1. Sets, Relations and Functions

- A **set** is a well-defined collection of objects.
- Sets are usually represented by capital letters A, B, C, D, X, Y, Z, etc. The objects inside a set are called **elements** or **members** of a set. They are denoted by small letters a, b, c, d, x, y, z, etc.
- If a is an element of a set A, then we say that "a belongs to A" and mathematically we write it as "$a \in A$"; if b is not an element of A, then we write "$b \notin A$".
- There are three different ways of representing a set:
  - **Description method:** Description about the set is made and it is enclosed in curly brackets { }.

  For example, the set of composite numbers less than 30 is written as follows:

  {Composite numbers less than 30}

  - **Roster method or tabular form:** Elements are separated by commas and enclosed within the curly brackets { }.

  For example, a set of all integers greater than 5 and less than 9 will be represented in roster form as {6, 7, 8}.

  - **Set-builder form or rule method:** All the elements of the set have a single common property that is exclusive to the elements of the set i.e., no other element outside the set has that property.

  For example, a set L of all integers greater than 5 and less than 9 in set-builder form can be represented as follows:

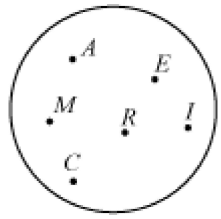
  $L = \{x : x$ is an integer greater than 5 and less than 9$\}$

- **Some important points:**
  1. The order of listing the elements in a set can be changed.
  2. If one or more elements in a set are repeated then the set remains the same.
  3. Each element of the set is listed once and only once.
- On the basis of number of elements, sets can be classified as:
  - **Finite set** − A set that contains limited (countable) number of different elements is called a finite set.
  - **Infinite set** − A set that contains unlimited (uncountable) number of different elements is called an infinite set.
  - **Empty set** − A set that contains no element is called an empty set. It is also called null (or void) set. An empty set is denoted by Φ or {}. Also, since an empty set has no element, it is regarded as a finite set.
- The number of distinct elements in a finite set A is called its **cardinal number**. It is denoted by $n(A)$.
- As the empty set has no elements, therefore, its cardinal number is 0 i.e., $n(\Phi) = 0$

- A set can also be represented using a venn diagram. **Venn diagrams** are closed figures such as square, rectangle, circle, etc. inside which some points are marked. The closed figure represents a set and the points marked inside it represent the elements of the set.

  For example, consider the set of all letters in the word AMERICA. This set consists of the letters A, M, E, R, I, and C.

This set can be represented by a Venn diagram as follows:



- If A and B are any two sets, then set A is said to be a subset of set B if every element of A is also an element of B. We write it as $A \subseteq B$ (read as 'A is a subset of B' or 'A is contained in B'). In this case, we say that B is a **superset** of A. We write it as $B \supset A$ (read as 'B contains A' or 'B is a superset of A').
- If there exists at least one element in A which is not an element of B, then A is not a subset of B. Mathematically, we write it as $A \not\subseteq B$.
- Let A be any set and B be a non-empty set. Set A is called a **proper subset** of B if and only if every member of A is also a member of B,and there exists at least one element in B which is not a member of A. We write it as $A \subset B$. Also, B is called the **superset** of A.
- Some important points:

  (a) Every set is a subset of itself.

  (b) A subset which is not a proper subset is called an improper subset. If A and B are two equal sets, then A and B are improper subsets of each other.

  (c) Every set has only one improper subset and that is itself.

  (d) An empty set is a subset of every set.

  (e) An empty set is a proper subset of every set except itself.

  (f) Every set is a subset of the universal set.

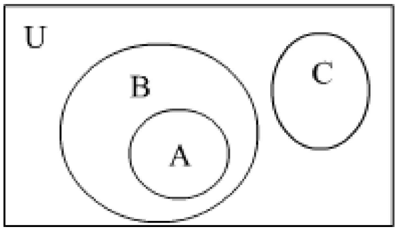
  (g) If $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$

- If cardinal number of the set A is m, i.e., $n(A) = m$, then

  The number of subsets of $A = 2^m$

  The number of proper subsets of $A = 2^m - 1$

- The collection of all subsets of a set A is called the **power set** of A. It is denoted by P(A). In P(A), every element is a set.
- If the number of elements in set A is m, then the number of elements in the power set of A is 2^m. i.e., $n\text{P}(A) = 2^m$, where $n(A) = m$
- A set that contains all the elements under consideration in a given problem is called **universal set** and it is denoted by U or ξ.
- Representing information using venn diagram:



Here, U, A, B and C are four sets.

From the diagram, following information is observed:

$A \subseteq B$ or $B \supseteq A$

Since $B \neq A$, $A \subset B$.

$C \not\subseteq B$ and $C \not\subseteq A$.

U is the universal set.

- Two finite sets are called equivalent, if they have the same number of elements.

Thus, two finite sets X and Y are equivalent, if $n(X) = n(Y)$. We write it as $X \leftrightarrow Y$ (read as "X is equivalent to Y")

For example, for sets $A = \{-9, -3, 0, 5, 12\}$, $B = \{-2, 1, 2, 4, 7\}$

$n(A) = 5$ and $n(B) = 5$

Therefore, sets A and B are equivalent sets

- Two sets are called equal, if they have same elements.

For example, for sets X = {all letters in the word STONE}, Y = {all letters in the word NOTES}

X = {S, T, O, N, E} and Y = {N, O, T, E, S}

Here, the sets X and Y have same elements. Therefore, in this case, we say that the sets X and Y are equal sets.

- Let X be any set and ξ be its universal set. The complement of set X is the set consisting of all the elements of ξ, which do not belong to X. It is denoted by X' or X^c (read as complement of set X).

Thus, $X' = \{x | x \in \xi \text{ and } x \notin X\}$ or $X' = \xi - X$

- $n(X') = n(\xi) - n(X)$
- Properties of a set and its complement.

(a) $X \cap X' = \phi$

(b) $X \cup X' = \xi$

(c) $\xi' = \phi$

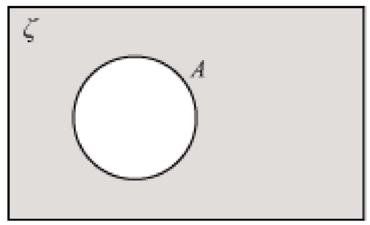
(d) $\phi' = \xi$

- **De Morgan's laws:**

  (a) $(A \cap B)' = A' \cup B'$

  (b) $(A \cup B)' = A' \cap B'$

- Compliment of a set, A denoted by A^c can be shown in Venn-diagram as follows:



A^C (Shaded portion)

The portion outside the set A, but inside the set ξ, represents the set A^c.

- The union of two sets A and B is the set that consists of all the elements of A, all the elements of B, and the common elements taken only once. The symbol '∪' is used for denoting the union.

  For example, if $X = \{2, 4, 6, 8, 10\}$ and $Y = \{4, 8, 12\}$, then the union of X and Y is given by $X \cup Y = \{2, 4, 6, 8, 10, 12\}$

- There are some properties of union of two sets:
  1. $A \cup B = B \cup A$
  2. $A \cup \Phi = A$
  3. $A \cup A = A$
  4. $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law)
  5. $U \cup A = U$ (Law of universal set, U)

- The intersection of sets A and B is the set of all elements that are common to both A and B. The symbol '∩' is used for denoting the intersection.

  For example, if $X = \{A, E, I, O, U\}$ and $Y = \{A, B, C, D, E\}$, then the intersection of the sets X and Y is given by $X \cap Y = \{A, E\}$

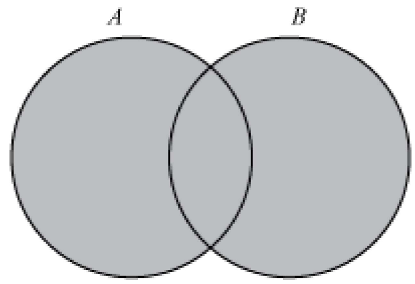
- The properties of the intersection of two sets are given as follows:
  1. $A \cap B = B \cap A$
  2. $\Phi \cap A = \Phi$
  3. $A \cap A = A$
  4. $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative law)
  5. $U \cap A = A$ (Law of U)
- Two sets are called overlapping (or joint) sets, if they have at least one element in common.
- If two sets A and B are such that $A \cap B = \Phi$ i.e., they have no element in common, then A and B are called disjoint sets.
- $\boldsymbol{n(A \cup B) = n(A) + n(B) - n(A \cap B)}$

  If A and B are two disjoint sets i.e., $A \cap B = \Phi$, then $n(A \cap B) = 0$

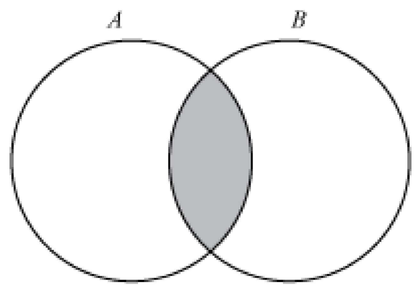
  In this case, the above formula will change into:

  $\boldsymbol{n(A \cup B) = n(A) + n(B)}$

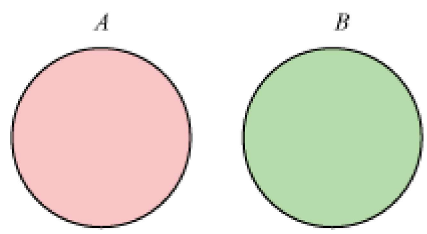
- Venn-diagram for union and intersection of sets are as follows:
  - When the sets A and B are overlapping, the Venn diagram representing $A \cup B$ can be shown as:



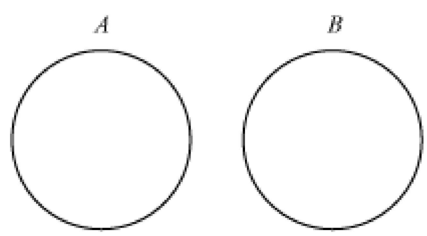
  - When the sets A and B are overlapping, the set $A \cap B$ is the shaded portion of the following the Venn diagram.



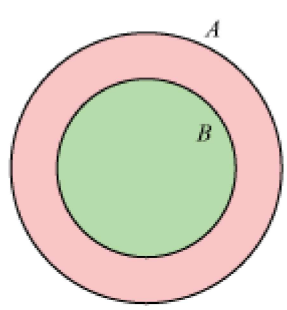
  - When the sets A and B are disjoint, the Venn diagrams representing $A \cup B$ can be shown as:



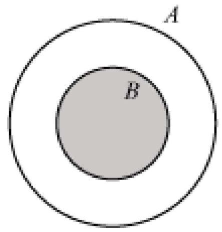
  - When the sets A and B are disjoint, the Venn diagrams representing $A \cap B$ can be shown as:



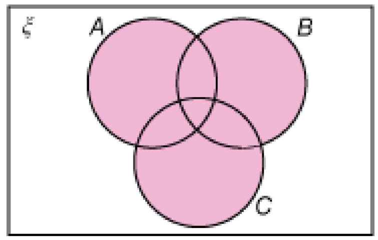
  - When set B is fully contained in set A, the Venn diagrams representing $A \cup B$ can be shown as:



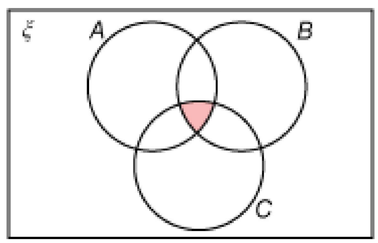
  - When set B is fully contained in set A, the Venn diagrams representing $A \cap B$ can be shown as:



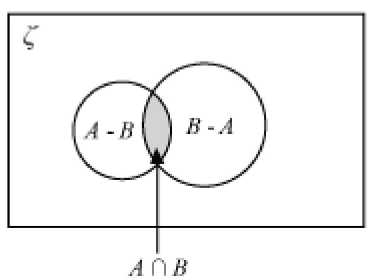
- The union of the three sets A, B and C, i.e., $A \cup B \cup C$, is represented by the shaded portion of the following Venn diagram.



- The intersection of the three sets A, B and C, i.e., $A \cap B \cap C$ is represented by the shaded portion of the following Venn diagram.



- The difference between sets A and B (in that order), i.e., $A - B$ is the set of elements belonging to A, but not to B. Thus, $A - B = \{x : x \in A \text{ and } x \notin B\}$.
- If U is the universal set for the sets A, B and C, then the sets $A - B$, $A \cap B$ and $B - A$ can be shown diagrammatically as



- If A and B are two sets, the their symmetric difference is $(A - B) \cup (B - A)$ and denoted by $A \Delta B$. Thus, $A \Delta B = (A - B) \cup (B - A) = \{x : x \notin A \cap B\}$.
- If A and B are two sets, then

  1. $n(A - B) = n(A \cup B) - n(B) = n(A) - n(A \cap B)$

  2. $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$

- Let X and Y are two non-empty sets. A **relation** R from X to Y is a rule, which associates elements of set X to elements of set Y.
- A relation can be represented in two forms:

(1) Roster form

(2) Arrow diagram

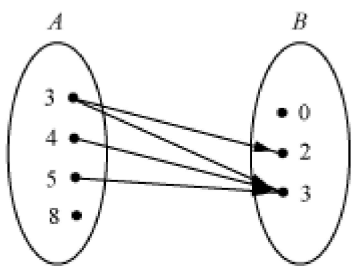
- An **ordered pair** is a pair of objects taken in a specific order. In the ordered pair (*a*, *b*), *a* is called the first member or the first component and *b* is called the second member or the second component.
    - If two order pairs (*a*, *b*) and (*c*, *d*) are equal, then $a = c$ and $b = d$.

- The set of all components of *X* which satisfy *R* is the **domain**.
- The set of all components of *Y* which satisfy *R* is the **range**.

  For example, if a relation R from *A* to *B* is given by "is less than twice of" where $A = \{3, 4, 5, 8\}$, $B = \{0, 2, 3\}$ then the first components are taken from *A* and second components are taken from *B* in such a way that:

  First component < 2 × Second component

  ∴ Relation *R* is written in roster form as $R = \{(3, 2), (3, 3), (4, 3), (5, 3)\}$ and is represented by an arrow diagram as:



  Here, domain of $R = \{3, 4, 5\}$

  Range of $R = \{2, 3\}$

- **Cartesian product of two sets:** Two non-empty sets P and Q are given. The Cartesian product P × Q is the set of all ordered pairs of elements from P and Q, i.e.,

$P \times Q = \{(p, q) : p \in P \text{ and } q \in Q\}$

Example: If $P = \{x, y\}$ and $Q = \{-1, 1, 0\}$, then $P \times Q = \{(x, -1), (x, 1), (x, 0), (y, -1), (y, 1), (y, 0)\}$

If either P or Q is a null set, then P × Q will also be a null set, i.e., $P \times Q = \varphi$.

In general, if A is any set, then $A \times \varphi = \varphi$.

- **Property of Cartesian product of two sets:**
    - If $n(A) = p$, $n(B) = q$, then $n(A \times B) = pq$.
    - If A and B are non-empty sets and either A or B is an infinite set, then so is the case with A × B
    - .$A \times A \times A = \{(a, b, c) : a, b, c \in A\}$. Here, (a, b, c) is called an ordered triplet.
    - $A \times (B \cap C) = (A \times B) \cap (A \times C)$
    - $A \times (B \cup C) = (A \times B) \cup (A \times C)$

- Two ordered pairs are equal if and only if the corresponding first elements are equal and the second elements are also equal. In other words, if $(a, b) = (x, y)$, then $a = x$ and $b = y$.

Example: Show that there does not exist $x, y \in$ R if $(x - y + 1, 4x - 2y - 6) = (y - x - 4, 7x - 5y - 2)$.
Solution: It is given that
$(x - y + 1, 4x - 2y - 6) = (y - x - 4, 7x - 5y - 2)$.
$\Rightarrow x - y + 1 = y - x - 4$ and $4x - 2y - 6 = 7x - 5y - 2$
$\Rightarrow 2x - 2y + 5 = 0$ … (1)
And $- 3x + 3y - 4 = 0$ … (2)
Now,
$\frac{2}{-3} = -\frac{2}{3}, \frac{-2}{3} = -\frac{2}{3}$ and $\frac{5}{-4} = -\frac{5}{4}$
Since $\frac{2}{-3} = \frac{-2}{3} \neq \frac{5}{-4}$, equations (1) and (2) have no solutions. This shows that there does not exist $x, y \in$ R if $(x - y + 1, 4x - 2y - 6) = (y - x - 4, 7x - 5y - 2)$.

In general, for any two sets A and B, A × B ≠ B × A.

- **Relation:** A relation *R* from a set A to a set B is a subset of the Cartesian product A × B, obtained by describing a relationship between the first element *x* and the second element *y* of the ordered pairs (*x*, *y*) in A × B.
- The image of an element *x* under a relation *R* is *y*, where $(x, y) \in$ R
- **Domain:** The set of all the first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R.
- **Range and Co-domain:** The set of all the second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the co-domain of the relation *R*. Range ⊆Co-domain

Example: In the relation X from **W** to **R**, given by X = {(x, y): $y = 2x + 1$; $x \in$ W, $y \in$ R}, we obtain X = {(0, 1), (1, 3), (2, 5), (3, 7) …}. In this relation X, domain is the set of all whole numbers, i.e., domain = {0, 1, 2, 3 …}; range is the set of all positive odd integers, i.e., range = {1, 3, 5, 7 …}; and the co-domain is the set of all real numbers. In this relation, 1, 3, 5 and 7 are called the images of 0, 1, 2 and 3 respectively.

- The total number of relations that can be defined from a set A to a set B is the number of possible subsets of A × B.

If n(A) = p and n(B) = q, then n(A × B) = pq and the total number of relations is 2^{pq}.

Types of Relation

- **One-One Relation :** A relation *R* from *A* to *B* is said to be one-one if every element of *A* has at most one image in *B* and distinct elements in *A* have distinct images in *B*.
- **Many-one Relation :** A relation *R* from *A* to *B* is said to be many-one if two or more than two elements in *A* have the same image in *B*.
- **Into Relation :** A relation *R* from *A* to *B* is said to be an into relation if there exists at least one element in *B* which has no pre-image in A.
- **Onto Relation :** A relation *R* from *A* to *B* is said to be an onto relation if every element of *B* is the image of some element of *A*.

- A relation R from a set A to a set B is a subset of $A \times B$ obtained by describing a relationship between the first element a and the second element b of the ordered pairs in $A \times B$. That is, $R \subseteq \{(a, b) \in A \times B, a \in A, b \in B\}$
- The domain of a relation R from set A to set B is the set of all first elements of the ordered pairs in R.
- The range of a relation R from set A to set B is the set of all second elements of the ordered pairs in R. The whole set B is called the co-domain of R. Range $\subseteq$ Co-domain
- A relation R in a set A is called an empty relation, if no element of A is related to any element of A. In this case, $R = \Phi \subset A \times A$

  Example: Consider a relation R in set $A = \{3, 4, 5\}$ given by $R = \{(a, b): a^b < 25$, where $a, b \in A\}$. It can be observed that no pair (a, b) satisfies this condition. Therefore, R is an empty relation.

- A relation R in a set A is called a universal relation, if each element of A is related to every element of A. In this case, $R = A \times A$
  Example: Consider a relation R in the set $A = \{1, 3, 5, 7, 9\}$ given by $R = \{(a, b)$: $a + b$ is an even number$\}$.
  Here, we may observe that all pairs (a, b) satisfy the condition R. Therefore, R is a universal relation.

- Both the empty and the universal relation are called trivial relations.
- A relation R in a set A is called reflexive, if $(a, a) \in R$ for every $a \in R$.

  Example: Consider a relation R in the set A, where $A = \{2, 3, 4\}$, given by $R = \{(a, b): a^b = 4, 27$ or $256\}$. Here, we may observe that $R = \{(2, 2), (3, 3)$, and $(4, 4)\}$. Since each element of R is related to itself (2 is related 2, 3 is related to 3, and 4 is related to 4), R is a reflexive relation.

- A relation R in a set A is called symmetric, if $(a_1, a_2) \in R \Rightarrow (a_2, a_1) \in R, \forall (a_1, a_2) \in R$
  Example: Consider a relation R in the set A, where A is the set of natural numbers, given by $R = \{(a, b): 2 \leq ab < 20\}$. Here, it can be observed that $(b, a) \in R$ since $2 \leq ba < 20$ [since for natural numbers a and b, $ab = ba$]
  Therefore, the relation R is symmetric.

- A relation R in a set A is called transitive, if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R \Rightarrow (a_1, a_3) \in R$ for all $a_1, a_2, a_3 \in A$
  Example: Let us consider a relation R in the set of all subsets with respect to a universal set U given by $R = \{(A, B)$: A is a subset of $B\}$
  Now, if A, B, and C are three sets in R, such that $A \subset B$ and $B \subset C$, then we also have $A \subset C$. Therefore, the relation R is a symmetric relation.

- A relation R in a set A is said to be an equivalence relation, if R is altogether reflexive, symmetric, and transitive.
  Example: Let (a, b) and (c, d) be two ordered pairs of numbers such that the relation between them is given by $a + d = b + c$. This relation will be an equivalence relation. Let us prove this.
  (a, b) is related to (a, b) since $a + b = b + a$. Therefore, $R$is reflexive.
  If (a, b) is related to (c, d), then $a + d = b + c \Rightarrow c + b = d + a$. This shows that (c, d) is related to (a, b). Hence, R is symmetric.
  Let (a, b) is related to (c, d); and (c, d) is related to (e, f), then $a + d = b + c$ and $c + f = d + e$. Now, $(a + d) + (c + f) = (b + c) + (d + e) \Rightarrow a + f = b + e$. This shows that (a, b) is related to (e, f). Hence, R is transitive.
  Since R is reflexive, symmetric, and transitive, R is an equivalence relation.

- Given an arbitrary equivalence relation R in an arbitrary set X, R divides X into mutually disjoint subsets Ai called partitions or subdivisions of X satisfying:
  - All elements of Ai are related to each other, for all i.
  - No element of Ai is related to any element of Aj , $i \neq j$
  - $\cup Aj = X$ and $Ai \cap Aj = \varnothing$, $i \neq j$

The subsets Ai are called equivalence classes.

- A relation R from a set A to a set B is said to be a **function** if for every a in A, there is a unique b in B such that $(a, b) \in R$.
- If R is a function from A to B and $(a, b) \in R$, then b is called the **image** of a under the relation R and a is called the **preimage** of b under R.
- For a function R from set A to set B, set A is the **domain** of the function; the images of the elements in set A or the second elements in the ordered pairs form the **range**,while the whole of set B is the **codomain** of the function.

For example, in relation $f = \{(-1,3)(0,2),(1,3),(2,6),(3,11)\}$ since each element in A has a unique image, therefore f is a function.

Each image in B is 2 more than the square of pre-image.

Hence, the formula for f is $f(x) = x^2 + 2$ Or $f : x \to x^2 + 2$

Domain = $\{-1, 0, 1, 2, 3\}$

Co-domain = $\{2, 3, 6, 11, 13\}$

Range = $\{2, 6, 3, 11\}$

- **Real-valued Function:** A function having either R (real numbers) or one of its subsets as its range is called a real-valued function. Further, if its domain is also either R or a subset of R, it is called a real function.

Types of functions:

- **Identity function:** The function f: R → R defined by $y = f(x) = x$, for each $x \in$ R, is called the identity function.

Here, R is the domain and range of f.

- **Constant function:** The function f: R → R defined by $y = f(x) = x$, for each $x \in$ R, where c is a constant, is a constant function.

Here, the domain of f is R and its range is $\{c\}$.

- **Polynomial function:** A function f: R → R is said to be a polynomial function if for each $x \in$ R, $y = f(x) = a^0 + a_1 x + \_\_\_ + a_n x^n a$ where n is a non-negative integer and $a_0, a_1, \ldots, a_n \in$ R .
- **Rational function:** The functions of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$, are called rational functions.
- **Modulus function:** The function f: R → R^+ defined by $f(x) = |x|$, for each $x \in$ R, is called the modulus function.

In other words, $f(x) = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$

- **Signum function:** The function f: R → R defined by

$$f(x)=\left\{\begin{array}{l} 1, \text{if } x>0 \\ 0, \text{if } x=0 \\ -1, \text{if } x<0 \end{array}\right\}$$

is called the signum function. Its domain is R and its range is the set {–1, 0, 1}.

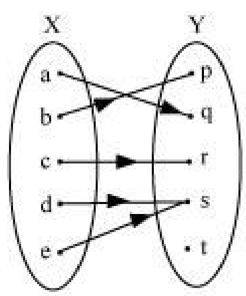
- **Greatest Integer function**: The function f: R → R defined by $f(x) = [x]$, $x \in$ R, assuming the value of the greatest integer less than or equal to x, is called the greatest integer function.

Example: [–2.7] = –3, [2.7] = 2, [2] = 2

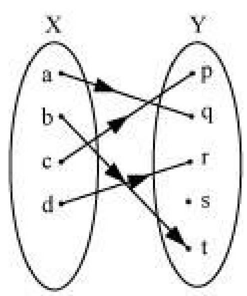
- **Linear function:** The function f defined by $f(x) = mx + c$, for $x \in$ R, where m and c are constants, is called the linear function. Here, R is the domain and range of f.

- A function f from set X to Y is a specific type of relation in which every element x of X has one and only one image y in set Y. We write the function f as f: $X \to Y$, where $f(x) = y$
- A function f: $X \to Y$ is said to be one-one or injective, if the image of distinct elements of X under f are distinct. In other words, if $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$, then $x_1 = x_2$. If the function f is not one-one, then f is called a many-one function.

  The one-one and many-one functions can be illustrated by the following figures:



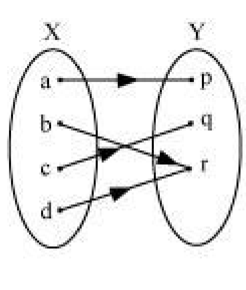
many-one



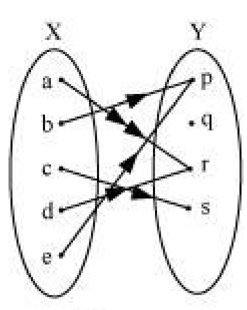
one-one

- A function f: $X \to Y$ can be defined as an onto (surjective) function, if $\forall\, y \in Y$, there exists $x \in X$ such that $f(x) = y$.

  The onto and many-one (not onto) functions can be illustrated by the following figures:

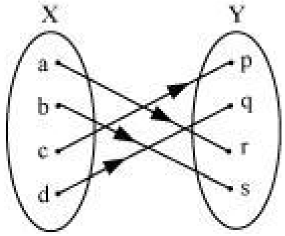


Onto



Not onto

- A function f: $X \to Y$ is said to be bijective, if it is both one-one and onto. A bijective function can be illustrated by the following figure:



Example: Show that the function f: **R** → **N** given by $f(x) = x^3 - 1$ is bijective.

Solution:
Let $x_1, x_2 \in \mathbf{R}$
For $f(x_1) = f(x_2)$, we have
$x_1^3 - 1 = x_2^3 - 1$
$\Rightarrow x_1^3 = x_2^3$
$\Rightarrow x_1 = x_2$
Therefore, f is one-one.
Also, for any y in **N**, there exists $\sqrt[3]{y+1}$ in **R** such that
$f\left(\sqrt[3]{y+1}\right) = \left(\sqrt[3]{y+1}\right)^3 - 1 = y$.
Therefore, f is onto.
Since f is both one-one and onto, f is bijective.

- **Addition and Subtraction of functions:** For functions f: X → R and g: X → R, we define

  - Addition of Functions

  $(f + g)$: X → R by $(f + g)(x) = f(x) + g(x)$, $x \in X$

  - Subtraction of Functions

  $(f - g)$: X → R by $(f - g)(x) = f(x) - g(x)$, $x \in X$

Example: Let $f(x) = 2x - 3$ and $g(x) = x^2 + 3x + 2$ be two real functions, then

$(f + g)(x) = f(x) + g(x)$
$= (2x - 3) + (x^2 + 3x + 2)$
$= x^2 + 5x - 1$

$(f - g) = f(x) - g(x)$
$= (2x - 3) - (x^2 + 3x + 2)$
$= -x^2 - x - 5$

- **Multiplication of real functions:** For functions f: X → R and g: X → R, we define

  - Multiplication of two real functions

  (fg): X → R by $(fg)(x) = f(x).\ g(x)$ $x \in X$

- Multiplication of a function by a scalar

  (af): X → R by $(a f)(x) = af(x)$ $x \in X$ and a is a real number

Example: Let $f(x) = 2x – 3$ and $g(x) = x^2 + 3x + 2$ be two real functions, then

$(fg)(x) = f(x) \times g(x)$
$= (2x – 3) \times (x^2 + 3x + 2)$
$= 2x^3 + 3x^2 – 5x – 6$

$(2f)(x) = 2.f(x)$
$= 2\times(2x – 3)$
$= 4x – 6$

- **Addition and Subtraction of functions:** For functions f: X → R and g: X → R, we define
  - Addition of Functions

    $(f + g)$: X → R by $(f + g)(x) = f(x) + g(x)$, $x \in X$
  - Subtraction of Functions

    $(f - g)$: X → R by $(f - g)(x) = f(x) - g(x)$, $x \in X$

Example: Let $f(x) = 2x – 3$ and $g(x) = x^2 + 3x + 2$ be two real functions, then

$(f + g)(x) = f(x) + g(x)$

$= (2x – 3) + (x^2 + 3x + 2)$

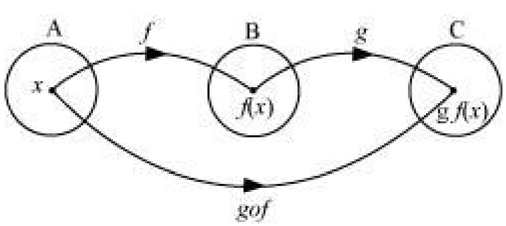
$= x^2 + 5x – 1$

$(f – g) = f(x) – g(x)$

$= (2x – 3) – (x^2 + 3x + 2)$

$= – x^2 – x – 5$

- **Composite function:** Let $f: A \to B$ and $g: B \to C$ be two functions. The composition of f and g, i.e. gof, is defined as a function from A to C given by $gof(x) = g(f(x))$, $\forall x \in A$



Example: Find gof and fog, if f: **R** → **R** and g: **R** → **R** are given by $f(x) = x^2 – 1$ and $g(x) = x^3 + 1$.

Solution:

$gof(x) = g(f(x))$

$= g\left(x^2 - 1\right)$

$= \left(x^2 - 1\right)^3 + 1$

$= x^6 - 1 - 3x^4 + 3x^2 + 1$

$= x^2\left(x^4 - 3x^2 + 3\right)$

$fog(x) = f(g(x))$

$= f\left(x^3 + 1\right)$

$= \left(x^3 + 1\right)^2 - 1$

$= x^6 + 2x^3 + 1 - 1$

$= x^3\left(x^3 + 2\right)$

- A function $f: X \rightarrow Y$ is said to be invertible, if there exists a function $g: Y \rightarrow X$ such that $gof = I_X$ and $fog = I_Y$. In this case, g is called inverse of f and is written as $g = f^{-1}$
- A function f is invertible, if and only if f is bijective.

Example: Show that $f: \mathbf{R}^+ \cup \{0\} \rightarrow \mathbf{N}$ defined as $f(x) = x^3 + 1$ is an invertible function. Also, find f^{-1}.

Solution:
Let $x_1, x_2 \in \mathbf{R}^+ \cup \{0\}$ and $f(x_1) = f(x_2)$

$\therefore x_1^3 + 1 = x_2^3 + 1$
$\Rightarrow x_1^3 = x_2^3$
$\Rightarrow x_1 = x_2$
Therefore, f is one-one.
Also, for any y in $\mathbf{N}$, there exists $\sqrt[3]{y-1} \in \mathbf{R}^+ \cup \{0\}$ such that $f\left(\sqrt[3]{y-1}\right) = y$.
$\therefore f$ is onto.
Hence, f is bijective.
This shows that, f is invertible.
Let us consider a function $g: \mathbf{N} \rightarrow \mathbf{R}^+ \cup \{0\}$ such that $g(y) = \sqrt[3]{y-1}$
Now,
$gof(x) = g(f(x)) = g\left(x^3 + 1\right) = \sqrt[3]{\left(x^3 + 1\right) - 1} = x$
$fog(y) = f(g(y)) = f\left(\sqrt[3]{y-1}\right) = \left(\sqrt[3]{y-1}\right)^3 + 1 = y$
Therefore, we have
$gof(x) = I_{R^+ \cup \{0\}}$ and $fog(y) = I_{\mathbf{N}}$
$\therefore f^{-1}(y) = g(y) = \sqrt[3]{y-1}$

- **Relation:** A relation R from a set A to a set B is a subset of the Cartesian product A × B, obtained by describing a relationship between the first element x and the second element y of the ordered pairs (x, y) in A × B.
- The image of an element x under a relation R is y, where $(x, y) \in$ R
- **Domain:** The set of all the first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R.
- **Range and Co-domain:** The set of all the second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the co-domain of the relation R. Range ⊆Co-domain

Example: In the relation X from **W** to **R**, given by X = $\{(x, y): y = 2x + 1; x \in W, y \in R\}$, we obtain X = {(0, 1), (1, 3), (2, 5), (3, 7) …}. In this relation X, domain is the set of all whole numbers, i.e., domain = {0, 1, 2, 3 …}; range is the set of all positive odd integers, i.e., range = {1, 3, 5, 7 …}; and the co-domain is the set of all real numbers. In this relation, 1, 3, 5 and 7 are called the images of 0, 1, 2 and 3 respectively.

- The total number of relations that can be defined from a set A to a set B is the number of possible subsets of A × B.

If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$ and the total number of relations is 2^{pq}.