 1$ and 3. $f(x)=\begin{cases} |x-3| & ; \quad x \geq 1 \\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4} & ; \quad x<1 \end{cases}$,

16. Find the values of a, b and c when the following is continuous at $x = 0$

$$f(x)=\begin{cases} \frac{\sin(a+1)x+\sin x}{x}, & \text{if} \quad x<0 \\ c, & \text{if} \quad x=0 \\ \frac{\sqrt{x+bx^2}-\sqrt{x}}{bx\sqrt{x}}, & \text{if} \quad x>0 \end{cases}$$

17. Examine the continuity of $f(x)=\frac{|3x-4|}{3x-4}$ at $x=\frac{4}{3}$

18. Examine the continuity of $f(x)=|x|+|x-1|$ in the interval $[-1, \ 2]$

19. Find the value of $f(0)$ if the following function is continuous at $x = 0$; $f(x)=\frac{\sqrt{1+x}-\sqrt[3]{1+x}}{x}$

20. Examine the continuity of $f(x)$ at $x = 0$ when $f(x)=\begin{cases} \frac{e^{1/x}-1}{e^{-1/x}+1}, & \text{when} \quad x \neq 0 \\ 1 & \text{when} \quad x=0 \end{cases}$

21. For what vlaues of x, $f(x)=\sin x, x$ is not differentiable.

22. Examine the differentiability of $f(x)=\begin{cases} x^2 \sin x & ; \quad x \neq 0 \\ 0 & ; \quad x=0 \end{cases}$, when $x \in R$ also find the value of $f'(0)$.

23. Examine the differetiability of following function at $x = a$

$$f(x)=\begin{cases} (x-a)^2 \sin\left(\frac{1}{x-a}\right) & ; \quad x \neq a \\ 0 & ; \quad x=a \end{cases},$$

24. Prove that the following function uis not differentiable at $x = 1$

$$f(x)=\begin{cases} x^2-1 & ; \quad x \geq 1 \\ 1-x & ; \quad x<1 \end{cases},$$

25. Examine the diferentiablity of following function at $x = 0$

$$f(x)=\begin{cases} -x & ; \quad x \leq 0 \\ x & ; \quad x>0 \end{cases},$$

26. Prove that the following function is differentiable at $x = 0$

$$f(x) = \begin{cases} \dfrac{x \log_e \cos x}{\log_e (1+x^2)} & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$$

27. Examine the differentiability of $f(x) = |x-2| + 2|x-3|$ in the interval [1, 3].

28. If the function $f(x) = x^3, x = 2$, is differentiable at $x = 2$ then find the vlaue of $f'(2)$

29. Prove that the greatest integer function $f(x) = [x]$ is not differentiable at $x = 2$

30. If $f(x) = \begin{cases} x-1 & ; \quad x < 2 \\ 2x-3 & ; \quad x \geq 2 \end{cases}$ then find $f'(2-0)$.

IMPORTANT POINTS

1. **Cauchy's definition of continuity**

   Let $f(x)$ be a function, then it is continuous at a point a in its domain if for a small positive number $\in$ there exists a positive δ such that $|f(x) - f(a)| < \in$ when $|x - a| < \delta$.

2. Alternate definition of continuity:

   A function $f(x)$ is continuous at a point a in its domain if $\lim_{x \to a} f(x) = f(a)$

   i.e. $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

   or $f(a-0) = f(a+0) = f(a)$

3. Continuous function in domain.

   Any function $f(x)$ is called continuous in its domain if $f(x)$ is continuous at every point of domain D.

4. Now continuous function

   (i) A function $f(x)$ is called non continuous at a point a if $f(x)$ is not continuous at this point.

   (ii) Function $f(x)$ is called non continuous in its domain D if it is not continuous at at least one point of D.

5. **Properties of continuity**

   (i) If $f(x)$ and $g(x)$ are two continuous functions in domain D then $f(x) \pm g(x)$, $c \cdot f(x)$, will be continuous in D. How ever $\dfrac{f(x)}{g(x)}, D$ will be continuous for all points in D where $g(x) \neq 0$.

   (ii) If $f(x)$ and $g(x)$ are two continuous functions in their respective domains then their composit function $(f \circ g)(x)$ will be continuous.

6. **Differentiability.**

   A function $f(x)$ is derivable at $x = a$, if

   or, $f'(c-0) = f'(c+0)$

   or, $\lim_{h \to 0} \dfrac{f(c-h) - f(c)}{-h} = \lim_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$

7. **Non-differentiablity at a point**

$f(x)$ is not differentiable at a point c if

(i) $f'(c-0) \neq f'(c+0)$

or

(ii) $f'(c-0)$ and $f'(c+0)$ either or both infinite

or

(iii) $f'(c-0)$ or $f'(c+0)$ either or both do not exist.

Answers

Exercise 6.1

1. (a) continuous ; (b) not continuous ; (c) continuous ; (d) not continuous ;
(e) not continuous ; (f) not continuous ; (g) continuous

2. Not continuous 3. $k = 7$ 4. Not continuous

Exercise 6.2

3. Not differentiable 4. Not differentiable 5. Not differentiable
6. Not differentiable 7. Not differentiable 8. Not differentiable
9. Not differentiable 10. Not differentiable 11. $m = 3, n = 5$

Miscellaneous Exercise - 6

1. (a) 2. (a) 3. (b) 4. (d) 5. (c) 6. (b) 7. (d)
8. (b) 9. (a) 10. (b)

11. Everywhere continuous in R 12. $m = \frac{-3}{2}$ 13. $m = 2, n = -1$ 14. continuous

15. continuous 16. $a = -3/2, c = 1/2$ and $b \in R$ 17. not continuous

18. continuous in $[-1, 2]$ 19. $1/6$ 20. Not continuous 21. R

22. differentiable for every $x \in R$ and $f'(0) = 0$ 23. Not differentiable

25. Not differentiable 27. Not differentiable at $x = 2$ 28. 12 30. 1