EXERCISE 27 A [Pg.No.: 1121]

1. A line passes through the point $(3,4,5)$ and is parallel to the vector $\left(2\hat{i}+2\hat{j}-3\hat{k}\right)$. Find the equations of the line in the vector as well as Cartesian forms

Sol. Position vector of point A(3, 4 , 5) is $\overrightarrow{OA} = 3\hat{i}+4\hat{j}+5\hat{k}$

Eqn. of live passing through A and parallel to the vector

$\vec{b} = 2\hat{i}+2\hat{j}-3\hat{k}$ is given by,

$\vec{r} = \overrightarrow{OA} + r\vec{b}$

i.e., $\vec{r} = (3\hat{i}+4\hat{j}+5\hat{k}) + r\left(2\hat{i}+2\hat{j}-3\hat{k}\right)$.

Cortesion form,

Eqn. of line passing through (α , β, y) and parallel to the vector $\vec{b} = a\hat{i}+b\hat{j}+c\hat{k}$ is given by,

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-y}{c}$$

Hence, required equation is, $\frac{x-3}{2} = \frac{y-4}{2} = \frac{z-5}{-3}$

2. A line passes through the point $(2,1,-3)$ and is parallel to the vector $\left(\hat{i}-2\hat{j}+3\hat{k}\right)$. Find the equations of the line in vector and Cartesian forms.

Sol. Vector equation of the given.

The line passes through the point $A(2, 1, -3)$ and is parallel to the vector $\vec{m} = \left(\hat{i}-2\hat{j}+3\hat{k}\right)$ also the position vector of A is $\vec{r_1} = 2\hat{i}+\hat{j}-3\hat{k}$.

$\therefore$ Vector equation of the given line is $\vec{r} = \vec{r_1} + \lambda\vec{m}$

$\Rightarrow \vec{r} = \left(2\hat{i}+\hat{j}-3\hat{k}\right) + \lambda\left(\hat{i}-2\hat{j}+3\hat{k}\right)$ ...(i)

Therefore, Cartesian equation of the given line, $\frac{x-2}{1} = \frac{y-1}{-2} = \frac{z+3}{3}$

3. Find the vector equation of the line passing through the point with position vector $\left(2\hat{i}+\hat{j}-5\hat{k}\right)$ and parallel to the vector $\left(\hat{i}+3\hat{j}-\hat{k}\right)$. Deduce the Cartesian equations of the line.

Sol. Vector equations of the given line passes through the point $A(2,1,-5)$ and is parallel to the vector $\vec{m} = \left(\hat{i}+3\hat{j}-\hat{k}\right)$. Also the position vector of A is $\vec{r_1} = \left(2\hat{i}+\hat{j}-5\hat{k}\right)$.

$\therefore$ Vector equation of the given line is $\vec{r} = \vec{r_1} + \lambda\vec{m}$

$\vec{r} = \left(2\hat{i}+\hat{j}-5\hat{k}\right) + \lambda\left(\hat{i}+3\hat{j}-\hat{k}\right)$ ...(i)

Cartesian equation of the given line, $\frac{x-2}{1} = \frac{y-1}{3} = \frac{z-(-5)}{-1}$ $\Rightarrow \frac{x-2}{1} = \frac{y-1}{3} = \frac{z+5}{-1}$

Hence $\frac{x-2}{1}=\frac{y-1}{3}=\frac{z+5}{-1}$ are the required equations of the given line in the Cartesian form.

4. A line is drawn in the direction of $\left(\hat{i}+\hat{j}-2\hat{k}\right)$ and it passes through a point with position vector $\left(2\hat{i}-\hat{j}-4\hat{k}\right)$. Find the equations of the line in the vector as well as Cartesian forms

Sol. Eqn. of line passing through the point having position vector $\vec{a}$ and parallel to the vector $\vec{b}$ is given by, $\vec{r}=\vec{a}+\vec{b}$

Hence, required vector equation of line is, $\vec{r}=\left(2\hat{i}-\hat{j}-4\hat{k}\right)+\mu\left(\hat{i}+\hat{j}-2\hat{k}\right)$

Now, $x\hat{i}+y\hat{j}+z\hat{k}=(2+\mu)\hat{i}+(-1+\mu)\hat{j}+(-4-2\mu)\hat{k}$

$\Rightarrow x=2+\mu, y=-1+\mu \ \& \ z=-4-2\mu \Rightarrow \frac{x-2}{1}=\frac{y+1}{1}=\frac{z+4}{-2}=\mu$

Hence, Cortes ion equation of line is $\frac{x-2}{1}=\frac{y+1}{1}=\frac{z+4}{-2}$

5. The Cartesian equations of a line are $\frac{x-3}{2}=\frac{y+2}{-5}=\frac{z-6}{4}$. Find the vector equation of the line.

Sol. Cartesian equations of the line is $\frac{x-3}{2}=\frac{y+2}{-5}=\frac{z-6}{4} \Rightarrow \frac{x-3}{2}=\frac{y-(-2)}{-5}=\frac{z-6}{4}$

Here, $x_1=3, y_1=-2, z_1=6$. So, $\vec{r_1}=3\hat{i}-2\hat{j}+6\hat{k}$

Here, $a=2, b=-5, c=4 \quad \therefore \vec{m}=2\hat{i}-5\hat{j}+4\hat{k}$

Vector equation of the given line is $\vec{r}=\vec{r_1}+\lambda\vec{m} \quad \Rightarrow \vec{r}=\left(3\hat{i}-2\hat{j}+6\hat{k}\right)+\lambda\left(2\hat{i}-5\hat{j}+4\hat{k}\right)$

6. The Cartesian equations of a line are $3x+1=6y-2=1-z$. Find the fixed point through which it passes, its direction ratios and also its vector equation.

Sol. Cartesian equations of the line is, $3x+1=6y-2=1-z$

$$\Rightarrow 3\left(x+\frac{1}{3}\right)=6\left(y-\frac{2}{6}\right)=-(z-1) \quad \Rightarrow \frac{x-\left(\frac{-1}{3}\right)}{\frac{1}{3}}=\frac{y-\frac{1}{3}}{\frac{1}{6}}=\frac{z-1}{-1}$$

Multiplying the denominator by 6, then $\frac{x-\left(-\frac{1}{3}\right)}{2}=\frac{y-\frac{1}{3}}{1}=\frac{z-1}{-6}$

$\Rightarrow \vec{r_1}=\left(-\frac{1}{3}\hat{i}+\frac{1}{3}\hat{j}+\hat{k}\right), \ \vec{m}=\left(2\hat{i}+\hat{j}-6\hat{k}\right)$. So, the fixed point $\left(-\frac{1}{3},\frac{1}{3},1\right)$

The vector equations of the line is $\vec{r}=\vec{r_1}+\lambda\vec{m} \quad \Rightarrow \vec{r}=\left(-\frac{1}{3}\hat{i}+\frac{1}{3}\hat{j}+\hat{k}\right)+\lambda\left(2\hat{i}+\hat{j}-6\hat{k}\right)$

7. Find the Cartesian equations of the line which passes through the point $(1,3,-2)$ and is parallel to the line given by $\frac{x+1}{3}=\frac{y-4}{5}=\frac{z+3}{-6}$. Also, find the vector form of the equations so obtained.

Sol. The given equations of parallel line is $\frac{x+1}{3}=\frac{y-4}{5}=\frac{z+3}{-6}$

Here, $a=3, b=5, c=-6$

Let be the points $\vec{r_1} = \left(\hat{i} + 3\hat{j} - 2\hat{k}\right)$, $\vec{m} = \left(3\hat{i} + 5\hat{j} - 6\hat{k}\right)$.

The vector equation of the line is $\vec{r} = \vec{r_1} + \lambda\vec{m}$.

$\Rightarrow \vec{r} = \left(\hat{i} + 3\hat{j} - 2\hat{k}\right) + \lambda\left(3\hat{i} + 5\hat{j} - 6\hat{k}\right)$ ...(i)

Cartesian equation of the given line is, $\frac{x-1}{3} = \frac{y-3}{5} = \frac{z+2}{-6}$

Hence $\frac{x-1}{3} = \frac{y-3}{5} = \frac{z+2}{-6}$ are the required equation of the given line in Cartesian form.

8. Find the equations of the line passing through the point $(1,-2,3)$ and parallel to the line $\frac{x-6}{3} = \frac{y-2}{-4} = \frac{z+7}{5}$.

Sol. Let the points be $\vec{r_1} = \left(\hat{i} - 2\hat{j} + 3\hat{k}\right)$

$\frac{x-6}{3} = \frac{y-2}{-4} = \frac{z+7}{5} \quad \Rightarrow \vec{m} = \left(3\hat{i} - 4\hat{j} + 5\hat{k}\right)$

$\Rightarrow$ Vector equation of a line is, $\vec{r} = \vec{r_1} + \lambda\vec{m} \quad \Rightarrow \vec{r} = \left(\hat{i} - 2\hat{j} + 3\hat{k}\right) + \lambda\left(3\hat{i} - 4\hat{j} + 5\hat{k}\right)$

Cartesian equation of the given line is, $\frac{x-1}{3} = \frac{y+2}{-4} = \frac{z-3}{5}$

Hence, $\frac{x-1}{3} = \frac{y+2}{-4} = \frac{z-3}{5}$ are the Cartesian equation of the given line.

9. Find the Cartesian and vector equations of a line which passes through the point $(1,2,3)$ and is parallel to the line $\frac{-x-2}{1} = \frac{y+3}{7} = \frac{2z-6}{3}$.

Sol. Let be the points $\vec{r_1} = \hat{i} + 2\hat{j} + 3\hat{k}$ and parallel to the line $\frac{-(x+2)}{1} = \frac{y+3}{7} = \frac{2(z-3)}{3}$

$\Rightarrow \frac{-x-2}{1} = \frac{y+3}{7} = \frac{2z-6}{3} \quad \Rightarrow \frac{(x+2)}{-1} = \frac{(y+3)}{7} = \frac{(z-3)}{3/2}$

$\Rightarrow \frac{x+2}{-1} = \frac{y+3}{7} = \frac{z-3}{3/2} \quad \Rightarrow \frac{x+2}{-2} = \frac{y+3}{14} = \frac{z-3}{3} \quad \Rightarrow \vec{m} = -2\hat{i} + 14\hat{j} + 3\hat{k}$

Vector equation of the line $\vec{r} = \vec{r_1} + \lambda\vec{m} \quad \Rightarrow \vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda\left(-2\hat{i} + 14\hat{j} + 3\hat{k}\right)$

Hence, $\frac{x-1}{-2} = \frac{y-2}{14} = \frac{z-3}{3}$ are Cartesian equation of given the line.

10. Find the equations of the line passing through the point $(-1,3,-2)$ and perpendicular to each of the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and $\frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}$

Sol. Let the direction ratios of the required line be a, b, c

This line being perpendicular to each of the given lines, we have

$a + 2b + 3c = 0$

$-3a + 2b + 5c = 0$

Cross multiplying (i) and (iii), we have $\frac{a}{-10-6} = \frac{b}{-9-5} = \frac{c}{2+6} = k(\text{let})$

$\Rightarrow a = 4k, b = -14k \ \& \ c = 8k$

Since, the line passes through (-1, 3, -2) $\therefore$ eqn. of line is, $\frac{x+1}{4k}=\frac{y-3}{-14k}=\frac{z+2}{8k}$

i.e, $\frac{x+1}{4}=\frac{y-3}{-14}=\frac{z+2}{8}$, this is the required eqn. of line.

11. Find the equations of the line passing through the point $(1,2,-4)$ and perpendicular to each of the lines $\frac{x-8}{8}=\frac{y+9}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$.

Sol. Let the direction ratio of the required line be a, b, c then, this line being perpendicular to each at the given lines, we have

$8a-16b+7c=0$ ...(i) $3a+8b-5c=0$ ...(ii)

Cross multiplying (i) and (ii) we get, $\frac{a}{80-56}=\frac{b}{21+40}=\frac{c}{64+48}=\lambda$

$\Rightarrow \frac{a}{24}=\frac{b}{61}=\frac{c}{112}=\lambda \Rightarrow a=24\lambda, b=61\lambda, c=112\lambda$

Thus the required lien has direction ratio $24\lambda,\ 61\lambda,\ 112\lambda$ and it passes through the point $(1,\ 2,\ -4)$.

Hence the required line equation is, $\frac{x-1}{24\lambda}=\frac{y-2}{61\lambda}=\frac{z+4}{112\lambda} \Rightarrow \frac{x-1}{24}=\frac{y-2}{61}=\frac{z+4}{112}$

12. Prove that the lines $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$ and $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$ intersect each other and find the point of their intersection.

Sol. The given line are $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}=\lambda(\text{say})$ ...(i)

and $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}=\mu(\text{say})$ ...(ii)

$P(\lambda+4,\ -4\lambda-3,\ 7\lambda-1)$ in any point on (i)

$Q(2\mu+1,-3\mu-1,8\mu-10)$ in any point on (ii)

Thus, the given lines will interest then $\lambda+4=2\mu+1,\ -4\lambda-3=-3\mu-1,\ 7\lambda-1=8\mu-10$

$\Rightarrow \lambda-2\mu=-3$ ...(i), $-4\lambda+3\mu=2$ ...(ii), $7\lambda-8\mu=-9$ ...(iii)

Solving equation (i) and (ii), we get $\lambda=1$, and $\mu=2$

Also these value of λ and μ satisfy (iii) hence the given lines intersect.

Putting $\lambda=1$ in P or $\mu=2$ in Q, we get the point of intersection of the given line as $(5,\ -7,\ 6)$

13. Show that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$ intersect each other. Also, find the point of their intersection.

Sol. The given equation of a line are $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=\lambda$ (say) ...(1)

and $\frac{x-4}{5}=\frac{y-1}{2}=z=k$ (say) ...(2)

$\Rightarrow x=2\lambda+1,\ y=3\lambda+2,\ z=4\lambda+3$

$\therefore$ $P(2\lambda+1,3\lambda+2,4\lambda+3)$ in any point on (1) and $Q(5k+4,2k+1,k)$ in any point on (2).

Thus, the given lines will interest if

$\Rightarrow 2\lambda+1=5k+4 \Rightarrow 2\lambda-5k=3$ ...(i)

$\Rightarrow 3\lambda+2=2k+1 \Rightarrow 3\lambda-2k=-1$ ...(ii) $\Rightarrow 4\lambda+3=k \Rightarrow 4\lambda-k=-3$ ...(iii)

Solving equation (i) and (ii), then we get $\lambda=-1, k=-1$.

Also these value of λ and k satisfy (iii) hence the given lines intersect.

Hence, two lines intersect to each other putting the value of λ in point P, then putting $\lambda=-1$ in P or $k=-1$ in Q.

We get the point of intersection of the given line as

$$P=\{2(-1)+1,\ 3(-1)+2,\ 4(-1)+3\}=(-2+1,-3+2,\ -4+3)=(-1,\ -1,\ -1)$$

Hence required point is $(-1,\ -1,\ -1)$.

14. Show that the lines $\frac{x-1}{2}=\frac{y+1}{3}=z$ and $\frac{x+1}{5}=\frac{y-2}{1}, z=2$ do not intersect each other.

Sol. The given line are $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z}{1}=\lambda(\text{say})$ ...(1)

$$\frac{x+1}{5}=\frac{y-2}{1}=\frac{z-2}{1}=\mu(\text{say}) \quad ...(2)$$

$P(2\lambda+1, 3\lambda-1, \lambda)$ in any point on (1). $Q(5\mu-1,\ \mu+2,\ \mu+2)$ in any point on (2).

If possible, let the given lines intersected.

Then, P and Q coincide form some particular value of λ and μ, in that case, we have,

$$2\lambda+1=5\mu-1,\ 3\lambda-1=\mu+2, \text{ and } \lambda=\mu+2$$

$\Rightarrow 2\lambda-5\mu=-2$ ...(i), $3\lambda-\mu=3$ ...(ii) & $\lambda-\mu=2$ ...(iii)

Solving equation (i) and (ii), we get $\lambda=\frac{17}{13}$ and $\mu=\frac{12}{13}$

However, these value of λ and μ do not satisfy (ii). Hence, the given lines do not intersect.

15. The Cartesian equations of a line are $3x+1=6y-2=1-z$. Find the fixed point through which it passes, its direction ratios and also its vector equation.

Sol. Cartesian equations of the line is, $3x+1=6y-2=1-z$

$$\Rightarrow 3\left(x+\frac{1}{3}\right)=6\left(y-\frac{2}{6}\right)=-(z-1) \quad \Rightarrow \frac{x-\left(\frac{-1}{3}\right)}{\frac{1}{3}}=\frac{y-\frac{1}{3}}{\frac{1}{6}}=\frac{z-1}{-1}$$

Multiplying the denominator by 6, then $\frac{x-\left(-\frac{1}{3}\right)}{2}=\frac{y-\frac{1}{3}}{1}=\frac{z-1}{-6}$

$\Rightarrow \vec{r_1}=\left(-\frac{1}{3}\hat{i}+\frac{1}{3}\hat{j}+\hat{k}\right), \vec{m}=\left(2\hat{i}+\hat{j}-6\hat{k}\right)$. So, the fixed point $\left(-\frac{1}{3}, \frac{1}{3}, 1\right)$

The vector equations of the line is $\vec{r}=\vec{r_1}+\lambda\vec{m} \Rightarrow \vec{r}=\left(-\frac{1}{3}\hat{i}+\frac{1}{3}\hat{j}+\hat{k}\right)+\lambda\left(2\hat{i}+\hat{j}-6\hat{k}\right)$

16. Find the length and the foot of the perpendicular drawn from the point $(2,-1,5)$ to the line $\frac{x-11}{10}=\frac{y+2}{-4}=\frac{z+8}{-11}$.

Sol. The given equation of the line $\frac{x-11}{10}=\frac{y+2}{-4}=\frac{z+8}{-11}=\lambda$

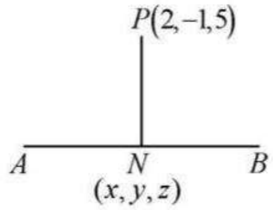
$\Rightarrow x=10\lambda+11,\ y=-4\lambda-2,\ z=-11\lambda-8$

$\therefore$ Co-ordinate of $N(10\lambda+11,-4\lambda-2,-11\lambda-8)$

$\therefore$ Direction ratios of PN

$=(10\lambda+11-2),(-4\lambda-2+1),(-11\lambda-8-5)$

$=(10\lambda+9,-4\lambda-1,-11\lambda-13)$



$PN \perp$ given line AB

$10(10\lambda+9)+(-4)(-4\lambda-1)+(-11)(-11\lambda-13)=0$

$\Rightarrow 100\lambda+90+16\lambda+4+121\lambda+143=0 \quad \Rightarrow 237\lambda+237=0 \quad \Rightarrow \lambda=-1$

Co-ordinates of $N=10(-1)+11,-4(-1)-2,-11(-1)-8=(-10+11,4-2,11-8)=(1,2,3)$

$\Rightarrow$ length of $PN=\sqrt{(1-2)^2+(2+1)^2+(3-5)^2}=\sqrt{(-1)^2+(3)^2+(-2)^2}=\sqrt{1+9+4}=\sqrt{14}$ units

17. Find the vector and Cartesian equations of the line passing through the points $A(3,4,-6)$ and $B(5,-2,7)$.

Sol. Let, $\vec{A}=\vec{r_1}=(3\hat{i}+4\hat{j}-6\hat{k})$, $\vec{B}=\vec{r_2}=(5\hat{i}-2\hat{j}+7\hat{k})$

$\Rightarrow (\vec{r_2}-\vec{r_1})=(5\hat{i}-2\hat{j}+7\hat{k})-(3\hat{i}+4\hat{j}-6\hat{k})$

$\therefore$ Vector equation of line is, $\vec{r}=\vec{r_1}+\lambda(\vec{r_2}-\vec{r_1}) \Rightarrow \vec{r}=(3\hat{i}+4\hat{j}-6\hat{k})+\lambda(2\hat{i}-6\hat{j}+13\hat{k})$

$\therefore$ Cartesian equation of a line is, $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

$\Rightarrow \frac{x-3}{5-3}=\frac{y-4}{-2-4}=\frac{z-(-6)}{7+6} \quad \Rightarrow \frac{x-3}{2}=\frac{y-4}{-6}=\frac{z+6}{13}$

18. Find the vector and Cartesian equations of the line passing through the points $A(2,-3,0)$ and $B(-2,4,3)$.

Sol. Vector equations of the given line

Let the position vector of A & B be $\vec{r_1}$ & $\vec{r_2}$ be the direction ratios respectively then

$\vec{r_1}=(2\hat{i}-3\hat{j}+0\hat{k})$ and $\vec{r_2}=(-2\hat{i}+4\hat{j}+3\hat{k})$

$\Rightarrow (\vec{r_2}-\vec{r_1})=(-2\hat{i}+4\hat{j}+3\hat{k})-(2\hat{i}-3\hat{j})=(-4\hat{i}+7\hat{j}+3\hat{k})$

$\therefore$ Vector equation of a line AB is $\vec{r}=\vec{r_1}+\lambda(\vec{r_2}-\vec{r_1})$ for some scalar λ

i.e., $\vec{r}=(2\hat{i}-3\hat{j})+\lambda(-4\hat{i}+7\hat{j}+3\hat{k}) \quad \ldots$(i)

Cartesian equation of a line is, $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

$\Rightarrow \frac{x-2}{-2-2}=\frac{y+3}{4+3}=\frac{z-0}{3-0} \quad \Rightarrow \frac{x-2}{-4}=\frac{y+3}{7}=\frac{z}{3}$

Hence, $\frac{x-2}{-4}=\frac{y+3}{7}=\frac{z}{3}$ are the Cartesian equation of the given line.

19. Find the vector and Cartesian equations of the line joining the points whose position vectors are $(\hat{i}-2\hat{j}+\hat{k})$ and $(\hat{i}+3\hat{j}-2\hat{k})$.

Sol. Let the position vector of A & B be $\vec{r_1}$ & $\vec{r_2}$ respectively then, $\vec{r_1}=(\hat{i}-2\hat{j}+\hat{k})$ & $\vec{r_2}=(\hat{i}+3\hat{j}-2\hat{k})$

$\Rightarrow (\vec{r}_2 - \vec{r}_1) = (\hat{i} + 3\hat{j} - 2\hat{k}) - (\hat{i} - 2\hat{j} + \hat{k}) = (5\hat{j} - 3\hat{k})$

$\therefore$ Vector equation of a line is, $\vec{r} = \vec{r}_1 + \lambda(\vec{r}_2 - \vec{r}_1)$ for some scalar λ.

$\Rightarrow \vec{r} = (\hat{i} - 2\hat{j} + \hat{k}) + \lambda(5\hat{j} - 3\hat{k})$

Cartesian equation of the given line is, $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$

$\Rightarrow \dfrac{x-1}{1-1} = \dfrac{y-(-2)}{3-(-2)} = \dfrac{z-1}{-2-1} \quad \Rightarrow \dfrac{x-1}{0} = \dfrac{y+2}{5} = \dfrac{z-1}{-3}$

Hence, $\dfrac{x-1}{0} = \dfrac{y+2}{5} = \dfrac{z-1}{-3}$ are the Cartesian equation of the given line.

20. Find the vector equation of a line passing through the point $A(3, -2, 1)$ and parallel to the line joining the points $B(-2, 4, 2)$ and $C(2, 3, 3)$. Also, find the Cartesian equations of the line.

Sol. Let be the points $\vec{r}_1 = (3\hat{i} - 2\hat{j} + \hat{k})$ and parallel to the line joining the points $\vec{B} = (-2\hat{i} + 4\hat{j} + 2\hat{k})$ & $\vec{C} = (2\hat{i} + 3\hat{j} + 3\hat{k})$

$\Rightarrow \overrightarrow{BC} =$ position vector of C – position vector of B

$= (2\hat{i} + 3\hat{j} + 3\hat{k}) - (-2\hat{i} + 4\hat{j} + 2\hat{k}) = (4\hat{i} - \hat{j} + \hat{k}) \quad \therefore \ \vec{m} = (4\hat{i} - \hat{j} + \hat{k})$

$\therefore$ Vector equation of a line is $\vec{r} = \vec{r}_1 + \lambda\vec{m} \quad \Rightarrow \vec{r} = (3\hat{i} - 2\hat{j} + \hat{k}) + \lambda(4\hat{i} - \hat{j} + \hat{k})$

Cartesian equation of the given line, $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c} \quad \Rightarrow \dfrac{x-3}{4} = \dfrac{y+2}{-1} = \dfrac{z-1}{1}$

Hence $\dfrac{x-3}{4} = \dfrac{y+2}{-1} = \dfrac{z-1}{1}$ are the Cartesian equations of the given line.

21. Find the vector equation of a line passing through the point having the position vector $(\hat{i} + 2\hat{j} - 3\hat{k})$ and parallel to the line joining the points with position vector $(\hat{i} - \hat{j} + 5\hat{k})$ and $(2\hat{i} + 3\hat{j} - 4\hat{k})$. Also, find the Cartesian equivalents of this equation.

Sol. Let be the points $\vec{r}_1 = (\hat{i} + 2\hat{j} - 3\hat{k})$ and parallel to the line joining the position vector $\vec{A} = (\hat{i} - \hat{j} + 5\hat{k})$ and $\vec{B} = (2\hat{i} + 3\hat{j} - 4\hat{k})$.

$\Rightarrow \overrightarrow{AB} =$ Position vector of B – of position vector of A

$= (2\hat{i} + 3\hat{j} - 4\hat{k}) - (\hat{i} - \hat{j} + 5\hat{k}) = (\hat{i} + 4\hat{j} - 9\hat{k}) = \vec{m}$

$\therefore$ Vector equation of a line is, $\vec{r} = \vec{r}_1 + \lambda\vec{m} \quad \Rightarrow \vec{r} = (\hat{i} + 2\hat{j} - 3\hat{k}) + \lambda(\hat{i} + 4\hat{j} - 9\hat{k})$

$\therefore$ Cartesian equation of the given line is, $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c} \quad \Rightarrow \dfrac{x-1}{1} = \dfrac{y-2}{4} = \dfrac{z+3}{-9}$

Hence, $\dfrac{x-1}{1} = \dfrac{y-2}{4} = \dfrac{z+3}{-9}$ are the Cartesian equations of the given line.

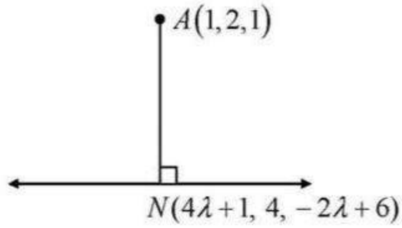
22. Find the coordinates of the foot of the perpendicular drawn from the point $A(1, 2, 1)$ to the line joining the points $B(1, 4, 6)$ and $C(5, 4, 4)$.

Sol. The equation of line BC is

$$\frac{x-1}{5-1}=\frac{y-4}{4-4}=\frac{z-6}{4-6}=\lambda$$

$$\Rightarrow \frac{x-1}{4}=\frac{y-4}{0}=\frac{z-6}{-2}=\lambda \qquad \ldots(1)$$

The general point on this line is, $(4\lambda+1,\ 4,\ -2\lambda+6)$.



Let N be the foot of the perpendicular drawn from the point $A(1,\ 2,\ 1)$ to the given line.

Any point on line BC will be, $N(4\lambda+1,\ 4,\ -2\lambda+6)$ for some value of λ.

Direction ratio of AN are $(4\lambda+1-1,\ 4-2,\ -2\lambda+6-1) \quad \Rightarrow (4\lambda,\ 2,\ -2\lambda+5)$

Direction of given line (1) are 4, 0, –2.

Since, AN perpendicular to given line (1), we have,

$$4(4\lambda)+0.2-2(-2\lambda+5)=0 \quad \Rightarrow 16\lambda+4\lambda-10=0 \quad \Rightarrow 20\lambda=10 \quad \Rightarrow \lambda=\frac{1}{2}$$

So, the required point of $N(3,\ 4,\ 5)$.

Hence, the required coordinates of foot of the perpendicular is $(3,\ 4,\ 5)$.

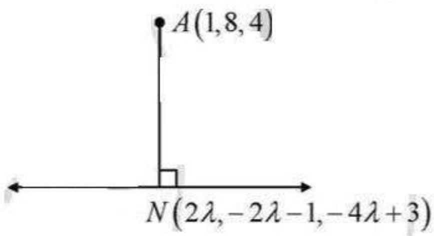
23. Find the coordinates of the foot of the perpendicular drawn from the point $A(1,8,4)$ to the line joining the points $B(0,-1,3)$ and $C(2,-3,-1)$.

Sol. The given line BC is

$$\frac{x-0}{2-0}=\frac{y-(-1)}{-3-(-1)}=\frac{z-3}{-1-3}=\lambda$$

$$\frac{x}{2}=\frac{y+1}{-2}=\frac{z-3}{-4}=\lambda(\text{say})\ldots(\text{i})$$

The general point on this line is $(2\lambda,\ -2\lambda-1,\ -4\lambda+3)$.



Let N be the foot of the perpendicular drawn from the point $A(1,8,4)$ to the given line.

Then, this point is $N(2\lambda,-2\lambda-1,-4\lambda+3)$ for some value of λ.

Direction ratio of AN are $(2\lambda-1,-2\lambda-1-8,-4\lambda+3-4) \quad \Rightarrow (2\lambda-1),(-2\lambda-9),(-4\lambda-1)$

Direction ratio of given line (i) are $(2,\ -2,\ -4)$

Since $AN \perp$ given line (i) we have, $2(2\lambda-1)-2(-2\lambda-9)-4(-4\lambda-1)=0$

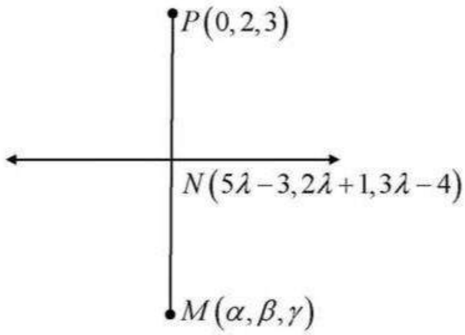
$$\Rightarrow 4\lambda-2+4\lambda+18+16\lambda+4=0 \quad \Rightarrow 24\lambda+20=0 \quad \Rightarrow \lambda=\frac{-20}{24}=\frac{-5}{6}$$

So, the required point of $N\left(\frac{-5}{3},\frac{2}{3},\frac{19}{3}\right)$.

Hence the required co–ordinate foot of the perpendicular is $\left(\frac{-5}{3},\frac{2}{3},\frac{19}{3}\right)$.

24. Find the image of the point $(0,2,3)$ in the line $\frac{x+3}{5}=\frac{y-1}{2}=\frac{z+4}{3}$.

Sol. The given line is $\frac{x+3}{5}=\frac{y-1}{2}=\frac{z+4}{3}$ ...(1)



Let N be the foot of the perpendicular drawn from the point $P(0,2,3)$ to the given line.

$\therefore N$ has the co–ordinate $(5\lambda-3,2\lambda+1,3\lambda-4)$

The direction ratio of PN are

$5\lambda-3-0, 2\lambda+1-2, 3\lambda-4-3$

$\Rightarrow (5\lambda-3),(2\lambda-1),(3\lambda-7)$

Also the direction ratio of the given line (i) are $5,2,3$ since PN is perpendicular to the given line (i) we have $5(5\lambda-3)+2(2\lambda-1)+3(3\lambda-7)=0$

$\Rightarrow 25\lambda-15+4\lambda-2+9\lambda-21=0 \quad \Rightarrow 38\lambda-38=0 \quad \Rightarrow \lambda=1$

Putting $\lambda=1$, we get the point $N(2,3,-1)$

Let $M(\alpha,\beta,\gamma)$ be the image of $P(0,2,3)$, in the given line.

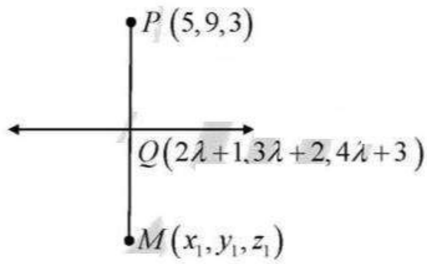
Then $N(2,3,-1)$ is the mid point of PM.

$\therefore \frac{\alpha+0}{2}=2, \frac{\beta+2}{2}=3$ and $\frac{\gamma+3}{2}=-1 \quad \Rightarrow \alpha=4, \beta=4$ and $\gamma=-5$

Hence, the image of $P(0,2,3)$ is $M(4,4,-5)$.

25. Find the image of the point $(5,9,3)$ in the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$

Sol. The given equation of the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=\lambda$ ...(i)



$\Rightarrow x=2\lambda+1, y=3\lambda+2, z=4\lambda+3$

co-ordinate of Q, $(2\lambda+1-5),(3\lambda+2-9),(4\lambda+3-3)$,

$Q=(2\lambda-4,3\lambda-7,4\lambda)$

Since, PQ is perpendicular to the given line (i) we have

$2(2\lambda-4)+3(3\lambda-7)+4(4\lambda)=0$

$\Rightarrow 4\lambda-8+9\lambda-21+16\lambda=0 \Rightarrow 29\lambda-29=0 \Rightarrow 29\lambda=29 \Rightarrow \lambda=1$

Putting the value of $\lambda=1$,

we get the point $Q=(2+1,3+2,4+3)=(3,5,7)$.

Let $M(x_1,y_1,z_1)$ be the image of $P(5,9,3)$ in the given line then $N(3,5,7)$ is the mid point of P.M.

$\Rightarrow \frac{5+x_1}{2}=3 \quad \Rightarrow 5+x_1=6 \quad \Rightarrow x_1=6-5 \quad \Rightarrow x_1=1$

and $\frac{9+y_1}{2}=5 \quad \Rightarrow 9+y_1=10 \quad \Rightarrow y_1=10-9 \quad \Rightarrow y_1=1$

and $\frac{3+z_1}{2}=7 \quad \Rightarrow 3+z_1=14 \quad \Rightarrow z_1=14-3 \quad \Rightarrow z_1=11$

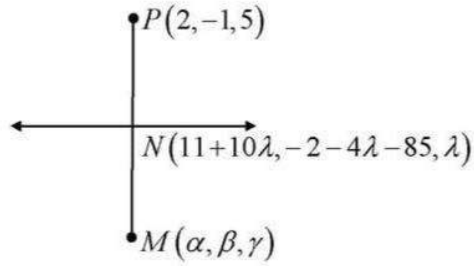
Hence image of point is $(1, 1, 11)$

26. Find the image of the point $(2,-1,5)$ in the line $\vec{r}=(11\hat{i}-2\hat{j}-8\hat{k})+\lambda(10\hat{i}-4\hat{j}-11\hat{k})$.

Sol. The given line is $\vec{r}=(11\hat{i}-2\hat{j}-8\hat{k})+\lambda(10\hat{i}-4\hat{j}-11\hat{k})$ ...(i)

Cartesian equation the given line.

$$\frac{x-11}{10}=\frac{y+2}{-4}=\frac{z+8}{-11}=\lambda \quad \text{...(ii)}$$



Let N be the foot at the perpendicular drawn from the point

$P(2,-1,5)$ to the given line.

$\therefore$ M has the coordinate $(11+10\lambda,-2-4\lambda,-8-11\lambda)$

The direction ratio of PN are,

$11+10\lambda-2,-2-4\lambda+1,-8-11\lambda-5$ i.e., $(9+10\lambda),(-1-4\lambda),(-13-11\lambda)$

Also the direction ratio at the given line are $10,-4,-11$.

Since PN is perpendicular to the given line (i), we have

$10(9+10\lambda)-4(-1-4\lambda)-11(-13-11\lambda)=0$

$\Rightarrow 90+100\lambda+4+16\lambda+143+121\lambda=0 \quad \Rightarrow 237\lambda+237=0 \quad \Rightarrow 237\lambda=-237 \quad \Rightarrow \lambda=-1$

Putting $\lambda=-1$, we get the point $N(1,\ 2,\ 3)$.

Let $M(\alpha,\ \beta,\ \gamma)$ be the image of $P(2,\ -1,\ 5)$ in the given line.

The $N(1,\ 2,\ 3)$ is the mid point of P.M.

$\therefore \dfrac{\alpha+2}{2}=1,\ \dfrac{\beta-1}{2}=2$ and $\dfrac{\gamma+5}{2}=3 \quad \Rightarrow \alpha+2=2,\ \beta-1=4$ and $\gamma+5=6$

$\Rightarrow \alpha=0,\ \beta=5,$ and $\gamma=1$. Hence, the image of $P(2,\ -1,\ 5)$ is $M(0,\ 5,\ 1)$.

EXERCISE 27 B [Pg.No.: 1129]

1. Prove that the points $A(2,1,3), B(-4,3,-1)$ and $C(5,0,5)$ are collinear.

Sol. Let $A=(2,1,3)$, $B=(-4,3,-1)$ & $C=(5,0,5)$

The equation of the line AB are

$$\Rightarrow \frac{x-2}{-4-2}=\frac{y-1}{3-1}=\frac{z-3}{-1-3} \quad \Rightarrow \frac{x-2}{-6}=\frac{y-1}{2}=\frac{z-3}{-4} \quad \text{...(i)}$$

The given points A, B, C are collinear

$\Rightarrow$ lies on the line AB $\Rightarrow C(5,0,5)$ satisfied (i)

$$\Rightarrow \frac{5-2}{-6}=\frac{0-1}{2}=\frac{5-3}{-4} \quad \Rightarrow \frac{3}{-6}=\frac{-1}{2}=\frac{2}{-4} \quad \Rightarrow \frac{1}{-2}=\frac{-1}{2}=\frac{1}{-2}$$

Hence the given points A, B and C are collinear.

2. Show that the points $A(2,3,-4), B(1,-2,3)$ and $C(3,8,-11)$ are collinear

Sol. The eqn. of line AB is. $\dfrac{x-2}{1-2}=\dfrac{y-3}{-2-3}=\dfrac{z+4}{3-(-4)}$

$\Rightarrow \dfrac{x-2}{-1}=\dfrac{y-3}{-5}=\dfrac{z+4}{7}$........(i)

Putting x = 3, y = 8 & z = -11 in eqn. (i), we have $\frac{3-2}{-1}=\frac{8-3}{-5}=\frac{-11+4}{7}$, which is true

Thus, the point e(3, 8, -11) lies on the line AB.

$\therefore$ Hence, the given points A, B and C are collinear.

3. Find the value of λ for which the points $A(2,5,1), B(1,2,-1)$ and $C(3,\lambda,3)$ are collinear

Sol. The equation of AB is,

$\frac{x-2}{1-2}=\frac{y-5}{2-5}=\frac{z-1}{-1-1}$ .... (i)

$\because$ points A, B & C are collinear

$\therefore$ c (3, λ, 3) satisfies the equation (i).

$\therefore \frac{3-2}{1-2}=\frac{\lambda-5}{2-5}=\frac{3-1}{-1-1} \Rightarrow -1=\frac{\lambda-5}{-3} \Rightarrow 3=\lambda-5 \quad \Rightarrow \lambda=8$ Ans.

4. Find the values of λ and μ so that points $A(3,2,-4), B(9,8,-10)$ and $C(\lambda,\mu,-6)$ are collinear

Sol. Eqn. of line passing through A(3, 2, -4) and B(9, 8, -10) is given by

$\frac{x-3}{9-3}=\frac{y-2}{8-2}=\frac{z+4}{-10+4}$

Since, the line passes through $c(\lambda,\mu,-6)$

$\because \frac{\lambda-3}{6}=\frac{\mu-2}{6}=\frac{-6+4}{-6}$

$\Rightarrow \lambda-3=2$ & $\mu-2=2$

$\Rightarrow \lambda=5$ & $\mu=4$ Ans.

5. Using the vector method, find the values of λ and μ so that the points $A(-1, 4, -2)$, $B(\lambda, \mu, 1)$ and $C(0,2,-1)$ are collinear.

Sol. Let , $\vec{a}, \vec{b}, \vec{c}$ be the position vector of the given point A, B, C respectively then

$\vec{a}=\left(-\hat{i}+4\hat{j}-2\hat{k}\right)$, $\vec{b}=\left(\lambda\hat{i}+\mu\hat{j}+\hat{k}\right)$ & $\vec{c}=\left(0\hat{i}+2\hat{j}-\hat{k}\right)$

$\Rightarrow \overrightarrow{AC}=$ Position vector of C – Position vector of A

$=\left(2\hat{j}-\hat{k}\right)-\left(-\hat{i}+4\hat{j}-2\hat{k}\right)=\left(\hat{i}-2\hat{j}+\hat{k}\right)$

$\therefore$ Vector equation of a line $\vec{r}=\vec{r_1}+t\left(\overrightarrow{AC}\right)$

$\Rightarrow \vec{r}=\left(-\hat{i}+4\hat{j}-2\hat{k}\right)+\left(t\hat{i}-2\hat{j}+\hat{k}\right) \qquad \Rightarrow \vec{r}=\hat{i}\left(-i+t\right)+\hat{j}\left(4-2t\right)+\hat{k}\left(-2+t\right)$ ...(i)

If the line AC passes through the point B we have

$\vec{b}=\hat{i}\left(-i+t\right)+\hat{j}\left(4-2t\right)+\hat{k}\left(-2+t\right)$ (for some scalar t)

$\Rightarrow \lambda\hat{i}+\mu\hat{j}+\hat{k}=\hat{i}\left(-1+t\right)+\hat{j}\left(4-2t\right)+\hat{k}\left(-2+t\right)$

Equating coefficient both side $\hat{i}, \hat{j}$ and $\hat{k}$ we get

$\Rightarrow \lambda=-1+t$ ...(A) $\qquad \mu=4-2t$ ...(B)

$\Rightarrow 1=-2+t \quad \Rightarrow t=3$

From equation (A), $\lambda=-1+3 \Rightarrow \lambda=2$

From equation (B), $\mu=4-2t \Rightarrow \mu=4-2(3) \Rightarrow \mu=-2$. Hence $\lambda=2$ and $\mu=-2$.

6. The position vectors of three points A, B and C are $\left(-4\hat{i}+2\hat{j}-3\hat{k}\right),\left(\hat{i}+3\hat{j}-2\hat{k}\right)$ and $\left(-9\hat{i}+\hat{j}-4\hat{k}\right)$ respectively. Show that the points A, B and C are collinear

Sol. The co-ordinates of given points are A(-4, 2, -3), B(1, 3, -2) and c(-9, 1, -4)

Eqn. of line passing through A (-4, 2, -3) and B(1, 3, -2) is

$$\frac{x+4}{1+4}=\frac{y-2}{3-2}=\frac{z+3}{-2+3}$$

$$\Rightarrow \frac{x+4}{5}=\frac{y-2}{1}=\frac{z+3}{1}\text{.........(i)}$$

Putting x = -9, y = 1 & z = -4, we get

$\frac{-9+4}{5}=\frac{1-2}{1}=\frac{-4+3}{1}$ which is true.thus, C(-9, 1, -4) lies on line AB

Hence, the points A, B & C are Collinear.

EXERCISE 27 C [Pg.No.: 1134]

Find the angle between each of the following pairs of lines

1. $\vec{r}=\left(3\hat{i}+\hat{j}-2\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}-2\hat{k}\right)$ and $\vec{r}=\left(2\hat{i}-\hat{j}-5\hat{k}\right)+\mu\left(3\hat{i}-5\hat{j}-4\hat{k}\right)$

Sol. The given lives are of the form.

$\vec{r}=\vec{a_1}+r\vec{b_1}$ and $\vec{r}=\vec{a_2}+\delta\vec{b_2}$

where, $\vec{b_1}=\hat{i}-\hat{j}\ 2\hat{k}$ and $\vec{b_2}=3\hat{i}-5\hat{j}-4\hat{k}$

Let, θ be the angle between the lines. $\therefore \theta\cos^{-1}\frac{|\vec{b_1}\cdot\vec{b_2}|}{|\vec{b_1}||\vec{b_2}|}$

$$\Rightarrow \theta=\cos^{-1}\frac{|3+5+8|}{\sqrt{6}\sqrt{50}} \Rightarrow \theta=\cos^{-1}\frac{16}{\sqrt{300}} \Rightarrow \theta=\cos^{-1}\frac{16}{10\sqrt{3}} \Rightarrow \theta=\cos^{-1}\frac{8\sqrt{3}}{15}$$

2. $\vec{r}=\left(3\hat{i}-4\hat{j}+2\hat{k}\right)+\lambda\left(\hat{i}+3\hat{k}\right)$ and $\vec{r}=5\hat{i}+\mu\left(-\hat{i}+\hat{j}+\hat{k}\right)$

Sol. $\vec{r}=\left(3\hat{i}-4\hat{j}+2\hat{k}\right)+\lambda\left(\hat{i}+3\hat{k}\right)$ and $\vec{r}=\left(5\hat{i}\right)+\mu\left(-\hat{i}+\hat{j}+\hat{k}\right)$

Let, $\vec{m}_1=\left(\hat{i}+3\hat{k}\right)$ and $\vec{m}_2=\left(-\hat{i}+\hat{j}+\hat{k}\right)$

$$\therefore \cos\theta=\left(\frac{\vec{m}_1.\vec{m}_2}{|\vec{m}_1||\vec{m}_2|}\right) \Rightarrow \cos\theta=\left(\frac{\left(\hat{i}+3\hat{k}\right).\left(-\hat{i}+\hat{j}+\hat{k}\right)}{\sqrt{(1)^2+(3)^2}\sqrt{(-1)^2+(1)^2(1)^2}}\right)$$

$$\Rightarrow \cos\theta=\left(\frac{-1+3}{\sqrt{1+9}\sqrt{1+1+1}}\right) \Rightarrow \cos\theta=\left(\frac{2}{\sqrt{10}\sqrt{3}}\right) \Rightarrow \cos\theta=\left(\frac{2}{\sqrt{30}}\right)$$

$$\Rightarrow \theta=\cos^{-1}\left(\frac{2}{\sqrt{30}}\right) \Rightarrow \theta=\cos^{-1}\left(\frac{2}{\sqrt{30}}\times\frac{\sqrt{30}}{\sqrt{30}}\right) \Rightarrow \theta=\cos^{-1}\left(\frac{2\sqrt{30}}{30}\right) \Rightarrow \theta=\cos^{-1}\left(\frac{\sqrt{30}}{15}\right)$$

Hence, the angle between the given line is $\cos^{-1}\left(\frac{\sqrt{30}}{15}\right)$.

3. $\vec{r}=\left(\hat{i}-2\hat{j}\right)+\lambda\left(2\hat{i}-2\hat{j}+\hat{k}\right)$ and, $\vec{r}=\left(3\hat{k}\right)+\mu\left(\hat{i}+2\hat{j}-2\hat{k}\right)$

Sol. $\vec{r}=\left(\hat{i}-2\hat{j}\right)+\lambda\left(2\hat{i}-2\hat{j}+\hat{k}\right)$ and $\vec{r}=\left(3\hat{k}\right)+\mu\left(\hat{i}+2\hat{j}-2\hat{k}\right)$

Let $\vec{m}_1=\left(2\hat{i}-2\hat{j}+\hat{k}\right)$ & $\vec{m}_2=\left(\hat{i}+2\hat{j}-2\hat{k}\right)$

$$\therefore\ \cos\theta=\left(\frac{\vec{m}_1.\vec{m}_2}{|\vec{m}_1||\vec{m}_2|}\right)\quad\Rightarrow\cos\theta=\left(\frac{\left(2\hat{i}-2\hat{j}+\hat{k}\right).\left(\hat{i}+2\hat{j}-2\hat{k}\right)}{\sqrt{(2)^2+(-2)^2+(1)^2}\sqrt{(1)^2+(2)^2+(-2)^2}}\right)$$

$$\Rightarrow\ \cos\theta=\left(\frac{2-4-2}{\sqrt{4+4+1}\sqrt{1+4+4}}\right)\quad\Rightarrow\cos\theta=\left(\frac{-4}{3\times3}\right)\quad\Rightarrow\cos\theta=\left(\frac{-4}{9}\right)\quad\Rightarrow\theta=\cos^{-1}\left(\frac{-4}{9}\right)$$

Hence, the angle between the given line is $\cos^{-1}\left(\frac{-4}{9}\right)$.

Find the angle between each of the following pairs of lines:

4. $\frac{x+1}{1}=\frac{y-4}{1}=\frac{z-5}{2}$ and $\frac{x+3}{3}=\frac{y-2}{5}=\frac{z+5}{4}$

Sol. $\frac{x+1}{1}=\frac{y-4}{1}=\frac{z-5}{2}$ and $\frac{x+3}{3}=\frac{y-2}{5}=\frac{z+5}{4}$

The directions ratios of the given line (1, 1, 2) & (3, 5, 4)

$$\therefore\ \cos\theta=\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)\quad\Rightarrow\cos\theta=\left(\frac{1(3)+1(5)+2(4)}{\sqrt{(1)^2+(1)^2+(2)^2}\sqrt{(3)^2+(5)^2+(4)^2}}\right)$$

$$\Rightarrow\ \cos\theta=\left(\frac{3+5+8}{\sqrt{1+1+4}\sqrt{9+25+16}}\right)\quad\Rightarrow\cos\theta=\left(\frac{16}{\sqrt{6}\sqrt{50}}\right)\quad\Rightarrow\cos\theta=\left(\frac{16}{\sqrt{300}}\right)$$

$$\Rightarrow\ \cos\theta=\left(\frac{16}{10\sqrt{3}}\right)\quad\Rightarrow\cos\theta=\left(\frac{8}{5\sqrt{3}}\right)\quad\Rightarrow\theta=\cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

$$\Rightarrow\ \theta=\cos^{-1}\left(\frac{8}{5\sqrt{3}}\times\frac{5\sqrt{3}}{5\sqrt{3}}\right)\quad\Rightarrow\theta=\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$$

Hence the angle between the given line $\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$.

5. $\frac{x-4}{3}=\frac{y+1}{4}=\frac{z-6}{5}$ and $\frac{x-5}{1}=\frac{2y+5}{-2}=\frac{z-3}{1}$

Sol. $\frac{x-4}{3}=\frac{y+1}{4}=\frac{z-6}{5}$ and $\frac{x-5}{1}=\frac{2y+5}{-2}=\frac{z-3}{1}$

The given equation of a line is $\frac{x-4}{3}=\frac{y+1}{4}=\frac{z-6}{5}$ & $\frac{x-5}{1}=\frac{y+5/2}{-1}=\frac{z-3}{1}$

The directions ratio of the given line are (3, 4, 5) & (1, –1, 1)

$$\therefore\ \cos\theta=\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)\quad\Rightarrow\ \cos\theta=\left(\frac{3(1)+4(-1)+5(1)}{\sqrt{(3)^2+(4)^2+(5)^2}\sqrt{(1)^2+(-1)^2+(1)^2}}\right)$$

$$\Rightarrow\ \cos\theta=\left(\frac{3-4+5}{\sqrt{9+16+25}\sqrt{1+1+1}}\right)\quad\Rightarrow\cos\theta\left(\frac{4}{\sqrt{50}\sqrt{3}}\right)\quad\Rightarrow\cos\theta=\left(\frac{4}{\sqrt{150}}\right)$$

$$\Rightarrow\ \cos\theta=\frac{4}{5\sqrt{6}}\quad\Rightarrow\ \cos\theta=\frac{4}{5\times\sqrt{6}}\times\frac{\sqrt{6}}{\sqrt{6}}\quad\Rightarrow\ \cos\theta=\frac{2\sqrt{6}}{15}\quad\Rightarrow\ \cos^{-1}\left(\frac{2\sqrt{6}}{15}\right)$$

Hence the angles between the given lines is $\cos^{-1}\left(\frac{2\sqrt{6}}{15}\right)$

6. $\frac{3-x}{-2}=\frac{y+5}{1}=\frac{1-z}{3}$ and $\frac{x}{3}=\frac{1-y}{-2}=\frac{z+2}{-1}$

Sol. $\frac{3-x}{-2}=\frac{y+5}{1}=\frac{1-z}{3}$ and $\frac{x}{3}=\frac{1-y}{-2}=\frac{z+2}{-1}$

The given equation of the line are, $\frac{3-x}{-2}=\frac{y+5}{1}=\frac{1-z}{3}$

$\Rightarrow \frac{-(x-3)}{-2}=\frac{y+5}{1}=\frac{-(z-1)}{3} \quad \Rightarrow \frac{x-3}{2}=\frac{y+5}{1}=\frac{z-1}{-3}$

Another given equation of the line are, $\frac{x}{3}=\frac{1-y}{-2}=\frac{z+2}{-1} \quad \Rightarrow \frac{x}{3}=\frac{-(y-1)}{-2}=\frac{z+2}{-1}$

The direction ratio of the given line are (2,1,–3) & (3,–2,–1)

$\therefore \cos\theta=\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)$

$\Rightarrow \cos\theta=\left(\frac{2(3)+1(2)+(-3)(-1)}{\sqrt{(2)^2+(1)^2+(-3)^2}\sqrt{(3)^2+(2)^2+(-1)^2}}\right) \quad \Rightarrow \cos\theta=\left(\frac{6+2+3}{\sqrt{4+1+9}\sqrt{9+4+1}}\right)$

$\Rightarrow \cos\theta=\left(\frac{11}{\sqrt{14}\sqrt{14}}\right) \quad \Rightarrow \cos\theta=\left(\frac{11}{14}\right) \quad \Rightarrow \theta=\cos^{-1}\left(\frac{11}{14}\right)$

Hence the angles between the given lines is $\cos^{-1}\left(\frac{11}{14}\right)$.

7. $\frac{x}{1}=\frac{z}{-1}, y=0$ and $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}$

Sol. Direction ratios of line $\frac{x}{1}=\frac{z}{-1}, \quad y=0$ are 1, 0, -1

Direction ratios of line $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}$ are 3, 4, 5

Let, θ = Angle b/ω the lines

$\therefore \theta=\cos^{-1}\frac{|1\times3+0\times4+(-1)\times5|}{\sqrt{1^2+0^2+(-1)^2}\sqrt{3^2+4^2+5^2}}$

$\Rightarrow \theta=\cos^{-1}\frac{2}{\sqrt{2}\sqrt{50}} \Rightarrow \theta=\cos^{-1}\frac{2}{10} \Rightarrow \theta=\cos^{-1}\frac{1}{5}$ Ans.

8. $\frac{5-x}{3}=\frac{y+3}{-2}, z=5$ and $\frac{x-1}{1}=\frac{1-y}{3}=\frac{z-5}{2}$

Sol. Given lines are,

$\frac{5+x}{3}=\frac{y+3}{-2}, z=5$ ....(i)

and. $\frac{x-1}{1}=\frac{1-y}{3}=\frac{z-5}{2}$ ....(ii)

from (i) $\frac{x-5}{-3}=\frac{y+3}{-2}=\frac{z-5}{0}$

Here, Direction ratios are -3, -2, 0

from (ii), $\frac{x-1}{1}=\frac{y-1}{-3}=\frac{z-5}{2}$

Here, Direction ratios are 1, -3, 2

Let, θ = Angle between the lines.

$\therefore\ \theta=\cos^{-1}\frac{-3\times1+(-2)\times(-3)+0\times2}{\sqrt{(-3)^2+(-2)^2}\sqrt{1^2+(-3)^2+22}}\Rightarrow\theta=\cos^{-1}\frac{-3+6}{\sqrt{13}\sqrt{14}}$

$\Rightarrow\theta=\cos^{-1}\frac{3}{\sqrt{182}}$ Ans.

9. Show that the lines $\frac{x-3}{2}=\frac{y+1}{-3}=\frac{z-2}{4}$ and $\frac{x+2}{2}=\frac{y-4}{4}=\frac{z+5}{2}$ are perpendicular to each other.

Sol. The given equation of the line are $\frac{x-3}{2}=\frac{y+1}{-3}=\frac{z-2}{4}$

Another equation of the line are $\frac{x+2}{2}=\frac{y-4}{4}=\frac{z+5}{2}$

The directions ratio of the given line are (2, –3, 4) & (2, 4, 2).

Two lines are perpendicular to each other. So, $a_1a_2+b_1b_2+c_1c_2=0$

$\Rightarrow\ 2(2)+(-3)(4)+(4)(2)=0\quad\Rightarrow4-12+8=0\quad\Rightarrow12-12=0$

Hence, the given lines are perpendicular to each other.

10. Find the value of k for which the lines $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$ and $\frac{x-1}{3k}=\frac{y-1}{1}=\frac{6-z}{5}$ are perpendicular to each other.

Sol. The given equation of the line is $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$

$\Rightarrow$ Another equation of the line is $\frac{x-1}{3k}=\frac{y-1}{1}=\frac{6-z}{5}$

$\Rightarrow\ \frac{x-1}{3k}=\frac{y-1}{1}=\frac{-(z-6)}{5}\quad\Rightarrow\frac{x-1}{3k}=\frac{y-1}{1}=\frac{z-6}{-5}$

The direction ratios of the given line are –3, $2k$, 2 and $3k$, 1, -5.

Given lines are perpendicular to each other. So, $a_1a_2+b_1b_2+c_1c_2=0$

$\Rightarrow\ -3(3k)+2k(1)+2(-5)=0\quad\Rightarrow-9k+2k-10=0\quad\Rightarrow-7k=10\quad\Rightarrow k=\frac{-10}{7}$

11. Show that the lines $x=-y=2z$ and $x+2=2y-1=-z+1$ are perpendicular to each other

Hints : The given lines are $\frac{x}{2}=\frac{y}{-2}=\frac{z}{1}$ and $\frac{x+2}{2}=\frac{y-1}{2}=\frac{z-1}{-2}$

Sol. The given lines are x = -y = 2z $\Rightarrow\frac{x}{2}=\frac{y}{-2}=\frac{z}{1}$

and, x + 2 = 2y – 1 = -z + 1

$\Rightarrow\frac{x+2}{2}=\frac{y-1/2}{1}=\frac{z-1}{-2}$

Direction rations of line x = -y = 2z are 2, -2, 1

Direction rations of live x + 2 = 2y – 1 = -z + 1 are $2,1,-2$

Now, 2 × 2 + (-2) × 1 + 1 × (-2)= 4 – 2 – 2= 0

Hence, both the lines are perpendicular.

12. Find the angle between two lines whose direction ratios are

(i) $2, 1, 2,$ and $4, 8, 1$ (ii) $5, -12, 13,$ and $-3, 4, 5$

(iii) $1, 1, 2,$ and $(\sqrt{3}-1), (-\sqrt{3}-1), 4$ (iv) a, b, c and $(b-c), (c-a), (a-b)$

Sol. **(i)** Let, $A=(2,1,2)$ i.e., $(a_1=2, b_1=1, c_1=2)$ and $B=(4, 8, 1)$ i.e. $(a_2=4, b_2=8, c_2=1)$

$$\therefore \cos\theta = \left(\frac{2(4)+1(8)+2(1)}{\sqrt{(2)^2+(1)^2+(2)^2}\sqrt{(4)^2+(8)^2+(1)^2}}\right) \Rightarrow \cos\theta = \left(\frac{8+8+2}{\sqrt{4+1+4}\sqrt{16+64+1}}\right)$$

$$\Rightarrow \cos\theta = \left(\frac{18}{\sqrt{9}\sqrt{81}}\right) \Rightarrow \cos\theta = \left(\frac{18}{3\times 9}\right) \Rightarrow \cos\theta = \left(\frac{2}{3}\right) \Rightarrow \theta = \cos^{-1}\left(\frac{2}{3}\right)$$

Hence, the angle between the given line is $\cos^{-1}\left(\frac{2}{3}\right)$.

(ii) $5, -12, 13$ and $-3, 4, 5$

Direction ratio of the first line are $5, -12, 13$

$$\cos\theta = \frac{5\times(-3)+(-12)(4)+13\times 5}{\sqrt{(5)^2+(-12)^2+(13)^2}\sqrt{(-3)^2+(4)^2+(5)^2}} = \frac{-15-48+65}{13\sqrt{2}\times 5\sqrt{2}}$$

$$\Rightarrow \cos\theta = \frac{2}{65.2} = \frac{1}{65} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{65}\right)$$

Hence, the angle between the given line is $\cos^{-1}\left(\frac{1}{65}\right)$.

(iii) Let A $(1,1,2)$ i.e. $(a_1=1, b_1=1, c_1=2)$ and $\left[(\sqrt{3}-1), (-\sqrt{3}-1), 4\right]$

i.e., $\left[a_2=(\sqrt{3}-1), b_2=(-\sqrt{3}-1), c_2=4\right]$

$$\therefore \cos\theta = \left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)$$

$$\Rightarrow \cos\theta = \left(\frac{1(\sqrt{3}-1)+1(-\sqrt{3}-1)+2(4)}{\sqrt{(1)^2+(1)^2+(2)^2}\sqrt{(\sqrt{3}-1)^2+(-\sqrt{3}-1)^2+(4)^2}}\right)$$

$$\Rightarrow \cos\theta = \left(\frac{\sqrt{3}-1-\sqrt{3}-1+8}{\sqrt{1+1+4}\sqrt{3+1-2\sqrt{3}+3+1+2\sqrt{3}+16}}\right)$$

$$\Rightarrow \cos\theta = \left(\frac{-2+8}{\sqrt{6}\sqrt{24}}\right) \Rightarrow \cos\theta = \left(\frac{6}{\sqrt{144}}\right) \Rightarrow \cos\theta = \left(\frac{6}{12}\right)$$

$$\Rightarrow \cos\theta = \left(\frac{1}{2}\right) \Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right) \Rightarrow \theta = \cos^{-1}\left(\cos\frac{\pi}{3}\right) \quad \therefore \theta = \frac{\pi}{3}$$

Hence, the angle between the given line is $\frac{\pi}{3}$.

(iv) $A(a, b, c)$ i.e., $(a_1=a, b_1=b, c_1=c)$

and $\left[(b-c), (c-a), (a-b)\right]$ i.e., $\left[a_2=(b-c), b_2=(c-a), c_2=(a-b)\right]$

$\therefore \quad \cos\theta = \left(\dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right) = \left(\dfrac{a(b-c) + b(c-a) + c(a-b)}{\sqrt{a^2+b^2+c^2}\sqrt{(b-c)^2 + (c-a)^2 + (a-b)^2}} \right)$

$\Rightarrow \cos\theta = \left(\dfrac{ab - ac + bc + ac - ab - bc}{\sqrt{a^2+b^2+c^2}\sqrt{b^2+c^2+c^2-2bc+a^2-2ac+a^2+b^2+2abb}} \right)$

$\Rightarrow \cos\theta = 0 \quad \Rightarrow \theta = \cos^{-1}(0) \quad \therefore \theta = \dfrac{\pi}{2}$

Hence, the angle between the given line is $\dfrac{\pi}{2}$.

13. If $A(1,2,3), B(4,5,7), C(-4,3,-6)$ and $D(2,9,2)$ are four given points then find the angle between the lines AB and CD

Sol. Direction ratios of AB are 4 – 1, 5 – 2, 7 – 3

i.e, 3, 3, 4

Direction ratios of CD are 2 – (-4), 9 – 3, 2 – (-6)

i.e, 6, 6, 8

Here, $\dfrac{3}{6} = \dfrac{3}{6} = \dfrac{4}{8}$ $\quad \therefore$ Both the lives are parallel

Hence, Angle between the lines = 0.

EXERCISE 27 D [Pg.No.: 1143]

In problems 1–8, find the shortest distance between the given lines.

1. $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k}),\ \vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$.

Sol. $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k}),\ \vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$

$$\text{Shortest distance } = \left|\frac{(\vec{r}_2 - \vec{r}_1).(\vec{m}_1 \times \vec{m}_2)}{|\vec{m}_1 \times \vec{m}_2|}\right|$$

$$\Rightarrow (\vec{r}_2 - \vec{r}_1) = (2\hat{i} + \hat{j} - \hat{k}) - (\hat{i} + \hat{j}) = (\hat{i} - \hat{k})$$

$$\Rightarrow (\vec{m}_1 \times \vec{m}_2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix} = \hat{i}\begin{vmatrix} -1 & 1 \\ -5 & 2 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -1 \\ 3 & -5 \end{vmatrix}$$

$$= \hat{i}(-2+5) - \hat{j}(4-3) + \hat{k}(-10+3) = (3\hat{i} - \hat{j} - 7\hat{k})$$

$$\Rightarrow |\vec{m}_1 \times \vec{m}_2| = \sqrt{(3)^2 + (-1)^2 + (-7)^2} = \sqrt{9+1+49} = \sqrt{59}$$

$$\therefore \text{ Shortest distance } = \left|\frac{(\hat{i} - \hat{k}).(3\hat{i} - \hat{j} - 7\hat{k})}{\sqrt{59}}\right| = \left|\frac{3+7}{\sqrt{59}}\right| = \frac{10}{\sqrt{59}} \text{ units.}$$

2. $\vec{r} = (-4\hat{i} + 4\hat{j} + \hat{k}) + \lambda(\hat{i} + \hat{j} - \hat{k}),\ \vec{r} = (-3\hat{i} - 8\hat{j} - 3\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 3\hat{k})$

Sol. $\vec{r} = (-4\hat{i} + 4\hat{j} + \hat{k}) + \lambda(\hat{i} + \hat{j} - \hat{k}),\ \vec{r} = (-3\hat{i} - 8\hat{j} - 3\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 3\hat{k})$

$$\text{Shortest distance } = \left|\frac{(\vec{r}_2 - \vec{r}_1).(\vec{m}_1 \times \vec{m}_2)}{|\vec{m}_1 \times \vec{m}_2|}\right|$$

$$\Rightarrow (\vec{r}_2 - \vec{r}_1) = (-3\hat{i} - 8\hat{j} - 3\hat{k}) - (-4\hat{i} + 4\hat{j} + \hat{k}) = (\hat{i} - 12\hat{j} - 4\hat{k})$$

$$\Rightarrow (\vec{m}_1 \times \vec{m}_2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 2 & 3 & 3 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & -1 \\ 3 & 3 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= \hat{i}(3+3) - \hat{j}(3+2) + \hat{k}(3-2) = (6\hat{i} - 5\hat{j} + \hat{k})$$

$$\Rightarrow |\vec{m}_1 \times \vec{m}_2| = \sqrt{(6)^2 + (-5)^2 + (1)^2} = \sqrt{36+25+1} = \sqrt{62}$$

$$\therefore \text{ Shortest distance } = \left|\frac{(\hat{i} - 12\hat{j} - 4\hat{k}).(6\hat{i} - 5\hat{j} + \hat{k})}{\sqrt{62}}\right| = \left|\frac{6+60-4}{\sqrt{62}}\right| = \left|\frac{62}{\sqrt{62}} \times \frac{\sqrt{62}}{\sqrt{62}}\right| = \left|\frac{62\sqrt{62}}{62}\right| = \sqrt{62} \text{ units}$$

3. $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}),\ \vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$

Sol. $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}),\ \vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$

$$\text{shortets distance } = \left|\frac{(\vec{r}_2 - \vec{r}_1).(\vec{m}_1 \times \vec{m}_2)}{|\vec{m}_1 \times \vec{m}_2|}\right|$$

$$\Rightarrow (\vec{r}_2 - \vec{r}_1) = (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = (3\hat{i} + 3\hat{j} + 3\hat{k})$$

$$\Rightarrow (\vec{m}_1 \times \vec{m}_2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = \hat{i}\begin{vmatrix} -3 & 2 \\ 3 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix}$$

$$= \hat{i}(-3-6) - \hat{j}(1-4) + \hat{k}(3+6) = (-9\hat{i} + 3\hat{j} + 9\hat{k})$$

$$\Rightarrow |\vec{m}_1 \times \vec{m}_2| = \sqrt{(-9)^2 + (3)^2 + (9)^2} = \sqrt{81+9+81} = \sqrt{171} = 3\sqrt{19}$$

$$\therefore \text{ Shortest distance } = \left|\frac{(3\hat{i}+3\hat{j}+3\hat{k}).(-9\hat{i}+3\hat{j}+9\hat{k})}{3\sqrt{19}}\right| = \left|\frac{-27+9+27}{3\sqrt{19}}\right| = \left|\frac{9}{3\sqrt{19}}\right| = \frac{3}{\sqrt{19}} \text{ units.}$$

4. $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$

$\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$

Sol. The give lines are

$L_1 : \vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$

$L_2 : \vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$

The equations are at the form $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, where

$\vec{a}_1 = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{a}_2 = 2\hat{i} - \hat{j} - \hat{k}$

$\vec{b}_1 = \hat{i} - \hat{j} + \hat{k}$ and $\vec{b}_2 = 2\hat{i} + \hat{j} + 2\hat{k}$

Now $\vec{a}_2 - \vec{a}_1 = (2\hat{i} - \hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + \hat{k}) = \hat{i} - 3\hat{j} - 2\hat{k}$

And $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix}\hat{i} - \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}\hat{j} + \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}\hat{k}$

$= (-1-2)\hat{i} - (2-2)\hat{j} + (1+2)\hat{k} = -3\hat{i} + 3\hat{k}$

$\Rightarrow |\vec{b}_1 \times \vec{b}_2| = \sqrt{(-3)^2 + 3^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$

Now $(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = (\hat{i} - 3\hat{j} - 2\hat{k})\cdot(-3\hat{i} + 3\hat{k}) = -3 - 6 = -9$

S.D. $\left|\frac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right| = \left|\frac{-9}{3\sqrt{2}}\right|$ units

$= \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$ units $= \frac{3\sqrt{2}}{2}$ units

5. $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}), \vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(-2\hat{i} + 3\hat{j} + 3\hat{k})$.

Sol. Comparing the given equation with the standard equations $\vec{r} = \vec{r}_1 + \lambda\vec{m}_1$ and $\vec{r} = \vec{r}_2 + \mu\vec{m}_2$

$\therefore \vec{r}_2 - \vec{r}_1 = (3\hat{i} + 3\hat{j} - 5\hat{k}) - (\hat{i} + 2\hat{j} - 4\hat{k}) = 2\hat{i} + \hat{j} - \hat{k}$

and $\vec{m}_1 \times \vec{m}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ -2 & 3 & 3 \end{vmatrix} = \hat{i}(9-18) - \hat{j}(6+12) + \hat{k}(6+6) = -9\hat{i} - 18\hat{j} + 12\hat{k}$

$$\therefore\ |\vec{m}_1 \times \vec{m}_2| = \sqrt{(-9)^2 + (-18)^2 + (12)^2} = \sqrt{81+324+144} = \sqrt{549}$$

$$\text{S.D.} = \left(\frac{(-9\hat{i} - 18\hat{j} + 12\hat{k})(2\hat{i} + \hat{j} - \hat{k})}{\sqrt{549}}\right) = \left|\frac{-9\times 2 + (-18)\times(1) + 12\times(-1)}{3\sqrt{61}}\right|$$

$$= \left|\frac{-18-18-12}{3\sqrt{61}}\right| = \left|\frac{-18-18-12}{3\sqrt{61}}\right| = \frac{16}{\sqrt{61}} \text{ units}$$

6. $\vec{r} = (6\hat{i} + 3\hat{k}) + \lambda(2\hat{i} - \hat{j} + 6\hat{k}),\ \vec{r} = (-9\hat{i} + \hat{j} - 10\hat{k}) + \mu(4\hat{i} + \hat{j} + 6\hat{k})$

Sol. $\vec{r} = (6\hat{i} + 3\hat{k}) + \lambda(2\hat{i} - \hat{j} + 6\hat{k}),\ \vec{r} = (-9\hat{i} + \hat{j} - 10\hat{k}) + \mu(4\hat{i} + \hat{j} + 6\hat{k})$

$$\text{Shortest distance} = \left|\frac{(\vec{r}_2 - \vec{r}_1).(\vec{m}_1 \times \vec{m}_2)}{|\vec{m}_1 \times \vec{m}_2|}\right|$$

$$\Rightarrow (\vec{r}_2 - \vec{r}_1) = (-9\hat{i} + \hat{j} - 10\hat{k}) - (6\hat{i} + 0\hat{j} + 3\hat{k}) = (-15\hat{i} + \hat{j} - 13\hat{k})$$

$$\Rightarrow (\vec{m}_1 \times \vec{m}_2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 4 \\ 4 & 1 & 6 \end{vmatrix} = \hat{i}\begin{vmatrix} -1 & 4 \\ 1 & 6 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 4 \\ 4 & 6 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix}$$

$$= \hat{i}(-6-4) - \hat{j}(12-16) + \hat{k}(2+4) = (-10\hat{i} + 4\hat{j} + 6\hat{k})$$

$$\Rightarrow |\vec{m}_1 \times \vec{m}_2| = \sqrt{(-10)^2 + (4)^2 + (6)^2} = \sqrt{100+16+36} = \sqrt{152} = 2\sqrt{38}$$

$$\therefore\ \text{Shortest distance} = \left|\frac{(-15\hat{i} + \hat{j} - 13\hat{k}).(-10\hat{i} + 4\hat{j} + 6\hat{k})}{2\sqrt{38}}\right| = \left|\frac{150+4-78}{2\sqrt{38}}\right| = \left|\frac{76}{2\sqrt{38}}\right| = \sqrt{38}$$

7. $\vec{r} = (3-t)\hat{i} + (4+2t)\hat{j} + (t-2)\hat{k},\ \vec{r} = (1+s)\hat{i} + (3s-7)\hat{i} + (2s-2)\hat{k}$

Sol. $\vec{r} = (3-t)\hat{i} + (4+2t)\hat{j} + (t-z)\hat{k}$ i.e., $\vec{r} = (3\hat{i} + 4\hat{j} - 2\hat{k}) + t(-\hat{i} + 2\hat{j} + \hat{k})$

$\vec{r} = (1+5)\hat{i} + (35+7)\hat{i} + (25-2)\hat{k}$ i.e., $\vec{r} = (\hat{i} - 7\hat{j} - 2\hat{k}) + s(\hat{i} + 3\hat{j} + 2\hat{k})$

$$\text{Shortest distance} = \left|\frac{(\vec{r}_2 - \vec{r}_1).(\vec{m}_1 \times \vec{m}_2)}{|\vec{m}_1 \times \vec{m}_2|}\right|$$

$$\Rightarrow (\vec{r}_2 - \vec{r}_1) = (\hat{i} - 7\hat{j} - 2\hat{k}) - (3\hat{i} + 4\hat{j} - 2\hat{k}) = (-2\hat{i} - 11\hat{j})$$

$$\Rightarrow (\vec{m}_1 \times \vec{m}_2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 1 \\ 1 & 3 & 2 \end{vmatrix} = \hat{i}(4-3) - \hat{j}(-2-1) + \hat{k}(-3-2) = (\hat{i} + 3\hat{j} - 5\hat{k})$$

$$\Rightarrow |\vec{m}_1 \times \vec{m}_2| = \sqrt{(1)^2 + (3)^2 + (-5)^2} = \sqrt{1+9+25} = \sqrt{35}$$

$$\therefore\ \text{Shortest distance} = \left|\frac{(-2\hat{i} - 11\hat{j}).(\hat{i} + 3\hat{j} - 5\hat{k})}{\sqrt{35}}\right| = \left|\frac{-2-33}{\sqrt{35}}\right| = \frac{35}{\sqrt{35}} = \sqrt{35} \text{ units}$$

8. $\vec{r} = (\lambda-1)\hat{i} + (\lambda+1)\hat{j} - (\lambda+1)\hat{k},\ \vec{r} = (1-\mu)\hat{i} + (2\mu-1)\hat{j} + (\mu+2)\hat{k}$

Sol. $\vec{r} = (\lambda-1)\hat{i} + (\lambda+1)\hat{j} - (\lambda+1)\hat{k}$ i.e. $\vec{r} = (-\hat{i} + \hat{j} - \hat{k}) + \lambda(\hat{i} + \hat{j} - \hat{k})$

$\vec{r} = (1-\mu)\hat{i} + (2\mu-1)\hat{j} + (\mu+2)\hat{k}$ i.e., $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(-\hat{i} + 2\hat{j} + \hat{k})$

Shortest distance $= \left|\dfrac{(\vec{r}_2 - \vec{r}_1).(\vec{m}_1 \times \vec{m}_2)}{|\vec{m}_1 \times \vec{m}_2|}\right|$

$\Rightarrow (\vec{r}_2 - \vec{r}_1) = (\hat{i} - \hat{j} + 2\hat{k}) - (-\hat{i} + \hat{j} - \hat{k}) = (2\hat{i} - 2\hat{j} + 3\hat{k})$

$\Rightarrow |\vec{m}_1 \times \vec{m}_2| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$

$= \hat{i}(1+2) - \hat{j}(1-1) + \hat{k}(2+1) = (3\hat{i} + 3\hat{k})$

$\Rightarrow |\vec{m}_1 \times \vec{m}_2| = \sqrt{(3)^2 + (3)^2} = \sqrt{9+9} = 3\sqrt{2}$

$\therefore$ Shortest distance $= \left|\dfrac{(2\hat{i} - 2\hat{j} + 3\hat{k}).(3\hat{i} + 3\hat{k})}{3\sqrt{2}}\right|$

$\therefore$ Shortest distance $= \left|\dfrac{(2\hat{i} - 2\hat{j} + 3\hat{k}).(3\hat{i} + 3\hat{k})}{3\sqrt{2}}\right| = \left|\dfrac{6+9}{3\sqrt{2}}\right| = \dfrac{15}{3\sqrt{2}} = \dfrac{5}{\sqrt{2}}$ i.e., $\dfrac{5}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{5\sqrt{2}}{2}$

9. Compute the shortest distance between the lines

$\vec{r} = (\hat{i} - \hat{j}) + \lambda(2\hat{i} - \hat{k})$ and $\vec{r} = (2\hat{i} - \hat{j}) + \mu(\hat{i} - \hat{j} - \hat{k})$

Determine whether these lines intersect or not

Sol. Competing the given equations with the standard equations

$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$

We get $\vec{a}_1 = \hat{i} - \hat{j}. \vec{b}_1 = 2\hat{i} - \hat{k}$, $\vec{a}_2 = 2\hat{i} - \hat{j}, \vec{b}_2 = \hat{i} - \hat{j} - \hat{k}$

Now $\vec{a}_2 - \vec{a}_1 = (2\hat{i} - \hat{j}) - (\hat{i} - \hat{j}) \Rightarrow \vec{a}_2 - \vec{a}_1 = \hat{i}$

And $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -1 \\ 1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ -1 & -1 \end{vmatrix}\hat{i} - \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix}\hat{j} + \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix}k = -\hat{i} + \hat{j} - 2\hat{k}$

$\Rightarrow |\vec{b}_1 \times \vec{b}_2| = \sqrt{(-1)^2 + 1^2 + (-2)^2} = \sqrt{1+1+4} = \sqrt{6}$

Here $(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = \hat{i}\cdot(-\hat{i} + \hat{j} - 2\hat{k}) = -1 \neq 0$

Hence the given lines do not intersect

Now S.D. $= \left|\dfrac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right| = \left|\dfrac{-1}{\sqrt{6}}\right| = \dfrac{1}{\sqrt{6}}$

10. Show that the lines $\vec{r} = (3\hat{i} - 15\hat{j} + 9\hat{k}) + \lambda(2\hat{i} - 7\hat{j} + 5\hat{k})$, and $\vec{r} = (-\hat{i} + \hat{j} + 9\hat{k}) + \mu(2\hat{i} + \hat{j} - 3\hat{k})$ do not intersect.

Sol. $\vec{r} = (3\hat{i} - 15\hat{j} + 9\hat{k}) + \lambda(2\hat{i} - 7\hat{j} + 5\hat{k})$ and $\vec{r} = (-\hat{i} + \hat{j} + 9\hat{k}) + \mu(2\hat{i} + \hat{j} - 3\hat{k})$

Shortest distance $=\left|\dfrac{(\vec{r}_2-\vec{r}_1).(\vec{m}_1\times\vec{m}_2)}{|\vec{m}_1\times\vec{m}_2|}\right|$

$\Rightarrow (\vec{r}_2-\vec{r}_1)=\left(-\hat{i}+\hat{j}+9\hat{k}\right)-\left(3\hat{i}-15\hat{j}+9\hat{k}\right)=\left(-4\hat{i}+16\hat{j}\right)$

$\Rightarrow (\vec{m}_1\times\vec{m}_2)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -7 & 5\\ 2 & 1 & -3\end{vmatrix}=\hat{i}(21-5)-\hat{j}(-6-10)+\hat{k}(2+14)=\left(16\hat{i}+16\hat{j}+16\hat{k}\right)$

$\Rightarrow |\vec{m}_1\times\vec{m}_2|=\sqrt{(16)^2+(16)^2+(16)^2}=16\sqrt{3}$

$\therefore$ Shortest distance $=\left|\dfrac{\left(-4\hat{i}+16\hat{j}\right).\left(16\hat{i}+16\hat{j}+16\hat{k}\right)}{16\sqrt{3}}\right|=\left|\dfrac{-64+256}{16\sqrt{3}}\right|=\dfrac{192}{16\sqrt{3}}\neq 0$

Hence the given lines are don't intersect proved.

11. Show that the lines $\vec{r}=\left(2\hat{i}-3\hat{k}\right)+\lambda\left(\hat{i}+2\hat{j}+3\hat{k}\right)$ and $\vec{r}=\left(2\hat{i}+6\hat{j}+3\hat{k}\right)+\mu\left(2\hat{i}+3\hat{j}+4\hat{k}\right)$ intersect

Also, find their point of intersection

Sol. Comparing the given equation with the standard equation

$\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ we get $\vec{a}_1=2\hat{i}-3\hat{k},\vec{b}_1=\hat{i}+2\hat{j}+3\hat{k}$

$\vec{a}_2=2\hat{i}+6\hat{j}+3\hat{k}$ and $\vec{b}_2=2\hat{i}+3\hat{j}+4\hat{k}$

Now, $\vec{a}_2-\vec{a}_1=\left(2\hat{i}+6\hat{j}+3\hat{k}\right)-\left(2\hat{i}-3\hat{k}\right)=6\hat{j}+6\hat{k}$

And $\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ 2 & 3 & 4\end{vmatrix}=\begin{vmatrix}2 & 3\\ 3 & 4\end{vmatrix}\hat{i}-\begin{vmatrix}1 & 3\\ 2 & 4\end{vmatrix}\hat{j}+\begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix}\hat{k}$

$=(8-9)\hat{i}-(4-6)\hat{j}+(3-4)\hat{k}=-\hat{i}+2\hat{j}-\hat{k}$

Here $\left(\vec{a}_2-\vec{a}_1\right)\cdot\left(\vec{b}_1\times\vec{b}_2\right)=\left(6\hat{j}+6\hat{k}\right)\cdot\left(-\hat{i}+2\hat{j}-\hat{k}\right)=12-6=6\neq 0$

Thus the lines do not interest

12. Show that the lines $\vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}+4\hat{k}\right)$ and $\vec{r}=\left(4\hat{i}+\hat{j}\right)+\mu\left(5\hat{i}+2\hat{j}+\hat{k}\right)$

Intersect

Also, find the their point of intersection

Sol. $\vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(2\hat{i}+3j+4\hat{k}\right)$ and $\vec{r}=\left(4\hat{i}+\hat{j}\right)+\mu\left(5\hat{i}+2\hat{j}+\hat{k}\right)$

Shortest distance $\left|\dfrac{(\vec{r}_2-\vec{r}_1).(\vec{m}_1\times\vec{m}_2)}{|\vec{m}_1\times\vec{m}_2|}\right|$

$\Rightarrow (\vec{r}_2-\vec{r}_1)=\left(4\hat{i}+\hat{j}\right)-\left(\hat{i}+2\hat{j}+3\hat{k}\right)=\left(3\hat{i}-\hat{j}-3\hat{k}\right)$

$\Rightarrow (\vec{m}_1\times\vec{m}_2)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 5 & 2 & 1\end{vmatrix}=\hat{i}\begin{vmatrix}3 & 4\\ 2 & 1\end{vmatrix}-\hat{j}\begin{vmatrix}2 & 4\\ 5 & 1\end{vmatrix}+\hat{k}\begin{vmatrix}2 & 3\\ 5 & 2\end{vmatrix}$

$=\hat{i}(3-8)-\hat{j}(2-20)+\hat{k}(4-15)=\left(-5\hat{i}+18\hat{j}-11\hat{k}\right)$

$\Rightarrow |\vec{m}_1 \times \vec{m}_2| = \sqrt{(-5)^2 + (18)^2 + (-11)^2} = \sqrt{25+324+121} = \sqrt{470}$

$\therefore$ Shortest distance $= \left| \dfrac{(3\hat{i} - \hat{j} - 3\hat{k}).(-5\hat{i} + 18\hat{j} - 11\hat{k})}{\sqrt{470}} \right| = \left| \dfrac{-15-18+33}{\sqrt{470}} \right| = 0$

Hence, the given lines are intersect to each other. $\vec{r} = \vec{r}$

$\Rightarrow (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k}) = (4\hat{i} + \hat{j}) + \mu(5\hat{i} + 2\hat{j} + \hat{k})$

$\Rightarrow \hat{i}(1+2\lambda) + \hat{j} + (2+3\lambda) + \hat{k}(3+4\lambda) = \hat{i}(4+5\mu) + \hat{j}(1+2\mu) + \hat{k}(\mu)$

Equating co-efficient both side $\hat{i}$, $\hat{j}$ and $\hat{k}$ we get

$1+2\lambda = 4+5\mu,\ 2+3\lambda = 1+2\mu,\ 3+4\lambda = \mu$

$\Rightarrow 2\lambda - 5\mu = 4-1,\ 3\lambda - 2\mu = 1-2,\ 4\lambda - \mu = 3$

$\Rightarrow 2\lambda - 5\mu = 3\ \ ...(A),\quad 3\lambda - 2\mu = -1\ \ ...(B),\quad 4\lambda - \mu = 3\ \ ...(C)$

Solving equation A and B, we get $\lambda = -1,\ \mu = -1$

Putting the value of λ in

$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k}) = (\hat{i} + 2\hat{j} + 3\hat{k}) - 1(2\hat{i} + 3\hat{j} + 4\hat{k})$

$= \hat{i}(1-2) + \hat{j}(2-3) + \hat{k}(3-4) = (-\hat{i} - \hat{j} - \hat{k})$ $\quad\therefore$ Point $(-1,-1,-1)$

13. Find the shortest distance between the lines L_1 and L_2 whose vector equations are

$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$ and $\vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 6\hat{k})$

Sol. The given lines are

$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

$\vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 6\hat{k})$

These equations are of the form $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$

Where $\vec{a}_1 = \hat{i} + 2\hat{j} - 4\hat{k}$, $\vec{a}_2 = 3\hat{i} + 3\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$

Clearly the given lines are parallel

Now, $\vec{a}_2 - \vec{a}_1 = (3\hat{i} + 3\hat{j} - 5\hat{k}) - (\hat{i} + 2\hat{j} - 4\hat{k}) = 2\hat{i} + \hat{j} - \hat{k}$

$$\therefore \vec{b} \times (\vec{a}_2 - \vec{a}_1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 6 \\ 1 & -1 \end{vmatrix}\hat{i} - \begin{vmatrix} 2 & 6 \\ 2 & -1 \end{vmatrix}\hat{j} + \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}\hat{k}$$

$= (-3-6)\hat{i} - (-2-12)\hat{j} + (2-6)\hat{k} \ = -9\hat{i} + 14\hat{j} - 4\hat{k}$

Now $\left|\vec{b} \times (\vec{a}_2 - \vec{a}_1)\right| = \sqrt{(-9)^2 + 14^2 + (-4)^2} \ = \sqrt{81+196+16} = \sqrt{293}$

And $\left|\vec{b}\right| = \sqrt{2^2 + 3^2 + 6^2} \ = \sqrt{49} = 7$

Shortest distance between L_1 and L_2,

S.D. $= \dfrac{\left|\vec{b} \times (\vec{a}_2 - \vec{a}_1)\right|}{\left|\vec{b}\right|} \ = \dfrac{\sqrt{293}}{7}$ units

14. Find the distance between the parallel lines L_1 and L_2 whose vector equations are

$\vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$ and $\vec{r}=\left(2\hat{i}-\hat{j}-\hat{k}\right)+\mu\left(\hat{i}-\hat{j}+\hat{k}\right)$.

Sol. $\vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$, $\vec{r}=\left(2\hat{i}-\hat{j}-\hat{k}\right)+\mu\left(\hat{i}-\hat{j}+\hat{k}\right)$

Shortest distance $=\dfrac{\left|\vec{m}\times\left(\vec{r}_2-\vec{r}_1\right)\right|}{|\vec{m}|}$ $\Rightarrow\left(\vec{r}_2-\vec{r}_1\right)=\left(2\hat{i}-\hat{j}-\hat{k}\right)-\left(\hat{i}+2\hat{j}+3\hat{k}\right)=\left(\hat{i}-3\hat{j}-4\hat{k}\right)$

$$\Rightarrow \vec{m}\times\left(\vec{r}_2-\vec{r}_1\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 1\\ 1 & -3 & -4\end{vmatrix}=\hat{i}\begin{vmatrix}-1 & 1\\ -3 & -4\end{vmatrix}-\hat{j}\begin{vmatrix}1 & 1\\ 1 & -4\end{vmatrix}+\hat{k}\begin{vmatrix}1 & -1\\ 1 & -3\end{vmatrix}$$

$$=\hat{i}\left(4+3\right)-\hat{j}\left(-4-1\right)+\hat{k}\left(-3+1\right)=\left(7\hat{i}+5\hat{j}-2\hat{k}\right)$$

$$\Rightarrow\left|\vec{m}\times\left(\vec{r}_2-\vec{r}_1\right)\right|=\sqrt{(7)^2+(5)^2+(-2)^2}=\sqrt{49+25+4}=\sqrt{78}$$

$\Rightarrow |\vec{m}|=\sqrt{(1)^2+(-1)^2+(1)^2}=\sqrt{3}$ $\quad\therefore$ Shortest distance $=\dfrac{\sqrt{78}}{\sqrt{3}}=\dfrac{\sqrt{3}\sqrt{26}}{\sqrt{3}}=\sqrt{26}$ units.

15. Find the vector equation of a line passing through the point $(2,3,2)$ and parallel to the line $\vec{r}=\left(-2\hat{i}+3\hat{j}\right)+\lambda\left(2\hat{i}-3\hat{j}+6\hat{k}\right)$. Also find the distance between these lines

Sol. The vector equation of line passing through the point $(2,3,2)$ and parallel to the line

$\vec{r}=\left(-2\hat{i}+3\hat{j}\right)+\lambda\left(2\hat{i}-3\hat{j}+6\hat{k}\right)$ is given by

$\vec{r}=\left(2\hat{i}+3\hat{j}+2\hat{k}\right)+\mu\left(2\hat{i}-3\hat{j}+6\hat{k}\right)$

Comparing the given equations with the standard equations

$\vec{r}=\vec{a}_1+\lambda\vec{b}$ and $\vec{r}=\vec{a}_2+\mu\vec{b}$ we get $\vec{a}_1=-2\hat{i}+3\hat{j}, \vec{a}_2=2\hat{i}+3\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}-3\hat{j}+6\hat{k}$

Now, $\vec{a}_2-\vec{a}_1=\left(2\hat{i}+3\hat{j}+2\hat{k}\right)-\left(-2\hat{i}+3\hat{j}\right)=4\hat{i}+2\hat{k}$

And $\vec{b}=2\hat{i}-3\hat{j}+6\hat{k}$

$$\therefore\ \vec{b}\times\left(\vec{a}_2-\vec{a}_1\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -3 & 6\\ 4 & 0 & 2\end{vmatrix}=\begin{vmatrix}-3 & 6\\ 0 & 2\end{vmatrix}\hat{i}-\begin{vmatrix}2 & 6\\ 4 & 2\end{vmatrix}\hat{j}+\begin{vmatrix}2 & -3\\ 4 & 0\end{vmatrix}\hat{k}$$

$$=-6\hat{i}-\left(4-24\right)\hat{j}+12\hat{k}=-6\hat{i}+20\hat{j}+12\hat{k}$$

Now $\left|\vec{b}\times\left(\vec{a}_2-\vec{a}_1\right)\right|=\sqrt{(-6)^2+20^2+12^2}=\sqrt{36+400+144}=\sqrt{580}$

And $\left|\vec{b}\right|=\sqrt{2^2+(-3)^2+6^2}=7$

S.D. $=\dfrac{\left|\vec{b}\times\left(\vec{a}_2-\vec{a}_1\right)\right|}{\left|\vec{b}\right|}=\dfrac{\sqrt{580}}{7}$ units

16. Write the vector equation of each of the following lines and hence determine the distance between them

$\dfrac{x-1}{2}=\dfrac{y-2}{3}=\dfrac{z+4}{6}$ and $\dfrac{x-3}{4}=\dfrac{y-3}{6}=\dfrac{z+5}{12}$

Sol. Given lines are

$L_1 : \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$ and $L_2 : \frac{x-3}{4} = \frac{y-3}{6} = \frac{z+5}{12}$

Vector equations of the lines are

$L_1 : \vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

And $L_2 : \vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + 2\mu(2\hat{i} + 3\hat{j} + 6\hat{k})$

Clearly the given lines are parallel Here $\vec{a}_1 = \hat{i} + 2\hat{j} - 4\hat{k}$, $\vec{a}_2 = 3\hat{i} + 3\hat{j} - 5\hat{k}$

And $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$

Now, $\vec{a}_2 - \vec{a}_1 = (3\hat{i} + 3\hat{j} - 5\hat{k}) - (\hat{i} + 2\hat{j} - 4\hat{k}) = 2\hat{i} + \hat{j} - \hat{k}$

$$\therefore \vec{b} \times (\vec{a}_2 - \vec{a}_1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 6 \\ 1 & -1 \end{vmatrix} \hat{i} - \begin{vmatrix} 2 & 6 \\ 2 & -1 \end{vmatrix} \hat{j} + \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} \hat{k}$$

$= (-3-6)\hat{i} - (-2-12)\hat{j} + (2-6)\hat{k} \quad = -9\hat{i} + 14\hat{j} - 4\hat{k}$

Now $\left|\vec{b} \times (\vec{a}_2 - \vec{a}_1)\right| = \sqrt{(-9)^2 + 14^2 + (-4)^2} = \sqrt{81 + 96 + 16} = \sqrt{293}$

And $\left|\vec{b}\right| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7$

17. Write the vector equations of the following lines and hence find the shortest distance between them

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-5}{5}$

Sol. Given lines are $L_1 : \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$, $L_2 : \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-5}{5}$

Vector equations of the lines are

$L_1 : \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 4\hat{k})$

And $L_2 : \vec{r} = (2\hat{i} + 3\hat{j} + 5\hat{k}) + \delta(3\hat{i} + 4\hat{j} + 5\hat{k})$

Here $\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}, \vec{a}_2 = 2\hat{i} + 3\hat{j} + 5\hat{k}$, $\vec{b}_1 = 2\hat{i} + 3\hat{j} + 4\hat{k}, \vec{b}_2 = 3\hat{i} + 4\hat{j} + 5\hat{k}$

Now $\vec{a}_2 - \vec{a}_1 = (2\hat{i} + 3\hat{j} + 5\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = \hat{i} + \hat{j} + 2\hat{k}$

$$\text{And } \vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} \hat{i} - \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} \hat{j} + \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \hat{k}$$

$= (15-16)\hat{i} - (10-12)\hat{j} + (8-9)\hat{k} \quad = -\hat{i} + 2\hat{j} - \hat{k}$

$\therefore (\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = (\hat{i} + \hat{j} + 2\hat{k}) \cdot (-\hat{i} + 2\hat{j} - \hat{k}) = -1 + 2 - 2 = -1$

And $\left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6}$

$$\text{S.D.} = \frac{\left|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)\right|}{\left|\vec{b}_1 \times \vec{b}_2\right|} = \frac{|-1|}{\sqrt{6}} = \frac{1}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{6}}{\sqrt{6}} \text{ units}$$

Find the shortest distance between the lines given below

18. $\frac{x-1}{-1}=\frac{y+2}{1}=\frac{z-3}{-2}$ and $\frac{x-1}{1}=\frac{y+1}{2}=\frac{z+1}{-2}$

Sol. Given lines are

$L_1:\frac{x-1}{-1}=\frac{y+2}{1}=\frac{z-3}{-2}$, $L_2:\ \frac{x-1}{1}=\frac{y+1}{2}=\frac{x+1}{-2}$

The shortest distance between the skew lines

$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$

Is given by S.D. $\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(a_1b_2-a_1b_1)^2+(b_1c_2-b_2c_1)^2+(a_1c_2-a_2c_1)^2}}$

$$\therefore\ \text{S.D.} = \frac{\begin{vmatrix} 1-1 & 1-2 & 1-(-3) \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}}{\sqrt{(-1\times 2-1\times 1)^2+(1\times -2-2\times -2)^2+(-1\times -2-1\times -2)^2}}$$

$$= \left|\frac{0\begin{vmatrix} 1 & -2 \\ 2 & -2 \end{vmatrix}+1\begin{vmatrix} -1 & -2 \\ 1 & -2 \end{vmatrix}+4\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix}}{\sqrt{9+4+16}}\right| = \left|\frac{(2+2)+4(-2-1)}{\sqrt{29}}\right| \text{ units}$$

$$= \left|\frac{4-12}{\sqrt{29}}\right| \text{ units} = \left|\frac{-8}{\sqrt{29}}\right| \text{ units} = \frac{8}{\sqrt{29}}\times\frac{\sqrt{29}}{\sqrt{29}} \text{ units} = \frac{8\sqrt{29}}{29} \text{ units}$$

19. $\frac{x-12}{-9}=\frac{y-1}{4}=\frac{z-5}{2}$ and $\frac{x-23}{-6}=\frac{y-19}{-4}=\frac{z-25}{3}$

Sol. Given lines are $L_1:\frac{x-12}{-9}=\frac{y-1}{4}=\frac{z-5}{2}$ and $L_2:\frac{x-23}{-6}=\frac{y-19}{-4}=\frac{z-25}{3}$

Vector equations of the lines are

$L_1:\vec{r}=(12\hat{i}+\hat{j}+5\hat{k})+\delta(-9\hat{i}+4\hat{j}+2\hat{k})$, $L_2:\vec{r}=(23\hat{i}+19\hat{j}+25\hat{k})+\mu(-6\hat{i}-4\hat{j}+3\hat{k})$

Here, $\vec{a}_1=12\hat{i}+\hat{j}+5\hat{k}$, $\vec{a}_2=23\hat{i}+19\hat{j}+25\hat{k}$, $\vec{b}_1=-9\hat{i}+4\hat{j}+2\hat{k}$ and $\vec{b}_2=-6\hat{i}-4\hat{j}+3\hat{k}$

Now, $\vec{a}_2-\vec{a}_1=(23\hat{i}+19\hat{j}+25\hat{k})-(12\hat{i}+\hat{j}+5\hat{k}) = 11\hat{i}+18\hat{j}+20\hat{k}$

And $\vec{b}_1\times\vec{b}_2=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -9 & 4 & 2 \\ -6 & -4 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ -4 & 3 \end{vmatrix}\hat{i}-\begin{vmatrix} -9 & 2 \\ -6 & 3 \end{vmatrix}\hat{j}+\begin{vmatrix} -9 & 4 \\ -6 & -4 \end{vmatrix}\hat{k}$

$=(12+8)\hat{i}-(-27+12)\hat{j}+(36+24)\hat{k} = 20\hat{i}+15\hat{j}+60\hat{k}$

Now $(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=(11\hat{i}+18\hat{j}+20\hat{k})\cdot(20\hat{i}+15\hat{j}+60\hat{k})$

$=11\times 20+18\times 15+20\times 60=220+270+1200=1690$

And $|\vec{b}_1\times\vec{b}_2|=\sqrt{20^2+15^2+60^2}=\sqrt{5^2(4^2+3^2+12^2)}=5\sqrt{169}=5\times 13=65$

S.D. $=\left|\dfrac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|=\left|\dfrac{1690}{65}\right|$ units $=26$ units

EXERCISE 27 E [Pg.No.: 1150]

Find the length and the equations of the line of shortest distance between the lines given by

1. $\dfrac{x-3}{3}=\dfrac{y-8}{-1}=z-3$ and $\dfrac{x+3}{-3}=\dfrac{y+7}{2}=\dfrac{z-6}{4}$

Sol. The given equations are

$\dfrac{x-3}{3}=\dfrac{y-8}{-1}=\dfrac{z-3}{1}=\lambda$ ….(i) $\dfrac{x+3}{-3}=\dfrac{y+7}{2}=\dfrac{z-6}{4}=\mu$ … (ii)

$P(3\lambda+3,-\lambda+8,\lambda+3)$ is any point on (i)

$Q(-3\mu-3,2\mu-7,4\mu+6)$ is any point on (ii)

The direction rations of PQ are $(-3\mu-3\lambda-6,2\mu+\lambda-15,4\mu-\lambda+3)$

It PQ is the shortest distance then PQ is perpendicular to each of (i) and (ii)

$\therefore 3(-3\mu-3\lambda-6)+(-1)\cdot(2\mu+\lambda-15)+1\cdot(4\mu-\lambda+3)=0$

And $-3(-3\mu-3\lambda-6)+2(2\mu+\lambda-15)+4\cdot(4\mu-\lambda+3)=0$

$\Rightarrow -11\lambda-7\mu=0$ and $7\lambda+29\mu=0$

Solving these equation we get $\lambda=0$ and $\mu=0$

Thus PQ will be the line of shortest distance when $\lambda=0$ and $\mu=0$

Substituting $\lambda=0$ and $\mu=0$ in P and Q respectively we get the points

$P(3,8,3)$ and $Q(-3,-7,6)$

$\therefore$ shortest distance $=PQ$

$=\sqrt{(-3-3)^2+(-7-8)^2+(6-3)^2}=\sqrt{36+225+9}=3\sqrt{30}$ units

Equation of line of shortest distance is $\dfrac{x-3}{-3-3}=\dfrac{y-8}{-7-8}=\dfrac{z-3}{6-3}$

$\Rightarrow \dfrac{x-3}{-6}=\dfrac{y-8}{-15}=\dfrac{z-3}{3}\ \Rightarrow \dfrac{x-3}{2}=\dfrac{y-8}{5}=\dfrac{z-3}{-1}$

2. $\dfrac{x-3}{-1}=\dfrac{y-4}{2}=\dfrac{z+2}{1}$ and $\dfrac{x-1}{1}=\dfrac{y+7}{3}=\dfrac{z+2}{2}$

Sol. The given equations are

$\dfrac{x-3}{-1}=\dfrac{y-4}{2}=\dfrac{z+2}{1}=\lambda(\text{say})$ …(i)

$\dfrac{x-1}{1}=\dfrac{y+7}{3}=\dfrac{z+2}{2}=\mu(\text{say})$ …(ii)

$P(-\lambda+3,2\lambda+4,\lambda-2)$ is any point on (i), $Q(\mu+1,3\mu-7,2\mu-2)$ is any point on (ii).

The direction ratio of PQ are $(-\lambda+3-\mu-1,2\lambda+4-3\mu+7,\lambda-2-2\mu+2)$

$\Rightarrow (-\lambda-\mu+2,\ 2\lambda-3\mu+11,\ \lambda-2\mu)$

If PQ is the line of shortest distance then PQ is perpendicular to each of (i) and (ii).

$\therefore\ -1(-\lambda-\mu+2)+2(2\lambda-3\mu+11)+1(\lambda-2\mu)=0$

$\Rightarrow \lambda+\mu-2+4\lambda-6\mu+22+\lambda-2\mu=0 \Rightarrow 6\lambda-7\mu+20=0$ ...(iii)

$\therefore\ 1(-\lambda-\mu+2)+3(2\lambda-3\mu+11)+2(\lambda-2\mu)=0$

$\Rightarrow -\lambda-\mu+2+6\lambda-9\mu+33+2\lambda-4\mu=0 \quad \Rightarrow 7\lambda-14\mu+35=0$

$\Rightarrow \lambda-2\mu+5=0$ ...(iv)

Solving equation (iii) and (iv) then we get $\lambda=-1,\ \mu=2$

Thus, PQ will be the line of shortest distance when $\lambda=-1$ and $\mu=2$.

Substituting $\lambda=-1$ and $\mu=2$ in P and Q respectively. We get the point $P(4,2,-3)$ and $Q(3,-1,2)$.

$\therefore$ S.D. $= PQ=\sqrt{(3-4)^2+(-1-2)^2+(2+3)^2}=\sqrt{1+9+25}=\sqrt{35}$ unit

Equation of the line of shortest distance means equation of PQ given by

$$\frac{x-4}{3-4}=\frac{y-2}{-1-2}=\frac{z+3}{2+3} \quad \Rightarrow \frac{x-4}{1}=\frac{y-2}{-3}=\frac{z+3}{5}$$

3. $\frac{x+1}{2}=\frac{y-1}{1}=\frac{z-9}{-3}$ and $\frac{x-3}{2}=\frac{y+15}{-7}=\frac{z-9}{5}$

Sol. The given equation are

$$\frac{x+1}{2}=\frac{y-1}{1}=\frac{z-9}{-3}=\lambda(\text{say}) \qquad \text{...(i)}$$

$$\frac{x-3}{2}=\frac{y+15}{-7}=\frac{z-9}{5}=\mu(\text{say}) \qquad \text{...(ii)}$$

$P(2\lambda-1,\lambda+1,-3\lambda+9)$ is any point on (i), $Q(2\mu+3,-7\mu-15,5\mu+9)$ is any point on (ii).

The direction ratio of PQ are $(2\lambda-1-2\mu-3,\lambda+1+7\mu+15,-3\lambda+9-5\mu-9)$.

$\Rightarrow (2\lambda-2\mu-4,\lambda+7\mu+16,-3\mu-5\mu)$

If PQ is the line of shortest distance then PQ is perpendicular to each of (i) and (ii).

$\therefore\ 2(2\lambda-2\mu-4)+1(\lambda+7\mu+16)-3(-3\lambda-5\mu)=0$

$\Rightarrow 4\lambda-4\mu-8+\lambda+7\mu+16+9\lambda+15\mu=0 \quad \Rightarrow 14\lambda+18\mu+8=0$

$\Rightarrow 7\lambda+9\mu+4=0$ ...(iii)

$\therefore\ 2(2\lambda-2\mu-4)-7(\lambda+7\mu+16)+5(-3\lambda-5\mu)=0$

$\Rightarrow 4\lambda-4\mu-8-7\lambda-49\mu-112-15\lambda-25\lambda=0 \Rightarrow -18\lambda-78\mu-120=0 \quad \Rightarrow 9\lambda+39\mu+60=0$

$\Rightarrow 3\lambda+13\mu+20=0$ ...(iv)

Solving equation (iii) and (iv) we get $\lambda=2$, and $\lambda=-2$.

Thus, PQ will be the line of shortest distance. When $\lambda=2$ and $\mu=-2$.

Substituting $\lambda=2,\mu=-2$ in P and Q respectively. We get the point $P(3,3,3)$ and $Q(-1,-1,-1)$.

$\therefore$ S.D. $= PQ=\sqrt{(-1-3)^2+(-1-3)^2+(-1-3)^2}=\sqrt{16+16+16}=\sqrt{48}=4\sqrt{3}$ units

Equation of the line of shortest distance means equation of PQ given by

$$\frac{x-3}{-1-3}=\frac{y-3}{-1-3}=\frac{z-3}{-1-3} \quad \Rightarrow \frac{x-3}{-4}=\frac{y-3}{-4}=\frac{z-3}{-4} \quad \Rightarrow \frac{x-3}{1}=\frac{y-3}{1}=\frac{z-3}{1}$$

$\therefore\ x=3,\ y=3,\ z=3$ Hence, $x=y=z$.

4. $\frac{x-6}{3}=\frac{y-7}{-1}=\frac{z-4}{1}$ and $\frac{x}{-3}=\frac{y+9}{2}=\frac{z-2}{4}$

Sol. The given equation are

$$\frac{x-6}{3}=\frac{y-7}{-1}=\frac{z-4}{1}=\lambda \text{ (say)} \qquad \ldots(i)$$

$$\frac{x}{-3}=\frac{y+9}{2}=\frac{z-2}{4}=\mu(\text{say}) \qquad \ldots(ii)$$

$P(3\lambda+6,-\lambda+7,\lambda+4)$ is any point on (i), $Q(-3\mu,2\mu-9,4\mu+2)$ is any point on (ii).

The direction ratio of PQ are, $(3\lambda+6+3\mu,\ -\lambda+7-2\mu+9,\ \lambda+4-4\mu-2)$.

$$PQ(3\lambda+3\mu+6,\ -\lambda-2\mu+16,\ \lambda-4\mu+2)$$

If PQ is the line of shortest distance then PQ is perpendicular to each of (i) and (ii).

$\therefore\ 3(3\lambda+3\mu+6)-1(-\lambda-2\mu+16)+1(\lambda-4\mu+2)=0$

$\Rightarrow 9\lambda+9\mu+18+\lambda+2\mu-16+\lambda-4\mu+2=0$

$\Rightarrow 11\lambda+7\mu+4=0 \qquad \ldots(iii)$

$\therefore\ -3(3\lambda+3\mu+16)+2(-\lambda-2\mu+16)+4(\lambda-4\mu+2)=0$

$\Rightarrow -9\lambda-9\mu-18-2\lambda-4\mu+32+4\lambda-16\mu+8=0$

$\Rightarrow -7\lambda-29\mu+22=0 \qquad \ldots(iv)$

Solving equation (iii) and (iv) then we get $\lambda=-1,\mu=1$.

Thus, PQ will be the line of shortest distance when $\lambda=-1,\mu=1$.

Substituting $\lambda=-1$, and $\lambda=1$ in P and Q respectively.

We get from point $P(3,8,3)$ and $Q(-3,-7,6)$.

$\therefore$ S.D. $=PQ=\sqrt{(-3-3)^2+(-7-8)^2+(6-3)^2}=\sqrt{36+225+9}=\sqrt{270}=3\sqrt{30}$ units

Equations of the line of shortest distance means equation of PQ given by

$$\frac{x-3}{-3-3}=\frac{y-8}{-7-8}=\frac{z-3}{6-3} \Rightarrow \frac{x-3}{-6}=\frac{y-8}{-15}=\frac{z-3}{3} \Rightarrow \frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}$$

5. Show that the lines $\frac{x}{1}=\frac{y-2}{2}=\frac{z+3}{3}$ and $\frac{x-2}{2}=\frac{y-6}{3}=\frac{z-3}{4}$ intersect and find their point of intersection.

Sol. The given lines are $\frac{x}{1}=\frac{y-2}{2}=\frac{z+3}{3}=\lambda \text{ (say)} \quad \ldots(i)$

$$\frac{x-2}{2}=\frac{y-6}{3}=\frac{z-3}{4}=\mu(\text{say}) \qquad \ldots(ii)$$

$P(\lambda,\ 2\lambda+2,\ 3\lambda-3)$ is any point on (1) $Q(2\mu+2,\ 3\mu+6,\ 4\mu+3)$ is any point on (ii) if the lines (i) and (ii) intersect. then P and Q must coincide for same particular value of λ and μ.

This give, $\lambda=2\mu+2,\ 2\lambda+2=3\mu+6,\ 3\lambda-3=4\mu+3$

$\lambda-2\mu=2 \qquad \ldots(i)$

$2\lambda-3\mu=4 \qquad \ldots(ii)$

$3\lambda-4\mu=6 \qquad \ldots(iii)$

Solving (i) and (ii) we get $\lambda=2,\mu=0$ and these value of λ and μ also satisfying (iii).

Hence, the gives lines intersect. The point of intersection of the given lines is $(2,6,3)$, which is obtained by putting $\lambda = 2$ in P or $\mu = 0$ in Q.

6. Show that the lines $\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5}$ and $\frac{x-2}{2} = \frac{y-1}{3} = \frac{z+1}{-2}$

Do not intersect each other

Sol. The equations of the given lines are

$L_1 : \frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{3} = \lambda$ ...... (i)

$L_2 : \frac{x-2}{2} = \frac{y-1}{3} = \frac{z+1}{-2} = \mu$ ...... (ii)

Any point on the line (i) is $P(3\lambda+1, 2\lambda-1, 3\lambda+1)$

Any point on the line (ii) is Q $\{2\mu+2, 3\mu+1, -2\mu+1\}$

It both the lines intersect the P and Q must loinlide for some particular values of λ and μ

Now $3\lambda+1 = 2\mu+2 \Rightarrow 3\lambda - 2\mu = 1$ .... (iii)

$2\lambda - 1 = 3\mu + 1 \Rightarrow 2\lambda - 3\mu = -2$ .... (iv)

$3\lambda + 1 = -2\mu + 1 \Rightarrow 3\lambda + 2\mu = 0$ .... (v)

Adding (iii) and (v) we have $6\lambda = 1 \Rightarrow \lambda = \frac{1}{6}$

Putting $\lambda = \frac{1}{6}$ in equation (iii) $3 \times \frac{1}{6} - 2\mu = 1$

$\Rightarrow \frac{1}{2} - 1 = 2\mu \Rightarrow -\frac{1}{2} = 2\mu \Rightarrow \mu = -\frac{1}{4}$

Putting $\lambda = \frac{1}{6}$ & $\mu = -\frac{1}{4}$ in (iv) $2 \times \frac{1}{6} - 3 \times \left(-\frac{1}{4}\right) = -2$

$\frac{1}{3} + \frac{3}{4} = -2$ which is false Hence, the given lines do not intersect each other

EXERCISE 27 F [Pg.No.: 1151]

1. If a line has direction rations $2, -, 1, -2$ then what are its direction cosines ?

Sol. Direction rations of the line are $2, -, 1-2$

Now $\sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3$ $\therefore \ell = \frac{2}{3}, m = -\frac{1}{3}$ and $n = -\frac{2}{3}$

Hence direction cosines are $\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}$

2. Find the direction cosines of the $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$

Sol. Given line is $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$ i.e. $\frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3}$

here, direction ratios of the line are $-2, 6, -3$

Now $\sqrt{(-2)^2 + 6^2 + (-3)^2} = 7$

$\therefore \ell = -\frac{2}{7}, m = \frac{6}{7}$ and $n = -\frac{3}{7}$ Hence direction cosines are $-\frac{2}{7}, \frac{6}{7}, -\frac{3}{7}$

3. if the equations of a line are $\frac{3-x}{-3} = \frac{y+2}{-2} = \frac{z+2}{6}$, find the direction cosines of line parallel to the given line

Sol. Given line is $\frac{3-x}{-3} = \frac{y+1}{-1} = \frac{z+2}{6}$ i.e. $\frac{x-3}{3} = \frac{y+2}{-2} = \frac{z+2}{6}$

Direction ratios at a line parallel to given line are $3, -2, 6$

Now $\sqrt{3^2 + (-2)^2 + 6^2} = \sqrt{49} = 7$

$\therefore \ell = \frac{3}{7}, m = \frac{-2}{7}$ and $n = \frac{6}{7}$ Hence direction cosines are $\frac{3}{7}, -\frac{2}{7}, \frac{6}{7}$

4. Write the equations of a line parallel to the line $\frac{x-2}{-2} = \frac{y+3}{2} = \frac{z+5}{6}$ and passing through the point $(1, -2, 3)$

Sol. Equation line parallel to the line $\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}$ and passing through the point (x, y, z) is given by

$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Therefore equation of line parallel the line $\frac{x-2}{-3} = \frac{y+3}{2} = \frac{z+5}{6}$ and passing through the point $(1, -2, 3)$ is

$\frac{x-1}{-3} = \frac{y+2}{2} = \frac{z-3}{6}$

Now, position vector of the point $(1, -2, 3), \vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

And a vector 11 to the line is $\vec{b} = -3\hat{i} + 2\hat{j} + 6\hat{k}$

Hence vector equation of line $\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \mu(-3\hat{i} + 2\hat{j} + 6\hat{k})$

5. find the Cartesian equations of the line which posses through the point $(-2, 4, -5)$ and which is parallel to the line $\frac{x+3}{3} = \frac{y-4}{-5} = \frac{z+5}{6}$

Sol. Equation of line parallel to the line $\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}$ passing through the point (x_1, y_1, z_1) is given by

$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

There force equation of line parallel to the line $\frac{x+3}{3} = \frac{y-4}{-5} = \frac{z+8}{6}$ and passing through the point m$(-2, 4, -5)$ is given by

$\frac{x+2}{3} = \frac{y-4}{-5} = \frac{z+5}{6}$

6. Write the vector equation of a line whose Cartesian equations are $\frac{x-5}{3}=\frac{y+4}{7}=\frac{6-z}{2}$

Sol. Given line is $L:\frac{x-5}{3}=\frac{y+4}{7}=\frac{6-z}{2}$ i.e $L:\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{-2}$

Clearly $(5,-4,6)$ line on the line position vector of $A(5,-4,6)$ is $\vec{a}=5\hat{i}-4\hat{j}+6\hat{k}$

A vector parallel to the line $\vec{b}=3\hat{i}+7\hat{j}-2\hat{k}$

Hence the vector equation of line is $\vec{r}=\left(5\hat{i}-4\hat{j}+6\hat{k}\right)+\mu\left(3\hat{i}+7\hat{j}-2\hat{k}\right)$

7. the Cartesian equations of a line are $\frac{3-x}{5}=\frac{y+4}{7}=\frac{2z-6}{4}$. write the vector equation of the line

Sol. Given line is $\frac{3-x}{5}=\frac{y+4}{7}=\frac{2z-6}{4}$ i.e. $\frac{x-3}{-5}=\frac{y+4}{7}=\frac{z-3}{2}$

Clearly $(3,-4,3)$ lies on the line

Now position vector of $(3,-4,3)$ is $\vec{a}=3\hat{i}-4\hat{j}+3\hat{k}$

A vector parallel to the given line $\vec{b}=-5\hat{i}+7\hat{j}+2\hat{k}$

Hence equation of line is $\vec{r}=\left(3\hat{i}-4\hat{j}+3\hat{k}\right)+\mu\left(-5\hat{i}+7\hat{j}+2\hat{k}\right)$

8. Write the vector equation of a line passing through the point $(1,-1,2)$ and parallel to the line whose equations are $\frac{x-3}{1}=\frac{y-1}{2}=\frac{z+1}{-2}$

Sol. Position vector of the point $(1,-1,2)$ is $\vec{a}=\hat{i}-\hat{j}+2\hat{k}$

A vector parallel to the line whose equations are $\frac{x-3}{1}=\frac{y-1}{2}=\frac{z+1}{-2}$ is $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$

Hence vector equation of line is $\vec{r}=\left(\hat{i}-\hat{j}+2\hat{k}\right)+\mu\left(\hat{i}+2\hat{j}-2\hat{k}\right)$

9. If $P(1,5,4)$ and $Q(4,1,-2)$ be two given points find the direction ratios of PQ

Sol. Given points are $P(1,5,4)$ and $Q(4,1,-2)$

Direction ratios of PQ are $4-1, 1-5, -2-4$ i.e. $3,-4,-6$

10. The equations of a line are $\frac{4-x}{2}=\frac{y+3}{2}=\frac{z+2}{1}$. Find the direction cosines of a line parallel to this line

Sol. Given line is $\frac{4-x}{2}=\frac{y+3}{2}=\frac{z+2}{1}$ i.e $\frac{x-4}{-2}=\frac{y+3}{2}=\frac{z+2}{1}$

D.r's of line parallel to the line are $-2,2,1$

Now $\sqrt{(-2)^2+2^2+1^2}=\sqrt{9}=3$

$\therefore \ell=-\frac{2}{3}, m=\frac{2}{3}$ and $n=\frac{1}{3}$ Hence direction cosines are $-\frac{2}{3},\frac{2}{3},\frac{1}{3}$

11. the Cartesian equations of a line are $\frac{x-1}{2}=\frac{y+2}{3}=\frac{5-z}{1}$ Find its vector equation

Sol. Given line is $\frac{x-1}{2}=\frac{y+2}{3}=\frac{5-z}{1}$ i.e. $\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-5}{-1}$

Clearly $(1,-2,5)$ lies on the position vector of point $(1,-2,5)$ is $\vec{a}=\hat{i}-2\hat{j}+5\hat{k}$

D.r.'s of line are $2,3,-1$

$\therefore$ A vector parallel to the line is $\vec{b}=2\hat{i}+3\hat{j}-\hat{k}$

Hence vector equation of line is $\vec{r}=\left(\hat{i}-2\hat{j}+5\hat{k}\right)+\mu\left(2\hat{i}+3\hat{j}-\hat{k}\right)$

12. Find the vector equation of a line passing through the point $(1,2,3)$ and parallel to the vector $\left(3\hat{i}+2\hat{j}-2\hat{k}\right)$

Sol. Position vector of the point $(1,2,3)$ is $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$

Equation $\vec{b}=3\hat{i}+2\hat{j}-2\hat{k}$ is vector parallel to the line

Hence required equation at line is $\vec{r}=\vec{a}+\mu\vec{b}$ i.e. $\vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\mu\left(3\hat{i}+2\hat{j}-2\hat{k}\right)$

13. The vector equation of a line is $\vec{r}=\left(2\hat{i}+\hat{j}-4\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}-\hat{k}\right)$. Find its cartesion equation

Sol. Given line is $\vec{r}=\left(2\hat{i}+\hat{j}-4\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}-\hat{k}\right)$

$\Rightarrow x\hat{i}+y\hat{j}+\hat{k}=(2+\lambda)\hat{i}+(1-\lambda)\hat{j}+\left(-4-\lambda\hat{k}\right)$

$\Rightarrow x=2+\lambda, y=1-\lambda$ and $z=-4-\lambda \Rightarrow \frac{x-2}{1}=\frac{y-1}{-1}=\frac{z+4}{-1}=\lambda.$

Hence Cartesian equation of line is $\frac{x-2}{1}=\frac{y-1}{-1}=\frac{z+4}{-1}$

14. Find the Cartesian equation of a line which passes through the point $(-2,4,-5)$ and which is parallel to the line $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$

Sol. Equation of line passes through the point (α,β,γ) and parallel to the line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ is given by

$\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{c}$

$\therefore$ equation of line passes through $(-2,4,-5)$ and parallel to the line $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+5}{6}$

15. Find the Cartesian equation of a line which passes through the point having position vector $\left(2\hat{i}-\hat{j}+4\hat{k}\right)$ and is in the direction of the vector $\left(\hat{i}+2\hat{j}-\hat{k}\right)$

Sol. The required line passes through the point $(2,-1,4)$ and it has direction ratios $1,2,-1$

$\therefore$ it equation is $\frac{x-2}{1}=\frac{y+1}{2}=\frac{z-4}{-1}$

16. Find the angle between the lines $\vec{r}=\left(2\hat{i}-5\hat{j}+\hat{k}\right)+\lambda\left(3\hat{i}+2\hat{j}+6\hat{k}\right)$ and $\vec{r}=\left(7\hat{i}-6\hat{k}\right)+\mu\left(\hat{i}+2\hat{j}+2\hat{k}\right)$

Sol. The angle between the lines and $\vec{r}=a_1+\lambda b_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$ is given by $\cos\theta=\frac{\vec{b}_1\cdot\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}$

$$\therefore \cos\theta=\frac{\left|\left(3\hat{i}+2\hat{j}+6\hat{k}\right)\cdot\left(\hat{i}+2\hat{j}+2\hat{k}\right)\right|}{\left\{\sqrt{3^2+2^2+6^2}\right\}\left\{\sqrt{1^2+2^2+2^2}\right\}}=\frac{(3+4+12)}{(7\times3)}=\frac{19}{21} \Rightarrow \theta=\cos^{-1}\left(\frac{19}{21}\right)$$

17. Find the angle between the lines $\frac{x+3}{3}=\frac{y-1}{5}=\frac{z+3}{4}$ and $\frac{x+1}{1}=\frac{y-4}{1}=\frac{z-5}{2}$

Sol. Hence $(a_1)=3, b_1=5, c_1=4$ and $(a_2=1, b_2=1, c_2=2)$

$$\therefore \cos\theta=\frac{|a_1a_2+b_1b_2+c_1c_2|}{\left\{\sqrt{a_1^2+b_1^2+c_1^2}\right\}\left\{\sqrt{a_2^2+b_2^2+c_2^2}\right\}}$$

$$=\frac{|(3\times1)+(5\times1)+(4\times2)|}{\left\{\sqrt{3^2+5^2+4^2}\right\}\left\{\sqrt{1^2+1^2+2^2}\right\}}=\frac{16}{\left(\sqrt{15}\times\sqrt{6}\right)}=\frac{16}{10\sqrt{3}}=\frac{8}{5\sqrt{3}}$$

$$\Rightarrow \theta=\cos^{-1}\left(\frac{8}{5\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\right)=\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$$

18. Show that the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ are at right angles

Sol. Here $(a_1=7, b_1=-5, c_1=1)$ and $(a_2=1, b_2=2, c_2=3)$

$(a_1a_2+b_1b_2+c_1c_2)=(7\times1)+(-5)\times2+(1\times3)=0$

Hence the given lines are at right angles

19. The direction ratios of a line are $2, 6, -6$. What are its direction cosines ?

Sol. We have $\sqrt{2^2+6^2+(-9)^2}=\sqrt{121}=11$

$\therefore$ d.c.'s of the given line are $\frac{2}{11}, \frac{6}{11}, \frac{-9}{11}$

20. A line makes angle $90°, 135°$ and $45°$ with the positive diectins of x-axis y-axis and z-axis respectively. What are the direction cosines of the line ?

Sol. D.c's of the line are $\cos 90°, \cos 135°$ and $\cos 45°$, i.e. $\left(0, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right)$

21. What are the direction cosines of the y-axis ?

Sol. Clearly, the y-axis makes an angle of $90°, 0°, 90°$ with the x-axis y-axis and z-axis respectively

So, its d.c.'s are $\cos 90°, \cos 0°, \cos 90°$, i.e. 0,1,0

22. What are the direction cosines of the vector $\left(2\hat{i}+\hat{j}-2\hat{k}\right)$?

Sol. D.r's of the given vector are $2, 1, -2$ and $\sqrt{2^2+1^2+(-2)^2}=\sqrt{9}=3$

$\therefore$ d.c's of the given vector are $\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$

23. What is the angle between the vector $\vec{r}=\left(4\hat{i}+8\hat{j}+\hat{k}\right)$ and the x-axis?

Sol. D.r.'s of the given vector are 4,8,1 and $\sqrt{4^2+8^2+1^2}=\sqrt{81}=9$

$\therefore$ d.c.'s of the given vector are $\frac{4}{9}, \frac{8}{9}, \frac{1}{9}$

Let α be the angle between the given vectors and the x-axis

The $\cos\alpha=\frac{4}{9} \Rightarrow \alpha=\cos^{-1}\left(\frac{4}{9}\right)$