For example,

$$\int \sqrt{x^2 - 25}\, dx = \int \sqrt{x^2 - 5^2}\, dx$$

$$= \frac{x}{2}\sqrt{x^2 - 5^2} - \frac{5^2}{2}\log\left|x + \sqrt{x^2 - 5^2}\right| + c$$

$$= \frac{x}{2}\sqrt{x^2 - 25} - \frac{25}{2}\log\left|x + \sqrt{x^2 - 25}\right| + c$$

(2) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log|x + \sqrt{x^2 + a^2}| + c$

Proof : $I = \int \sqrt{x^2 + a^2} \cdot 1\, dx$

$$= \sqrt{x^2 + a^2}\int 1\, dx - \int \left(\frac{d}{dx}\sqrt{x^2 + a^2}\int 1\, dx\right) dx$$

$$= x\sqrt{x^2 + a^2} - \int \frac{2x}{2\sqrt{x^2 + a^2}}\, x\, dx$$

$$= x\sqrt{x^2 + a^2} - \int \frac{x^2}{\sqrt{x^2 + a^2}}\, dx$$

$$= x\sqrt{x^2 + a^2} - \int \frac{(x^2 + a^2) - a^2}{\sqrt{x^2 + a^2}}\, dx$$

$$= x\sqrt{x^2 + a^2} - \int \sqrt{x^2 + a^2}\, dx + a^2\int \frac{dx}{\sqrt{x^2 + a^2}}$$

$$I = x\sqrt{x^2 + a^2} - I + a^2\log|x + \sqrt{x^2 + a^2}| + c'$$

$$\therefore\quad 2I = x\sqrt{x^2 + a^2} + a^2\log|x + \sqrt{x^2 + a^2}| + c'$$

$$\therefore\quad I = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log|x + \sqrt{x^2 + a^2}| + c \qquad \left(\frac{c'}{2} = c\right)$$

$$\therefore\quad \int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log|x + \sqrt{x^2 + a^2}| + c \qquad (a > 0)$$

This formula can also be obtained using substitution $x = a\tan\theta$ $(a > 0)$.

For example, $\int \sqrt{x^2 + 4}\, dx = \int \sqrt{x^2 + 2^2}\, dx$

$$= \frac{x}{2}\sqrt{x^2 + 2^2} + \frac{2^2}{2}\log\left|x + \sqrt{x^2 + 2^2}\right| + c$$

$$= \frac{x}{2}\sqrt{x^2 + 4} + 2\log\left|x + \sqrt{x^2 + 4}\right| + c$$

(3) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c \qquad (a > 0)$

Proof : $I = \int \sqrt{a^2 - x^2} \cdot 1\, dx$

$$= \sqrt{a^2 - x^2}\int 1\, dx - \int \left(\frac{d}{dx}\sqrt{a^2 - x^2}\int 1\, dx\right) dx$$

$$= x\sqrt{a^2 - x^2} - \int \left(\frac{1}{2\sqrt{a^2 - x^2}}(-2x)\cdot x\right) dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\sqrt{a^2 - x^2}} dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\sqrt{a^2 - x^2}} dx$$

$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$I = x\sqrt{a^2 - x^2} - I + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c'$$

$$\therefore \quad 2I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c'$$

$$\therefore \quad I = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c \qquad \left(\frac{c'}{2} = c\right)$$

$$\therefore \quad \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + c$$

Remark : What difference will it make if $a < 0$?

For example,

$$\int \sqrt{9 - x^2}\, dx = \int \sqrt{3^2 - x^2}\, dx$$

$$= \frac{x}{2}\sqrt{3^2 - x^2} + \frac{3^2}{2} \sin^{-1}\left(\frac{x}{3}\right) + c$$

$$= \frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2} \sin^{-1}\left(\frac{x}{3}\right) + c$$

This formula can be proved using substitution $x = a\sin\theta$ also.

(4) $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$

Proof : $I = \int e^x [f(x) + f'(x)]\, dx$

$$= \int e^x f(x)\, dx + \int e^x f'(x)\, dx$$

$$= f(x)\int e^x dx - \int \left(\frac{d}{dx} f(x) \int e^x dx\right) dx + \int e^x \cdot f'(x)\, dx$$

$$= f(x)\, e^x - \int f'(x)\, e^x\, dx + \int f'(x)\, e^x\, dx$$

$$= e^x f(x) + c$$

For example,

(1) $\int e^x \sec x\, (1 + \tan x)\, dx = \int e^x (\sec x + \sec x \tan x)\, dx$

$$= \int e^x \left[\sec x + \frac{d}{dx}(\sec x)\right] dx$$

$$= e^x \sec x + c$$

(2) $\int e^x \left(\frac{x-1}{x^2}\right) dx = \int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$

$$= \int e^x \left[\frac{1}{x} + \frac{d}{dx}\left(\frac{1}{x}\right)\right] dx$$

$$= e^x \cdot \frac{1}{x} + c$$

(3) $\int x \cdot e^x \, dx = \int [(x - 1) + 1] \, e^x \, dx$

$$= \int \left[(x - 1) + \frac{d}{dx}(x - 1)\right] e^x \, dx$$

$$= e^x (x - 1) + c$$

(5) $\int e^{ax} \cdot \sin(bx + k) \, dx = \frac{e^{ax}}{a^2 + b^2} [a \sin(bx + k) - b \cos(bx + k)] + c, \quad a, b \neq 0$

Proof : $I = \int e^{ax} \cdot \sin(bx + k) \, dx$

$$= \sin(bx + k) \int e^{ax} \, dx - \int \left(\frac{d}{dx} \sin(bx + k) \int e^{ax} \, dx\right) dx$$

$$= \sin(bx + k) \cdot \frac{e^{ax}}{a} - \int \left(b \cos(bx + k) \cdot \frac{e^{ax}}{a}\right) dx$$

$$= \frac{e^{ax}}{a} \sin(bx + k) - \frac{b}{a} \int \cos(bx + k) \, e^{ax} \, dx$$

$$= \frac{e^{ax}}{a} \sin(bx + k) - \frac{b}{a}\left[\cos(bx + k) \int e^{ax} \, dx - \int \left(\frac{d}{dx} \cos(bx + k) \int e^{ax} \, dx\right) dx\right]$$

$$= \frac{e^{ax}}{a} \sin(bx + k) - \frac{b}{a}\left[\cos(bx + k) \frac{e^{ax}}{a} - \int \left(-b \sin(bx + k) \frac{e^{ax}}{a}\right) dx\right]$$

$$= \frac{e^{ax}}{a} \sin(bx + k) - \frac{b}{a^2} e^{ax} \cos(bx + k) - \frac{b^2}{a^2} \int e^{ax} \cdot \sin(bx + k) \, dx$$

$\therefore \quad I = \frac{e^{ax}}{a^2} [a \sin(bx + k) - b \cos(bx + k)] - \frac{b^2}{a^2} I + c'$

$\therefore \quad I + \frac{b^2}{a^2} I = \frac{e^{ax}}{a^2} [a \sin(bx + k) - b \cos(bx + k)] + c'$

$\therefore \quad (a^2 + b^2) I = e^{ax} [a \sin(bx + k) - b \cos(bx + k)] + a^2 c'$

$\therefore \quad I = \frac{e^{ax}}{a^2 + b^2} [a \sin(bx + k) - b \cos(bx + k)] + c$, where $c = \frac{a^2 c'}{a^2 + b^2}$ **(i)**

Now, we will express this result in another form.

$$I = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \left[\frac{a}{\sqrt{a^2 + b^2}} \sin(bx + k) - \frac{b}{\sqrt{a^2 + b^2}} \cos(bx + k)\right] + c$$

Here $a \neq 0$, $b \neq 0$. Hence,

$$0 < \left|\frac{a}{\sqrt{a^2 + b^2}}\right| < 1, \; 0 < \left|\frac{b}{\sqrt{a^2 + b^2}}\right| < 1$$

Now $\left(\frac{a}{\sqrt{a^2 + b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2 + b^2}}\right)^2 = 1.$

So, there exists $\alpha \in (0, 2\pi)$, such that

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \ \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}.$$

$$\therefore \quad I = \frac{e^{ax}}{\sqrt{a^2+b^2}} [\sin(bx+k)\cos\alpha - \cos(bx+k)\sin\alpha] + c$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}} \sin(bx+k-\alpha) + c, \text{ where } \cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \ \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}.$$

$$\therefore \quad \int e^{ax} \cdot \sin(bx+k)\, dx = \frac{e^{ax}}{a^2+b^2} (a\sin(bx+k) - b\cos(bx+k)) + c, \quad a, b \neq 0$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}} \sin(bx+k-\alpha) + c$$

where $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \ \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}. \quad \alpha \in (0, 2\pi)$

For example, $\int e^{2x} \cdot \sin 3x\, dx = \frac{e^{2x}}{2^2+3^2} (2\sin 3x - 3\cos 3x) + c = \frac{e^{2x}}{13} (2\sin 3x - 3\cos 3x) + c$

Another form for $\int e^{2x} \cdot \sin 3x\, dx$.

Let $\cos\alpha = \frac{2}{\sqrt{13}}, \ \sin\alpha = \frac{3}{\sqrt{13}}$, so $\tan\alpha = \frac{3}{2}$

$$\therefore \quad \alpha = \tan^{-1}\frac{3}{2}, \quad 0 < \alpha < \frac{\pi}{2}$$

$$\therefore \quad \int e^{2x} \cdot \sin 3x\, dx = \frac{e^{2x}}{\sqrt{13}} \sin\left(3x - \tan^{-1}\frac{3}{2}\right) + c$$

(6) $\int e^{ax} \cos(bx+k)\, dx = \frac{e^{ax}}{a^2+b^2} [a\cos(bx+k) + b\sin(bx+k)] + c, \quad a \neq 0, b \neq 0$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}} \cos(bx+k-\alpha) + c$$

where $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \ \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}. \quad \alpha \in (0, 2\pi).$

Proof : $I = \int e^{ax} \cos(bx+k)\, dx$

$$= \cos(bx+k) \int e^{ax}\, dx - \int \left(\frac{d}{dx}\cos(bx+k) \int e^{ax}\, dx\right) dx$$

$$= \cos(bx+k) \cdot \frac{e^{ax}}{a} - \int \left(-b\sin(bx+k) \cdot \frac{e^{ax}}{a}\right) dx$$

$$= \frac{e^{ax}}{a} \cos(bx+k) + \frac{b}{a} \int e^{ax} \sin(bx+k)\, dx$$

$$= \frac{e^{ax}}{a} \cos(bx+k) + \frac{b}{a} \left[\sin(bx+k) \int e^{ax}\, dx - \int \left(\frac{d}{dx}\sin(bx+k) \int e^{ax}\, dx\right) dx\right]$$

$$= \frac{e^{ax}}{a} \cos(bx+k) + \frac{b}{a} \left[\sin(bx+k) \cdot \frac{e^{ax}}{a} - \int \left(b\cos(bx+k) \cdot \frac{e^{ax}}{a}\right) dx\right]$$

$$= \frac{e^{ax}}{a} \cos(bx+k) + \frac{b}{a^2} e^{ax} \sin(bx+k) - \frac{b^2}{a^2} \int e^{ax} \cdot \cos(bx+k)\, dx$$

$$\therefore \quad I = \frac{e^{ax}}{a} cos(bx + k) + \frac{b}{a^2} e^{ax} \cdot sin(bx + k) - \frac{b^2}{a^2} I + c'$$

$$\therefore \quad I + \frac{b^2}{a^2} I = \frac{e^{ax}}{a^2} [a\, cos(bx + k) + b\, sin(bx + k)] + c'$$

$$\therefore \quad (a^2 + b^2) I = e^{ax} [a\, cos(bx + k) + b\, sin(bx + k)] + a^2c'$$

$$\therefore \quad I = \frac{e^{ax}}{a^2 + b^2} [a\, cos(bx + k) + b\, sin(bx + k)] + c, \text{ where } c = \frac{a^2c'}{a^2 + b^2} \qquad \text{(i)}$$

Another Form :

There exists $\alpha \in (0, 2\pi)$, such that $cos\alpha = \frac{a}{\sqrt{a^2 + b^2}}$, $sin\alpha = \frac{b}{\sqrt{a^2 + b^2}}$.

$$\therefore \quad I = \frac{e^{ax}}{\sqrt{a^2 + b^2}} [cos(bx + k) \cdot cos\alpha + sin(bx + k) \cdot sin\alpha] + c$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} cos(bx + k - \alpha) + c$$

$$\therefore \quad \int e^{ax} cos(bx + k)\, dx = \frac{e^{ax}}{\sqrt{a^2 + b^2}} cos(bx + k - \alpha) + c$$

where $cos\alpha = \frac{a}{\sqrt{a^2 + b^2}}$, $sin\alpha = \frac{b}{\sqrt{a^2 + b^2}}$

For example : $\int e^{-x} cos \frac{x}{2}\, dx = \frac{e^{-x}}{(1)^2 + \left(\frac{1}{2}\right)^2} \left(-1\, cos \frac{x}{2} + \frac{1}{2} sin \frac{x}{2}\right) + c$

$$= \frac{4e^{-x}}{5} \left(-cos \frac{x}{2} + \frac{1}{2} sin \frac{x}{2}\right) + c$$

Another form for $\int e^{-x} cos \frac{x}{2}\, dx$.

Here $cos\alpha = \frac{-2}{\sqrt{5}}$, $sin\alpha = \frac{1}{\sqrt{5}}$. So $tan\alpha = -\frac{1}{2}$, $\quad \frac{\pi}{2} < \alpha < \pi$

$$\therefore \quad \alpha = \pi - tan^{-1}\left(\frac{1}{2}\right)$$

$$\therefore \quad \int e^{-x} cos \frac{x}{2}\, dx = \frac{2}{\sqrt{5}} e^{-x} \left[cos\left(\frac{x}{2} - \left(\pi - tan^{-1} \frac{1}{2}\right)\right)\right] + c$$

$$= \frac{2}{\sqrt{5}} e^{-x} cos\left(\frac{x}{2} + tan^{-1} \frac{1}{2} - \pi\right) + c$$

$$= -\frac{2}{\sqrt{5}} e^{-x} cos\left(\frac{x}{2} + tan^{-1} \frac{1}{2}\right) + c$$

2.4 Integrals of the type : (1) $\int \sqrt{ax^2 + bx + c}\, dx$ (2) $\int (Ax + B) \sqrt{ax^2 + bx + c}\, dx$

(1) If we express $ax^2 + bx + c$ in the form of a perfect square, the integral can be obtained using standard forms (1), (2), (3).

(2) We will find out two constants m, n such that

$Ax + B = m(\text{derivative of } ax^2 + bx + c) + n$

$$Ax + B = m\left(\frac{d}{dx}(ax^2 + bx + c)\right) + n$$

$$Ax + B = m(2ax + b) + n$$

Comparing coefficient of x on both sides we get

$m = \frac{A}{2a}$ and $n = B - mb$

Now, $\int (Ax + B)\sqrt{ax^2+bx+c}\, dx = \int [m(2ax + b) + n]\sqrt{ax^2+bx+c}\, dx$

$$= m\int (2ax+b)\sqrt{ax^2+bx+c}\, dx + n\int \sqrt{ax^2+bx+c}\, dx$$

$$= mI_1 + nI_2$$

where $I_1 = \int (ax^2 + bx + c)^{\frac{1}{2}}\,(2ax + b)\, dx$

$$= \int (ax^2 + bx + c)^{\frac{1}{2}} \frac{d}{dx}(ax^2 + bx + c)\, dx$$

$$= \frac{(ax^2+bx+c)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c_1$$

$$= \frac{2}{3}(ax^2 + bx + c)^{\frac{3}{2}} + c_1$$

and $I_2 = \int \sqrt{ax^2+bx+c}\, dx$

I_2 can be obtained using method (1).

Example 8 : Evaluate : $\int x\sqrt{x^4-25}\, dx$.

Solution : $I = \int x\sqrt{x^4-25}\, dx$

Let $x^2 = t$. So $2x\, dx = dt$ i.e. $x\, dx = \frac{1}{2}dt$

$$\therefore \quad I = \int \sqrt{(x^2)^2-(5)^2}\cdot x\, dx$$

$$= \int \sqrt{t^2-5^2}\,\frac{1}{2}\, dt$$

$$= \frac{1}{2}\left[\frac{t}{2}\sqrt{t^2-5^2} - \frac{5^2}{2}\log\left|t + \sqrt{t^2-5^2}\right|\right] + c$$

$$= \frac{t}{4}\sqrt{t^2-25} - \frac{25}{4}\log\left|t + \sqrt{t^2-25}\right| + c$$

$$= \frac{x^2}{4}\sqrt{x^4-25} - \frac{25}{4}\log\left|x^2 + \sqrt{x^4-25}\right| + c$$

$$= \frac{x^2}{4}\sqrt{x^4-25} - \frac{25}{4}\log\left(x^2 + \sqrt{x^4-25}\right) + c, \text{ as } x^2 > 0$$

Example 9 : Evaluate : $\int \sqrt{(x-3)(7-x)}\, dx$. $(3 < x < 7)$

Solution : $I = \int \sqrt{(x-3)(7-x)}\, dx$

$$= \int \sqrt{10x-x^2-21}\, dx$$

Now, $10x - x^2 - 21 = -[x^2 - 10x + 21]$

$= -[x^2 - 10x + 25 - 4]$

$= -[(x - 5)^2 - 4]$

$= 4 - (x - 5)^2$

$$\therefore \quad I = \int \sqrt{2^2 - (x-5)^2}\, dx$$

$$= \frac{x-5}{2}\sqrt{2^2 - (x-5)^2} + \frac{4}{2}\sin^{-1}\left(\frac{x-5}{2}\right) + c$$

$$= \frac{x-5}{2}\sqrt{(x-3)(7-x)} + 2\sin^{-1}\left(\frac{x-5}{2}\right) + c$$

Example 10 : Evaluate : $\int e^x \left(\frac{1 + \sin x \cos x}{\cos^2 x}\right) dx$

Solution : $I = \int e^x \left(\frac{1 + \sin x \cos x}{\cos^2 x}\right) dx$

$$= \int e^x \left(\frac{1}{\cos^2 x} + \frac{\sin x \cos x}{\cos^2 x}\right) dx$$

$$= \int e^x (\sec^2 x + \tan x)\, dx$$

$$= \int e^x \left(\tan x + \frac{d}{dx}(\tan x)\right) dx$$

$$= e^x \tan x + c$$

Example 11 : Evaluate : $\int \frac{\sqrt{1 - \sin x}}{1 + \cos x} e^{-\frac{x}{2}} dx,\ 0 < x < \frac{\pi}{2}$

Solution : $I = \int \frac{\sqrt{1 - \sin x}}{1 + \cos x} e^{-\frac{x}{2}} dx$

$$= \int \frac{\sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}}{2\cos^2\frac{x}{2}} e^{-\frac{x}{2}} dx$$

$$= \int \frac{\sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}{2\cos^2\frac{x}{2}} e^{-\frac{x}{2}} dx$$

$$= \int \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{2\cos^2\frac{x}{2}} e^{-\frac{x}{2}} dx \qquad \left(\text{since } 0 < \frac{x}{2} < \frac{\pi}{4},\ \cos\frac{x}{2} > \sin\frac{x}{2}\right)$$

Let $-\frac{x}{2} = t$, $-dx = 2dt$. So $dx = -2dt$.

$$\therefore \quad I = -\int \frac{\cos t + \sin t}{2\cos^2 t} e^t \cdot (2dt)$$

$$= -\int \left(\frac{1}{\cos t} + \frac{\sin t}{\cos^2 t}\right) e^t\, dt$$

$$= -\int (\sec t + \sec t \tan t)\, e^t\, dt$$

$$= -\int \left(\sec t + \frac{d}{dt}(\sec t)\right) e^t\, dt$$

$$= -\sec t \cdot e^t + c$$

$$= -e^{-\frac{x}{2}} \cdot \sec\left(\frac{x}{2}\right) + c \qquad \left(\sec\left(-\frac{x}{2}\right) = \sec\frac{x}{2}\right)$$

Example 12 : Evaluate : $\int e^x \sin^2 x\, dx$

Solution : $I = \int e^x \sin^2 x\, dx$

$$= \int e^x \frac{(1 - \cos 2x)}{2}\, dx$$

$$= \frac{1}{2}\int e^x\, dx - \frac{1}{2}\int e^x \cdot \cos 2x\, dx$$

$$= \frac{1}{2} e^x - \frac{1}{2}\left[\frac{e^x}{1^2 + 2^2}(\cos 2x + 2\sin 2x)\right] + c$$

$$= \frac{e^x}{2} - \frac{e^x}{10}(\cos 2x + 2\sin 2x) + c$$

Example 13 : Evaluate : $\int 2^x \cos^2 x\, dx$

Solution : $I = \int 2^x \cos^2 x\, dx$

$$= \int 2^x \left(\frac{1 + \cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\int 2^x\, dx + \frac{1}{2}\int 2^x \cos 2x\, dx$$

$$= \frac{1}{2}\int 2^x\, dx + \frac{1}{2}\int e^{x \cdot \log_e 2} \cos 2x\, dx$$

$$= \frac{1}{2} \cdot \frac{2^x}{\log_e 2} + \frac{1}{2} \cdot \frac{e^{x \log_e 2}}{4 + (\log_e 2)^2}\left[(\log_e 2)\cos 2x + 2\sin 2x\right] + c$$

$$\therefore \quad I = \frac{2^{x-1}}{\log_e 2} + \frac{1}{2} \cdot \frac{2^x}{4 + (\log_e 2)^2} \cdot \left[(\log_e 2)\cos 2x + 2\sin 2x\right] + c$$

Example 14 : Evaluate : $\int (x - 5)\sqrt{x^2 + x}\, dx$

Solution : Here, we find m and n such that,

$$x - 5 = m\left[\frac{d}{dx}(x^2 + x)\right] + n$$

$$= m(2x + 1) + n$$

$$\therefore \quad x - 5 = 2mx + m + n$$

Comparing coefficients of x and constant terms,

$2m = 1$ and $m + n = -5$

$\therefore \quad m = \frac{1}{2}$ and $n = -5 - \frac{1}{2} = -\frac{11}{2}$

$\therefore \quad x - 5 = \frac{1}{2}(2x + 1) - \frac{11}{2}$

$$\therefore \quad I = \int (x-5)\sqrt{x^2+x}\,dx$$

$$= \int \left[\tfrac{1}{2}(2x+1) - \tfrac{11}{2}\right]\sqrt{x^2+x}\,dx$$

$$= \tfrac{1}{2}\int (2x+1)\sqrt{x^2+x}\,dx - \tfrac{11}{2}\int \sqrt{x^2+x}\,dx$$

$$= \tfrac{1}{2}\int (x^2+x)^{\frac{1}{2}} \cdot \tfrac{d}{dx}(x^2+x)\,dx - \tfrac{11}{2}\int \sqrt{\left(x+\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2}\,dx$$

$$= \tfrac{1}{2} \cdot \frac{(x^2+x)^{\frac{3}{2}}}{\frac{3}{2}} - \tfrac{11}{2}\left[\frac{\left(x+\frac{1}{2}\right)}{2}\sqrt{x^2+x} - \tfrac{1}{8}\log\left|\left(x+\tfrac{1}{2}\right) + \sqrt{x^2+x}\right|\right] + c$$

$$= \tfrac{1}{3}(x^2+x)^{\frac{3}{2}} - \tfrac{11}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x} - \tfrac{1}{8}\log\left|x+\tfrac{1}{2}+\sqrt{x^2+x}\right|\right] + c$$

Exercise 2.2

Integrate the following functions *w.r.t.* x considering them well defined over proper domains :

1. $\sqrt{9-x^2}$

2. $\sqrt{2x^2+10}$

3. $\sqrt{5x^2-3}$

4. $\sqrt{4-3x-2x^2}$

5. $\sqrt{4x^2+4x-15}$

6. $x^2\sqrt{8-x^6}$

7. $\cos x\sqrt{4-\sin^2 x}$

8. $e^x(\log \sin x + \cot x)$

9. $e^x \dfrac{1-\sin x}{1-\cos x}$

10. $\dfrac{1+\sin 2x}{1+\cos 2x}e^{2x}$

11. $\dfrac{x^2 e^x}{(x+2)^2}$

12. $\dfrac{x^2-x+1}{(x^2+1)^{\frac{3}{2}}}e^x$

13. $e^x\left(\dfrac{1-x}{1+x^2}\right)^2$

14. $x\sqrt{1+x-x^2}$

15. $(3x-2)\sqrt{x^2+x+1}$

16. $(2x-5)\sqrt{2+3x-x^2}$

17. $e^{2x}\sin 4x$

18. $e^{-\frac{x}{2}}\cos^2 x$

19. $3^x\sin^2 x$

20. $e^{2x}\sin 3x\,\sin x$

*

2.5 Method of Partial Fractions

Now we shall study the method of integrating rational functions. If $p(x)$ and $q(x)$ are two polynomials, then $\dfrac{p(x)}{q(x)}$, $q(x) \neq 0$ is called a rational algebraic function or a rational function of x. We know how to simplify algebraic operations on rational functions.

For example, $\dfrac{5}{x-3} + \dfrac{1}{x-2} = \dfrac{5(x-2)+1(x-3)}{(x-3)(x-2)} = \dfrac{6x-13}{(x-3)(x-2)}$

Let us think the other way round. Can we put $\dfrac{6x-13}{(x-3)(x-2)}$ in the form $\dfrac{5}{x-3} + \dfrac{1}{x-2}$?

The method of expressing a rational function as a sum of other rational functions in this way is known as the **method of partial fractions**.

Expressing $\frac{6x-13}{(x-3)(x-2)}$ as $\frac{5}{x-3} + \frac{1}{x-2}$, its integration will become very simple.

Let us try to understand this method :

(1) If the degree of $p(x)$ < the degree of $q(x)$, then $\frac{p(x)}{q(x)}$ is called a **Proper Rational Function.**

For example, $\frac{5-3x}{x^3+3x+2}$, $\frac{2x^2+3x+7}{x^3-7x+2}$, $\frac{3x+2}{x^3-6x^2+11x-6}$ are proper rational functions.

(2) If the degree of $p(x) \geq$ the degree of $q(x)$, then $\frac{p(x)}{q(x)}$ is called an **Improper Rational Function.**

For example, $\frac{x^3+1}{x^2-2x+1}$, $\frac{x^2+x+1}{x^2+3x+2}$, $\frac{x^3-6x^2+10x-2}{x^2-5x+6}$ are improper rational functions.

If $\frac{p(x)}{q(x)}$ is an improper rational function, we divide $p(x)$ by $q(x)$ so that $p(x) = q(x)\, s(x) + r(x)$, where $r(x) = 0$ or degree of $r(x)$ is less then that of $q(x)$. The improper rational function $\frac{p(x)}{q(x)}$ is expressed in the form $s(x) + \frac{r(x)}{q(x)}$ where $r(x)$ and $s(x)$ are polynomials such that the degree of $r(x)$ is less than that of $q(x)$ or $r(x) = 0$. Thus, $\frac{r(x)}{q(x)}$ is a proper rational function or 0. For example, let us consider $\frac{4x^3-x^2+1}{x^2-2}$.

We should divide $p(x) = 4x^3 - x^2 + 1$ by $q(x) = x^2 - 2$.

$$
\begin{array}{r|l}
 & 4x \quad - 1 \\
\hline
x^2 - 2 & 4x^3 - x^2 + 1 \\
 & 4x^3 \quad - 8x \\
 & - \qquad + \\
\hline
 & -x^2 + 8x + 1 \\
 & -x^2 \qquad + 2 \\
 & + \qquad\quad - \\
\hline
 & 8x - 1
\end{array}
$$

$\therefore$ Quotient $s(x) = 4x - 1$ and remainder $r(x) = 8x - 1$

Thus, $\frac{4x^3-x^2+1}{x^2-2} = (4x-1) + \frac{8x-1}{x^2-2}$.

Here, the quotient $4x - 1$ is a polynomial function and $\frac{8x-1}{x^2-2}$ is a proper rational function. Now we study the method of integrating a proper rational function.

Suppose $\frac{p(x)}{q(x)}$ is a proper rational function. The resolution of $\frac{p(x)}{q(x)}$ into partial fraction depends mainly upon the nature of the factors of $q(x)$ as discussed below.

Case 1 : Real, Linear and Non-repeated Factors :

Let $q(x)$ have n real, linear and non-repeated factors $x - \alpha_1, x - \alpha_2, ..., x - \alpha_n$. i.e.

$q(x) = (x - \alpha_1)(x - \alpha_2)...(x - \alpha_n)$. **($\alpha_i \neq \alpha_j$ for $i \neq j$)**

Then we can express $\frac{p(x)}{q(x)}$ as

$\frac{p(x)}{q(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + ... + \frac{A_n}{x-\alpha_n}$, where $A_1, A_2, ..., A_n$ are constants. We can always determine A_i, $i = 1, 2, ..., n$ uniquely and integrate function on the right hand side easily. Let us take an example to understand this method.

Example 15 : Evaluate : $\int \frac{2x-3}{(x-1)(x-2)(x-3)}\,dx$

Solution : $I = \int \frac{2x-3}{(x-1)(x-2)(x-3)}\,dx$

We can see that given rational function is a proper rational function and in the denominator, we have real, linear and non-repeated factors.

Let $\frac{2x-3}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$. **(i)**

where A, B, C are constants. Multiplying both sides by $(x - 1)(x - 2)(x - 3)$ we get

$2x - 3 = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2)$ **(ii)**

Now we can find constants A, B, C by any one of the following three methods.

First Method :

Denominator of the rational function $(x - 1)(x - 2)(x - 3)$ has three zeros 1, 2, 3.

Let $x = 1, 2, 3$ in equation (ii) by turn and we get the values of A, B, C.

$x = 1$ gives $2(1) - 3 = A(-1)(-2)$. Hence $A = -\frac{1}{2}$.

$x = 2$ gives $2(2) - 3 = B(1)(-1)$. Hence $B = -1$.

$x = 3$ gives $2(3) - 3 = C(2)(1)$. Hence $C = \frac{3}{2}$.

Second Method :

We have $\frac{2x-3}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$. **(ii)**

To find A, we select the factor $x - 1$ in the denominator of A and put that factor equal to zero (i.e. $x - 1 = 0$) and obtain the value of x (i.e. $x = 1$). Replace x by that value in $\frac{2x-3}{(x-2)(x-3)}$, obtained after removing $x - 1$ from L.H.S. Then $A = \frac{2(1)-3}{(1-2)(1-3)} = -\frac{1}{2}$. Similarly to obtain the value of B, we substitute $x = 2$ in $\frac{2x-3}{(x-1)(x-3)}$. So $B = \frac{2(2)-3}{(2-1)(2-3)} = -1$. To obtain value of C, we substitute $x = 3$ in $\frac{2x-3}{(x-1)(x-2)}$. So $C = \frac{2(3)-3}{(3-1)(3-2)} = \frac{3}{2}$.

Thus, $A = -\frac{1}{2}$, $B = -1$ and $C = \frac{3}{2}$.

Third Method :

From (ii) we have,

$$(2x - 3) = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2)$$

$\therefore$ $2x - 3 = A(x^2 - 5x + 6) + B(x^2 - 4x + 3) + C(x^2 - 3x + 2)$

$\therefore$ $2x - 3 = (A + B + C)\,x^2 + (-5A - 4B - 3C)x + (6A + 3B + 2C)$

Comparing the coefficients of x^2 coefficients of x and constant terms on both sides we get,

$A + B + C = 0,\ -5A - 4B - 3C = 2,\ 6A + 3B + 2C = -3$

Solving these equations, we get $A = -\frac{1}{2}$, $B = -1$ and $C = \frac{3}{2}$.

We can use any of the above three methods, whichever seems simple for a particular problem. Now, substituting values of A, B and C in (i) we get,

$$\frac{2x-3}{(x-1)(x-2)(x-3)} = \frac{-\frac{1}{2}}{x-1} + \frac{-1}{x-2} + \frac{\frac{3}{2}}{x-3}.$$

$$\therefore \int \frac{2x-3}{(x-1)(x-2)(x-3)}\,dx = -\frac{1}{2}\int \frac{1}{x-1}\,dx - \int \frac{1}{x-2}\,dx + \frac{3}{2}\int \frac{1}{x-3}\,dx.$$

$$= -\frac{1}{2}\log|x-1| - \log|x-2| + \frac{3}{2}\log|x-3| + c$$

Case 2 : Real, Linear Repeated and Non-repeated Factors :

If $q(x) = (x - \alpha)^k (x - \alpha_1)(x - \alpha_2)...(x - \alpha_n)$, then let

$$\frac{p(x)}{q(x)} = \frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + ... + \frac{A_k}{(x-\alpha)^k} + \frac{B_1}{x-\alpha_1} + \frac{B_2}{x-\alpha_2} + ... + \frac{B_n}{(x-\alpha_n)}$$

Corresponding to non-repeated linear factors we assume as in case (1) and for each repeated factor $(x - \alpha)^k$,, we assume partial fractions,

$\frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \frac{A_3}{(x-\alpha)^3} + ... + \frac{A_k}{(x-\alpha)^k}$, where $A_1, A_2, A_3, ..., A_k$ are constants. Let us take an example to understand this method.

Example 16 : Evaluate : $\int \frac{x}{(x-1)^2(x+2)}\,dx$

Solution : $I = \int \frac{x}{(x-1)^2(x+2)}\,dx$

Let $\frac{x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$ **(i)**

Multiplying both sides by $(x - 1)^2 (x + 2)$, we get

$$x = A(x - 1)(x + 2) + B(x + 2) + C(x - 1)^2$$

Now, $x = 1$ gives $1 = B(3)$. So $B = \frac{1}{3}$

$x = -2$ gives $-2 = C(9)$. So $C = -\frac{2}{9}$

Comparing coefficient of x^2. $A + C = 0$. So $A = -C$.

$\therefore$ $A = \frac{2}{9}$

Substituting values of A, B, C in expression (i),

$$\frac{x}{(x-1)^2(x+2)} = \frac{2}{9(x-1)} + \frac{1}{3(x-1)^2} - \frac{2}{9(x+2)}$$

$$\therefore \quad \int \frac{x\,dx}{(x-1)^2(x+2)} = \frac{2}{9}\int \frac{1}{x-1}\,dx + \frac{1}{3}\int \frac{1}{(x-1)^2}\,dx - \frac{2}{9}\int \frac{1}{x+2}\,dx$$

$$= \frac{2}{9}\log|x-1| + \frac{1}{3}\frac{(x-1)^{-1}}{-1} - \frac{2}{9}\log|x+2| + c$$

$$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + c$$

Case 3 : One Real Quadratic and Other Linear non-repeated factors :

If $q(x) = (ax^2 + bx + c)(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n)$, then let

$$\frac{p(x)}{q(x)} = \frac{Ax + B}{ax^2 + bx + c} + \frac{A_1}{x - \alpha_1} + \frac{A_2}{x - \alpha_2} + \ldots + \frac{A_n}{x - \alpha_n}$$

where $A_1, A_2, A_3, \ldots, A_n$ are constants to be determined. Let us take an example to understand this method.

Example 17 : Evaluate : $\int \frac{x\,dx}{(3x^2+2)(x-2)}$

Solution : $I = \int \frac{x\,dx}{(3x^2+2)(x-2)}$

Let $\frac{x}{(3x^2+2)(x-2)} = \frac{A}{x-2} + \frac{Bx + C}{3x^2+2}$

Multiplying by $(3x^2 + 2)(x - 2)$ on both the sides,

$$x = A(3x^2 + 2) + (Bx + C)(x - 2)$$

$$\therefore \quad x = A(3x^2 + 2) + Bx(x - 2) + C(x - 2)$$

$x = 2$ gives $2 = 14A$. So $A = \frac{1}{7}$.

Comparing coefficients of x^2 on both sides,

$3A + B = 0$. So $B = -3A$

$\therefore \quad B = -\frac{3}{7}$

Comparing coefficients of x on both sides,

$C - 2B = 1$. So $C = 1 + 2B = 1 - \frac{6}{7} = \frac{1}{7}$

$\therefore \quad C = \frac{1}{7}$

$$\therefore \quad \int \frac{x\,dx}{(3x^2+2)(x-2)} = \int \frac{\frac{1}{7}dx}{x-2} + \int \frac{\left(-\frac{3}{7}x + \frac{1}{7}\right)dx}{3x^2+2}$$

$$= \frac{1}{7}\int \frac{dx}{x-2}\,dx - \frac{1}{7}\int \frac{(3x-1)\,dx}{3x^2+2}$$

$$= \frac{1}{7}\int \frac{1}{x-2}\,dx - \frac{1}{7}\int \frac{3x\,dx}{3x^2+2} + \frac{1}{7}\int \frac{dx}{3x^2+2}$$

$$= \frac{1}{7}\int \frac{1}{x-2}\,dx - \frac{1}{14}\int \frac{6x\,dx}{3x^2+2} + \frac{1}{7}\int \frac{dx}{(\sqrt{3}x^2)+(\sqrt{2})^2}\,dx$$

$$= \frac{1}{7}\log|x-2| - \frac{1}{14}\log|3x^2+2| + \frac{1}{7\sqrt{6}}\,tan^{-1}\left(\frac{\sqrt{3}x}{\sqrt{2}}\right) + c$$

$$= \frac{1}{7}\log|x-2| - \frac{1}{14}\log(3x^2+2) + \frac{1}{7\sqrt{6}}\,tan^{-1}\frac{\sqrt{3}x}{\sqrt{2}} + c \text{ as } x^2 \geq 0$$

Example 18 : Evaluate : $\int \frac{x^2\,dx}{(x^2+1)(x^2+4)}$

Solution : $I = \int \frac{x^2\,dx}{(x^2+1)(x^2+4)}$

Here all the indices of x are even. Write $x^2 = t$ in the integrand. (It is not a substitution).

$$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{t}{(t+1)(t+4)}$$

Let $\frac{t}{(t+1)(t+4)} = \frac{A}{t+1} + \frac{B}{t+4}$

$\therefore \quad t = A(t+4) + B(t+1)$

Taking $t = -1$, we get $-1 = 3A$. So $A = -\frac{1}{3}$.

Taking $t = -4$, we get $-4 = -3B$. So $B = \frac{4}{3}$.

Substituting values of A and B in (i)

$$\frac{t}{(t+1)(t+4)} = \frac{-\frac{1}{3}}{t+1} + \frac{\frac{4}{3}}{t+4}$$

Now, $t = x^2$ thus, $\frac{x^2}{(x^2+1)(x^2+4)} = \frac{-\frac{1}{3}}{x^2+1} + \frac{\frac{4}{3}}{x^2+4}$

$$\therefore \quad \int \frac{x^2}{(x^2+1)(x^2+4)}\,dx = -\frac{1}{3}\int \frac{dx}{x^2+1} + \frac{4}{3}\int \frac{dx}{x^2+4}$$

$$= -\frac{1}{3}\,tan^{-1}x + \frac{4}{3} \times \frac{1}{2}\,tan^{-1}\left(\frac{x}{2}\right) + c$$

$$\therefore \quad I = -\frac{1}{3}\,tan^{-1}x + \frac{2}{3}\,tan^{-1}\left(\frac{x}{2}\right) + c$$

Example 19 : Evaluate : $\int \frac{x^2}{(x^3+2)(x^3-5)}\,dx$

Solution : $I = \int \frac{x^2}{(x^3+2)(x^3-5)}\,dx$

Let $x^3 = t$, So $3x^2\,dx = dt$. Hence $x^2\,dx = \frac{1}{3}\,dt$

$\therefore \quad I = \frac{1}{3}\int \frac{dt}{(t+2)(t-5)}$.

Let $\frac{1}{(t+2)(t-5)} = \frac{A}{t+2} + \frac{B}{t-5}$

$1 = A(t-5) + B(t+2)$

$t = -2$ gives, $1 = -7A$. So $A = -\frac{1}{7}$

$t = 5$ gives, $1 = 7B$. So $B = \frac{1}{7}$

$$\therefore \quad \frac{1}{(t+2)(t-5)} = \frac{-\frac{1}{7}}{t+2} + \frac{\frac{1}{7}}{t-5}.$$

$$\therefore \quad I = \frac{1}{3}\int \frac{dt}{(t+2)(t-5)}$$

$$= -\frac{1}{21}\int \frac{1}{t+2}\,dt + \frac{1}{21}\int \frac{1}{t-5}\,dt$$

$$= -\frac{1}{21}\log|t+2| + \frac{1}{21}\log|t-5| + c$$

$$= \frac{1}{21}\log\left|\frac{t-5}{t+2}\right| + c$$

$$= \frac{1}{21}\log\left|\frac{x^3-5}{x^3+2}\right| + c$$

Example 20 : Evaluate : $\displaystyle\int \frac{x^2+x+1}{(x-1)^3}\,dx$

Solution : $I = \displaystyle\int \frac{x^2+x+1}{(x-1)^3}\,dx$

Let $x-1 = t$, $dx = dt$.

$$I = \int \frac{(t+1)^2+(t+1)+1}{t^3}\,dt$$

$$= \int \frac{t^2+3t+3}{t^3}\,dt$$

$$= \int \left(\frac{1}{t} + \frac{3}{t^2} + \frac{3}{t^3}\right) dt$$

$$= \int \frac{1}{t}\,dt + 3\int t^{-2}\,dt + 3\int t^{-3}\,dt$$

$$= \log|t| + 3\left(\frac{-1}{t}\right) + 3\left(\frac{1}{-2t^2}\right) + c$$

$$= \log|t| - \frac{3}{t} - \frac{3}{2t^2} + c$$

$$= \log|x-1| - \frac{3}{x-1} - \frac{3}{2(x-1)^2} + c$$

Note : This sum can also be done using partial fractions.

$$\frac{x^2+x+1}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

Example 21 : Evaluate : $\displaystyle\int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}\,d\theta$

Solution : $I = \displaystyle\int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}\,d\theta$

$$= \int \frac{\tan\theta\,(1+\tan^2\theta)}{1+\tan^3\theta}\,d\theta$$

$$= \int \frac{\tan\theta \cdot \sec^2\theta}{1+\tan^3\theta}\,d\theta$$

Let $tan\theta = t$. So $sec^2\theta \, d\theta = dt$

$$I = \int \frac{t\,dt}{1+t^3}$$

$$= \int \frac{t\,dt}{(t+1)(t^2-t+1)}$$

Let $\frac{t}{(t+1)(t^2-t+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2-t+1}$

$\therefore \quad t = A(t^2 - t + 1) + (Bt + C)(t + 1)$

$\therefore \quad t = A(t^2 - t + 1) + Bt(t + 1) + C(t + 1)$

$t = -1$ gives $-1 = 3A$. So $A = -\frac{1}{3}$

Comparing the coefficients of t^2 on both sides, we get $A + B = 0$. So $B = -A$.

$\therefore \quad B = \frac{1}{3}$

Comparing the constant terms on both sides, we get $A + C = 0$. So $C = -A$.

$\therefore \quad C = \frac{1}{3}$

$$\therefore \quad \frac{t}{(t+1)(t^2-t+1)} = \frac{-\frac{1}{3}}{t+1} + \frac{\frac{1}{3}t+\frac{1}{3}}{t^2-t+1}$$

$$\therefore \quad I = -\frac{1}{3}\int \frac{1}{t+1}\,dt + \frac{1}{3}\int \frac{t+1}{t^2-t+1}\,dt$$

$$= -\frac{1}{3}\int \frac{1}{t+1}\,dt + \frac{1}{6}\int \frac{2t+2}{t^2-t+1}\,dt$$

$$= -\frac{1}{3}\int \frac{1}{t+1}\,dt + \frac{1}{6}\int \frac{(2t-1)+3}{t^2-t+1}\,dt$$

$$= -\frac{1}{3}\int \frac{dt}{t+1} + \frac{1}{6}\int \frac{(2t-1)\,dt}{t^2-t+1} + \frac{3}{6}\int \frac{dt}{t^2-t+1}$$

$$= -\frac{1}{3}\int \frac{dt}{t+1} + \frac{1}{6}\int \frac{(2t-1)\,dt}{t^2-t+1} + \frac{1}{2}\int \frac{dt}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{1}{3}\log|t+1| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{2} \times \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$= -\frac{1}{3}\log|t+1| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{\sqrt{3}} tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + c$$

$$\therefore \quad I = -\frac{1}{3}\log|tan\theta + 1| + \frac{1}{6}\log|tan^2\theta - tan\theta + 1| + \frac{1}{\sqrt{3}} tan^{-1}\left(\frac{2tan\theta - 1}{\sqrt{3}}\right) + c$$

Exercise 2.3

Integrate the following functions defined over a proper domain *w.r.t.* x :

1. $\dfrac{x^2+4x-1}{x^3-x}$

2. $\dfrac{3x+2}{(x-1)(x-2)(x-3)}$

3. $\dfrac{x^3-6x^2+10x-2}{x^2-5x+6}$

4. $\dfrac{x^2}{(2x^2+1)(x^2-1)}$

5. $\dfrac{x^2+1}{(x^2+2)(2x^2+1)}$

6. $\dfrac{x^3}{(x^2+2)(x^2+5)}$

7. $\dfrac{x^2+x+1}{(x+1)^2(x+2)}$

8. $\dfrac{5x}{(x+1)(x^2+9)}$

9. $\dfrac{1}{6e^{2x}+5e^x+1}$

10. $\dfrac{sec^2\theta}{tan^2\theta-4tan\,\theta+3}$

11. $\dfrac{1}{(x+1)^2(x^2+1)}$

12. $\dfrac{x^2}{(x-1)^3(x+1)}$

13. $\dfrac{1}{sin\,x-sin\,2x}$

14. $\dfrac{1}{sin\,x(3+2cos x)}$

*

Miscellaneous Examples :

Example 22 : Evaluate : $\int (x+1)\sqrt{\dfrac{x+2}{x-2}}\,dx \qquad x > 2$ **(If $x < -2$?)**

Solution : $I = \int (x+1)\sqrt{\dfrac{x+2}{x-2}}\,dx$

$$= \int (x+1)\sqrt{\frac{x+2}{x-2}\times\frac{x+2}{x+2}}\,dx \qquad (x > 2)$$

$$= \int \frac{(x+1)(x+2)}{\sqrt{x^2-4}}\,dx$$

$$= \int \frac{x^2+3x+2}{\sqrt{x^2-4}}\,dx$$

$$= \int \frac{(x^2-4)+3x+6}{\sqrt{x^2-4}}\,dx$$

$$= \int \sqrt{x^2-4}\,dx + 3\int \frac{x}{\sqrt{x^2-4}}\,dx + 6\int \frac{dx}{\sqrt{x^2-4}}$$

$$= \int \sqrt{x^2-4}\,dx + \frac{3}{2}\int (x^2-4)^{-\frac{1}{2}}(2x)\,dx + 6\int \frac{dx}{\sqrt{x^2-4}}$$

$$= \frac{x}{2}\sqrt{x^2-4} - \frac{4}{2}\log|x+\sqrt{x^2-4}| + \frac{3}{2}\,\frac{(x^2-4)^{\frac{1}{2}}}{\frac{1}{2}} + 6\log|x+\sqrt{x^2-4}| + c$$

$$= \frac{x}{2}\sqrt{x^2-4} + 4\log\left|x + \sqrt{x^2-4}\right| + 3\sqrt{x^2-4} + c$$

$$= \left(\frac{x}{2} + 3\right)\sqrt{x^2-4} + 4\log\left|x + \sqrt{x^2-4}\right| + c$$

Example 23 : Evaluate : $\int \frac{(1+\sin x)\,dx}{\sin x(1+\cos x)}$

Solution : $I = \int \frac{(1+\sin x)\,dx}{\sin x(1+\cos x)}$

$$I = \int \frac{dx}{\sin x(1+\cos x)} + \int \frac{dx}{1+\cos x}$$

Let $I = I_1 + I_2$ where $I_1 = \int \frac{dx}{\sin x(1+\cos x)}$, $I_2 = \int \frac{dx}{1+\cos x}$

$$I_1 = \int \frac{dx}{\sin x(1+\cos x)}$$

$$= \int \frac{\sin x\,dx}{\sin^2 x(1+\cos x)}$$

$$= \int \frac{\sin x\,dx}{(1-\cos x)(1+\cos x)^2}$$

Now, $\cos x = t$ gives $\sin x\,dx = -dt$

$$I_1 = \int \frac{-dt}{(1-t)(1+t)^2}$$

Let $\frac{-1}{(1-t)(1+t)^2} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{(1+t)^2}$

$-1 = A(1+t)^2 + B(1-t)(1+t) + C(1-t)$

$t = 1$ gives $-1 = A(4)$. So $A = -\frac{1}{4}$

$t = -1$ gives $-1 = C(2)$. So $C = -\frac{1}{2}$

$t = 0$ gives (or any convenient value of t can be taken)

$-1 = A + B + C$

$\therefore \quad B = -1 + \frac{1}{4} + \frac{1}{2}$

$\therefore \quad B = -\frac{1}{4}$

$$\therefore \quad \frac{-1}{(1-t)(1+t)^2} = \frac{-\frac{1}{4}}{1-t} + \frac{-\frac{1}{4}}{1+t} + \frac{-\frac{1}{2}}{(1+t)^2}$$

$$I_1 = -\frac{1}{4}\int \frac{1}{1-t}\,dt - \frac{1}{4}\int \frac{1}{1+t}\,dt - \frac{1}{2}\int (1+t)^{-2}\,dt$$

$$= \frac{1}{4}\log\left|\frac{t-1}{t+1}\right| + \frac{1}{2(t+1)} + c_1$$

$$\therefore \quad I_1 = \frac{1}{4} \log \left| \frac{\cos x - 1}{\cos x + 1} \right| + \frac{1}{2(\cos x + 1)} + c_1$$

$$\text{Now, } I_2 = \int \frac{1}{1 + \cos x} \, dx = \int \frac{1}{2\cos^2 \frac{x}{2}} \, dx = \frac{1}{2} \int \sec^2 \frac{x}{2} \, dx$$

$$= \frac{1}{2} \frac{\tan \frac{x}{2}}{\frac{1}{2}} + c_2$$

$$\therefore \quad I_2 = \tan \frac{x}{2} + c_2$$

$$I = I_1 + I_2$$

$$\therefore \quad I = \frac{1}{4} \log \left| \frac{\cos x - 1}{\cos x + 1} \right| + \frac{1}{2(\cos x + 1)} + \tan \frac{x}{2} + c \qquad (c_1 + c_2 = c)$$

$$= \frac{1}{4} \log \left| \tan^2 \frac{x}{2} \right| + \frac{1}{4\cos^2 \frac{x}{2}} + \tan \frac{x}{2} + c$$

$$= \frac{1}{2} \log \left| \tan \frac{x}{2} \right| + \frac{1}{4} \sec^2 \frac{x}{2} + \tan \frac{x}{2} + c$$

Second Method :

Let $\tan \frac{x}{2} = t$, so $\sec^2 \frac{x}{2} \cdot \frac{1}{2} dx = dt$

Hence $dx = \frac{2dt}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

$$I = \int \frac{(1 + \sin x) \, dx}{\sin x (1 + \cos x)}$$

$$= \int \frac{1 + \frac{2t}{1+t^2}}{\left(\frac{2t}{1+t^2}\right)\left(1 + \frac{1-t^2}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{1+t^2+2t}{2t\,(1+t^2+1-t^2)} \cdot 2dt$$

$$= \int \frac{1 + 2t + t^2}{2t} \, dt$$

$$= \frac{1}{2} \int \left(\frac{1}{t} + 2 + t\right) dt$$

$$= \frac{1}{2} \left[\log |t| + 2t + \frac{t^2}{2}\right] + c$$

$$= \frac{1}{2} \log |t| + t + \frac{1}{4} t^2 + c$$

$$= \frac{1}{2} \log \left| \tan \frac{x}{2} \right| + \tan \frac{x}{2} + \frac{1}{4} \tan^2 \frac{x}{2} + c'$$

$$\text{Observe that } I = \frac{1}{2} \log \left| \tan \frac{x}{2} \right| + \tan \frac{x}{2} + \frac{1}{4} \left(\sec^2 \frac{x}{2} - 1\right) + c'$$

$$= \frac{1}{2} \log \left| \tan \frac{x}{2} \right| + \tan \frac{x}{2} + \frac{1}{4} \sec^2 \frac{x}{2} - \frac{1}{4} + c'$$

$$= \frac{1}{2} \log \left| \tan \frac{x}{2} \right| + \tan \frac{x}{2} + \frac{1}{4} \sec^2 \frac{x}{2} + c \qquad \left(c = c' - \frac{1}{4}\right)$$

Thus, we can see that answers obtained by both the methods are same.

Example 24 : Evaluate : $\int \left(\log(\log x) + \frac{1}{(\log x)^2}\right) dx, \quad x > 1$

Solution : $I = \int \left(\log(\log x) + \frac{1}{(\log x)^2}\right) dx$

Let $\log x = t$. So $x = e^t$

$\therefore \quad dx = e^t\, dt$

$$\therefore \quad I = \int \left(\log t + \frac{1}{t^2}\right) e^t\, dt$$

$$= \int \left(\log t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2}\right) e^t\, dt$$

$$= \int \left[\left(\log t + \frac{1}{t}\right) - \left(\frac{1}{t} - \frac{1}{t^2}\right)\right] e^t\, dt$$

$$= \int \left(\log t + \frac{1}{t}\right) e^t\, dt - \int \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t\, dt$$

$$= e^t \log t - e^t \frac{1}{t} + c$$

$$= x \log(\log x) - \frac{x}{\log x} + c$$

Example 25 : Evaluate : $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\, dx$

Solution : $I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\, dx$

$$= \int \frac{\sin^{-1}\sqrt{x} - (\frac{\pi}{2} - \sin^{-1}\sqrt{x})}{\frac{\pi}{2}}\, dx \qquad \left(\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}\right)$$

$$= \int \frac{2\sin^{-1}\sqrt{x} - \frac{\pi}{2}}{\frac{\pi}{2}}\, dx$$

$$= \frac{4}{\pi} \int \sin^{-1}\sqrt{x}\, dx - \int dx$$

Let $I_1 = \int \sin^{-1}\sqrt{x}\, dx$

Let $\sin^{-1}\sqrt{x} = \theta$. So $x = \sin^2\theta, \quad 0 < \theta < \frac{\pi}{2}$ $\qquad \left(\sqrt{x} > 0. \text{ So, } 0 < \theta < \frac{\pi}{2}\right)$

$\therefore \quad dx = 2\sin\theta \cdot \cos\theta\, d\theta$

$$\therefore \quad I_1 = \int \theta\, 2\sin\theta \cos\theta\, d\theta$$

$$= \int \theta \sin 2\theta\, d\theta$$

$$= -\frac{\theta \cos 2\theta}{2} + \frac{1}{2} \int \cos 2\theta\, d\theta$$

$$= -\frac{\theta}{2} \cos 2\theta + \frac{\sin 2\theta}{4}$$

$$= -\frac{\theta}{2}(1 - 2\sin^2\theta) + \frac{1}{2}\sin\theta \cdot \cos\theta$$

$$= -\frac{1}{2}\ sin^{-1}\sqrt{x}\ (1 - 2x) + \frac{1}{2}\sqrt{x}\ \sqrt{1-x}$$

$$= -\frac{1}{2}\ sin^{-1}\sqrt{x} + x\ sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x-x^2}$$

$$\therefore \quad I = \frac{4}{\pi}\int sin^{-1}\sqrt{x}\, dx - \int dx$$

$$= \frac{4}{\pi}\left[-\frac{1}{2}\ sin^{-1}\sqrt{x} + x sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x-x^2}\right] - x + c$$

Exercise 2

Integrate the following functions defined on proper domain *w.r.t.* x :

1. $x^2\ sin^{-1}x$

2. $tan^{-1}\sqrt{\frac{1-x}{1+x}}$

3. $\frac{x - sin\ x}{1 - cos\ x}$

4. $\frac{\sqrt{sin\,x}}{cos\,x}$

5. $\log\,(x + \sqrt{x^2 + a^2}\,)$

6. $sin^{-1}\sqrt{\frac{x}{x+a}}$

7. $\frac{sin^{-1}\sqrt{x}}{\sqrt{1-x}}$

8. $\frac{\sqrt{1+sin 2x}}{1+cos 2x}\ e^x$

9. $\frac{\log x - 1}{(\log x)^2}$

10. $\log\,(\log x) + \frac{1}{\log x}$

11. $x\sqrt{2ax - x^2}$

12. $(x - 5)\sqrt{x^2 + x}$

13. $\frac{1}{cos\ x\ cos\ 2x}$

14. $\frac{1}{sin\ x + sin\ 2x}$

15. $\frac{sin\ x}{sin\ 4x}$

16. $cot^{-1}(1 - x + x^2)$ **$(0 < x < 1)$**

17. $\frac{1}{sin\,x\sqrt{cos^3 x}}$

18. $\frac{sec\ x}{1 + cosec\ x}$

19. $\frac{1 + sin\ x}{sin\ x\,(1 + cos\ x)}$

20. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) $\int cos\,(\log x)\ dx = ... + c$ ☐

(a) $\frac{x}{2}\,[cos\,(\log x) + sin\,(\log x)]$

(b) $\frac{x}{4}\,[cos\,(\log x) + sin\,(\log x)]$

(c) $\frac{x}{2}\,[cos\,(\log x) - sin\,(\log x)]$

(d) $\frac{x}{2}\,[sin\,(\log x) - cos\,(\log x)]$

(2) $\int e^x\,sin\,x\ cos\,x\ dx = ... + c$ ☐

(a) $\frac{e^x}{2\sqrt{5}}\ cos\,(2x - tan^{-1}2)$

(b) $\frac{e^x}{2\sqrt{5}}\ sin\,(2x - tan^{-1}2)$

(c) $\frac{e^2}{2\sqrt{5}}\ sin\,(2x + tan^{-1}2)$

(d) $\frac{e^{2x}}{2\sqrt{5}}\ sin\,(2x + \pi - tan^{-1}2)$

(3) $\int e^x \sec x\ (1 + \tan x)\ dx$ = ... + c

(a) $e^x \sec x\ \tan x$ (b) $e^x \tan x$ (c) $e^x \sec x$ (d) $-e^x \sec x$

(4) $\int \frac{(5 + \log x)\, dx}{(6 + \log x)^2}$ = ... + c

(a) $\frac{x}{\log_e x + 6}$ (b) $\frac{1}{5 + \log_e x}$ (c) $\frac{x}{\log_e x + 5}$ (d) $\frac{e^x}{\log_e x + 6}$

(5) $\int \frac{e^{\tan^{-1}x}}{1 + x^2} (1 + x + x^2)\ dx$ = ... + c

(a) $e^{\tan^{-1}x}$ (b) $\frac{e^{\tan^{-1}x}}{1 + x^2}$ (c) $x \cdot e^{\tan^{-1}x}$ (d) $\frac{x}{1 + x}\ e^{\tan^{-1}x}$

(6) $\int e^x \left(\frac{1 + \sin x}{1 + \cos x}\right) dx$ = ... + c

(a) $e^x \cot x$ (b) $e^x \cot \frac{x}{2}$ (c) $e^x \tan \frac{x}{2}$ (d) $e^{\frac{x}{2}} \cdot \tan \frac{x}{2}$

(7) $\int e^x \left(\frac{1 + x \log x}{x}\right) dx$ = ... + c

(a) $e^x \log x$ (b) $x \cdot e^x$ (c) $\frac{1}{x} \log x$ (d) $e^{-x} \log x$

(8) $\int \left(\log x + \frac{1}{x^2}\right) e^x\ dx$ = ... + c

(a) $e^x \left(\log x + \frac{1}{x^2}\right)$ (b) $e^x \left(\log x + \frac{1}{x}\right)$ (c) $e^x \left(\log x - \frac{1}{x^2}\right)$ (d) $e^x \left(\log x - \frac{1}{x}\right)$

(9) $\int \left(\frac{x - 1}{x^2}\right) e^x\ dx$ = ... + c

(a) $\frac{1}{x^2}\ e^x$ (b) $\frac{1}{x}\ e^x$ (c) $-\frac{1}{x^2}\ e^x$ (d) $-\frac{1}{x}\ e^x$

(10) $\int (x^6 + 7x^5 + 6x^4 + 5x^3 + 4x^2 + 3x + 1)\ e^x\ dx$ = ... + c

(a) $\sum_{i=1}^{7} x^i e^x$ (b) $\sum_{i=1}^{6} x^i e^x$ (c) $\sum_{i=0}^{6} i\ e^x$ (d) $\sum_{i=0}^{6} (xe)^i$

(11) $\int \tan^{-1}x\ dx$ = ... + c

(a) $x \tan^{-1}x - \frac{1}{2} \log |1 + x^2|$ (b) $x \tan^{-1}x + \frac{1}{2} \log \frac{\tan^{-1}x}{1 + x^2}$

(c) $x \tan^{-1}x + \frac{1}{2} \log |x^2 + 1|$ (d) $\frac{1}{1 + x^2}$

Summary

We have studied the following points in this chapter :

1. Rule of Integration by Parts :

If (1) f and g are differentiable on I = (a, b) and

(2) f' and g' are continuous on I, then $\int f(x) \cdot g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx$

If we take $f(x) = u$ and $g'(x) = v$, then $f'(x) = \frac{du}{dx}$ and $g(x) = \int v\, dx$

Then the new form is $\int uv\, dx = u \int v\, dx - \int \left(\frac{du}{dx} \int v\, dx\right) dx$.

2. Standard Forms of Integration :

(1) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + c$

(2) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + c$

(3) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$ **($a > 0$)**

(4) $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$

(5) $\int e^{ax} \cdot \sin(bx + k)\, dx = \frac{e^{ax}}{a^2 + b^2} [a \sin(bx + k) - b \cos(bx + k)] + c$ **($a \neq 0, b \neq 0$)**

$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin(bx + k - \alpha) + c$

where $\cos\alpha = \frac{a}{\sqrt{a^2 + b^2}}$, $\sin\alpha = \frac{b}{\sqrt{a^2 + b^2}}$. $\alpha \in (0, 2\pi)$

(6) $\int e^{ax} \cos(bx + k)\, dx = \frac{e^{ax}}{a^2 + b^2} [a \cos(bx + k) + b \sin(bx + k)] + c$ **($a \neq 0, b \neq 0$)**

$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos(bx + k - \alpha) + c$

where $\cos\alpha = \frac{a}{\sqrt{a^2 + b^2}}$, $\sin\alpha = \frac{b}{\sqrt{a^2 + b^2}}$. $\alpha \in (0, 2\pi)$.

3. Integrals of the type : (1) $\int \sqrt{ax^2 + bx + c}\, dx$ (2) $\int (Ax + B)\sqrt{ax^2 + bx + c}\, dx$

4. Method of Partial Fractions.

3 DEFINITE INTEGRATION

Calculus required continuity and continuity was supposed to require the infinitely little; but nobody could discover what the infinitely little might be.

– Bertrand Russell

◆

All great theorems were discovered after midnight.

– Adrian Mathesis

3.1 Introduction

We have already studied integration (antiderivation) as an operation inverse to differentiation. From the historical point of view, the concept of integration originated earlier than the concept of differentiation. Infact the concept of integration owes its origin to the problem of finding areas of plane regions, surface areas and volumes of solid bodies etc. Firstly the definite integral was expressed as a limit of a certain sum expressing the area of some region. The word integration has originated from 'addition' and the verb 'to integrate' means 'to merge'. Later on, link between apparently two different concepts of differentiation and integration was established by well known mathematicians **Newton** and **Leibnitz** in 17th century. This relation is known as fundamental theorem of integral calculus and we will learn it in this chapter.

The calculations of area, volume are done using integration. In the 19th century, **Cauchy** and **Riemann** developed the concept of Riemann integration.

Now in this chapter we shall understand the idea of definite integration as the limit of a sum and how it is helpful to find out the area as well as how it can be linked with differentiation.

3.2 Definite Integral as the Limit of a Sum

You have studied in std. XI that restoring force acting on spring-mass system is given by $F = -kx$, where k is force constant of the spring. If we consider only magnitude, we may consider $F = kx$. If $k = 10$, then $F = 10x$. Here we would find the work done, if displacement occurs due to the force. As per definition of work, work done by the system at a particular moment is,

w = Force acting at a particular moment $\times$ displacement due to force.

Now $F = 10x$ shows that force changes with displacement. So, how would we find the work done during the displacement of 10 units ?

As per a common estimate for work done during the displacement,

Initial force $\times$ displacement $\leq w \leq$ final force $\times$ displacement

Let us calculate w for the above mentioned example. First displacement occurs in [0, 10]. In this case for $x = 10$, force is maximum i.e. 100 units and for $x = 0$, it is minimum i.e. zero. So in this interval, work w satisfies,

$$0 \times 0 \leq w \leq 100 \times 10 \qquad \textbf{(}w \times d = 0 \times 0 \textbf{ and } w \times d = 100 \times 10\textbf{)}$$

$\therefore$ For work done in interval [0, 10], $0 \leq w \leq 1000$ **(i)**

Now to get a better estimate of work (w), let us divide the interval [0, 10] into two congruent subintervals i.e. [0, 5] and [5, 10]. Suppose in the interval [0, 5], the work done is w_1, then since maximum force is 50 units and minimum force is 0 unit in this interval, so for interval [0, 5], work done w_1 satisfies,

$$0 \le w_1 \le 50 \times 5$$

$$\therefore \quad 0 \le w_1 \le 250$$

Similarly, if the work done in the second interval is w_2, $250 \le w_2 \le 500$

$$\therefore \quad \text{Total work done } w = w_1 + w_2$$

$$250 \le w_1 + w_2 \le 750$$

$$\therefore \quad 250 \le w \le 750 \qquad \textbf{(ii)}$$

Here it can be seen that result (ii) gives a better estimate than result (i). If the interval [0, 10] is divided into three subintervals $\left[0, \frac{10}{3}\right], \left[\frac{10}{3}, \frac{20}{3}\right], \left[\frac{20}{3}, 10\right]$, work done in each interval would be as follows :

Taking $x = \frac{10}{3}$ in F = 10x, we get maximum work $w = \frac{100}{3} \times \frac{10}{3} = \frac{1000}{9}$

$$0 \le w_1 \le \frac{1000}{9}$$

Similarly $\frac{1000}{9} \le w_2 \le \frac{2000}{9}$

and $\frac{2000}{9} \le w_3 \le \frac{3000}{9}$

As $w = w_1 + w_2 + w_3$, so, $\frac{3000}{9} \le w \le \frac{6000}{9}$

$$\therefore \quad 333\tfrac{1}{3} \le w \le 666\tfrac{2}{3} \qquad \textbf{(iii)}$$

It is seen that result (iii) is still a better estimate than result (ii). Thus more and more divisions of the intervals lead to better estimates of the work. If [0, 10] is divided into n equal intervals viz, $\left[0, \frac{10}{n}\right]$, $\left[\frac{10}{n}, \frac{20}{n}\right], \left[\frac{20}{n}, \frac{30}{n}\right], \ldots, \left[\frac{10(n-1)}{n}, 10\right]$.

ith interval in this partition would satisfy $\left[\frac{10(i-1)}{n}, \frac{10i}{n}\right]$.

Taking $x = \frac{10i}{n}$ in F = 10x, we get maximum work $w = 10 \times \frac{10i}{n} \times \frac{10}{n} = \frac{1000i}{n^2}$

The work done in this subinterval would satisfy $\frac{1000(i-1)}{n^2} \le w_i \le \frac{1000i}{n^2}$

$$\therefore \quad \text{Total work will satisfy } \frac{1000}{n^2} \sum_{i=1}^{n} (i-1) \le w \le \frac{1000}{n^2} \sum_{i=1}^{n} i.$$

Here, the difference between the maximum and minimum values of work is

$$\frac{1000}{n^2} \sum_{i=1}^{n} i - \frac{1000}{n^2} \sum_{i=1}^{n} (i-1) = \frac{1000}{n^2} \sum_{i=1}^{n} (1) = \frac{1000}{n^2} \times n = \frac{1000}{n}$$

As value of n increases, this decreases and the difference tends to zero. In other words

$$\lim_{n \to \infty} \frac{1000}{n^2} \sum_{i=1}^{n} i = \lim_{n \to \infty} \frac{1000}{n^2} \sum_{i=1}^{n} (i-1)$$

Since the value of w lies between these two, as per sandwich theorem, true value of w will be the value of this limit.

$$\therefore \quad w = \lim_{n \to \infty} \frac{1000}{n^2} \sum_{i=1}^{n} i = \lim_{n \to \infty} \frac{1000}{n^2} \left(\frac{n(n+1)}{2}\right).$$

$$= \lim_{n \to \infty} 500\left(1 + \frac{1}{n}\right) = 500$$

Thus $w = 500$ which is the correct value of work done. Thus we have carried out integration in the interval $[0, 10]$ *w.r.t.* x, which is known as $\int_0^{10} f(x)\,dx = \int_0^{10} 10x\,dx$.

> Here we are using the concept of the limit of a sequence. If (S_n) is a sequence and as n increases indefinitely $|S_n - l|$ becomes arbitrarily small for a definite real number l, we say the sequence is approching l as n tends to infinity and write $\lim_{n \to \infty} S_n = l$. We had intuitively seen this concept in the introduction of e in semester III. We will not study this concept in detail.

Generally, to evaluate $\int_a^b f(x)\,dx$, $[a, b]$ is divided into n congruent sub-intervals. Each interval will have length $h = \left(\frac{b-a}{n}\right)$. Now $[a, b]$ can be partitioned into $[a, a+h]$, $[a+h, a+2h]$,..., $[a+(n-1)h, a+nh]$.

$$\frac{b-a}{n} \sum_{i=1}^{n} f[a + (i-1)h] \le \int_a^b f(x)\,dx \le \frac{b-a}{n} \sum_{i=1}^{n} f(a+ih)$$

and we can take $\int_a^b f(x)\,dx = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f(a+ih)$. From these concepts and understanding, this conclusion will be accepted as a definition.

Definition : Let $f : [a, b] \to R$ be a continuous function. For positive interger n, let $h = \frac{b-a}{n}$. If we partition $[a, b]$ into n sub-intervals of equal length, then the dividing points are $a, a+h, a+2h, ..., a+nh = b$.

Figure 3.1

Let $S_n = \frac{b-a}{n} \sum_{i=1}^{n} f(a+ih)$

Thus we get a sequence $\{S_n\}$ based on function f and partition of $[a, b]$. We assume that for a continuous function, this sequence has a limit and this limit is called definite integral of f over $[a, b]$. It is denoted by $\int_a^b f(x)\,dx$.

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \left(\frac{b-a}{n}\right) \sum_{i=1}^{n} f(a+ih) \qquad \text{(i)}$$

***a* is called the lower limit and *b* is called the upper limit of definite integration.**

Also, we can prove that $\lim\limits_{n \to \infty} \dfrac{b-a}{n} \sum\limits_{i=0}^{n-1} f(a + ih)$ is also equal to $\int\limits_a^b f(x)\, dx$.

Above definition is called the definition of definite integral as the limit of a sum. The above process of linking a function f with its definite integral is called evaluation of definite integral as a limit of a sum.

Note : $\int\limits_a^b f(x)\, dx$ can be defined for certain functions which may not be continuous. But at present we will not discuss them.

Symbol :

Integrand ↓

Upper limit of integration → b

$\int f(x)\, dx$ ← dx suggests integration is carried out *w.r.t.* x.

Lower limit of integration → a

Integration of f from a to b.

3.3 Some Important Results

(1) $1 + 2 + 3 + ... + n = \sum\limits_{i=1}^{n} i = \dfrac{n(n+1)}{2}$

(2) $1^2 + 2^2 + 3^2 + ... + n^2 = \sum\limits_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$

(3) $1^3 + 2^3 + 3^3 + ... + n^3 = \sum\limits_{i=1}^{n} i^3 = \dfrac{n^2(n+1)^2}{4}$

(4) $a + ar + ar^2 + ... + ar^{n-1} = \dfrac{a(r^n - 1)}{r-1} \quad (r \neq 1)$

(5) Let $S_n = sin(a + h) + sin(a + 2h) + ... + sin(a + nh)$, where $h \neq 2n\pi$. $n \in Z$

To find this sum let us multiply both sides by $2sin\frac{h}{2}$. So, we have

$$2sin\frac{h}{2} \cdot S_n = \left[2sin(a+h)\, sin\frac{h}{2} + 2sin\,(a+2h)\, sin\frac{h}{2} + 2sin\,(a+3h)\, sin\frac{h}{2} + ... + 2sin(a+nh)\, sin\frac{h}{2}\right]$$

$$= \left[cos\left(a + \frac{h}{2}\right) - cos\left(a + \frac{3h}{2}\right)\right] + \left[cos\left(a + \frac{3h}{2}\right) - cos\left(a + \frac{5h}{2}\right)\right] + \left[cos\left(a + \frac{5h}{2}\right) - cos\left(a + \frac{7h}{2}\right)\right] + ... + \left[cos\left(a + nh - \frac{h}{2}\right) - cos\left(a + nh + \frac{h}{2}\right)\right]$$

$$2sin\frac{h}{2} \cdot S_n = \left[cos\left(a + \frac{h}{2}\right) - cos\left(a + nh + \frac{h}{2}\right)\right]$$

$$\therefore \quad S_n = \frac{cos\left(a + \frac{h}{2}\right) - cos\left(a + nh + \frac{h}{2}\right)}{2sin\frac{h}{2}} \qquad \left(sin\frac{h}{2} \neq 0\right)$$

If $h = 2n\pi$, $S_n = n\, sin\, na$

(6) Let $S_n = cos(a + h) + cos(a + 2h) + cos(a + 3h) + ... + cos(a + nh)$, where $h \neq 2n\pi$. $n \in Z$

To find this sum let us multiply both the sides by $2sin\frac{h}{2}$. So, we have

$$2sin\frac{h}{2} \cdot S_n = \left[2cos(a+h)\, sin\frac{h}{2} + 2cos\,(a+2h)\, sin\frac{h}{2} + 2cos\,(a+3h)\, sin\frac{h}{2} + ... + 2cos(a+nh)\, sin\frac{h}{2}\right]$$

$$= \left[\sin\left(a + \frac{3h}{2}\right) - \sin\left(a + \frac{h}{2}\right)\right] + \left[\sin\left(a + \frac{5h}{2}\right) - \sin\left(a + \frac{3h}{2}\right)\right] +$$

$$\left[\sin\left(a + \frac{7h}{2}\right) - \sin\left(a + \frac{5h}{2}\right)\right] + \ldots + \left[\sin\left(a + nh + \frac{h}{2}\right) - \sin\left(a + nh - \frac{h}{2}\right)\right]$$

$$2\sin\frac{h}{2} \cdot S_n = \left[\sin\left(a + nh + \frac{h}{2}\right) - \sin\left(a + \frac{h}{2}\right)\right]$$

$$\therefore \quad S_n = \frac{\sin\left(a + nh + \frac{h}{2}\right) - \sin\left(a + \frac{h}{2}\right)}{2\sin\frac{h}{2}} \qquad \left(\sin\frac{h}{2} \neq 0\right)$$

If $h = 2n\pi$, $S_n = n \cos na$

Example 1 : Obtain $\int_1^3 x\,dx$ as the limit of a sum.

Solution : Here, $f(x) = x$ is continuous on [1, 3]. Divide [1, 3] into n congruent sub-intervals and the length of each sub-interval is given by $h = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$.

Here, $a = 1$, $b = 3$ and $f(a + ih) = f(1 + ih) = 1 + ih$

According to the definition,

$$\int_1^3 x\,dx = \lim_{n \to \infty} h \sum_{i=1}^{n} f(a + ih)$$

$$= \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} f(1 + ih)$$

$$= \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} (1 + ih)$$

$$= \lim_{n \to \infty} \frac{2}{n} \left[\sum_{i=1}^{n} 1 + h \sum_{i=1}^{n} i\right]$$

$$= \lim_{n \to \infty} \frac{2}{n} \left[n + \frac{2}{n} \frac{n(n+1)}{2}\right]$$

$$= \lim_{n \to \infty} \left[2 + 2\left(1 + \frac{1}{n}\right)\right]$$

$$= 2 + 2(1 + 0)$$

$$= 4$$

Example 2 : Obtain $\int_0^2 (3x^2 - 2x + 4)dx$ as the limit of a sum.

Solution : Here, $f(x) = 3x^2 - 2x + 4$ is continuous on [0, 2]. Divide [0, 2] into n congruent sub-intervals and the length of each sub-interval is given by $h = \frac{b-a}{n}$.

$$\therefore \quad h = \frac{2-0}{n} = \frac{2}{n}$$

$$\therefore \quad h = \frac{2}{n}$$

Here $a = 0$, $b = 2$, $f(x) = 3x^2 - 2x + 4$

$f(a + ih) = f(0 + ih)$

$= f(ih)$

$= 3i^2h^2 - 2ih + 4$

According to the definition,

$$\int_0^2 (3x^2 - 2x + 4)dx = \lim_{n \to \infty} h \sum_{i=1}^{n} f(a + ih)$$

$$= \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} (3i^2h^2 - 2ih + 4)$$

$$= \lim_{n \to \infty} \frac{2}{n} \left[3h^2 \sum_{i=1}^{n} i^2 - 2h \sum_{i=1}^{n} i + \sum_{4=1}^{n} 1\right]$$

$$= \lim_{n \to \infty} \frac{2}{n} \left[3 \cdot \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} - 2 \cdot \frac{2}{n} \frac{n(n+1)}{2} + 4n\right]$$

$$= \lim_{n \to \infty} \left[4\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - 4\left(1 + \frac{1}{n}\right) + 8\right]$$

$$= 4(1 + 0)(2 + 0) - 4(1 + 0) + 8$$

$$= 8 - 4 + 8$$

$$= 12$$

Example 3 : Obtain $\int_{-1}^{1} a^x \, dx$ as the limit of a sum. $(a > 0)$

Solution : Here, $f(x) = a^x$ is continuous on $[-1, 1]$. Divide $[-1, 1]$ into n congruent sub-intervals. Length of each sub-interval is $h = \frac{b-a}{n} = \frac{1+1}{n} = \frac{2}{n}$. So $nh = 2$.

Here, $a = -1$, $b = 1$, $f(x) = a^x$

$f(a + ih) = f(-1 + ih)$

$= a^{-1 + ih}$

$= a^{-1} \cdot a^{ih}$

$f(a + ih) = \frac{a^{ih}}{a}$

As $n \to \infty$, $h \to 0$

Now, $$\int_{-1}^{1} a^x \, dx = \lim_{h \to 0} h \sum_{i=1}^{n} f(a + ih)$$

$$= \lim_{h \to 0} h \sum_{i=1}^{n} \frac{a^{ih}}{a}$$

$$= \lim_{h \to 0} \frac{h}{a} [a^h + a^{2h} + a^{3h} + \ldots + a^{nh}]$$

$$= \lim_{h \to 0} \frac{h}{a} \left[\frac{a^h(a^{nh} - 1)}{a^h - 1}\right]$$

$$= \lim_{h \to 0} \frac{1}{a} \frac{a^h(a^2 - 1)}{\left(\frac{a^h - 1}{h}\right)} \qquad (nh = 2)$$

$$= \frac{1}{a} \cdot \frac{a^0(a^2 - 1)}{\log_e a}$$

$$= \left(\frac{a^2 - 1}{a}\right) \log_a e$$

$$= \left(a - \frac{1}{a}\right) \log_a e$$

Example 4 : Obtain $\int_a^b \sin x \, dx$ as the limit of a sum.

Solution : Here, $f(x) = \sin x$ is a continuous function on $[a, b]$. Divide $[a, b]$ into n congruent sub-intervals. Length of each sub-interval is $h = \frac{b-a}{n}$.

$\therefore \quad nh = b - a, \ a + nh = b$

Also $f(a + ih) = \sin(a + ih)$

As $n \to \infty$, $h \to 0$.

Now, $\int_a^b \sin x \, dx = \lim_{h \to 0} h \sum_{i=1}^{n} f(a + ih)$

$$= \lim_{h \to 0} h \sum_{i=1}^{n} \sin(a + ih)$$

$$= \lim_{h \to 0} h \left[\sin(a + h) + \sin(a + 2h) + \sin(a + 3h) + \ldots + \sin(a + nh)\right]$$

$$= \lim_{h \to 0} h \left[\frac{\cos\left(a + \frac{h}{2}\right) - \cos\left(a + nh + \frac{h}{2}\right)}{2\sin\frac{h}{2}}\right]$$

$$= \lim_{h \to 0} \frac{\cos\left(a + \frac{h}{2}\right) - \cos\left(b + \frac{h}{2}\right)}{\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)} \qquad \textbf{(a + nh = b)}$$

$$= \frac{\cos a - \cos b}{1} \qquad \textbf{(as cosine is continuous)}$$

$$= \cos a - \cos b$$

Note : Since $h \to 0$, we can have $|h| < 2\pi < 2|k|\pi$, $k \in Z - \{0\}$.

Exercise 3.1

Obtain the following definite integrals as the limit of a sum :

1. $\int_0^2 (x + 3)dx$ **2.** $\int_2^4 (2x - 1)dx$ **3.** $\int_1^3 (2x^2 + 7)dx$

4. $\int_1^3 (x^2 + x)dx$ **5.** $\int_{-1}^1 e^x \, dx$ **6.** $\int_0^1 e^{2-3x} \, dx$

7. $\int_1^2 3^x \, dx$ **8.** $\int_{\log_e 2}^{\log_e 5} e^x \, dx$ **9.** $\int_0^2 (e^x - x) \, dx$

10. $\int_{\log_a 2}^{\log_a 4} a^x \, dx$ **11.** $\int_0^2 (6x^2 - 2x + 7)\, dx$ **12.** $\int_a^b \cos x \, dx$

13. $\int_0^{\pi} \sin x \, dx$ **14.** $\int_0^{\frac{\pi}{2}} \cos x \, dx$ **15.** $\int_1^3 x^3 \, dx$

*

3.4 Fundamental Principle of Definite Integration

From what we have learnt, we can definitely say that to obtain definite integral as the limit of a sum is not so simple. In fact it is tedious. We will see that this task becomes very simple using fundamental principle of definite integration.

The following principle is called **fundamental principle of definite integration.**

Principle : If function f is continuous on $[a, b]$ and F is a differentiable function on (a, b) such that $\forall x \in (a, b)$, $\frac{d}{dx}[F(x)] = f(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Here, $F(x)$ is a primitive of $f(x)$. $F(b) - F(a)$ is expressed as $[F(x)]_a^b$.

With the help of this result, we can obtain definite integral by taking difference of values of its primitive at the end-points of given interval. **Newton** and **Leibnitz** independently obtained this result. This principle establishes a relation between the process of differentiation and integration. This result is accepted without proof.

Note : (1) Here $\forall x \in (a, b)$, $\frac{d}{dx}[F(x)] = f(x)$.

So, $\int f(x)\, dx = F(x) + c$, where c is an arbitrary constant.

$$\begin{aligned} \text{But } \int_a^b f(x)\, dx &= [F(x) + c]_a^b \\ &= [F(b) + c] - [F(a) + c] \\ &= F(b) + c - F(a) - c \\ &= F(b) - F(a) \end{aligned}$$

Thus, in definite integration arbitrary constant is eliminated and we get the definite value of integral.

$\therefore$ Definite integral is a finite definite real number. Hence the process of obtaining such an integral is called definite integration.

(2) If $a > b$, then we define $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

Also, we will accept that for $a = b$,

$$\int_a^b f(x)\, dx = \int_a^a f(x)\, dx = 0$$

(3) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$, where f is continuous on $[a, b]$.

Let F(x) be a primitive of $f(x)$. Then by fundamental principle of definite integration,

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a) \text{ and}$$

$$\int_a^b f(t)\,dt = [F(t)]_a^b = F(b) - F(a).$$

Hence, $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$.

Thus, the value of definite integral does not depend upon variable with respect to which integration is carried out.

Earlier in this chapter, we have learnt how to obtain value of definite integral as the limit of a sum. Now we will see how easily we can obtain the value of definite integral using the fundamental principle of definite integration.

Now, we will review examples 1 to 4 using the fundamental principle of definite integration.

(1) $\int_1^3 x\,dx = \left[\frac{x^2}{2}\right]_1^3 = \left[\frac{3^2}{2} - \frac{1^2}{2}\right] = \left[\frac{9}{2} - \frac{1}{2}\right] = \frac{8}{2} = 4$

(2) $\int_0^2 (3x^2 - 2x + 4)\,dx = \left[\frac{3x^3}{3} - \frac{2x^2}{2} + 4x\right]_0^2 = [8 - 4 + 8] = 12$

(3) $\int_{-1}^1 a^x\,dx = \left[\frac{a^x}{\log_e a}\right]_{-1}^1 = \frac{1}{\log_e a}\,(a^1 - a^{-1}) = \left(a - \frac{1}{a}\right)\log_a e.$

(4) $\int_a^b \sin x\,dx = [-\cos x]_a^b = -[\cos b - \cos a] = \cos a - \cos b$

3.5 Working Rules of Definite Integration

(1) If functions f and g are continuous on $[a, b]$, then

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Proof : Let F(x) and G(x) be primitives of $f(x)$ and $g(x)$ respectively on $[a, b]$.

∴ F(x) + G(x) is a primitive $f(x) + g(x)$.

∴ According to the fundamental principle of definite integration,

$$\begin{aligned}\int_a^b [f(x) + g(x)]\,dx &= [F(x) + G(x)]_a^b \\ &= [F(b) + G(b)] - [F(a) + G(a)] \\ &= [F(b) - F(a)] + [G(b) - G(a)] \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx\end{aligned}$$

(2) If f is continuous on $[a, b]$ and k is a constant, then $\int_a^b k f(x)\, dx = k \int_a^b f(x)\, dx$.

Proof : Let F(x) be a primitive of $f(x)$ on $[a, b]$ and k is any constant,

$\therefore$ kF(x) is a primitive of $kf(x)$.

$\therefore$ According to the fundamental principle of definite integration.

$$\begin{aligned} \int_a^b k f(x)\, dx &= [kF(x)]_a^b \\ &= kF(b) - kF(a) \\ &= k[F(b) - F(a)] \\ &= k \int_a^b f(x)\, dx \end{aligned}$$

(3) If function f is continuous on $[a, b]$ and $a < c < b$, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Proof : Let F(x) be a primitive of $f(x)$ over $[a, b]$. Then by the fundamental principle of definite integration,

$$\int_a^b f(x)\, dx = [F(x)]_a^b = F(b) - F(a)$$

$$\int_a^c f(x)\, dx = [F(x)]_a^c = F(c) - F(a)$$

$$\int_c^b f(x)\, dx = [F(x)]_c^b = F(b) - F(c)$$

$$\begin{aligned} \text{Now, } \int_a^c f(x)\, dx + \int_c^b f(x)\, dx &= F(c) - F(a) + F(b) - F(c) \\ &= F(b) - F(a) \\ &= \int_a^b f(x)\, dx \end{aligned}$$

Thus, if $a < c < b$, then $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$.

The same result holds for any finite partition of $[a, b]$. For instance, if $a < c < d < b$, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^d f(x)\, dx + \int_d^b f(x)\, dx.$$

Even if c is not in between a and b, and $a < c$ and f is continuous on $[a, c]$, then also this result is true. If $a < b < c$, then

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

$$\therefore \int_a^b f(x)\,dx = \int_a^c f(x)\,dx - \int_b^c f(x)\,dx$$

$$\therefore \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example 5 : Evaluate : **(1)** $\int_0^{\frac{\pi}{2}} cos^3 x\ dx$ **(2)** $\int_0^{\frac{\pi}{4}} \sqrt{1 - sin\,2x}\ dx$

Solution : (1) $I = \int_0^{\frac{\pi}{2}} cos^3 x\ dx$

$$= \int_0^{\frac{\pi}{2}} \frac{cos3x + 3cos\,x}{4}\ dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} (cos\ 3x + 3cos\ x)\ dx$$

$$= \frac{1}{4}\left[\frac{sin\,3x}{3} + 3sin\ x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left[\left(\frac{1}{3}sin\ \frac{3\pi}{2} + 3sin\ \frac{\pi}{2}\right) - \left(\frac{1}{3}sin\ 0 + 3sin\ 0\right)\right]$$

$$= \frac{1}{4}\left[\left(-\frac{1}{3} + 3\right) - (0 + 0)\right]$$

$$= \frac{1}{4}\left(\frac{8}{3}\right) = \frac{2}{3}$$

(2) $I = \int_0^{\frac{\pi}{4}} \sqrt{1 - sin\,2x}\ dx$

$$= \int_0^{\frac{\pi}{4}} \sqrt{sin^2 x + cos^2 x - 2\,sinx\,cosx}\ dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{(cos\,x - sin\,x)^2}\ dx$$

$$= \int_0^{\frac{\pi}{4}} |\ cos\ x - sin\ x\ |\ dx$$

$$= \int_0^{\frac{\pi}{4}} (cos\ x - sin\ x)\ dx$$

$\left(\textbf{As, } 0 < x < \frac{\pi}{4};\ \ cos\ x > sin\ x\right)$

$$= [\sin x + \cos x]_0^{\frac{\pi}{4}}$$

$$= \left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4}\right) - (\sin 0 + \cos 0)$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0 + 1) = \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$$

Example 6 : Evaluate : **(1)** $\int_0^3 \frac{1}{\sqrt{x^2 + 2x + 3}} dx$ **(2)** $\int_0^2 \frac{5x + 2}{x^2 + 4} dx$

Solution : (1) $I = \int_0^3 \frac{1}{\sqrt{x^2 + 2x + 3}} dx$

$$= \int_0^3 \frac{1}{\sqrt{(x + 1)^2 + (\sqrt{2})^2}} dx$$

$$= \left[\log \left| x + 1 + \sqrt{(x+1)^2 + (\sqrt{2})^2} \right|\right]_0^3$$

$$= \left[\log \left(x + 1 + \sqrt{x^2 + 2x + 3}\right)\right]_0^3 \qquad (x \in (0, 3))$$

$$= \log (4 + \sqrt{9 + 6 + 3}) - \log (1 + \sqrt{3})$$

$$= \log (4 + 3\sqrt{2}) - \log (\sqrt{3} + 1)$$

$$= \log \left(\frac{4 + 3\sqrt{2}}{\sqrt{3} + 1}\right)$$

(2) $I = \int_0^2 \frac{5x + 2}{x^2 + 4} dx$

$$= \int_0^2 \frac{5x}{x^2 + 4} dx + \int_0^2 \frac{2}{x^2 + 4} dx$$

$$= \frac{5}{2} \int_0^2 \frac{2x}{x^2 + 4} dx + 2 \int_0^2 \frac{1}{x^2 + 2^2} dx$$

$$= \frac{5}{2} [\log (x^2 + 4)]_0^2 + \frac{2}{2}\left[\tan^{-1} \frac{x}{2}\right]_0^2$$

$$= \frac{5}{2} [\log 8 - \log 4] + \left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= \frac{5}{2} \log \left(\frac{8}{4}\right) + \left[\frac{\pi}{4} - 0\right]$$

$$= \left(\frac{5}{2} \log 2 + \frac{\pi}{4}\right)$$

Example 7 : Evaluate : $\int_0^{2\pi} f(x)\,dx$, where $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 1+\cos x, & \pi \le x \le 2\pi \end{cases}$

Solution : (1) $\int_0^{2\pi} f(x)\,dx = \int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx$

$$= \int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} (1 + \cos x)\,dx$$

$$= [-\cos x]_0^{\pi} + [x + \sin x]_{\pi}^{2\pi}$$

$$= -[\cos\pi - \cos 0] + [(2\pi + \sin 2\pi) - (\pi + \sin \pi)]$$

$$= -[-1 - 1] + [(2\pi + 0) - (\pi + 0)]$$

$$= 2 + \pi = \pi + 2$$

3.6 Method of Substitution for Definite Integration

We have learnt the method of substitution for indefinite integration. We have seen that if the integrand is not in standard form, then the method of substitution is very useful to obtain certain integrals.

We can use it in combination with the fundamental principle for definite integration. Let us see the rule of substitution for definite integration.

Rule of substitution for definite integration :

Let $f : [a, b] \to R$ be a continuous function and $g : [\alpha, \beta] \to [a, b]$ be increasing or decreasing (monotonic) function. $x = g(t)$ is continuous in $[\alpha, \beta]$ and differentiable in (α, β). $g'(t)$ is continuous in (α, β) and $g'(t) \ne 0$, $\forall\, t \in (\alpha, \beta)$. $a = g(\alpha)$ and $b = g(\beta)$.

Then, $\int_a^b f(x)\,dx = \int_{\alpha}^{\beta} f(g(t))\,g'(t)\,dt$

Let us understand this method by some illustrations.

Example 8 : Evaluate : **(1)** $\int_1^9 \frac{dx}{x+\sqrt{x}}$ **(2)** $\int_0^{\frac{\pi}{2}} \frac{dx}{2\cos x + 4\sin x}$ **(3)** $\int_0^{\frac{\pi}{2}} \frac{\sin 2t}{\sin^4 t + \cos^4 t}\,dt$

Solution : (1) $I = \int_1^9 \frac{dx}{x+\sqrt{x}}$

Let $x = t^2$ $(t \ge 1)$, $dx = 2t\,dt$

When, $x = 1$, $t = 1$ since $x = t^2$ and $t \ge 1$ and if $x = 9$, $t = 3$ as $x = t^2$

$x = g(t) = t^2$ is increasing for $t \ge 1$. It is continuous in $[1, 3]$ and differentiable in $(1, 3)$.

$g'(t) = 2t \ne 0$ in $(1, 3)$

$\therefore$ $\alpha = 1$, $\beta = 3$

$$\therefore \quad I = \int_1^9 \frac{dx}{x+\sqrt{x}}$$

$$= \int_1^3 \frac{2t\,dt}{t^2+t} \qquad (\sqrt{x} = t \geq 1 \text{ as } t \geq 1)$$

$$= 2\int_1^3 \frac{1}{t+1}\,dt \qquad (t \neq 0)$$

$$= 2[\log (t+1)]_1^3$$

$$= 2[\log 4 - \log 2]$$

$$= 2 \log 2$$

(2) $$I = \int_0^{\frac{\pi}{2}} \frac{dx}{2\cos x + 4\sin x}$$

Let $tan\,\frac{x}{2} = t$, $dx = \frac{2dt}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$

When, $x = 0$, $t = tan\,0 = 0$ and when $x = \frac{\pi}{2}$, $t = tan\,\frac{\pi}{4} = 1$ $\qquad (\alpha = 0, \beta = 1)$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{dx}{2\cos x + 4\sin x}$$

$$= \int_0^1 \frac{1}{2\left(\frac{1-t^2}{1+t^2}\right)+4\left(\frac{2t}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{dt}{1-t^2+4t}$$

$$= \int_0^1 \frac{dt}{5-(t^2-4t+4)}$$

$$= \int_0^1 \frac{dt}{(\sqrt{5})^2-(t-2)^2}$$

$$= \frac{1}{2\sqrt{5}}\left[\log\left|\frac{\sqrt{5}+(t-2)}{\sqrt{5}-(t-2)}\right|\right]_0^1$$

$$= \frac{1}{2\sqrt{5}}\left[\log\left|\frac{\sqrt{5}-1}{\sqrt{5}+1}\right| - \log\left|\frac{\sqrt{5}-2}{\sqrt{5}+2}\right|\right]$$

$$= \frac{1}{2\sqrt{5}}\log\left(\frac{\sqrt{5}-1}{\sqrt{5}+1} \times \frac{\sqrt{5}+2}{\sqrt{5}-2}\right) \qquad (\sqrt{5} - 2 > 0)$$

$$= \frac{1}{2\sqrt{5}} \log \left(\frac{3+\sqrt{5}}{3-\sqrt{5}} \right)$$

$$= \frac{1}{2\sqrt{5}} \log \left(\frac{3+\sqrt{5}}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}} \right)$$

$$= \frac{1}{2\sqrt{5}} \log \left(\frac{3+\sqrt{5}}{2} \right)^2$$

$$= \frac{1}{\sqrt{5}} \log \left(\frac{3+\sqrt{5}}{2} \right)$$

Note : $I = \frac{1}{\sqrt{5}} \log \left(\frac{6+2\sqrt{5}}{4} \right)$

$$= \frac{1}{\sqrt{5}} \log \left(\frac{\sqrt{5}+1}{2} \right)^2$$

$$= \frac{2}{\sqrt{5}} \log \left(\frac{\sqrt{5}+1}{2} \right)$$

(3) $I = \int_0^{\frac{\pi}{2}} \frac{\sin 2t}{\sin^4 t + \cos^4 t} \, dt$

Let $\sin^2 t = x$, $2 \sin t \cos t \, dt = dx$. So $\sin 2t \, dt = dx$

When, $t = 0$, $x = 0$ and when $t = \frac{\pi}{2}$, $x = 1$ **($\alpha = 0, \beta = 1$)**

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{\sin 2t}{\sin^4 t + \cos^4 t} \, dt$$

$$= \int_0^1 \frac{dx}{x^2 + (1-x)^2}$$

$$= \int_0^1 \frac{dx}{2x^2 - 2x + 1}$$

$$= \frac{1}{2} \int_0^1 \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{2} \left[2 \tan^{-1} \left(\frac{x - \frac{1}{2}}{\frac{1}{2}} \right) \right]_0^1$$

$$= \left[\tan^{-1} (2x - 1) \right]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(-1)$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4} \right) = \frac{\pi}{2}$$

3.7 Method of Integration by Parts for Definite Integration

We have studied the method of integration by parts to obtain indefinite integral of product of two functions. We can also use integration by parts for definite integration.

To use method of integration by parts in definite integration, we use following formula.

$f(x)$, $g(x)$, $f'(x)$ and $g'(x)$ all are continuous on $[a, b]$.

$$\int_a^b f(x)\ g'(x)\, dx = [f(x)\ g(x)]_a^b - \int_a^b f'(x)\ g(x)\, dx$$

$$\therefore \int_a^b f(x)\ g'(x)\, dx = [f(b)\ g(b) - f(a)\ g(a)] - \int_a^b f'(x)\ g(x)\, dx$$

Now, we will understand this method by some examples.

Example 9 : Evaluate : **(1)** $\int_0^1 x\, tan^{-1}x\ dx$ **(2)** $\int_0^{\frac{1}{\sqrt{2}}} \frac{sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}\, dx$ **(3)** $\int_0^1 \frac{x\, dx}{(1+x^2)(2+x^2)}$

Solution : (1) $I = \int_0^1 x\, tan^{-1}x\ dx$

$$= \left[tan^{-1}x \cdot \frac{x^2}{2}\right]_0^1 - \int_0^1 \left(\frac{1}{1+x^2} \cdot \frac{x^2}{2}\right) dx$$

$$= \left(tan^{-1}(1) \cdot \frac{1}{2} - 0\right) - \frac{1}{2}\int_0^1 \frac{x^2}{x^2+1}\, dx$$

$$= \left(\frac{\pi}{4} \cdot \frac{1}{2} - 0\right) - \frac{1}{2}\int_0^1 \frac{(x^2+1)-(1)}{x^2+1}\, dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\int_0^1 \left(1 - \frac{1}{x^2+1}\right) dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[x - tan^{-1}x\right]_0^1$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[(1 - tan^{-1}1) - (0 - tan^{-1}0)\right]$$

$$= \frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

(2) $I = \int_{0}^{\frac{1}{\sqrt{2}}} \frac{sin^{-1}x}{(1-x^2)^{\frac{3}{2}}} dx$

Let $sin^{-1}x = t$, $x = sin\, t$, $dx = cos\, t\, dt$, $x \in \left[0, \frac{1}{\sqrt{2}}\right]$, $t \in \left[0, \frac{\pi}{4}\right]$

When, $x = 0$, $t = sin^{-1}0 = 0$ and when $x = \frac{1}{\sqrt{2}}$, $t = sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$ $\left(\alpha = 0, \beta = \frac{\pi}{4}\right)$

$$\therefore \quad I = \int_{0}^{\frac{1}{\sqrt{2}}} \frac{sin^{-1}x}{(1-x^2)^{\frac{3}{2}}} dx$$

$$= \int_{0}^{\frac{\pi}{4}} \frac{t}{(1-sin^2t)^{\frac{3}{2}}} \cdot cost\ dt$$

$$= \int_{0}^{\frac{\pi}{4}} t\, sec^2t\, dt \qquad \left(cos\, t > 0 \text{ in } \left[0, \frac{\pi}{4}\right]\right)$$

$$= [t \cdot tan\, t]_{0}^{\frac{\pi}{4}} - \int_{0}^{\frac{\pi}{4}} tan\, t\, dt$$

$$= [t \cdot tan\, t]_{0}^{\frac{\pi}{4}} + [\log |cos\, t|]_{0}^{\frac{\pi}{4}}$$

$$= \left(\frac{\pi}{4} tan\frac{\pi}{4} - 0\right) + \left[\log\left(cos\frac{\pi}{4}\right) - \log(cos\, 0)\right]$$

$$= \frac{\pi}{4} + \log\frac{1}{\sqrt{2}} - \log 1$$

$$= \frac{\pi}{4} - \frac{1}{2}\log 2$$

(3) $I = \int_{0}^{1} \frac{x\, dx}{(1+x^2)(2+x^2)}$

For $x \geq 0$, let $x^2 = t$, $2x\, dx = dt$. So $x\, dx = \frac{1}{2}\, dt$

When, $x = 0$, $t = 0$ and when $x = 1$, $t = 1$

$$\therefore \quad I = \int_{0}^{1} \frac{x\, dx}{(1+x^2)(2+x^2)} = \frac{1}{2}\int_{0}^{1} \frac{dt}{(t+1)(t+2)}$$

Now let $\dfrac{1}{(t+1)(t+2)} = \dfrac{A}{t+1} + \dfrac{B}{t+2}$

$\therefore \quad 1 = A(t+2) + B(t+1)$

If $t = -2$, $1 = -B$. So $B = -1$

If $t = -1$, $1 = A$. So $A = 1$

$\therefore \quad \dfrac{1}{(t+1)(t+2)} = \dfrac{1}{t+1} + \dfrac{-1}{t+2}$

$$\therefore \quad I = \frac{1}{2}\int_0^1 \frac{dt}{(t+1)(t+2)} = \frac{1}{2}\int_0^1 \left(\frac{1}{t+1} + \frac{-1}{t+2}\right) dt$$

$$= \frac{1}{2}\left[\log|t+1| - \log|t+2|\right]_0^1$$

$$= \frac{1}{2}\left[\log\left|\frac{t+1}{t+2}\right|\right]_0^1$$

$$= \frac{1}{2}\left[\log\frac{2}{3} - \log\frac{1}{2}\right]$$

$$= \frac{1}{2}\log\left(\frac{4}{3}\right)$$

Example 10 : Evaluate : $\int_0^{2\pi} \sin ax \cdot \sin bx \; dx, \quad a, b \in N$

Solution : $I = \int_0^{2\pi} \sin ax \cdot \sin bx \; dx$

Case 1 : $a \neq b$

$$I = \frac{1}{2}\int_0^{2\pi} 2\sin ax \cdot \sin bx \; dx$$

$$= \frac{1}{2}\int_0^{2\pi} [\cos(a-b)x - \cos(a+b)x]\, dx$$

$$= \frac{1}{2}\left[\frac{\sin(a-b)x}{a-b} - \frac{\sin(a+b)x}{a+b}\right]_0^{2\pi} \qquad \textbf{(}a \neq b \textbf{ and } a+b \neq 0 \textbf{ as } a, b \in N\textbf{)}$$

$$= \frac{1}{2}(0 - 0) \qquad \textbf{(Why ?)}$$

$\therefore \quad I = 0$

Case 2 : $a = b$

$$I = \int_0^{2\pi} \sin^2 ax \; dx$$

$$= \int_0^{2\pi} \left(\frac{1-\cos 2ax}{2}\right) dx$$

$$= \frac{1}{2}\left[x - \frac{sin2ax}{2a}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[\left(2\pi - \frac{sin4\pi a}{2a}\right) - (0 - 0)\right]$$

$$= \frac{1}{2}(2\pi)$$ **(Why $sin\ 4\pi a = 0$?)**

$\therefore \quad I = \pi$

$$\therefore \int_0^{2\pi} sin\ ax \cdot sin\ bx \ dx = \begin{cases} 0 & \text{if } a \neq b \\ \pi & \text{if } a = b \end{cases}$$

Example 11 : For $\alpha > 0$, if $f(x + \alpha) = f(x)$, $\forall x \in R$ i.e. if f has period α, prove that $\int_0^{n\alpha} f(x)\,dx = n\int_0^{\alpha} f(x)\,dx$, where $n \in N$ and hence obtain $\int_0^{10\pi} |sin\,x|\ dx$.

Solution : $I = \int_0^{n\alpha} f(x)\,dx,\ n \in N$

$$= \int_0^{\alpha} f(x)\,dx + \int_{\alpha}^{2\alpha} f(x)\,dx + \int_{2\alpha}^{3\alpha} f(x)\,dx + ... + \int_{k\alpha}^{(k+1)\alpha} f(x)\,dx + ... + \int_{(n-1)\alpha}^{n\alpha} f(x)\,dx$$

We shall prove that each of these integrals is equal to $\int_0^{\alpha} f(x)\,dx$

Now let $I_k = \int_{k\alpha}^{(k+1)\alpha} f(x)\,dx$ **[$k = 1, 2,..., (n - 1)$]**

Let $x = k\alpha + t$, $dx = dt$

When $x = k\alpha$, $t = 0$ and $x = (k + 1)\alpha$, $t = \alpha$

$$\therefore \quad I_k = \int_0^{\alpha} f(k\alpha + t)\,dt$$

If α is a period of f, then $k\alpha$ are periods of f. **($k \in N$)**

$\therefore \quad f(k\alpha + t) = f(t)$

$$\therefore \quad I_k = \int_0^{\alpha} f(t)\,dt = \int_0^{\alpha} f(x)\,dx$$

i.e. $\int_{k\alpha}^{(k+1)\alpha} f(x)\,dx = \int_0^{\alpha} f(x)\,dx$ **[$k = 1, 2, 3,..., n - 1$]**

$$\therefore \quad I = \int_0^{\alpha} f(x)\,dx + \int_0^{\alpha} f(x)\,dx + ... + \int_0^{\alpha} f(x)\,dx \ (n \text{ times})$$

$$= n\int_0^{\alpha} f(x)\,dx$$

Now $I = \int_0^{10\pi} |\sin x| \, dx.$

$= 10 \int_0^{\pi} |\sin x| \, dx$ **(period of $|\sin x|$ is π)**

$= 10 \int_0^{\pi} \sin x \, dx$ **(for $0 \le x \le \pi, \sin x \ge 0$)**

$= 10 [-\cos x]_0^{\pi}$

$= -10 [\cos\pi - \cos 0]$

$= -10 (-1 - 1)$

$= -10 (-2)$

$= 20$

Example 12 : Evaluate $\int_{-1}^{3} |2x - 1| \, dx$

Solution : $2x - 1 \ge 0 \Leftrightarrow x \ge \frac{1}{2}$

$\therefore \quad |2x - 1| = \begin{cases} 2x - 1 & x \ge \frac{1}{2} \\ 1 - 2x & x < \frac{1}{2} \end{cases}$

Now $-1 < \frac{1}{2} < 3$

$\therefore \quad I = \int_{-1}^{3} |2x - 1| \, dx$

$= \int_{-1}^{\frac{1}{2}} |2x - 1| \, dx + \int_{\frac{1}{2}}^{3} |2x - 1| \, dx$

$= \int_{-1}^{\frac{1}{2}} (1 - 2x) \, dx + \int_{\frac{1}{2}}^{3} (2x - 1) \, dx$

$= [x - x^2]_{-1}^{\frac{1}{2}} + [x^2 - x]_{\frac{1}{2}}^{3}$

$= \left[\left(\frac{1}{2} - \frac{1}{4}\right) - (-1 - 1)\right] + \left[(9 - 3) - \left(\frac{1}{4} - \frac{1}{2}\right)\right]$

$= \left(\frac{1}{4} + 2\right) + \left(6 + \frac{1}{4}\right)$

$= \frac{17}{2}$

Exercise 3.2

Evaluate (1 to 35) :

1. $\int_0^1 \frac{1}{\sqrt{1+x}-\sqrt{x}}\,dx$

2. $\int_0^{\frac{\pi}{4}} tan^2 x\ dx$

3. $\int_0^{\frac{\pi}{2}} sin^2 x\ dx$

4. $\int_0^{\frac{\pi}{4}} tan\,x\ dx$

5. $\int_0^{\frac{\pi}{2}} \sqrt{1-cos\,2x}\ dx$

6. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1-sin\,2x}\ dx$

7. $\int_0^{\sqrt{2}} \sqrt{2-x^2}\ dx$

8. $\int_2^5 \frac{2x}{5x^2+1}\,dx$

9. $\int_0^1 \frac{2x+3}{5x^2+1}\,dx$

10. $\int_0^{\frac{\pi}{2}} \frac{sin^2 x}{(1+cosx)^2}\,dx$

11. $\int_0^1 \frac{dx}{x^2+x+1}$

12. $\int_0^9 \frac{dx}{1+\sqrt{x}}$

13. $\int_0^{\frac{\pi}{4}} \frac{cosx}{\sqrt{2-sin^2 x}}\,dx$

14. $\int_0^1 \frac{dx}{e^x+e^{-x}}$

15. $\int_0^1 tan^{-1}x\ dx$

16. $\int_0^4 \frac{dx}{\sqrt{12+4x-x^2}}$

17. $\int_0^{\frac{\pi}{2}} x^2 cos\,2x\,dx$

18. $\int_0^{\frac{1}{2}} \frac{x sin^{-1}x}{\sqrt{1-x^2}}\,dx$

19. $\int_0^{\frac{\pi}{2}} \frac{dx}{3+2sinx+cosx}$

20. $\int_0^{\frac{\pi}{4}} \frac{dx}{2+3cos^2 x}$

21. $\int_0^1 sin^{-1}\sqrt{\frac{x}{x+1}}\ dx$

22. $\int_0^1 \sqrt{\frac{1-x}{1+x}}\ dx$

23. $\int_0^{\frac{\pi}{2}} \frac{cos\,x}{(1+sinx)(2+sinx)(3+sinx)}\,dx$

24. $\int_0^{\frac{\pi}{4}} \frac{sin\,x+cos\,x}{1+cos2x}\,dx$

25. $\int_1^2 \frac{1}{x(1+x^2)}\,dx$

26. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} cos\,2x \cdot \log\ sin\,x\ dx$

27. $\int_0^{\frac{\pi}{2}} \frac{1}{3+2cosx}\,dx$

28. $\int_0^{\frac{\pi}{4}} \frac{1}{4sin^2 x+5cos^2 x}\,dx$

29. $\int_0^{2\pi} |\,cosx\,|\ dx$

30. $\int_1^4 f(x)\,dx$, where $f(x) = \begin{cases} 2x+8 & 1 \le x \le 2 \\ 6x & 2 < x \le 4 \end{cases}$

31. $\int_0^9 f(x)\,dx$, where $f(x) = \begin{cases} \sin x & 0 \le x \le \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x \le 5 \\ e^{x-5} & 5 < x \le 9 \end{cases}$

32. $\int_0^1 |5x - 3|\,dx$

33. $\int_0^2 |x^2 + 2x - 3|\,dx$

34. $\int_0^{2\pi} \sin ax \cos bx\,dx \quad \forall a, b \in Z - \{0\}$

35. $\int_0^{2\pi} \sin mx \cos nx\,dx \quad \forall m, n \in N$

36. If $\int_{\sqrt{2}}^{k} \frac{dx}{x\sqrt{x^2 - 1}} = \frac{\pi}{12}$, obtain k.

37. If $\int_0^k \frac{dx}{2 + 8x^2} = \frac{\pi}{16}$, then find k.

38. If $\int_0^a \sqrt{x}\,dx = 2a \int_0^{\frac{\pi}{2}} \sin^3 x\,dx$, then obtain $\int_a^{a+1} x\,dx$.

*

3.8 Some Useful Results about Definite Integration

Theorem 3.1 : If f is continuous on $[0, a]$, then $\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$

Proof : Let $I = \int_0^a f(x)\,dx$

Let $x = a - t$. So $dx = -dt$

$x = g(t)$ is monotonic decreasing and continuous in $[0, a]$.

Now, $\frac{dx}{dt} = -1$ is continuous on $(0, a)$.

Also, $x = 0 \Rightarrow t = a$ and $x = a \Rightarrow t = 0$. So $\alpha = a$, $\beta = 0$

$$\therefore \quad I = \int_a^0 f(a - t)(-dt)$$

$$= -\int_a^0 f(a - t)\,dt$$

$$= \int_0^a f(a - t)\,dt$$

$$= \int_0^a f(a - x)\,dx$$

i.e. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Now to understand this theorem, let us take an example.

Evaluate : $\int_0^{2\pi} \cos^3 x \sin^5 x \, dx$

$$I = \int_0^{2\pi} \cos^3 x \sin^5 x \, dx$$

$$= \int_0^{2\pi} \cos^3(2\pi - x) \sin^5(2\pi - x) \, dx$$

$$= \int_0^{2\pi} (\cos^3 x)(-\sin^5 x) \, dx$$

$$= -\int_0^{2\pi} \cos^3 x \sin^5 x \, dx = -I$$

$\therefore \quad I = -I$

$\therefore \quad 2I = 0$

$\therefore \quad I = 0$

Theorem 3.2 : If f is continuous over $[a, b]$, then $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Proof : Let $I = \int_a^b f(x)\,dx$

Let $x = a + b - t$. So $dx = -dt$

$\therefore \quad x = g(t) = a + b - t$ is decreasing and continuous in $[a, b]$.

Also, $\frac{dx}{dt} = -1$ is continuous on (a, b).

Here, $x = a \Rightarrow t = b$ and $x = b \Rightarrow t = a$. So $\alpha = b$ and $\beta = a$

$$\therefore \quad I = \int_b^a f(a+b-t)(-dt)$$

$$= -\int_b^a f(a+b-t)\,dt$$

$$= \int_a^b f(a+b-t)\,dt$$

$$= \int_a^b f(a+b-x)\,dx$$

i.e. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(See that in theorem 3.2, if $a = 0$ and b is replaced by a, we get theorem 3.1)

Now, to understand this theorem, let us take an example.

Evaluate : $\int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\, dx$

$$I = \int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\, dx \qquad \text{(i)}$$

$$= \int_1^2 \frac{\sqrt{(1+2)-x}}{\sqrt{3-(1+2-x)}+\sqrt{1+2-x}}\, dx \qquad (a + b = 1 + 2 = 3)$$

$$= \int_1^2 \frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}\, dx \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_1^2 \frac{\sqrt{x}+\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}\, dx$$

$$= \int_1^2 1\, dx = [x]_1^2 = 2 - 1 = 1$$

$\therefore \quad 2I = 1$

$\therefore \quad I = \frac{1}{2}$

Theorem 3.3 : If the function f is continuous over $[0, 2a]$, then

$$\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(2a - x)\, dx$$

Proof : Here $0 < a < 2a$

$$\therefore \quad \int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_a^{2a} f(x)\, dx \qquad \text{(i)}$$

Now, let $I = \int_a^{2a} f(x)\, dx$

Let $x = g(t) = 2a - t$. So $dx = -dt$

$x = g(t)$ is decreasing in $[a, 2a]$. $\frac{dx}{dt} = -1$ is continuous in $(a, 2a)$.

Now, if $x = a$, $t = a$ and if $x = 2a$, $t = 0$ $\qquad (\alpha = a$ and $\beta = 0)$

$$I = \int_a^0 f(2a - t)(-dt)$$

$$= -\int_a^0 f(2a - t)\, dt$$

$$= \int_0^a f(2a - t)\, dt$$

$$I = \int_0^a f(2a - x)\, dx$$

Substituting the value of I in (i), we get

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

Corollary : If $\forall x \in [0, 2a]$, $f(2a-x) = f(x)$, then $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$

If $\forall x \in [0, 2a]$, $f(2a-x) = -f(x)$, then $\int_0^{2a} f(x)\,dx = 0$

Proof : $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$ **(i)**

Now, taking $f(2a-x) = f(x)$ in (i), we get

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

Now, if $f(2a-x) = -f(x)$, we get

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(x)\,dx = 0$$

i.e. $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0 & , \text{ if } f(2a-x) = -f(x) \end{cases}$

Note : (1) We will see in chapter 4 that the area enclosed by the curve $y = f(x)$, lines $x = a$, $x = b$ and X-axis is $\left|\int_a^b f(x)\,dx\right|$. With this reference we interpret the above theorems as follows.

(2) If $f(2a-x) = f(x)$, then the graph of $f(x)$ is symmetric about $x = a$ as shown in figure 3.2.

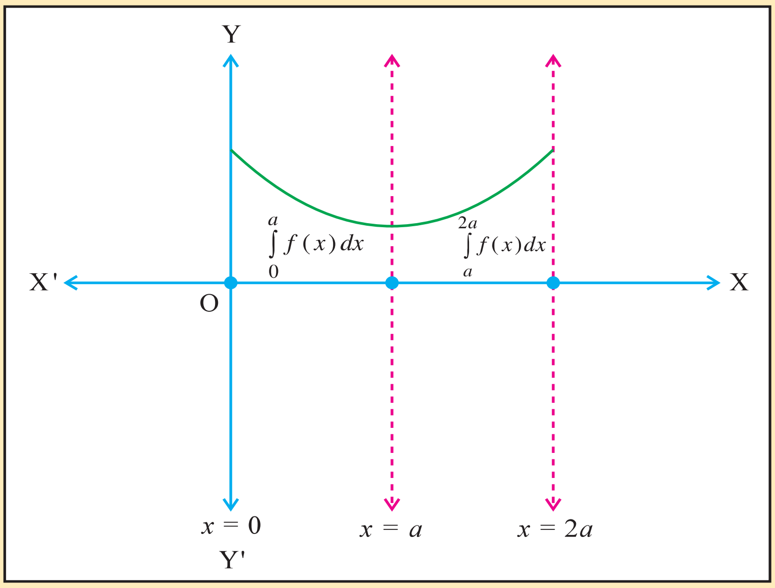

Figure 3.2

$\therefore \int_a^{2a} f(x)\,dx = \int_0^a f(x)\,dx$

$\therefore \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$

If $f(2a - x) = -f(x)$, then the graph of $f(x)$

$$\therefore \int_0^{\frac{\pi}{2}} f(x)\, dx = -\int_{\frac{\pi}{2}}^{\pi} f(x)\, dx$$

$$\therefore \int_0^{\frac{\pi}{2}} f(x)\, dx + \int_{\frac{\pi}{2}}^{\pi} f(x)\, dx = 0$$

$$\therefore \int_0^{\pi} f(x)\, dx = 0$$

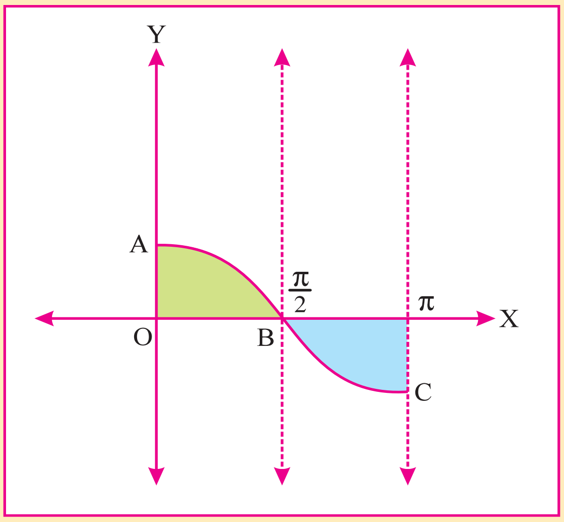

Figure 3.3

Now, to understand this theorem, let us take an example.

Evaluate : $\int_0^{2\pi} cos^3x\, dx.$

$$I = \int_0^{2\pi} cos^3x\, dx.$$

Let $f(x) = cos^3x$. Then

$$f(2\pi - x) = cos^3(2\pi - x) = cos^3x = f(x)$$

$$\therefore \int_0^{2\pi} cos^3x\, dx = 2\int_0^{\pi} cos^3x\, dx \qquad \mathbf{(a = \pi,\ f(2a - x) = f(x))}$$

Now, $f(\pi - x) = cos^3(\pi - x) = -cos^3x = -f(x)$ $\qquad \mathbf{\left(a = \frac{\pi}{2},\ f(2a - x) = -f(x)\right)}$

$$\therefore \int_0^{\pi} cos^3x\, dx = 0$$

Hence, $\int_0^{2\pi} cos^3x\, dx = 2\int_0^{\pi} cos^3x\, dx = 2 \times 0 = 0$

We have studied about even and odd functions. Let us recall them. Let $f : A \longrightarrow R$ be a real function of a real variable. Let $\forall x \in A, -x \in A$.

(i) If $f(-x) = f(x)$, $\forall x \in A$, then f is called an even function.

(ii) If $f(-x) = -f(x)$, $\forall x \in A$, then f is called an odd function.

For example $cosx$, $secx$, x^2 are even functions and $sinx$, $tanx$, x^3 are odd functions.

Theorem 3.4 : If f is an even continuous function defined on $[-a, a]$ $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

Proof : Here $-a < 0 < a$.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \qquad \text{(i)}$$

Let $I = \int_{-a}^{0} f(x)\,dx$

Let $x = -t$, $dx = -dt$

Also when $x = -a$, $t = a$ and when $x = 0$, $t = 0$

Here $\frac{dx}{dt} = -1$ is continuous and non-zero on $(-a, 0)$

$$\therefore \quad I = \int_{a}^{0} f(-t)(-dt)$$

$$= -\int_{a}^{0} f(-t)\,dt$$

$$= \int_{0}^{a} f(-t)\,dt$$

$$= \int_{0}^{a} f(t)\,dt, \text{ as } f \text{ is an even function.}$$

$$\therefore \quad I = \int_{0}^{a} f(x)\,dx$$

Substituting the value of I in (i), we get

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= 2\int_{0}^{a} f(x)\,dx$$

Now, let us understand this by an example.

$y = cos\,x$ is continuous even function in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cosx\,dx = [sin\,x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = sin\,\frac{\pi}{2} - sin\left(-\frac{\pi}{2}\right) = 1 + 1 = 2$$

$$2\int_0^{\frac{\pi}{2}} \cos x \, dx = 2[\sin x]_0^{\frac{\pi}{2}} = 2\left[\sin\frac{\pi}{2} - \sin 0\right] = 2(1) = 2$$

Thus, $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx = 2\int_0^{\frac{\pi}{2}} \cos x \, dx$

Theorem 3.5 : If f is an odd continuous function on $[-a, a]$, $\int_{-a}^{a} f(x)\, dx = 0$.

Proof : Here $-a < 0 < a$.

$$\therefore \int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_{0}^{a} f(x)\, dx \qquad \text{(i)}$$

Let $I = \int_{-a}^{0} f(x)\, dx$

Let $x = -t$, $dx = -dt$

Also, when $x = -a$, $t = a$ and when $x = 0$, $t = 0$

Here $\frac{dx}{dt} = -1$ is continuous and non-zero on $(-a, 0)$

$$\therefore \quad I = \int_{-a}^{0} f(x)\, dx$$

$$= \int_{a}^{0} f(-t)\,(-dt)$$

$$= -\int_{a}^{0} f(-t)\, dt$$

$$= \int_{0}^{a} f(-t)\, dt$$

$$= -\int_{0}^{a} f(t)\, dt, \text{ since } f \text{ is an odd function.}$$

$$= -\int_{0}^{a} f(x)\, dx$$

Substituting the value of I in (i), we get,

$$\int_{-a}^{a} f(x)\, dx = -\int_{0}^{a} f(x)\, dx + \int_{0}^{a} f(x)\, dx$$

$$\therefore \int_{-a}^{a} f(x)\, dx = 0$$

Now, let us understand this by an example.

$y = \sin x$ is an odd continuous function on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \, dx = [-\cos x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -\left[\cos\frac{\pi}{2} - \cos\left(-\frac{\pi}{2}\right)\right] = -\left[\cos\frac{\pi}{2} - \cos\frac{\pi}{2}\right] = -(0-0) = 0$$

Hence $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \, dx = 0$

Example 13 : Evaluate **(i)** $\int_{-1}^{1} \sin^3 x \cos^4 x \, dx$ **(ii)** $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}} \, dx$ **$(a > 0)$**

Solution : (i) $I = \int_{-1}^{1} \sin^3 x \cos^4 x \, dx$

Here $f(x) = \sin^3 x \cos^4 x$

$\therefore \quad f(-x) = \sin^3(-x) \cos^4(-x)$

$= -\sin^3 x \cdot \cos^4 x$

$= -f(x)$

$\therefore \quad f(x) = \sin^3 x \cos^4 x$ is an odd function on $[-1, 1]$

$\therefore \quad \int_{-1}^{1} \sin^3 x \cos^4 x \, dx = 0$

(ii) $I = \int_{-a}^{a} \sqrt{\frac{a-x}{a+x}} \, dx$

$= \int_{-a}^{a} \sqrt{\frac{a-x}{a+x} \times \frac{a-x}{a-x}} \, dx$

$= \int_{-a}^{a} \frac{a-x}{\sqrt{a^2-x^2}} \, dx$ **(Since $x < a$, $\sqrt{(a-x)^2} = |x-a| = a-x$)**

$= \int_{-a}^{a} \frac{a}{\sqrt{a^2-x^2}} \, dx - \int_{-a}^{a} \frac{x}{\sqrt{a^2-x^2}} \, dx$

$I = aI_1 - I_2$, where $I_1 = \int_{-a}^{a} \frac{1}{\sqrt{a^2-x^2}} \, dx$ and $I_2 = \int_{-a}^{a} \frac{x}{\sqrt{a^2-x^2}} \, dx$

Let $f(x) = \frac{1}{\sqrt{a^2-x^2}}$ and $g(x) = \frac{x}{\sqrt{a^2-x^2}}$

Then $f(-x) = \frac{1}{\sqrt{a^2-(-x)^2}} = \frac{1}{\sqrt{a^2-x^2}} = f(x)$ and

$g(-x) = \frac{-x}{\sqrt{a^2-(-x)^2}} = \frac{-x}{\sqrt{a^2-x^2}} = -g(x)$

$\therefore \quad f(x)$ is an even function and $g(x)$ is an odd function.

$\therefore \quad I_1 = 2\int_{0}^{a} \frac{1}{\sqrt{a^2-x^2}} \, dx$ and $I_2 = 0$

$$\therefore \quad I = 2a \int_0^a \frac{1}{\sqrt{a^2 - x^2}}\, dx$$

$$= 2a \left[\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 2a\, [\sin^{-1}1 - \sin^{-1}0]$$

$$= 2a\left[\frac{\pi}{2}\right]$$

$$= a\pi$$

Example 14 : Evaluate **(i)** $\int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}\, dx$ **(ii)** $\int_0^1 x^2(1 - x)^{\frac{1}{2}}\, dx$ **(iii)** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x \log \tan x\, dx$

Solution : (i) $I = \int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}\, dx$

$$= \int_0^{\pi} \frac{x\sin x}{1 + \sin x}\, dx \qquad \text{(i)}$$

Replace x by $\pi - x$ in (i)

$$\therefore \quad I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \sin(\pi - x)}\, dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \sin x}\, dx$$

$$= \int_0^{\pi} \frac{\pi\sin x}{1 + \sin x}\, dx - \int_0^{\pi} \frac{x\sin x}{1 + \sin x}\, dx$$

$$\therefore \quad I = \pi\int_0^{\pi} \frac{\sin x}{1 + \sin x}\, dx - I \qquad \text{(by (i))}$$

$$\therefore \quad 2I = \pi\int_0^{\pi} \frac{\sin x}{1 + \sin x}\, dx$$

$$= \pi\int_0^{\pi} \frac{1 + \sin x - 1}{1 + \sin x}\, dx$$

$$= \pi\int_0^{\pi} dx - \pi\int_0^{\pi} \frac{dx}{1 + \sin x}$$

$$= \pi [x]_0^{\pi} - \pi \int_0^{\pi} \frac{dx}{1+sinx}$$

$$= \pi^2 - \pi \int_0^{\pi} \frac{dx}{1+sinx}$$

Now let, $I_1 = \int_0^{\pi} \frac{dx}{1+sinx}$

$$= 2 \int_0^{\frac{\pi}{2}} \frac{dx}{1+sinx}$$ **(since $f(2a - x) = f(\pi - x) = f(x)$)**

Let, $tan\frac{x}{2} = t$, $dx = \frac{2dt}{1+t^2}$, $sinx = \frac{2t}{1+t^2}$ in $[0, \pi]$

when $x = \frac{\pi}{2}$, $t = tan\frac{\pi}{4} = 1$ and when $x = 0$, $t = 0$.

$$\therefore \quad I_1 = 2 \int_0^1 \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}}$$

$$= 4 \int_0^1 \frac{dt}{(1+t)^2}$$

$$= 4 \left[-\frac{1}{1+t}\right]_0^1$$

$$= 4 \left[-\frac{1}{2} + 1\right] = 4\left(\frac{1}{2}\right) = 2$$

$$\therefore \quad 2I = \pi^2 - 2\pi = \pi(\pi - 2)$$

$$\therefore \quad I = \frac{\pi}{2} (\pi - 2)$$

Note : Multiplying and dividing I_1 by $1 - sinx$, the calculation seems to become simpler but at $x = \frac{\pi}{2}$, $1 - sinx = 0$

(ii) $I = \int_0^1 x^2(1 - x)^{\frac{1}{2}} dx$

Replace x by $1 - x$.

$$\therefore \quad I = \int_0^1 (1 - x)^2 [1 - (1 - x)]^{\frac{1}{2}} dx$$

$$= \int_0^1 (1 - 2x + x^2) \cdot x^{\frac{1}{2}} dx$$

$$= \int_0^1 (x^{\frac{1}{2}} - 2x^{\frac{3}{2}} + x^{\frac{5}{2}}) dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{4}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}}\right]_0^1$$

$$= \left(\frac{2}{3} - \frac{4}{5} + \frac{2}{7}\right) = \frac{16}{105}$$

(iii) $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x \,\log \tan x \; dx$ **(i)**

Replace x by $\frac{\pi}{6} + \frac{\pi}{3} - x = \frac{\pi}{2} - x$ in (i).

$$\therefore \quad I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2\left(\frac{\pi}{2} - x\right) \log \tan\left(\frac{\pi}{2} - x\right) dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x \log \cot x \; dx \qquad \textbf{(ii)}$$

Adding (i) and (ii), we get,

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x\, [\log \tan x + \log \cot x] \; dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x \log\,(\tan x \cdot \cot x) \; dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x \cdot \log 1 \; dx$$

$\therefore \quad 2I = 0$

$\therefore \quad I = 0$

Exercise 3.3

1. Evaluate :

(1) $\int_{-1}^{1} \frac{x}{\sqrt{a^2 - x^2}} \, dx \quad (a > 1)$ **(2)** $\int_{-a}^{a} \frac{x}{2 + x^8} \, dx$ **(3)** $\int_{-\pi}^{\pi} \sqrt{5 + x^4} \, \sin^3 x \; dx$

(4) $\int_{-1}^{1} \log\left(\frac{3-x}{3+x}\right) dx$ **(5)** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \; dx$ **(6)** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \; dx$

2. Evaluate :

(1) $\int_0^{\pi} sin^2x\, cos^3x \; dx$ **(2)** $\int_0^{2\pi} sin^3x \; cos^2x \; dx$

Prove the following (3 to 15)

3. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{sin\,x}}{\sqrt{cos\,x} + \sqrt{sin\,x}}\, dx = \frac{\pi}{4}$ **4.** $\int_0^{\frac{\pi}{2}} \frac{sin^n x}{sin^n x + cos^n x}\, dx = \frac{\pi}{4} \; (n \in N)$ **5.** $\int_1^{4} \frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}}\, dx = \frac{3}{2}$

6. $\int_0^{1} x(1-x)^{\frac{3}{2}}\, dx = \frac{4}{35}$ **7.** $\int_0^{\pi} \frac{e^{cosx}}{e^{cosx} + e^{-cosx}}\, dx = \frac{\pi}{2}$ **8.** $\int_0^{3} x^2(3-x)^{\frac{1}{2}}\, dx = \frac{144\sqrt{3}}{35}$

9. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{cotx}}\, dx = \frac{\pi}{12}$ **10.** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log\left(\frac{1+sin\,x}{1+cos\,x}\right) dx = 0$ **11.** $\int_0^{\pi} \frac{x\,dx}{1+sinx} = \pi$

12. $\int_0^{\frac{\pi}{4}} \log\,(1 + tan\,x)\, dx = \frac{\pi}{8} \log 2$ **13.** $\int_0^{\pi} \frac{x\,sinx}{1+cos^2x}\, dx = \frac{\pi^2}{4}$

14. $\int_0^{\pi} x\,sin^3x\, dx = \frac{2\pi}{3}$ **15.** $\int_0^{\frac{\pi}{2}} \frac{sin^2x}{sin\,x + cosx}\, dx = \frac{1}{\sqrt{2}} \log\,(\sqrt{2} + 1)$

*

Miscellaneous Examples :

Example 15 : Prove that $\int_0^{\frac{\pi}{2}} \frac{x}{sinx + cosx}\, dx = \frac{\pi}{2\sqrt{2}} \log\,(\sqrt{2} + 1)$

Solution : $I = \int_0^{\frac{\pi}{2}} \frac{x}{cos\,x + sin\,x}\, dx$ **(i)**

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)}{cos\left(\frac{\pi}{2} - x\right) + sin\left(\frac{\pi}{2} - x\right)}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)}{cos\,x + sin\,x}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}}{cos\,x + sin\,x}\, dx - \int_0^{\frac{\pi}{2}} \frac{x}{cos\,x + sin\,x}\, dx$$

$$\therefore \quad I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\cos x + \sin x}\, dx - I$$

$$\therefore \quad 2I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\cos x + \sin x}\, dx$$

$$\therefore \quad I = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)}\, dx$$

$$\therefore \quad I = \frac{\pi}{4\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{1}{\left(\cos x \cos\frac{\pi}{4} + \sin x \sin\frac{\pi}{4}\right)}\, dx$$

$$= \frac{\pi}{4\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{1}{\cos\left(x - \frac{\pi}{4}\right)}\, dx$$

$$= \frac{\pi}{4\sqrt{2}}\int_0^{\frac{\pi}{2}} \sec\left(x - \frac{\pi}{4}\right) dx$$

$$= \frac{\pi}{4\sqrt{2}}\left[\log\left|\sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right)\right|\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4\sqrt{2}}\left[\log\left|\sec\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + \tan\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right| - \log\left|\sec\left(-\frac{\pi}{4}\right) + \tan\left(-\frac{\pi}{4}\right)\right|\right]$$

$$= \frac{\pi}{4\sqrt{2}}\left[\log\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \log\left|\sec\frac{\pi}{4} - \tan\frac{\pi}{4}\right|\right]$$

$$= \frac{\pi}{4\sqrt{2}}\left(\log|\sqrt{2} + 1| - \log|\sqrt{2} - 1|\right)$$

$$= \frac{\pi}{4\sqrt{2}} \log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}\right) \qquad (\sqrt{2} > 1)$$

$$= \frac{\pi}{4\sqrt{2}} \log(\sqrt{2} + 1)^2$$

$$= \frac{\pi}{2\sqrt{2}} \log(\sqrt{2} + 1)$$

Example 16 : Prove that $\int_0^{\frac{\pi}{4}} \tan^n x\, dx + \int_0^{\frac{\pi}{4}} \tan^{n-2} x\, dx = \frac{1}{n-1}$, $n \in \mathrm{N} - \{1\}$.

Solution : $I = \int_0^{\frac{\pi}{4}} \tan^n x\, dx + \int_0^{\frac{\pi}{4}} \tan^{n-2} x\, dx$

$$= \int_0^{\frac{\pi}{4}} (\tan^n x + \tan^{n-2} x)\, dx$$

$$= \int_0^{\frac{\pi}{4}} tan^{n-2}x \ (tan^2x + 1) \ dx$$

$$= \int_0^{\frac{\pi}{4}} tan^{n-2}x \ (sec^2x) \ dx$$

$$= \int_0^{\frac{\pi}{4}} (tan\,x)^{n-2} \ \frac{d}{dx}(tan\,x) \ dx$$

$$= \left[\frac{(tan\,x)^{n-1}}{n-1}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{n-1}\left[\left(tan\,\frac{\pi}{4}\right)^{n-1} - (tan\,0)^{n-1}\right]$$

$$= \frac{1}{n-1}$$

Example 17 : Evaluate : $\int_0^1 cot^{-1}(1 - x + x^2)\ dx$

Solution : $I = \int_0^1 cot^{-1}(1 - x + x^2)\ dx$

$\therefore \quad 0 < x < 1$

$\therefore \quad 0 < 1 - x < 1$

$\therefore \quad 0 < x(1 - x) < 1$

$\therefore \quad 0 < x - x^2 < 1$

$\therefore \quad 0 < 1 - x + x^2$

$$\therefore \quad I = \int_0^1 tan^{-1}\left(\frac{1}{1 - x + x^2}\right) dx \qquad \left(cot^{-1}x = tan^{-1}\frac{1}{x} \text{ for } x > 0\right)$$

$$= \int_0^1 tan^{-1}\left(\frac{1}{1 - x\,(1 - x)}\right) dx$$

$$= \int_0^1 tan^{-1}\left(\frac{x + (1 - x)}{1 - x\,(1 - x)}\right) dx$$

$$= \int_0^1 (tan^{-1}x + tan^{-1}(1 - x))\ dx \qquad (0 < x < 1,\ 0 < 1 - x < 1,\ 0 < x(1 - x) < 1)$$

$$= \int_0^1 tan^{-1}x\ dx + \int_0^1 tan^{-1}(1 - x)\ dx$$

$$= \int_0^1 tan^{-1}x\ dx + \int_0^1 tan^{-1}(1 - (1 - x))\ dx$$

$$= \int_0^1 \tan^{-1}x \, dx + \int_0^1 \tan^{-1}x \, dx$$

$$= 2\int_0^1 \tan^{-1}x \cdot 1 \, dx$$

$$= 2\,[x \tan^{-1}x]_0^1 - 2\int_0^1 \frac{x}{1+x^2}\, dx$$

$$= 2\,[x \tan^{-1}x]_0^1 - \int_0^1 \frac{2x}{x^2+1}\, dx$$

$$= 2\,[x \tan^{-1}x]_0^1 - [\log |x^2+1|]_0^1$$

$$= 2\,[\tan^{-1} 1 - 0] - [\log (1+1) - \log (0+1)]$$

$$= 2\left(\frac{\pi}{4}\right) - (\log 2 - \log 1)$$

$$= \frac{\pi}{2} - \log 2$$

Example 18 : Evaluate : $\displaystyle\int_{-\pi}^{\pi} \frac{2x\,(1+\sin x)}{1+\cos^2 x}\, dx$

Solution : $\displaystyle I = \int_{-\pi}^{\pi} \frac{2x\,(1+\sin x)}{1+\cos^2 x}\, dx$

$$= \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\, dx + \int_{-\pi}^{\pi} \frac{2x \sin x}{1+\cos^2 x}\, dx$$

$\therefore\ I = I_1 + I_2$, where $\displaystyle I_1 = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\, dx$ and $\displaystyle I_2 = \int_{-\pi}^{\pi} \frac{2x \sin x}{1+\cos^2 x}\, dx$

Let $\displaystyle f(x) = \frac{2x}{1+\cos^2 x}$ and $\displaystyle g(x) = \frac{2x \sin x}{1+\cos^2 x}$

Then $\displaystyle f(-x) = \frac{2\,(-x)}{1+\cos^2(-x)} = \frac{-2x}{1+\cos^2 x} = -f(x)$ and

$$g(-x) = \frac{2\,(-x)\sin(-x)}{1+\cos^2(-x)} = \frac{2x \sin x}{1+\cos^2 x} = g(x)$$

$\therefore$ $f(x)$ is an odd function and $g(x)$ is an even function.

$\therefore\ \displaystyle I_1 = 0$ and $\displaystyle I_2 = 2\int_0^{\pi} \frac{2x \sin x}{1+\cos^2 x}\, dx$

$$\therefore \quad I_2 = 4\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx \qquad \text{(i)}$$

$$= 4\int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\,dx$$

$$= 4\int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\,dx$$

$$I_2 = 4\int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}\,dx - 4\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$$

$$\therefore \quad I_2 = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx - I_2 \qquad \text{(Re (i))}$$

$$\therefore \quad 2I_2 = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

Let $\cos x = t$, $-\sin x\, dx = dt$, $\sin x\, dx = -dt$. When $x = 0$, $t = 1$ and when $x = \pi$, $t = -1$

$$\therefore \quad 2I_2 = 4\pi\int_1^{-1} \frac{-dt}{1+t^2}$$

$$= 4\pi\int_{-1}^{1} \frac{dt}{1+t^2}$$

$$= 4\pi\left[\tan^{-1} t\right]_{-1}^{1}$$

$$= 4\pi\left[\tan^{-1}(1) - \tan^{-1}(-1)\right]$$

$$= 4\pi\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

$\therefore \quad 2I_2 = 2\pi^2$

$\therefore \quad I_2 = \pi^2$

Now, $I = I_1 + I_2$

$\therefore \quad I = 0 + \pi^2$

$\therefore \quad I = \pi^2$

Example 19 : Prove that : $\int_0^{\frac{\pi}{2}} \log \sin x\, dx = -\frac{\pi}{2}\log 2$.

Solution : $I = \int_0^{\frac{\pi}{2}} \log \sin x\, dx \qquad \text{(i)}$

Then, $I = \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx$

$\therefore \quad I = \int_0^{\frac{\pi}{2}} \log \cos x \, dx$ **(ii)**

Adding (i) and (ii) we get

$$2I = \int_0^{\frac{\pi}{2}} \log \sin x \, dx + \int_0^{\frac{\pi}{2}} \log \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} \log (\sin x \cdot \cos x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} \log \left(\frac{2 \sin x \cos x}{2} \right) dx$$

$$= \int_0^{\frac{\pi}{2}} \log \left(\frac{\sin 2x}{2} \right) dx$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \int_0^{\frac{\pi}{2}} \log 2 \, dx$$

Let $I_1 = \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx$

$\therefore \quad 2I = I_1 - \log 2 \int_0^{\frac{\pi}{2}} dx$ **(iii)**

Now, $I_1 = \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx$

Let $2x = t$, we get $dx = \frac{1}{2} dt$

When $x = 0$, $t = 0$ and when $x = \frac{\pi}{2}$, $t = \pi$.

$$\therefore \quad I_1 = \int_0^{\pi} \log \sin t \cdot \frac{1}{2} dt$$

$$= \frac{1}{2} \int_0^{\pi} \log \sin t \, dt$$

$$= \frac{1}{2} \cdot 2 \cdot \int_0^{\frac{\pi}{2}} \log \sin t \, dt$$

$$\left(\log \sin (\pi - t) = \log \sin t. \text{ So } \int_0^{\pi} \log \sin t \, dt = 2 \int_0^{\frac{\pi}{2}} \log \sin t \, dt \right)$$

$\therefore \quad I_1 = \int_0^{\frac{\pi}{2}} \log \sin t \, dt$

$\therefore \quad I_1 = \int_0^{\frac{\pi}{2}} \log \sin x \, dx = I$ **(Definite integral does not depend upon variable)**

So, from (iii) we get,

$$2I = I - \frac{\pi}{2} \log 2$$

$$\therefore \quad I = -\frac{\pi}{2} \log 2$$

Not for examination :

Infact $\int_0^{\frac{\pi}{2}} \log \sin x \, dx$ is not a definite integral in usual sense. The function $\log \sin x$ is unbounded near end point 0 of $\left[0, \frac{\pi}{2}\right]$. Such integrals are called improper integrals.

Actually $\lim_{t \to 0+} \int_t^{\frac{\pi}{2}} \log sinx \, dx = \int_0^{\frac{\pi}{2}} \log \sin x \, dx$.

It is improper integral of first kind. Integrals like $\int_0^{\infty} \frac{sinx}{x} dx$ are called improper integrals of second kind.

If either function is unbounded in $[a, b]$, $a \in R$, $b \in R$ or interval is unbounded like $(-\infty, a)$, (a, ∞), $(-\infty, \infty)$ the integral is an improper definite integral as against definite integral studied in the chapter.

Sometimes regarding an improper integral as a definite integral would give incorrect results.

We could get $\int_{-2}^{3} \frac{dx}{x} = [\log |x|]_{-2}^{3} = \log 3 - \log 2 = \log\frac{3}{2}$

But $\frac{1}{x}$ is unbounded at $x = 0$.

$$\therefore \quad \int_{-2}^{3} \frac{dx}{x} = \int_{-2}^{0} \frac{dx}{x} + \int_{0}^{3} \frac{dx}{x}$$

$$= \lim_{t_1 \to 0-} \int_{-2}^{t_1} \frac{dx}{x} + \lim_{t_2 \to 0+} \int_{t_2}^{3} \frac{dx}{x}$$

does not exist.

$\int_0^{\pi} sec^2 x \, dx = [tanx]_0^{\pi} = 0 - 0 = 0$ is incorrect.

sec is unbounded at $x = \frac{\pi}{2}$

Exercise 3

1. If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx$, then prove that $n\,(I_{n-1} + I_{n+1}) = 1$

2. If $f(x) = f(a + b - x)$, prove that $\int_a^b x\, f(x)\, dx = \frac{a+b}{2} \int_a^b f(x)\, dx$.

3. Prove that $\int_0^{\pi} x\, f(\sin x)\, dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x)\, dx$ and using this evaluate

(i) $\int_0^{\pi} x \sin^2 x\, dx$ **(ii)** $\int_0^{\pi} \frac{x}{1+\sin x}\, dx$

4. Prove that $\int_0^n f(x)\, dx = \sum_{r=1}^{n} \int_0^1 f(t + r - 1)\, dt$

5. If $\int_n^{n+1} f(x)\, dx = n^3$, then find $\int_{-4}^{4} f(x)\, dx,\ n \in Z$

6. Prove that : $\int_0^{\frac{\pi}{2}} \log \cos x\, dx = -\frac{\pi}{2} \log 2$

7. Prove that : $\int_0^a x^2 (a - x)^n\, dx = \frac{2a^{n+3}}{(n+1)(n+2)(n+3)}$

Evaluate (8 to 17) :

8. $\int_0^{\log 2} x e^{-x}\, dx$

9. $\int_0^{\frac{\pi}{4}} \frac{dx}{a^2 \cos^2 x - b^2 \sin^2 x}$ **($a > b > 0$)**

10. $\int_1^2 \frac{x^2+1}{x^4+1}\, dx$

11. $\int_0^{\frac{\pi}{2}} \left(\frac{\pi x}{2} - x^2\right) \cos 2x\, dx$

12. $\int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{1 + \sin x \cos x}\, dx$

13. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \cos x + \sin x}\, dx$

14. $\int_0^1 \frac{\log(1+t)}{1+t^2}\, dt$

15. $\int_1^3 \frac{dx}{x^2(x+1)}$

16. $\int_0^{\pi} \frac{x^2 \sin x}{(2x - \pi)(1 + \cos^2 x)}\, dx$

17. $\int_0^{\frac{\pi}{2}} |\sin x - \cos x|\, dx$

18. Evaluate : $\int_1^3 (x^2 + x)\, dx$ as the limit of a sum.

19. Evaluate : $\int_0^4 (x + e^{2x})\, dx$ as the limit of a sum.

20. Prove that $\int_0^{\frac{\pi}{2}} \log \tan x \, dx = 0$

21. Prove that $\int_0^{\frac{\pi}{2}} (2 \log \sin x - \log \sin 2x)\, dx = -\frac{\pi}{2} \log 2$

22. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}}\, dx = ...$. ☐

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{12}$ (d) $\frac{\pi}{2}$

(2) $\int_1^e \log x \, dx = ...$. ☐

(a) 1 (b) $e + 1$ (c) $e - 1$ (d) 0

(3) $\int_0^{\frac{\pi}{2}} \frac{1}{1+\cot x}\, dx = ...$. ☐

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π

(4) If $\int_0^a \frac{1}{1+4x^2}\, dx = \frac{\pi}{8}$, then $a = ...$. ☐

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{1}{2}$ (d) 1

(5) $\int_0^3 \frac{3x+1}{x^2+9}\, dx = ...$. ☐

(a) $\frac{\pi}{12} + \log(2\sqrt{2})$ (b) $\frac{\pi}{3} + \log(2\sqrt{2})$ (c) $\frac{\pi}{12} + \log\sqrt{2}$ (d) $\frac{\pi}{6} + \log(2\sqrt{2})$

(6) $\int_{-1}^1 |1 - x|\, dx = ...$. ☐

(a) -2 (b) 2 (c) 0 (d) 4

(7) If $\int_0^1 (3x^2 + 2x + k)\, dx = 0$, then $k = ...$. ☐

(a) 1 (b) 2 (c) -2 (d) 4

(8) If $\int_1^a (3x^2 + 2x + 1)\, dx = 11$, then $a = ...$. ☐

(a) 2 (b) 3 (c) -3 (d) $\frac{2}{3}$

(9) $\int_{-1}^{0} |x| \, dx = ...$.

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) 2

(10) $\int_{-1}^{1} \log\left(\frac{7-x}{7+x}\right) dx = ...$.

(a) 1 (b) 0 (c) 2 (d) -2

(11) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} cotx \; dx = ...$.

(a) $\frac{1}{2}\log\left(\frac{3}{2}\right)$ (b) $\log\left(\frac{3}{2}\right)$ (c) $\frac{1}{2}\log\frac{\sqrt{3}}{2}$ (d) $2\log\frac{3}{2}$

(12) $\int_{1}^{k} f(x)\, dx = 47$, $f(x) = \begin{cases} 2x+8 & 1 \le x \le 2 \\ 6x & 2 < x \le k \end{cases}$, then k ....

(a) 4 (b) -4 (c) 2 (d) -2

(13) $\int_{1}^{\sqrt{3}} \frac{dx}{1+x^2} = ...$.

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{12}$ (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

(14) $\int_{1}^{4} \left(\frac{x^2+1}{x}\right)^{-1} = ...$.

(a) $\log\left(\frac{17}{2}\right)$ (b) $\frac{1}{2}\log\left(\frac{17}{2}\right)$ (c) $2\log(17)$ (d) $\log(17)$

(15) $\int_{0}^{\sqrt{2}} \sqrt{2-x^2}\, dx = ...$.

(a) $-\frac{\pi}{2}$ (b) π (c) 0 (d) $\frac{\pi}{2}$

(16) $\int_{0}^{2a} \frac{f(x)\,dx}{f(x)+f(2a-x)} = ...$.

(a) $-a$ (b) a (c) $\frac{a}{2}$ (d) $-\frac{a}{2}$

(17) $\int_{0}^{\pi} sin^3x \; cos^3x \;\; dx = ...$.

(a) π (b) $-\pi$ (c) $\frac{\pi}{2}$ (d) 0

(18) If $\int_{2}^{k} (2x + 1)\, dx = 6$, then $k = ...$.

(a) 3 (b) 4 (c) -4 (d) -2

(19) $\int_{0}^{1} \frac{dx}{x+\sqrt{x}} = ...$.

(a) log 2 (b) log 4 (c) log 3 (d) $-$ log 2

(20) If $\int_0^{\frac{\pi}{3}} \frac{\cos x}{3 + 4\sin x}\, dx = k \log \left(\frac{3 + 2\sqrt{3}}{3}\right)$, then $k =$ ....

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{8}$

Summary

We have studied the following points in this chapter :

1. $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function. Divide $[a, b]$ into n sub-intervals of equal length given by $h = \frac{b-a}{n}$. Then $\int_a^b f(x)\, dx = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f(a + ih)$.

2. Fundamental theorem of integral calculus : If the function f is continuous on $[a, b]$ and F is a differentiable in (a, b) such that

$\forall x \in (a, b), \frac{d}{dx}[F(x)] = f(x)$, then $\int_a^b f(x)\, dx = F(b) - F(a)$

3. $\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$, i.e. definite integral is independent of variable.

4. $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx \quad (a > b)$

5. If f is continuous on $[a, b]$ and $a < c < b$, then $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$.

6. If f is continuous on $[a, b]$, then $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$

7. If f is continuous on $[a, b]$, then $\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$

8. If f is continuous on $[0, 2a]$, then $\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(2a - x)\, dx$

If $f(2a - x) = f(x), \forall x \in [0, 2a]$, then $\int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx$

If $f(2a - x) = -f(x), \forall x \in [0, 2a]$, then $\int_0^{2a} f(x)\, dx = 0$

9. If f is an even continuous function on $[-a, a]$, then $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$

10. If f is an odd continuous function on $[-a, a]$, then $\int_{-a}^{a} f(x)\, dx = 0$.

AN APPLICATION OF INTEGRALS 4

Music is the pleasure the human soul experiences from counting without being aware that it is counting.
– Gottfried Wilhelm

◆

There are no deep theorems - only theorems that we have not understood very well.
– Nicholas Goodman

4.1 Introduction

Integration and differentiation are basic operations of calculus having numerous applications in science and engineering. Integrals appear in many practical applications.

If the archways of a building has triangular shape or semi-circular shape or rectangular shape and we need to fix glass in the archways, then we can use formulae of elementary geometry to decide how much glass material is needed. But if the archways are in section of an elliptic shape, then we have to resort to integration to find out the quantity of glass material needed.

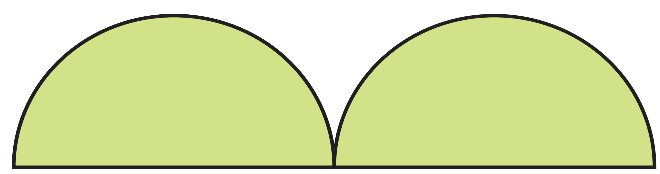

Figure 4.1

We need to know the area under a curve for this purpose. Before integration was developed, one could only approximate the area. Method of approximation was known to the ancient Greeks. A Greek mathematician **Archimedes**, worked-out good approximation to the area of a circle. Finding the area of a region is one of the most fundamental applications of the definite integral. The concept of integration was developed by **Newton** and **Leibnitz**.

4.2 Area Under Simple Curves

In the previous chapter, we have studied how to find the value of a definite integral as the limit of a sum. Let us study how integration is useful to find the area enclosed by simple curves, area between lines and arcs of circles, parabolas and ellipses. We shall also discuss how to find the area between two curves.

We will assume following property of a continuous function defined on a closed interval : A continuous function defined on a closed interval attains maximum value at some point of interval as well as minimum value at some point of interval.

Case 1 : Curves which are entirely above X-axis :

Let f be a continuous function defined over $[a, b]$. Assume that $f(x) \geq 0$ for all $x \in [a, b]$. We want to find the area A enclosed by the curve $y = f(x)$, the X-axis and the lines $x = a$ and $x = b$. (The coloured region in the figure 4.2(a) and 4.2(b).)

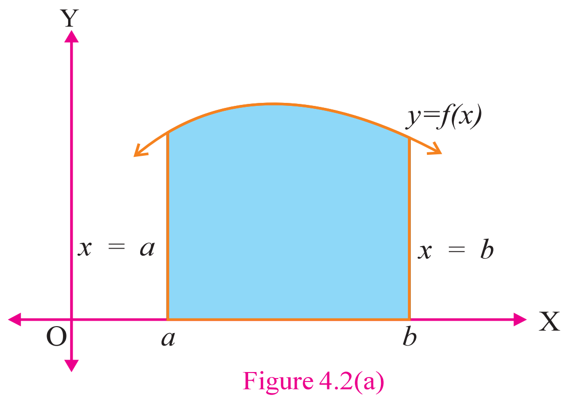

Figure 4.2(a)

We first divide the interval $[a, b]$ into n subintervals determined by the end-points $a = x_0, x_1, x_2,..., x_n = b$. Since $f(x)$ is continuous on each subinterval $[x_{i-1}, x_i]$, $i = 1, 2,..., n$, there exists a point $x_i' \in [x_{i-1}, x_i]$ such that $f(x_i')$ is minimum value of $f(x)$ in this subinterval. Also, there exists a point $x_i^* \in [x_{i-1}, x_i]$ such that $f(x_i^*)$ is maximum value of $f(x)$ in this subinterval. Let $\Delta x_i = x_i - x_{i-1}$. We construct a rectangle with $f(x'_i)$ as its height and Δx_i $(i = 1, 2,..., n)$ as its breadth. (as in the figure 4.3). The sum of the areas of these rectangles is clearly less than the area A we are trying to find.

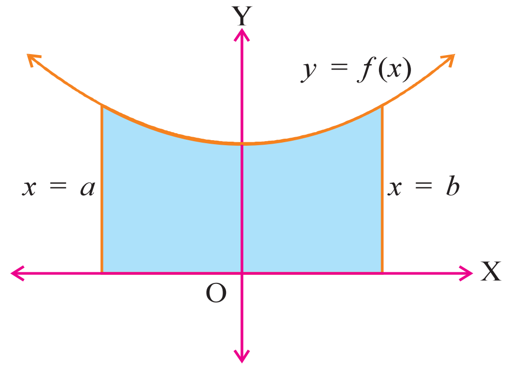

Figure 4.2(b)

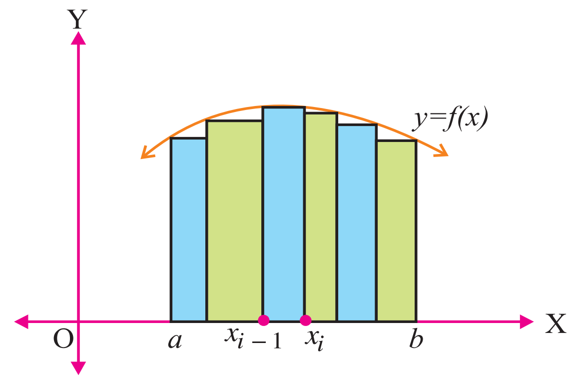

Figure 4.3

i.e., $\sum_{i=1}^{n} f(x'_i)\,\Delta x_i \leq A$ **(i)**

This sum $\sum_{i=1}^{n} f(x'_i)\,\Delta x_i$ is called a **lower sum**.

We construct a rectangle with $f(x_i^*)$ as its height and $\Delta x_i = x_i - x_{i-1}$ $(i = 1, 2,..., n)$ as its breadth. (as in the figure 4.4)

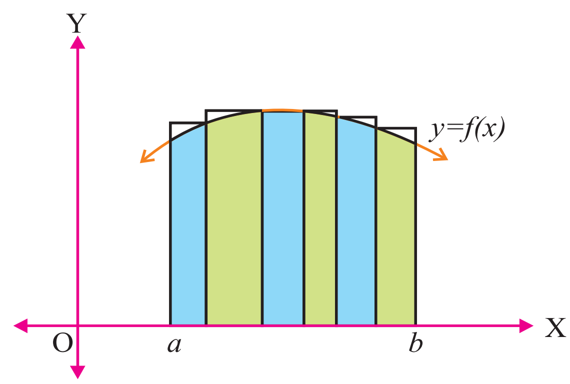

Figure 4.4

The sum of the areas of these rectangles is clearly greater than the area A we are trying to find.

i.e., $\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i \geq A$ **(ii)**

This sum $\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$ is called an **upper sum**.

Thus, from (i) and (ii) we have

$$\sum_{i=1}^{n} f(x'_i)\,\Delta x_i \leq A \leq \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

The area is equal to the limit of the lower sum or of the upper sum as the number of subdivisions tend to infinity and maximum of $\Delta x_i \to 0$ provided upper sums and lower sums tend to a common limit and can be written as follows :

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x'_i)\,\Delta x_i = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

As discussed in previous chapter, the above expression is $\int_a^b f(x)\,dx$.

Thus, area A = $\int_a^b f(x)\,dx$.

Case 2 : Curves which are entirely below the X-axis

If the curve under consideration lies below the X-axis, then $f(x) < 0$ from $x = a$ to $x = b$ as shown in figure 4.5. Then the sum defined in (i) and (ii) will be negative but the area bounded by the curve $y = f(x)$, X-axis and the lines $x = a$, $x = b$ is positive. In this case we take absolute value of the integral i.e., $\left| \int_a^b f(x)\,dx \right|$ as the area enclosed.

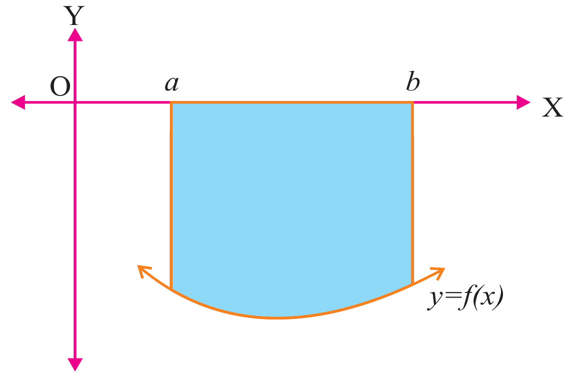

Figure 4.5

Thus, area A = | I | where I = $\int_a^b f(x)\,dx$.

Case 3 : Curves which intersect X-axis at one point :

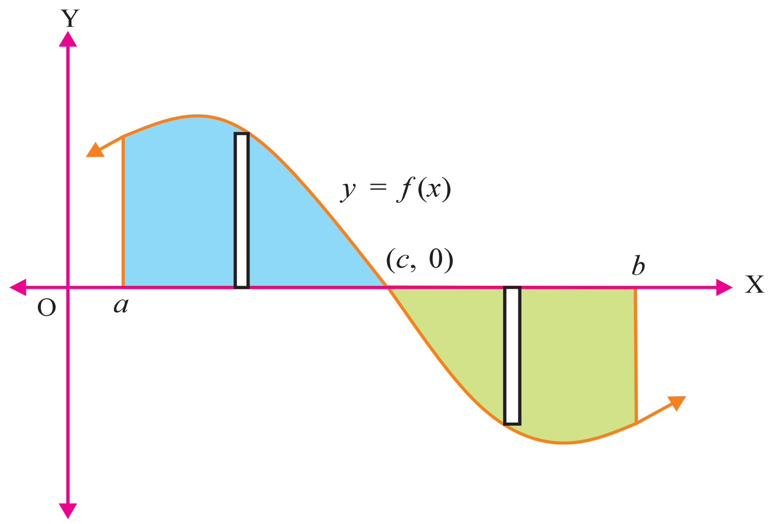

Figure 4.6

Let the graph of $y = f(x)$ intersect X-axis at $(c, 0)$ only and $a < c < b$. Let $f(x) \geq 0 \ \forall x \in [a, c]$, $f(x) \leq 0 \ \forall x \in [c, b]$. Then the area of the region bounded by $y = f(x)$, $x = a$, $x = b$ and X-axis is given by A = $|I_1| + |I_2|$.

where $I_1 = \int_a^c f(x)\,dx$, $I_2 = \int_c^b f(x)\,dx$.

Even if the curve intersects X-axis at finite number of points $c_1, c_2, ..., c_n$, we can have

$$I_k = \int_{c_k}^{c_{k+1}} f(x)dx \text{ and Area } = \sum_{k=0}^{n} |I_k|. \ (c_0 = a, c_{n+1} = b)$$

As above,

(1) Let $x = g(y)$ be continuous function of y over $[c, d]$ and $g(y) \geq 0$ or $g(y) \leq 0$, $\forall y \in [c, d]$. Then the area of the region bounded by $x = g(y)$, $y = c$, $y = d$ and Y-axis is $A = | I |$.

$$\text{where } I = \int_c^d x dy = \int_c^d g(y)\, dy.$$

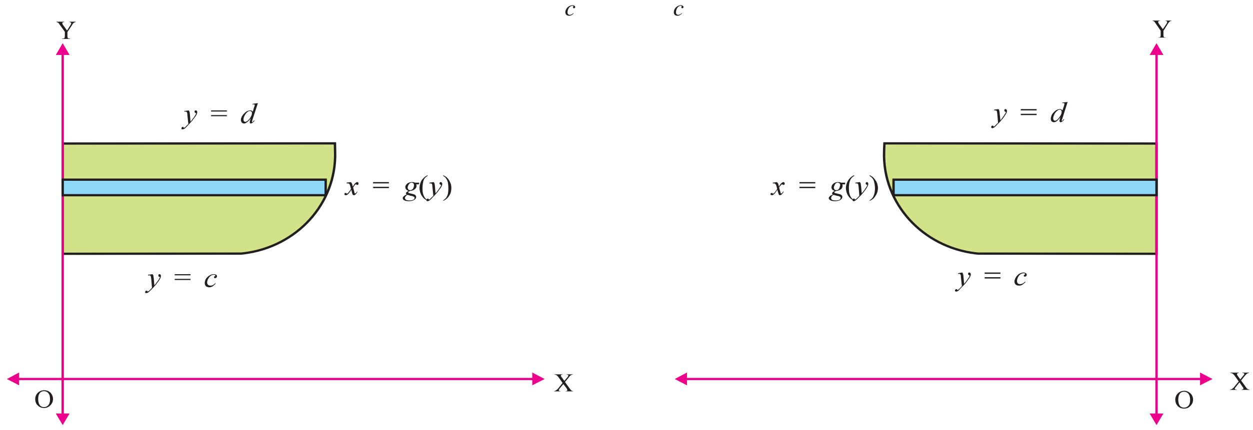

Figure 4.7

(2) Let the graph of $x = g(y)$ intersect Y-axis at only $(0, e)$ and $c < e < d$. Then the area of the region bounded by $x = g(y)$, $y = c$, $y = d$ and Y-axis is given by $A = | I_1 | + | I_2 |$,

$$\text{where } I_1 = \int_c^e g(y)\, dy \text{ and}$$

$$I_2 = \int_e^d g(y)\, dy.$$

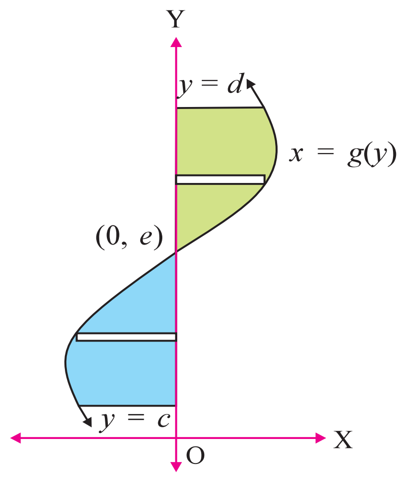

Figure 4.8

(3) If the curve and the region bounded by the curve are symmetric about X-axis and if one part of the area is in upper semi-plane of X-axis and the second one is in the lower semi-plane of X-axis, then the total area of the region will be two times the area in the upper semi-plane. This method can also be applied to calculate the area of a region symmetric about Y-axis.

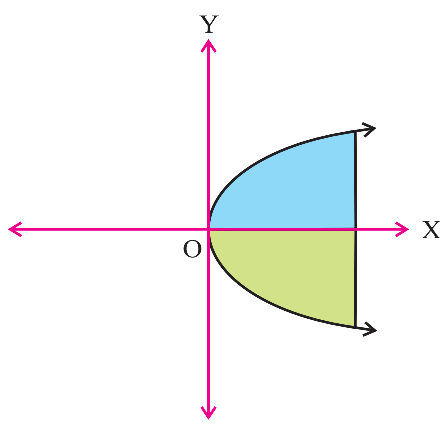

Figure 4.9

(4) If the curve and the region bounded by the curve are symmetric about both the axes, then its area can be calculated by considering the area of the region in the first quadrant and multiplying the same by four.

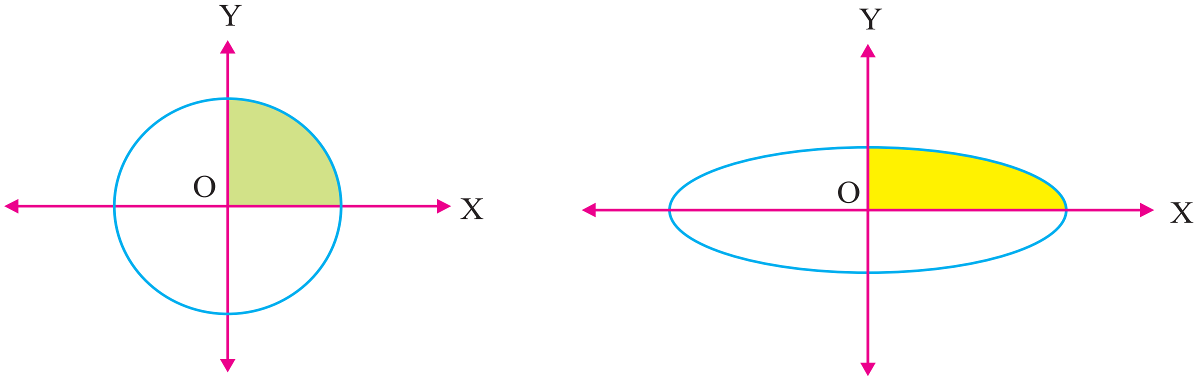

Figure 4.10

Region bounded by circle, ellipse are examples of this type.

Example 1 : Using integration, find the area of the region bounded by the line $2y = -x + 8$, X-axis and the lines $x = 2$ and $x = 4$.

Solution : Required area $= |\,\text{I}\,|$, where

$$\text{I} = \int_2^4 y\,dx$$

$$= \int_2^4 \left(\frac{-x}{2} + 4\right) dx$$

$$= \left[\frac{-x^2}{4} + 4x\right]_2^4$$

$$= \left[\frac{-(4)^2}{4} + 16\right] - \left[\frac{-(2)^2}{4} + 8\right]$$

$$= (-4 + 16) - (-1 + 8)$$

$$= 12 - 7$$

$$= 5$$

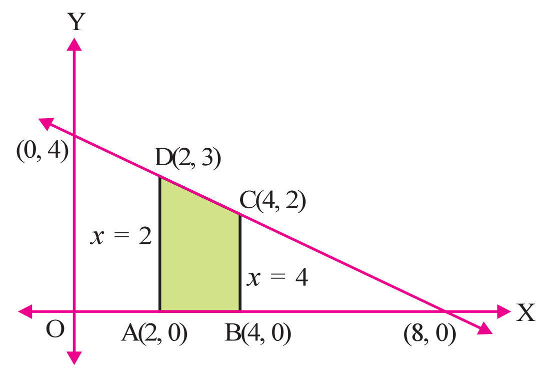

Figure 4.11

$\therefore$ Required area $= 5$

Note : Area of trapezium ABCD

$= \frac{1}{2}$ (Distance between parallel sides)(Sum of lengths of parallel sides)

$= \frac{1}{2}(4 - 2)(3 + 2) = 5$

Example 2 : Find the area of the region bounded by the curve $y = 4 - x^2$, X-axis and the lines $x = 0$ and $x = 2$.

Solution : Here $y = 4 - x^2$

$\therefore \quad x^2 = -(y - 4)$ which represents a parabola. Its vertex is (0, 4). Parabola opens downwards. Required area A = | I |, where

$$I = \int_0^2 y\,dx$$

$$= \int_0^2 (4 - x^2)\,dx$$

$$= \left[4x - \frac{x^3}{3}\right]_0^2$$

$$= 8 - \frac{8}{3} = \frac{16}{3}$$

$$\therefore \quad A = \frac{16}{3}$$

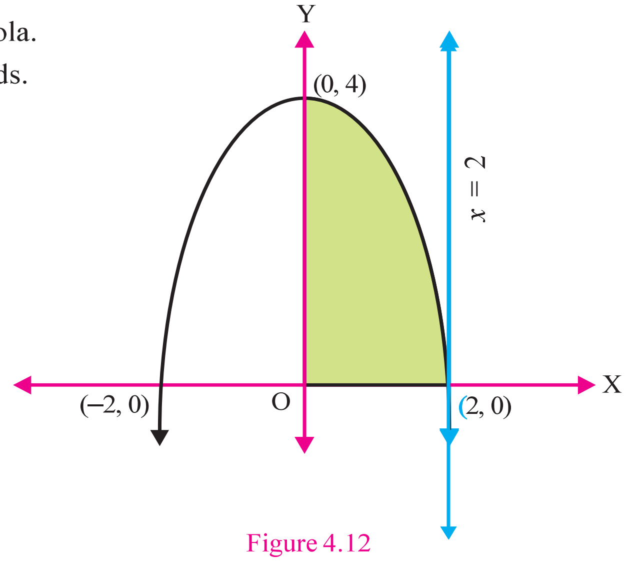

Figure 4.12

Example 3 : Find the area of the region bounded by $y = x^2 - 1$, X-axis and $y = 8$.

Solution : Here the curve $y = x^2 - 1$ is symmetric about Y-axis. So its area can be calculated by calculating the area enclosed by the arc in the first quadrant and then multiplying the same by 2.

Now, $y = x^2 - 1$. So $x^2 = y - (-1)$

This is a parabola whose vertex is $(0, -1)$ and it opens upwards. The limits of the region bounded by the curve in the first quadrant and Y-axis are $y = 0$ and $y = 8$.

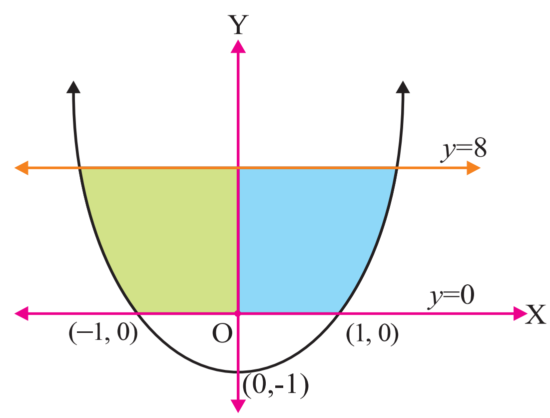

Figure 4.13

$\therefore \quad$ Area A = 2 | I |

$$\text{where } I = \int_0^8 x\,dy$$

$$= \int_0^8 \sqrt{y+1}\,dy \qquad (x > 0 \text{ in the first quadrant})$$

$$= \frac{2}{3}\left[(y + 1)^{\frac{3}{2}}\right]_0^8$$

$$= \frac{2}{3}\left((9)^{\frac{3}{2}} - 1\right) = \frac{52}{3}$$

$$\therefore \quad A = 2\,|\,I\,| = 2\left(\frac{52}{3}\right) = \frac{104}{3}$$

Example 4 : Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution : The ellipse is symmetrical about both X-axis and Y-axis.

Required area = 4 × Area OAB in the 1st quadrant

$$= 4\,|\,I\,|, \text{ where } I = \int_0^a y\,dx$$