CHAPTER 5

Application of Derivatives: Tangents and Normals, Rate Measure

- Equation of Tangents and Normals
- Length of Tangent, Normal, Sub-tangent and Sub-normal
- Angle between the Curves
- Miscellaneous Applications
- Interpretation of dy/dx as a Rate Measurer
- Approximations
- Mean Value Theorems

EQUATION OF TANGENTS AND NORMALS

Let $P(x_1, y_1)$ be any point on the curve $y = f(x)$.

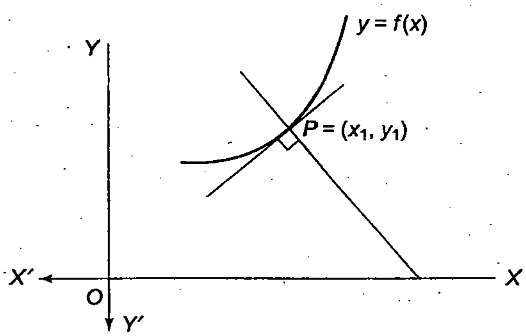


Fig. 5.1

If a tangent at P makes an angle θ with the positive direction of the x-axis, then $\frac{dy}{dx} = \tan\theta$.

Equation of Tangent:

Equation of a tangent at point $P(x_1, y_1)$ is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

Equation of Normal:

Equation of a normal at point $P(x_1, y_1)$ is $y - y_1 = \left(-\frac{dx}{dy}\right)_{(x_1, y_1)} (x - x_1)$

Note:
- *The point $P(x_1, y_1)$ will satisfy the equation of the curve and the equation of tangent and normal line.*
- *If the tangent at any point P on the curve is parallel to the axis of x, then $dy/dx = 0$ at the point P.*
- *If the tangent at any point on the curve is parallel to the axis of y, then $dy/dx = \infty$ or $dx/dy = 0$.*
- *If the tangent at any point on the curve is equally inclined to both the axes, then $dy/dx = \pm 1$.*
- *If the tangent at any point makes an equal intercept on the coordinate axes, then $dy/dx = -1$.*
- *Tangent to a curve at point $P(x_1, y_1)$ can be drawn, even though dy/dx at P does not exist. e.g., $x = 0$ is a tangent to $y = x^{2/3}$ at (0, 0).*

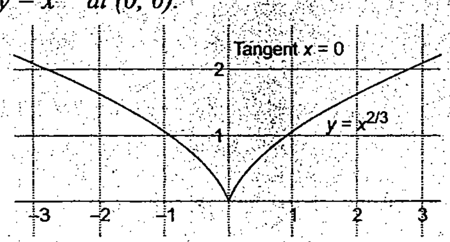


Fig. 5.2

- *If there is a tangent to an even function at $x = 0$, then it is always parallel to the x-axis*

Example 5.1 Find the total number of parallel tangents of $f_1(x) = x^2 - x + 1$ and $f_2(x) = x^3 - x^2 - 2x + 1$.

Sol. Here,

$f_1(x) = x^2 - x + 1$ and $f_2(x) = x^3 - x^2 - 2x + 1$

$\Rightarrow f_1'(x_1) = 2x_1 - 1$ and $f_2'(x_2) = 3x_2^2 - 2x_2 - 2$

Let the tangents drawn to the curves $y = f_1(x)$ and $y = f_2(x)$ at $(x_1, f_1(x_1))$ and $(x_2, f_2(x_2))$ are parallel.

$\Rightarrow 2x_1 - 1 = 3x_2^2 - 2x_2 - 2$ or $2x_1 = (3x_2^2 - 2x_2 - 1)$

which is possible for infinite numbers of ordered pairs;

$\Rightarrow$ Infinite solutions.

Example 5.2 Prove that the tangent drawn at any point to the curve $f(x) = x^5 + 3x^3 + 4x + 8$ would make an acute angle with the x-axis.

Sol. $f(x) = x^5 + 3x^3 + 4x + 8$

$\Rightarrow f'(x) = 5x^4 + 9x^2 + 4$

Clearly, $f'(x) > 0\ \forall x \in R$

Thus, the tangent drawn at any point would have positive slope and hence would make an acute angle with the x-axis.

Example 5.3 (a) Find the equation of the normal to the curve $y = |x^2 - |x||$ at $x = -2$.

(b) Find the equation of tangent to the curve $y = \sin^{-1}\frac{2x}{1+x^2}$ at $x = \sqrt{3}$

Sol. (a) In the neighbourhood of $x = -2$, $y = x^2 + x$.
Hence, the point on curve is $(-2, 2)$.

$$\frac{dy}{dx} = 2x + 1 \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=-2} = -3.$$

So the slope of the normal at $(-2, 2)$ is $\frac{1}{3}$.

Hence, the equation of the normal is $\frac{1}{3}(x + 2) = y - 2$.

$\Rightarrow \quad 3y = x + 8$

(b) $y = \sin^{-1}\frac{2x}{1+x^2} = \pi - 2\tan^{-1}x$, for $x > 1$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{1+x^2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=\sqrt{3}} = -\frac{2}{1+3} = -\frac{1}{2}$$

Also when $x = \sqrt{3}$, $y = \pi - 2\frac{\pi}{3} = \frac{\pi}{3}$

Hence, equation of tangent is $y - \frac{\pi}{3} = -\frac{1}{2}(x - \sqrt{3})$

Example 5.4 Find the equation of tangent and normal to the curve $x = 2at^2/(1 + t^2)$, $y = 2at^3/(1 + t^2)$ at the point for which $t = 1/2$.

Sol. Given that

$x = 2at^2/(1+t^2), y = 2at^3/(1+t^2)$

At $t = 1/2, x = 2a/5, y = a/5$

Also $\dfrac{dx}{dt} = \dfrac{4at}{(1+t^2)^2}$ and $\dfrac{dy}{dt} = \dfrac{2at^2(3+t^2)}{(1+t^2)^2}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1}{2}t(3+t^2)$

When $t = \dfrac{1}{2}, \dfrac{dy}{dx} = \dfrac{1}{2}\dfrac{1}{2}\left(3+\dfrac{1}{4}\right) = \dfrac{13}{16}$.

$\therefore$ The equation of the tangent when $t = 1/2$ is

$y - a/5 = (13/16)(x - 2a/5) \Rightarrow 13x - 16y = 2a$.

And the equation of the normal is

$(y - a/5)(13/16) + x - 2a/5 = 0$

$\Rightarrow \quad 16x + 13y = 9a$

Example 5.5 Find the equation of the normal to $y = x^3 - 3x$, which is parallel to $2x + 18y = 9$.

Sol. The curve is $y = x^3 - 3x$ (1)

$\Rightarrow \quad dy/dx = 3x^2 - 3$

The normal is parallel to the line $2x + 18y = 9$, then the slope of the normal $= -\dfrac{1}{(dy/dx)} = -\dfrac{1}{9}$ (Slope of the line)

$\Rightarrow \quad dy/dx = 9 \quad \Rightarrow 3x^2 - 3 = 9 \quad \Rightarrow x = \pm 2$

From equation (1), when $x = 2, y = 2$ and when $x = -2, y = -2$.

Hence, the required normals are

$y - 2 = -(1/9)(x-2)$ and $y + 2 = -(1/9)(x+2)$

$\Rightarrow \quad x + 9y = 20$ and $x + 9y + 20 = 0$

Example 5.6 If the tangent at any point $(4m^2, 8m^3)$ of $x^3 - y^2 = 0$ is a normal to the curve $x^3 - y^2 = 0$, then find the value of m.

Sol. Here, $y^2 = x^3$ (1)

$\Rightarrow \quad 2y\dfrac{dy}{dx} = 3x^2$

$\therefore \quad$ slope at $(4m^2, 8m^3) = \left(\dfrac{3x^2}{2y}\right)_{(4m^2, 8m^3)} = 3m$

$\therefore \quad$ Equation of the tangent at $(4m^2, 8m^3)$; $\dfrac{y - 8m^3}{x - 4m^2} = 3m$

$\Rightarrow \quad y = 3mx - 4m^3$ (2)

For another point, solving equations (1) and (2), we get

$x^3 = (3mx - 4m^3)^2$

$\Rightarrow \quad x = 4m^2, m^2$

$\therefore \quad A(4m^2, 8m^3)$ and $B(m^2, -m^3)$

$\Rightarrow \quad$ Slope of the tangent at B,

$$\left(\frac{dy}{dx}\right)_{(m^2, -m^3)} = \left(\frac{3x^2}{2y}\right)_{(m^2, -m^3)} = \frac{-3}{2}m$$

$\Rightarrow \quad$ Slope of the normal at $B = \dfrac{2}{3m}$

Since tangent and normal coincide, we get

$$\therefore \quad \frac{2}{3m} = 3m \Rightarrow m^2 = \frac{2}{9} \Rightarrow m = \pm\sqrt{\frac{2}{9}}$$

Example 5.7 For the curve $xy = c$, prove that the portion of the tangent intercepted between the coordinate axes is bisected at the point of contact.

Sol. Let the point at which the tangent is drawn be (α, β) on the curve $xy = c$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right) = -\frac{\beta}{\alpha}$$

Thus, the equation of the tangent is

$$y - \beta = -\frac{\beta}{\alpha}(x - \alpha)$$

$$\Rightarrow \quad y\alpha - \alpha\beta = -x\beta + \alpha\beta$$

$$\Rightarrow \quad \frac{x}{2\alpha} + \frac{y}{2\beta} = 1$$

It is clear that the tangent line cuts x- and y-axes at $A(2\alpha, 0)$ and $B(0, 2\beta)$, respectively and the point (α, β) bisects AB.

Tangent from an External Point

Given a point $P(a, b)$ which does not lie on the curve $y = f(x)$, then the equation of the possible tangents to the curve $y = f(x)$, passing through (a, b) can be found by solving for the point of contact Q.

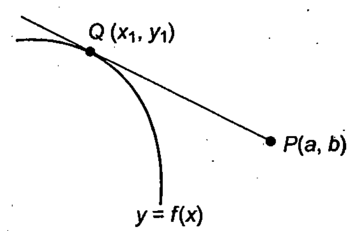


Fig. 5.3

Let point Q be (x_1, y_1). Since Q lies on the curve $y_1 = f(x_1)$ (1)

Also, the slope of $PQ = \left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} \Rightarrow \dfrac{y_1 - b}{x_1 - a} = \left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}$

Example 5.8 Find the equation of all possible normals to the parabola $x^2 = 4y$ drawn from the point $(1, 2)$.

Sol. Let point Q be $\left(h, \dfrac{h^2}{4}\right)$ and point P be the point of contact on the curve.

Now, m_{PQ} = slope of the normal at Q. (1)

$x^2 = 4y$

Differentiating w.r.t. $x \Rightarrow 2x = 4\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{x}{2}$

$\Rightarrow \quad$ Slope of the normal at $Q = -\left.\dfrac{dx}{dy}\right|_{x=h} = -\dfrac{2}{h}$

$$\Rightarrow \frac{\frac{h^2}{4}-2}{h-1}=-\frac{2}{h} \qquad \text{[from equation (1)]}$$

$$\Rightarrow \frac{h^3}{4}-2h=-2h+2 \Rightarrow h^3=8 \Rightarrow h=2$$

Hence, the co-ordinates of point Q is (2, 1), and so the equation of the required normal becomes $x+y=3$.

Example 5.9 Find the point on the curve where tangents to the curve $y^2-2x^3-4y+8=0$ pass through (1, 2).

Sol. $y^2-2x^3-4y+8=0$

Let a tangent is drawn to the curve at point $Q(\alpha, \beta)$ on the curve which passes through $P(1, 2)$.

Differentiating w.r.t. x, $2y\frac{dy}{dx}-6x^2-4\frac{dy}{dx}=0$

$$\Rightarrow \frac{dy}{dx}=\frac{3x^2}{y-2}$$

Now, slope of line $PQ=\frac{dy}{dx}_{(\alpha,\beta)}$

$$\Rightarrow \frac{\beta-2}{\alpha-1}=\frac{3\alpha^2}{\beta-2}$$

$\Rightarrow (\beta-2)^2=3\alpha^2(\alpha-1)$ (1)

Also (α, β) satisfies the equation of the curve.

$\Rightarrow \beta^2-2\alpha^3-4\beta+8=0$ or $(\beta-2)^2=2\alpha^3-4$ (2)

From equations (1) and (2), $3\alpha^2(\alpha-1)=2\alpha^3-4$

$\Rightarrow \alpha^3-3\alpha^2+4=0$ or $(\alpha-2)(\alpha^2-\alpha-2)=0$ or $(\alpha-2)^2(\alpha+1)=0$

When $\alpha=2$, $(\beta-2)^2=12$ or $\beta=2\pm2\sqrt{3}$

When $\alpha=-1$, $(\beta-2)^2=-6$ (not possible)

$\Rightarrow (\alpha,\beta)\equiv(2, 2\pm2\sqrt{3})$

Example 5.10 Find the equation of the normal to the curve $x^3+y^3=8xy$ at the point where it meets the curve $y^2=4x$ other than the origin.

Sol. The curves are $x^3+y^3=8xy$ (1)

and $y^2=4x$ (2)

Solving equations (1) and (2), we get $x^3+y\,4x=8xy$

$\Rightarrow \quad x^3=4xy$

$\Rightarrow \quad x^3=4x\;2\sqrt{x}$

$\Rightarrow \quad x^{3/2}(x^{3/2}-8)=0$

$\Rightarrow \quad x=0$ or $x^{3/2}=8=2^3$

$\Rightarrow \quad x=0$ or $x=2^2=4$.

Now when $x=0$, we get $y=0$

and when $x=4$, we get $y^2=16$ or $y\pm4$.

But $x=4$ and $y=-4$ does not satisfy equation (1)

Thus, (0, 0) and (4, 4) are the points of intersection of equations (1) and (2).

Differentiating equation (1), we get $\frac{dy}{dx}=\frac{8y-3x^2}{3y^2-8x}$

at (4, 4), $\frac{dy}{dx}=-1$

Hence, the equation of the normal to (1) at (4, 4) is

$(y-4)=1(x-4)$ or $y-x=0$

Condition for Which Given Line is Tangent or Normal to the Given Curve

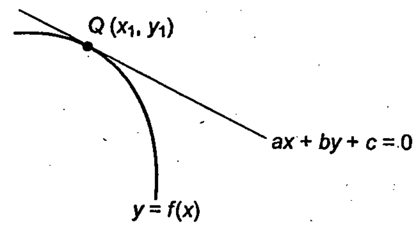


Fig. 5.4

Let the point on the curve be $P(x_1, y_1)$ where a line touches the curve.

Then P lies on the curve $\Rightarrow y_1=f(x_1)$ (1)

Also P lies on the line $\Rightarrow ax_1+by_1+c=0$ (2)

Further, slope of the line

= slope of tangent to the curve at point P

$\Rightarrow \quad -\frac{a}{b}=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}$ (3)

Eliminating x_1 and y_1 from the above three equations, we get the required condition.

Example 5.11 Show that the straight line $x\cos\alpha+y\sin\alpha=p$ touches the curve $xy=a^2$, if $p^2=4a^2\cos\alpha\sin\alpha$.

Sol. Let the line touches the curve at point $P(x_1, y_1)$ on the curve

$\Rightarrow x_1\cos\alpha+y_1\sin\alpha=p$ (1)

and $x_1y_1=a^2$ (2)

Differentiating $xy=a^2$ w.r.t. x, we get $\frac{dy}{dx}=-\frac{y}{x}$

Now, slope of the line = slope of the tangent to the curve at $P(x_1, y_1)$

$\Rightarrow -\frac{y_1}{x_1}=-\frac{\cos\alpha}{\sin\alpha}$ (3)

From equations (1) and (3), $x_1\cos\alpha+x_1\cos\alpha=p$

$\Rightarrow 2\cos\alpha\,x_1=p$

And $2\sin\alpha\,y_1=p$

$\Rightarrow (2\cos\alpha)(2\sin\alpha)(x_1, y_1)=p^2$

$\Rightarrow p^2=4a^2\cos\alpha\sin\alpha$

Example 5.12 If the line $x\cos\theta+y\sin\theta=P$ is the normal to the curve $(x+a)y=1$, then show

$$\theta\in\left(2n\pi+\frac{\pi}{2}, (2n+1)\pi\right)$$

$$\cup\left(2n\pi+\frac{3\pi}{2}, (2n+2)\pi\right), n\in Z$$

Sol. Here, $y=\dfrac{1}{x+a} \Rightarrow \dfrac{dy}{dx}=-\dfrac{1}{(x+a)^2}$

Slope of the normal is $(x+a)^2>0$ (for all x)

$\therefore\ x\cos\theta+y\sin\theta=P$ is normal if $-\dfrac{\cos\theta}{\sin\theta}>0$

or $\cot\theta<0$, i.e., θ lies in II or IV quadrant.

So, $\theta\in\left(2n\pi+\dfrac{\pi}{2},(2n+1)\pi\right)\cup\left(2n\pi+\dfrac{3\pi}{2},(2n+2)\pi\right)$

where $n\in Z$.

Concept Application Exercise 5.1

1. Show that the tangent to the curve $3xy^2-2x^2y=1$ at (1, 1) meets the curve again at the point (– 16/5, – 1/20).
2. The normal to the curve $x=a(1+\cos\theta), y=a\sin\theta$ at θ. Prove that it always passes through a fixed point and find that fixed point.
3. If the curve $y=ax^2-6x+b$ passes through (0, 2) and has its tangent parallel to the x-axis at $x=\dfrac{3}{2}$, then find the values of a and b.
4. Find the equation of the tangent to the curve $(1+x^2)y=2-x$, where it crosses the x-axis.
5. If the equation of the tangent to the curve $y^2=ax^3+b$ at point (2, 3) is $y=4x-5$, then find the values of a and b.
6. Find the value of $n\in N$ such that the curve $\left(\dfrac{x}{a}\right)^n+\left(\dfrac{y}{b}\right)^n=2$ touches the straight line $\dfrac{x}{a}+\dfrac{y}{b}=2$ at the point (a, b).
7. Find the condition that the line $Ax+By=1$ may be normal to the curve $a^{n-1}y=x^n$.

LENGTH OF TANGENT, NORMAL, SUB-TANGENT AND SUB-NORMAL

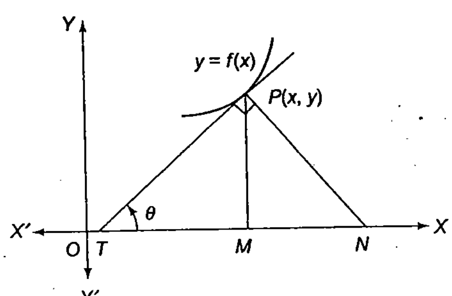


Fig. 5.5

(a) Length of Tangent:

PT is defined as the length of the tangent.

In ΔPMT, $PT=|y\operatorname{cosec}\theta|$

$$=|y\sqrt{(1+\cot^2\theta)}|$$

$$=\left|y\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}\right|$$

$\Rightarrow$ Length of tangent $=\left|y\sqrt{\left\{1+\left(\dfrac{dx}{dy}\right)^2\right\}}\right|$

(b) Length of Normal:

PN is defined as the length of the normal.

In ΔPMN, $PN=|y\operatorname{cosec}(90°-\theta)|$

$$=|y\sec\theta|$$

$$=|y\sqrt{1+\tan^2\theta}|=\left|y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\right|$$

$\Rightarrow$ Length of normal $=\left|y\sqrt{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}}\right|$

(c) Length of Sub-tangent:

TM is defined as sub-tangent.

In ΔPTM, $TM=|y\cot\theta|=\left|\dfrac{y}{\tan\theta}\right|=\left|y\dfrac{dx}{dy}\right|$

$\Rightarrow$ Length of sub-tangent $=\left|y\dfrac{dx}{dy}\right|$

(d) Length of Sub-normal:

MN is defined as sub-normal.

In ΔPMN, $MN=|y\cot(90°-\theta)|=|y\tan\theta|=\left|y\dfrac{dy}{dx}\right|$

$\Rightarrow$ Length of sub-normal $=\left|y\dfrac{dy}{dx}\right|$

Example 5.13 Find the length of sub-tangent to the curve $y=e^{x/a}$.

Sol. Here, $y=e^{x/a}$ (1)

$\Rightarrow \dfrac{dy}{dx}=e^{x/a}\dfrac{1}{a}$ (2)

And we know that the length of the sub-tangent $=y\dfrac{dx}{dy}$

$=e^{x/a}\dfrac{a}{e^{x/a}}=a$ [using (1) and (2)]

Example 5.14 Determine p such that the length of the sub-tangent and sub-normal is equal for the curve $y=e^{px}+px$ at the point (0, 1).

Sol. $\dfrac{dy}{dx}=pe^{px}+p$ at point $(0,1)=2p$

Sub-tangent $=\left|y\dfrac{dx}{dy}\right|$, Sub-normal $=\left|y\dfrac{dy}{dx}\right|$

Given, sub-tangent = sub-normal

$\Rightarrow \dfrac{dy}{dx}=\pm1 \Rightarrow 2p=\pm1 \Rightarrow p=\pm\dfrac{1}{2}$

Example 5.15 Find the length of normal to the curve,

$x=a(\theta+\sin\theta), y=a(1-\cos\theta)$, at $\theta=\dfrac{\pi}{2}$.

Sol. Here, $\dfrac{dx}{d\theta}=a(1+\cos\theta)$ and $\dfrac{dy}{d\theta}=a(\sin\theta)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=\pi/2} = \frac{\sin\pi/2}{1+\cos\pi/2} = 1$$

and the length of normal is

$$\left\{ y\sqrt{1+\left(\frac{dy}{dx}\right)^2} \right\}_{\theta=\pi/2} = a\left(1-\cos\frac{\pi}{2}\right)\sqrt{1+1^2} = \sqrt{2}a$$

Example 5.16 In the curve $x^{m+n} = a^{m-n} y^{2n}$. Prove that the mth power of the sub-tangent varies as the nth power of the sub-normal.

Sol. Given $x^{m+n} = a^{m-n} y^{2n}$ (1)

Taking logarithm of both sides, we get

$(m+n)\ln x = (m-n)\ln a + 2n \ln y$

Differentiating both sides w.r.t. x, we get

$$\frac{(m+n)}{x} = 0 + \frac{2n}{y}\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{(m+n)}{2n}\frac{y}{x}$$

$$\text{Now } \frac{(\text{Sub-tangent})^m}{(\text{Sub-normal})^n} = \frac{\left(y\frac{dx}{dy}\right)^m}{\left(y\frac{dy}{dx}\right)^n} = \frac{y^{m-n}}{\left(\frac{dy}{dx}\right)^{m+n}}$$

$$= \frac{y^{m-n}}{\left\{\frac{(m+n)}{2n}\frac{y}{x}\right\}^{m+n}} = \frac{x^{m+n}}{\left(\frac{m+n}{2n}\right)^{m+n} y^{2n}} = \frac{a^{m-n}}{\left(\frac{m+n}{2n}\right)^{m+n}}$$

{from (1)}

= constant (independent of x and y)

$\Rightarrow$ (Sub-tangent)$^m \propto$ (Sub-normal)^n

Concept Application Exercise 5.2

1. Find the length of the tangent for the curve $y = x^3 + 3x^2 + 4x - 1$ at point $x = 0$.
2. For the curve $y = a \ln (x^2 - a^2)$, show that the sum of lengths of tangent and sub-tangent at any point is proportional to product of coordinates of point of tangency.
3. For the curve $y = f(x)$, prove that
$$\frac{(\text{length of normal})^2}{(\text{length of tangent})^2} = \frac{\text{sub-normal}}{\text{sub-tangent}}.$$
4. If the sub-normal at any point on $y = a^{1-n}x^n$ is of constant length, then find the value of n.

ANGLE BETWEEN THE CURVES

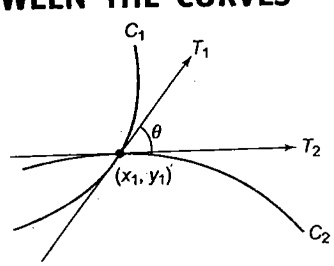


Fig. 5.6

Angle between two intersecting curves is defined as the acute angle between their tangents or the normals at the point of intersection of two curves.

$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$, where m_1 and m_2 are the slopes of tangents at the intersection point (x_1, y_1).

Note:

- *The curves must intersect for the angle between them to be defined. This can be ensured by finding their point of intersection analytically or graphically.*
- *If the curves intersect at more than one point, then the angle between the curves is written with references to the point of intersection.*
- *Two curves are said to be orthogonal if angle between them at each point of intersection is right angle, i.e., $m_1 m_2 = -1$.*

Example 5.17 Find the angle between curves $y^2 = 4x$ and $y = e^{-x/2}$.

Sol.

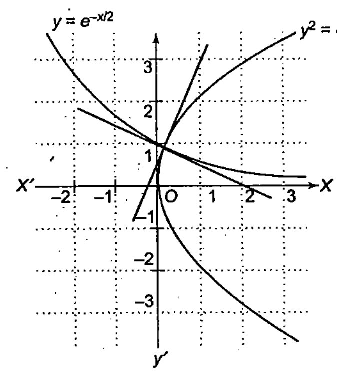


Fig. 5.7

Let the curves intersect at point (x_1, y_1)

$$\text{for } y^2 = 4x, \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = \frac{2}{y_1}$$

$$\text{and for } y = e^{-x/2}, \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = -\frac{1}{2}e^{-x_1/2} = -\frac{y_1}{2}$$

$\Rightarrow \quad m_1 m_2 = -1 \quad \Rightarrow$ Hence, $\theta = 90°$.

Example 5.18 The cosine of the angle of intersection of curves $f(x) = 2^x \log_e x$ and $g(x) = x^{2x} - 1$ is

Sol. Clearly, (1, 0) is the point of intersection of the given curves

$$\text{Now, } f'(x) = \frac{2^x}{x} + 2^x(\log_e 2)(\log_e x)$$

$\therefore$ Slope of tangent to the curve $f(x)$ at $(1, 0) = m_1 = 2$

$$\text{Similarly, } g'(x) = \frac{d}{dx}\left(e^{2x\log x} - 1\right) = x^{2x}\left(2x \times \frac{1}{x} + 2\log_e x\right)$$

$\therefore$ Slope of tangent to the curve $g(x)$ at $(1, 0) = m_2 = 2$

since $m_1 = m_2 = 2$

$\Rightarrow$ Two curves touch each other, so the angle between them is 0.

Hence, $\cos\theta = \cos 0 = 1$

Note:
Here, we have not actually found the intersection point but geometrically we can see that the curves intersect.

Example 5.19 Find the values of a if the curves $x^2/a^2 + y^2/4 = 1$ and $y^3 = 16x$ cut orthogonally.

Sol. The two curves are

$$x^2/a^2 + y^2/4 = 1 \quad (1)$$

$$y^3 = 16x \quad (2)$$

Differentiating (1), $dy/dx = -4x/(a^2y) = m_1$

Differentiating (2), $dy/dx = 16/(3y^2) = m_2$

The two curves cut orthogonally,

$\therefore\ m_1m_2 = -1$

$\Rightarrow\ [-4x/(a^2y)]\,[16/(3y^2)] = -1$

$\Rightarrow\ 64x = 3a^2y^3 \Rightarrow 64x = 3a^2\,16x$, using (2)

$\Rightarrow\ a^2 = 4/3$

$a = \pm\, 2/\sqrt{3}$

Example 5.20 Find the acute angle between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection.

Sol. $y = |x^2 - 1|$ (1)

and $y = |x^2 - 3|$ (2)

They intersect when $|x^2 - 1| = |x^2 - 3|$

$\Rightarrow\ 1 - x^2 = x^2 - 3 \ \Rightarrow\ x^2 = 2 \ \Rightarrow\ x = \pm\sqrt{2}$

$\Rightarrow$ The points of intersection are $(\pm\sqrt{2}, 1)$.

Since the curves are symmetrical about the y-axis, the angle of intersection at $(-\sqrt{2}, 1)$ = the angle of intersection at $(\sqrt{2}, 1)$

At $(\sqrt{2}, 1)$, $m_1 = 2x = 2\sqrt{2}$, $m_2 = -2x = -2\sqrt{2}$

$$\therefore\ \tan\theta = \left|\frac{4\sqrt{2}}{1-8}\right| = \frac{4\sqrt{2}}{7} \ \Rightarrow\ \theta = \tan^{-1}\frac{4\sqrt{2}}{7}$$

Example 5.21 Find the angle at which the two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$ intersect.

Sol. We have $x^3 - 3xy^2 + 2 = 0$ (1)

and, $3x^2y - y^3 - 2 = 0$ (2)

Differentiating equations (1) and (2) with respect to x, we obtain

$$\left(\frac{dy}{dx}\right)_{c_1} = \frac{x^2 - y^2}{2xy} \text{ and } \left(\frac{dy}{dx}\right)_{c_2} = \frac{-2xy}{x^2 - y^2}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{c_1} \times \left(\frac{dy}{dx}\right)_{c_2} = -1$$

Hence, the two curves cut at right angles.

Concept Application Exercise 5.3

1. Find the angle of intersection of $y = a^x$ and $y = b^x$.
2. Find the angle of intersection of the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$.
3. Find the angle at which the curve $y = Ke^{Kx}$ intersects the y-axis.
4. If the curve $ay + x^2 = 7$ and $x^3 = y$ cut orthogonally at (1, 1), then find the value a.
5. Find the angle between the curves $x^2 - \dfrac{y^2}{3} = a^2$ and $C_2: xy^3 = c$
6. Find the angle between the curves $2y^2 = x^3$ and $y^2 = 32x$.

MISCELLANEOUS APPLICATIONS

Example 5.22 Find possible values of p such that the equation $px^2 = \log_e x$ has exactly one solution.

Sol. Two curves $y = px^2$ and $y = \log_e x$ must intersect at only one point.

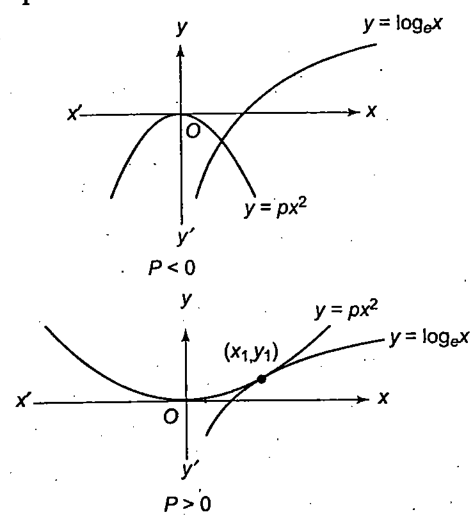


Fig. 5.8

Case 1: If $p \le 0$, then there is only one solution (see Fig. 5.8).

Case 2: If $p > 0$, then the two curves must only touch each other, i.e., tangent at $y = px^2$ and $y = \ln x$ must have the same slope at point (x_1, y_1).

Differentiating the given relation on both sides w.r.t. x, we get

$$2px_1 = \frac{1}{x_1} \ \Rightarrow\ x_1^2 = \frac{1}{2p} \quad (1)$$

Also (x_1, y_1) lies on the curves

$$\Rightarrow\ y_1 = px_1^2 \ \Rightarrow\ y_1 = p\left(\frac{1}{2p}\right) \quad \text{[(from (1)]}$$

$$\Rightarrow\ y_1 = \frac{1}{2} \quad (2)$$

$$\text{and } y_1 = \log_e x_1 \ \Rightarrow\ \frac{1}{2} = \log_e x_1$$

$$\Rightarrow\ x_1 = e^{1/2} \quad (3)$$

Hence, $x_1^2 = \dfrac{1}{2p} \Rightarrow e = \dfrac{1}{2p} \Rightarrow p = \dfrac{1}{2e}$.

Hence, possible values of p are $(-\infty, 0] \cup \left\{\dfrac{1}{2e}\right\}$.

Example 5.23 Find the values of a if equation $1 - \cos x = \frac{\sqrt{3}}{2}|x| + a, x \in (0, \pi)$ has exactly one solution.

Sol.

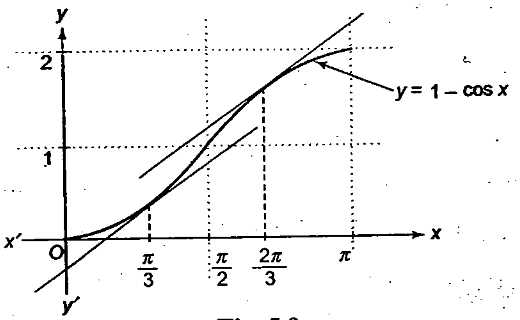


Fig. 5.9

$1 - \cos x = \frac{\sqrt{3}}{2}|x| + a$ has root when $y = 1 - \cos x$ and $y = \frac{\sqrt{3}}{2}|x| + a$ intersect.

For one real solution, consider the case when two curves touch each other.

Slope of C_1 is $\sin x$ and for $x > 0$ slope of C_2 is $\frac{\sqrt{3}}{2}$. Thus, for the point of contact

$$\sin x = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}.$$

Hence, the point of contact is $\left(\frac{\pi}{3}, \frac{1}{2}\right)$ or $\left(\frac{2\pi}{3}, \frac{3}{2}\right)$

For $\left(\frac{\pi}{3}, \frac{1}{2}\right)$, we get $a = \frac{1}{2} - \frac{\pi}{2\sqrt{3}}$.

For $\left(\frac{2\pi}{3}, \frac{3}{2}\right)$, we get $a = \frac{3}{2} - \frac{\pi}{\sqrt{3}}$.

Example 5.24 Find the locus of point on the curve $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$ where tangents are parallel to the axis of x.

Sol. We have $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$ (1)

Differentiating w.r.t. x, we get $2y\frac{dy}{dx} = 4a\left[1 + \cos\frac{x}{a}\right]$

For points at which the tangents are parallel to x-axis,

$$\frac{dy}{dx} = 0 \Rightarrow 4a\left(1 + \cos\frac{x}{a}\right) = 0$$

$$\Rightarrow \cos\frac{x}{a} = -1 \Rightarrow \frac{x}{a} = (2n+1)\pi$$

For these values of x, $\sin\frac{x}{a} = 0$.

Therefore, all these points lie on the parabola $y^2 = 4ax$.
[Putting $\sin x/a = 0$ in equation (1)]

Shortest Distance between Two Curves

The shortest distance between two non-intersecting curves is always along the common normal (wherever defined).

Example 5.25 Find the shortest distance between the line $y = x - 2$ and the parabola $y = x^2 + 3x + 2$.

Sol. Let $P(x_1, y_1)$ be a point closest to the line $y = x - 2$.

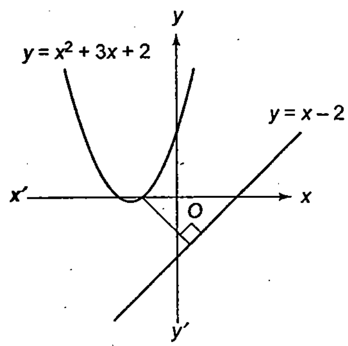


Fig. 5.10

Then $\left.\frac{dy}{dx}\right|_{(x_1, y_1)}$ = slope of line

$\Rightarrow \quad 2x_1 + 3 = 1$

$\Rightarrow \quad x_1 = -1$

$\Rightarrow \quad y_1 = 0.$

Hence, point $(-1, 0)$ is the closest and its perpendicular distance from the line $y = x - 2$ will give the shortest distance.

$\Rightarrow$ Shortest distance $= \frac{3}{\sqrt{2}}$

Example 5.26 Find the point on the curve $3x^2 - 4y^2 = 72$ which is nearest to the line $3x + 2y + 1 = 0$.

Sol.

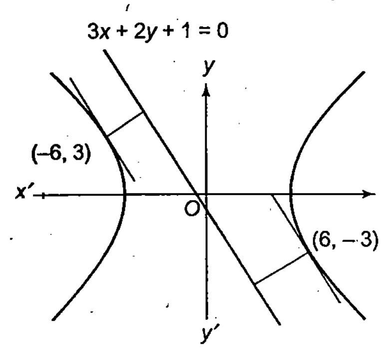


Fig. 5.11

Slope of the given line $3x + 2y + 1 = 0$ is $(-3/2)$.

Let us locate the point on the curve at which the tangent is parallel to given line.

Differentiating the curve on both sides w.r.t. to x, we get

$$6x - 8y\frac{dy}{dx} = 0$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{3x_1}{4y_1} = -\frac{3}{2}$$

[since parallel to $3x + 2y + 1 = 0$]

$\Rightarrow \quad \dfrac{x_1}{y_1} = -2$ (1)

Also the point (x_1, y_1) lies on $3x^2 - 4y^2 = 72$

$\Rightarrow \quad 3x_1^2 - 4y_1^2 = 72 \quad \Rightarrow \quad 3\dfrac{x_1^2}{y_1^2} - 4 = \dfrac{72}{y_1^2}$ (2)

$\Rightarrow \quad 3(4) - 4 = \dfrac{72}{y_1^2}$ [from (1)]

$\Rightarrow \quad y_1^2 = 9 \quad \Rightarrow \quad y_1 = \pm 3.$

The required points are (– 6, 3) and (6, – 3).

Distance (– 6, 3) from the given line

$$= \frac{|-18+6+1|}{\sqrt{13}} = \frac{11}{\sqrt{13}}.$$

and distance of (6, –3) from the given line

$$= \frac{|18-6+1|}{\sqrt{13}} = \frac{13}{\sqrt{13}} = \sqrt{13}$$

Thus, (– 6, 3) is the required point.

Example 5.27 The tangent at any point on the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ meets the axes in P and Q. Prove that the locus of the mid-point of PQ is a circle.

Sol. The given curve is $x = a\cos^3\theta, y = a\sin^3\theta$.

$$\text{Then } \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta(-\sin\theta)} = -\tan\theta$$

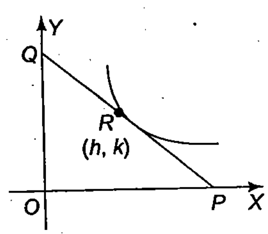


Fig. 5.12

$\therefore$ Equation of tangent at θ is

$y - a\sin^3\theta = -\tan\theta(x - a\cos^3\theta)$

$$\Rightarrow \quad \frac{y}{\sin\theta} - a\sin^2\theta = -\frac{x}{\cos\theta} + a\cos^2\theta$$

$$\Rightarrow \quad \frac{x}{\cos\theta} + \frac{y}{\sin\theta} = a \text{ or } \frac{x}{(a\cos\theta)} + \frac{y}{(a\sin\theta)} = 1$$

$\therefore \quad P \equiv (a\cos\theta, 0)$ and $Q = (0, a\sin\theta)$

If mid-point of PQ is $R(h, k)$, then

$2h = a\cos\theta$ and $2k = a\sin\theta$

$\therefore \quad (2h)^2 + (2k)^2 = a^2$ or $h^2 + k^2 = a^2/4$

Hence, the locus of mid-point is $x^2 + y^2 = a^2/4$, which is a circle.

INTERPRETATION OF dy/dx AS A RATE MEASURER

Recall that by the derivative ds/dt, we mean the rate of change of distance s with respect to the time t. In a similar fashion, whenever one quantity y varies with another quantity x, satisfying some rule $y = f(x)$, then $\dfrac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y with respect to x and $\left(\dfrac{dy}{dx}\right)_{x=x_0}$ (or $f'(x_0)$) represents the rate of change of y with respect to x at $x = x_0$.

Further, if two variables x and y vary with respect to another variable t, i.e., if $x = f(t)$ and $y = g(t)$, then by chain rule $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$, if $\dfrac{dx}{dt} \neq 0$. Thus, the rate of change of y with respect to x can be calculated using the rate of change of y and that of x, both with respect to t.

Example 5.28 Displacement s of a particle at time t is expressed as $s = \dfrac{1}{2}t^3 - 6t$. Find the acceleration at the time when the velocity vanishes (i.e., velocity tends to zero).

Sol. $s = \dfrac{1}{2}t^3 - 6t$

Thus velocity, $v = \dfrac{ds}{dt} = \left(\dfrac{3t^2}{2} - 6\right)$

and acceleration, $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} = 3t$

Velocity vanishes when $\dfrac{3t^2}{2} - 6 = 0$

$\Rightarrow \quad t^2 = 4 \quad \Rightarrow t = 2$

Thus, the acceleration when the velocity vanishes is $a = 3t = 6$ units.

Example 5.29 On the curve $x^3 = 12y$, find the interval of values of x for which the abscissa changes at a faster rate than the ordinate?

Sol. Given $x^3 = 12y$; differentiating w.r.t. y, we get

$$3x^2\frac{dx}{dy} = 12$$

$$\therefore \quad \frac{dx}{dy} = \frac{12}{3x^2}$$

Now if abscissa changes at a faster rate than the ordinate then we must have $\left|\dfrac{dx}{dy}\right| > 1$

$\Rightarrow \quad \left|\dfrac{12}{3x^2}\right| > 1$

$\Rightarrow \quad |x^2| < 4, x \neq 0$

$\Rightarrow \quad -2 < x < 2, x \neq 0$

$\Rightarrow \quad x \in (-2, 2) - (0)$

Example 5.30 A man 1.6 m high walks at the rate of 30 m/min away from a lamp which is 4 m above the ground. How fast does the man's shadow lengthen?

Sol.

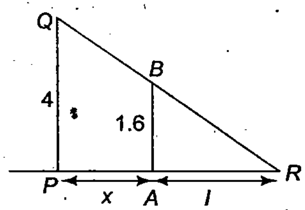


Fig. 5.13

Let $PQ = 4$ m be the height of pole and $AB = 1.6$ m be the height of the man.

Let the end of a shadow is R, and it is at a distance of l from A when the man is at a distance x from PQ at some instant.

Since, ΔPQR and ΔABR are similar, we have $\frac{PQ}{AB} = \frac{PR}{AR}$

$$\Rightarrow \quad \frac{4}{1.6} = \frac{x+l}{l}$$

$$\Rightarrow \quad 2x = 3l$$

$$\Rightarrow \quad 2\frac{dx}{dt} = 3\frac{dl}{dt} \qquad \left[\text{given } \frac{dx}{dt} = 30 \text{ m/min}\right]$$

$$\Rightarrow \quad \frac{dl}{dt} = \frac{2}{3} \times 30 \text{ m/min} = 20 \text{ m/min}$$

Example 5.31 If water is poured into an inverted hollow cone whose semi-vertical angle is 30°, show that its depth (measured along the axis) increases at the rate of 1 cm/s. Find the rate at which the volume of water increases when the depth is 24 cm.

Sol.

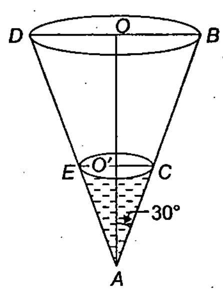


Fig. 5.14

Let A be the vertex and AO the axis of the cone.

Let $O'A = h$ be the depth of water in the cone.

In $\Delta AO'C$, $\tan 30° = \frac{O'C}{h}$

$$\Rightarrow \quad O'C = \frac{h}{\sqrt{3}} = \text{radius}$$

$$V = \text{volume of water in the cone} = \frac{1}{3}\pi (O'C)^2 \times AO'$$

$$= \frac{1}{3}\pi\left(\frac{h^2}{3}\right) \times h$$

$$\Rightarrow \quad V = \frac{\pi}{9}h^3 \quad \Rightarrow \quad \frac{dV}{dt} = \frac{\pi}{3}h^2\frac{dh}{dt} \qquad (1)$$

But given that depth of water increases at the rate of 1 cm/sec

$$\Rightarrow \quad \frac{dh}{dt} = 1 \text{ cm/s} \qquad (2)$$

From (1) and (2), $\frac{dV}{dt} = \frac{\pi h^2}{3}$

When $h = 24$ cm, the rate of increase of volume

$$\frac{dV}{dt} = \frac{\pi(24)^2}{3} = 192 \text{ cm}^3\text{/s}$$

Example 5.32 Let x be the length of one of the equal sides of an isosceles triangle, and let θ be the angle between them. If x is increasing at the rate (1/12) m/h, and θ is increasing at the rate of $\pi/180$ radius/h, then find the rate in m^2/h at which the area of the triangle is increasing when $x = 12$ m and $\theta = \pi/4$.

Sol.

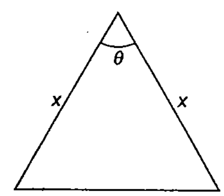


Fig. 5.15

$$A = \frac{1}{2}x^2 \sin\theta \quad \Rightarrow \quad 2A = x^2 \sin\theta$$

$$\Rightarrow \quad 2\frac{dA}{dt} = x^2 \cos q \,\frac{d\theta}{dt} + \sin\theta \; 2x\frac{dx}{dt}$$

$$\Rightarrow \quad 2\frac{dA}{dt} = (144)\left(\frac{1}{\sqrt{2}}\right)\frac{\pi}{180} + \frac{1}{\sqrt{2}} \times 2 \times 12 \times \frac{1}{12}$$

$$= \frac{12\pi}{15\sqrt{2}} + \frac{2}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{2\pi}{5\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}\pi}{5} + \frac{\sqrt{2}}{2} = \sqrt{2}\left(\frac{\pi}{5} + \frac{1}{2}\right)$$

Example 5.33 A horse runs along a circle with a speed of 20 km/h. A lantern is at the centre of the circle. A fence is along the tangent to the circle at the point at which the horse starts. Find the speed with which the shadow of the horse moves along the fence at the moment when it covers 1/8 of the circle in km/h is

Sol.

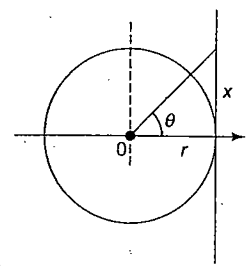


Fig. 5.16

$\tan\theta = x/r \Rightarrow x = r\tan\theta$

$\Rightarrow \quad dx/dt = r\sec^2\theta\,(d\theta/dt) = r\omega\sec^2\theta = v\sec^2\theta$

where $\theta = 2\pi/8 \Rightarrow dx/dt = v\sec^2(\pi/4) = 2v = 40$ km/h;

$\theta = 45°$

Concept Application Exercise 5.4

1. The distance covered by a particle moving in a straight line from a fixed point on the line is s, where $s^2 = at^2 + 2bt + c$, then prove that acceleration is proportional to s^{-3}.
2. Tangent of an angle increases four times as the angle itself. At what rate the sine of the angle increases w.r.t. the angle?
3. Two cyclists start from the junction of two perpendicular roads, their velocities being $3u$ m/min and $4u$ m/min, respectively. Find the rate at which the two cyclists separate.
4. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm³/min. When the thickness of ice is 5 cm, then find the rate at which the thickness of ice decreases.
5. x and y are the sides of two squares such that $y = x - x^2$. Find the rate of the change of the area of the second square with respect to the first square.
6. Two men P and Q start with velocities u at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, find the rate at which they are being separated.

APPROXIMATIONS

Let $f: A \to R, A \subset R$, be a given function and let $y = f(x)$. Let Δx denotes a small increment in x. Recall that the increment in y corresponding to the increment in x, denoted by Δy, is given by $\Delta y = f(x + \Delta x) - f(x)$. We define the following

(i) The differential of x, denoted by dx, is defined by $dx = \Delta x$.

(ii) The differential of y, denoted by dy, is defined by

$$dy = f'(x)\,dx \text{ or } dy = \left(\frac{dy}{dx}\right)\Delta x$$

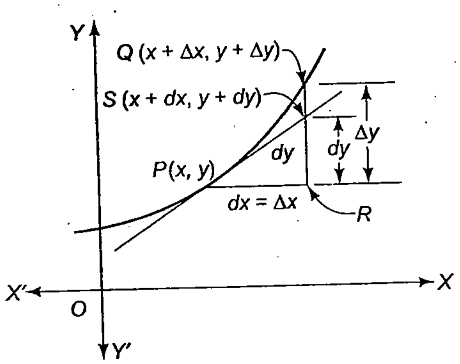


Fig. 5.17

In case $dx = \Delta x$ is relatively small when compared with x, dy is a good approximation of Δy and we denote it by $dy \approx \Delta y$.

Example 5.34 Find the approximate value of $\sqrt{36.6}$.

Sol. Consider the function $y = \sqrt{x}$.

Let $x = 36$ and $\Delta x = 0.6$.

Then $\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{36.6} - \sqrt{36} = \sqrt{36.6} - 6$

or $\sqrt{36.6} = 6 + \Delta y$

Now dy is approximately equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)_{x=36}\Delta x = \frac{1}{2\sqrt{36}}(0.6) = 0.05 \qquad (\text{as } y = \sqrt{x})$$

Thus, the approximate value of $\sqrt{36.6}$ is $6 + 0.05 = 6.05$.

Example 5.35 Find the approximate value of $(25)^{\frac{1}{3}}$.

Sol. Consider the function $y = x^{\frac{1}{3}}$

Let $x = 27$ and let $\Delta x = -2$.

then $\Delta y = (x + \Delta x)^{\frac{1}{3}} - x^{\frac{1}{3}} = (25)^{\frac{1}{3}} - (27)^{\frac{1}{3}} = (25)^{\frac{1}{3}} - 3$

or $(25)^{\frac{1}{3}} = 3 + \Delta y$

Now dy is approximately equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3x^{\frac{2}{3}}}(-2) \qquad (\text{as } y = x^{\frac{1}{3}})$$

At $x = 27$

$$dy = \frac{1}{3((27)^{\frac{1}{3}})^2}(-2) = \frac{-2}{27} = -0.074$$

Thus, the approximate value of $(25)^{\frac{1}{3}}$ is given by

$3 + (-0.074) = 2.926$.

Example 5.36 Find the approximate change in the volume V of a cube of side x metres caused by increasing the side by 2%.

Sol. We have volume $V = x^3$

$$\Rightarrow \quad dV = \left(\frac{dV}{dx}\right)\Delta x$$

$= (3x^2)\,\Delta x$

$= (3x^2)(0.02x)$

$= 0.06x^3\ \text{m}^3$ (as 2% of x is $0.02x$)

Thus, the approximate change in the volume is $0.06x^3$ m³.

Example 5.37 In an acute triangle ABC if sides a, b are constants and the base angles A and B vary, then show that $\dfrac{dA}{\sqrt{a^2 - b^2\sin^2 A}} = \dfrac{dB}{\sqrt{b^2 - a^2\sin^2 B}}$

Sol. $\frac{a}{\sin A} = \frac{b}{\sin B}$

or $\quad b \sin A = a \sin B$

$$b \cos A \, dA = a \cos B \, dB$$

$$\frac{dA}{a \cos B} = \frac{dB}{b \cos A}$$

$$\Rightarrow \quad \frac{dA}{a\sqrt{1-\sin^2 B}} = \frac{dB}{b\sqrt{1-\sin^2 A}}$$

$$\Rightarrow \quad \frac{dA}{a\sqrt{1-\frac{b^2 \sin^2 A}{a^2}}} = \frac{dB}{b\sqrt{1-\frac{a^2 \sin^2 B}{b^2}}}$$

$$\Rightarrow \quad \frac{dA}{\sqrt{a^2 - b^2 \sin^2 A}} = \frac{dB}{\sqrt{b^2 - a^2 \sin^2 B}}$$

Concept Application Exercise 5.5

1. Find the approximate value of $f(3.02)$, where $f(x) = 3x^2 + 5x + 3$.
2. If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.
3. Find the approximate value of $(1.999)^6$.
4. If $1° = \alpha$ radians, then find the approximate value of $\cos 60° 1'$.

MEAN VALUE THEOREMS

In calculus, the *mean value theorem*, roughly, states that in a given section of a smooth curve, there is a point at which the derivative (slope) of the curve is equal to the "average" derivative of the section.

This theorem can be understood concretely by applying it to motion: if a car travels 100 miles in 1 hour, so that its *average* speed during that time is 100 miles per hour, then at some time its *instantaneous* speed must have been exactly 100 miles per hour.

Rolle's Theorem

Statement:

If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$, i.e., continuous at each point in the interval $[a, b]$

(ii) differentiable in an open interval (a, b), i.e., differentiable at each point in the open interval (a, b)

(iii) $f(a) = f(b)$,

then there will be at least one point c in the interval (a, b) such that $f'(c) = 0$.

Geometrical Meaning of Rolle's Theorem

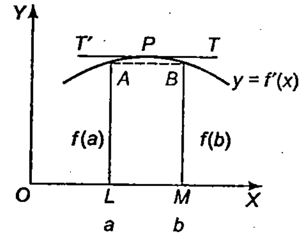


Fig. 5.18

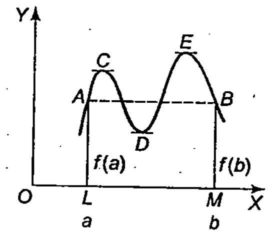


Fig. 5.19

If the graph of a function $y = f(x)$ is continuous at each point from the point $A(a, f(a))$ to the point $B(b, f(b))$ and the tangent at each point between A and B is unique, i.e., tangent at each point between A and B exists and ordinates, i.e., y co-ordinates of points A and B are equal, then there will be at least one point P on the curve between A and B at which tangent will be parallel to the x-axis.

In Fig. 5.18, there is only one such point P where tangent is parallel to the x-axis; however, in Fig. 5.19, there are more than one such points where tangents are parallel to the x-axis.

Note:

Converse of Rolle's theorem is not true, i.e., if a function $f(x)$ is such that $f'(c) = 0$ for at least one c in the open interval (a, b), then it is not necessary that

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is differentiable in (a, b)

(iii) $f(a) = f(b)$.

For example, we consider the function $f(x) = x^3 - x^2 - x + 1$ and the interval $[-1, 2]$

Here, $f'(x) = 3x^2 - 2x - 1$

$\therefore \quad f'(1) = 3 - 2 - 1 = 0$ *and* $1 \in (-1, 2)$

But condition (iii) of Rolle's theorem is not satisfied since $f(-1) \neq f(2)$.

Example 5.38 Discuss the applicability of Rolle's theorem for the following functions on the indicated intervals:

(i) $f(x) = |x|$ in $[-1, 1]$

(ii) $f(x) = 3 + (x-2)^{2/3}$ in $[1, 3]$

(iii) $f(x) = \tan x$ in $[0, \pi]$

(iv) $f(x) = \log\left\{\frac{x^2 + ab}{x(a+b)}\right\}$ in $[a, b]$, where $0 < a < b$.

Sol. (i) $f(x) = |x|$ is continuous but non-differentiable in $[-1, 1]$, hence Rolle's theorem is not applicable.

(ii) $f(x) = 3 + (x-2)^{2/3} \Rightarrow f'(x) = \frac{2}{3(x-2)^{1/3}}$. Thus $f(x)$ is continuous but derivative does not exist at $x = 2$. Hence, Rolle's theorem is not applicable.

(iii) $f(x) = \tan x$ in $[0, \pi]$ is discontinuous at $x = \pi/2$. Hence Rolle's theorem is not applicable.

(iv) $f(x) = \log\left\{\frac{x^2 + ab}{x(a+b)}\right\}$ in $[a, b]$, where $0 < a < b$.

For $0 < a < b$, $f(x)$ is continuous and differentiable.

$$f(a) = \log\left\{\frac{a^2+ab}{a(a+b)}\right\} = \log\left\{\frac{a(a+b)}{a(a+b)}\right\} = \log 1 = 0$$

and

$$f(b) = \log\left\{\frac{b^2+ab}{b(a+b)}\right\} = \log\left\{\frac{b(a+b)}{b(a+b)}\right\} = \log 1 = 0$$

Hence $f(a) = f(b)$, and Rolle's theorem is applicable.

Example 5.39 If the function $f(x) = x^3 - 6x^2 + ax + b$ defined on [1, 3] satisfies the Rolle's theorem for $c = \frac{2\sqrt{3}+1}{\sqrt{3}}$, then find the values of a and b.

Sol. Since $f(x)$ satisfies conditions of Roll's theorem on [1, 3]

$\therefore\ f(1) = f(3)$

$\therefore\ 1 - 6 + a + b = 27 - 54 + 3a + b$

$\Rightarrow\ 2a = 22$ or $a = 11$

Since $f(1) = f(3)$ is independent of b

$\therefore\ a = 11$ and $b \in R$

Example 5.40 Let $f(x) = (x-a)(x-b)(x-c)$, $a < b < c$, show that $f'(x) = 0$ has two roots one in (a, b) and the other in (b, c).

Sol. Here, $f(x)$ being a polynomial is continuous and differentiable for all real values of x. We also have $f(a) = f(b) = f(c)$. Rolle's theorem is applicable to $f(x)$ in $[a, b]$ and $[b, c]$. We observe that $f'(x) = 0$ has at least one root in (a, b) and at least one root in (b, c). But it is a polynomial of degree two, hence $f'(x) = 0$ cannot have more than two roots. It implies that exactly one root of $f'(x) = 0$ would lie in (a, b) and exactly one root of $f'(x) = 0$ would lie in (b, c).

Note: *Let $y = f(x)$ be a polynomial function of degree n. If $f(x) = 0$ has real roots only, then $f'(x) = 0$, $f''(x) = 0$, ..., $f^{n-1}(x) = 0$ would have only real roots. It is so because if $f(x) = 0$ has all real roots, then between two consecutive roots of $f(x) = 0$, exactly one root of $f'(x) = 0$ would lie.*

Example 5.41 If $2a + 3b + 6c = 0$, then prove that at least one root of the equation $ax^2 + bx + c = 0$ lies in the interval (0, 1).

Sol. Consider the function $f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + d$.

We have $f(0) = d$ and

$$f(1) = \frac{a}{3} + \frac{b}{2} + c + d = \frac{2a+3b+6c}{6} + d = 0 + d = d$$

$(\because 2a + 3b + 6c = 0)$

Thus $f(0) = f(1) = d$. Consequently, there exists at least one root of the polynomial $f'(x) = ax^2 + bx + c$ lying between 0

Example 5.42 Show that between any two roots of $e^{-x} - \cos x = 0$, there exists at least one root of $\sin x - e^{-x} = 0$.

Sol. Let $f(x) = e^{-x} - \cos x$ and let α and β be two of many roots of the equation $e^{-x} - \cos x = 0$

$\Rightarrow\ f(\alpha) = 0$ and $f(\beta) = 0$

Also $f(x)$ is continuous and differentiable.

Then, according to the Rolle's theorem, there exists at least one $c \in (a, b)$ such that $f'(c) = 0$, or $e^{-c} - \sin c = 0$ or c is root of the equation $e^{-x} - \sin x = 0$.

Example 5.43 Let $P(x)$ be a polynomial with real coefficients. Let $a, b \in R$, $a < b$, be two consecutive roots of $P(x)$. Show that there exists c such that $a \le c \le b$ and $P'(c) + 100\,P(c) = 0$.

Sol. Consider $f(x) = e^{100x} P(x)$.

Now $f(a) = f(b) = 0$ (as $P(a) = P(b) = 0$)

Also as $P(x)$ is polynomial $\Rightarrow f(x)$ is continuous and differentiable in $[a, b]$

$\Rightarrow$ Rolle's theorem can be applied

$\Rightarrow\ \exists\, c \in (a, b)$ such that $f'(c) = 0$

Now $f'(x) = e^{100x}(P'(x) + 100\,P(x))$

$\Rightarrow\ e^{100c}(P'(c) + 100\,P(c)) = 0$,

$\Rightarrow\ P'(c) + 100 \times P(c) = 0$ (as $e^{100c} \ne 0$)

Example 5.44 If the equation $ax^2 + bx + c = 0$ has two positive and real roots, then prove that the equation $ax^2 + (b + 6a)x + (c + 3b) = 0$ has atleast one positive real root.

Sol. Consider $f(x) = e^{3x}(ax^2 + bx + c)$

$f(x) = 0$ has two positive real roots.

Using Rolle's theorem,

We can say $f'(x) = 0$ has at least one real root between two roots of $f(x) = 0$.

$\Rightarrow\ e^{3x}(ax^2 + (b + 6a)x + c + 3b) = 0$ has at least one positive real root.

$\Rightarrow\ ax^2 + (b + 6a)x + c + 3b = 0$ has at least one positive real root.

Example 5.45 Let $f(x)$ and $g(x)$ be differentiable functions such that $f'(x)\,g(x) \ne f(x)\,g'(x)$ for any real x. Show that between any two real solutions of $f(x) = 0$, there is at least one real solution of $g(x) = 0$.

Sol. Let a, b be the solutions of $f(x) = 0$.

Suppose $g(x)$ is not equal to zero for any x belonging to $[a,b]$

Now consider $h(x) = f(x)/g(x)$

Since $g(x)$ not equal to zero, $h(x)$ is differentiable and continuous in (a, b)

$h(a) = h(b) = 0$ (as $f(a) = 0$ and $f(b) = 0$ but $g(a)$ or $g(b) \ne 0$)

Applying Rolle's theorem

$h'(c) = 0$ for some c belonging to (a, b)

$f(x)g'(x) = f'(x)g(x)$

This gives the contradiction.

Lagrange's Mean Value Theorem

Statement:

If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$, i.e., continuous at each point in the interval $[a, b]$

(ii) differentiable in the open interval (a, b), i.e., differentiable at each point in the interval (a, b)

Then, there will be at least one point c, where $a < c < b$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proof:

(Using Rolle's theorem)

Let $\quad F(x) = Ax + f(x) \qquad (1)$

where A is a constant. We choose A such that $F(a) = F(b)$

$$\Rightarrow \quad Aa + f(a) = Ab + f(b) \quad \text{or} \quad A = -\frac{f(b)-f(a)}{b-a} \qquad (2)$$

Now since $f(x)$ is continuous in the closed interval $[a, b]$ and x is continuous everywhere; therefore, $F(x)$ is continuous in $[a, b]$.

Again since $f(x)$ is differentiable in (a, b) and x is differentiable everywhere; therefore, $F(x)$ is also differentiable in (a, b).

Also for the value of A given by (2), $F(a) = F(b)$. Hence, all the conditions of Rolle's theorem are satisfied for $F(x)$ in $[a, b]$; therefore, there exists at least one c, where $a < c < b$, such that

$$F'(c) = 0 \qquad (3)$$

From (1), differentiating w.r.t. to x, we get

$$F'(x) = A.1 + f'(x) \Rightarrow F'(c) = A + f'(c)$$

From (3), $F'(c) = 0 \Rightarrow A + f'(c) = 0$

or, $f'(c) = -A = \dfrac{f(b)-f(a)}{b-a}$, where $a < c < b$. [from (2)]

Another Form of Lagrange's Mean Value Theorem

Statement:

If a function $f(x)$ is

(i) continuous in the closed interval $[a, a+h]$

(ii) differentiable in the open interval $(a, a+h)$

then there exists at least one value θ, where $0 < \theta < 1$, such that

$$f(a+h) = f(a) + hf'(a+\theta h)$$

Proof:

Putting $b = a + h$ in the above theorem, there will be at least one c, $a < c < a + h$ such that

$$f'(c) = \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h} \qquad (1)$$

Let $c = a + \theta h$

Then, $a < c < a + h \Rightarrow a < a + \theta h < a + h$

$\Rightarrow 0 < \theta h < h \Rightarrow 0 < \theta < 1$

$\therefore$ from (1), $f'(a+\theta h) = \dfrac{f(a+h)-f(a)}{h}$

or, $f(a+h) = f(a) + hf'(a+\theta h)$, where $0 < \theta < 1$

Geometrical Meaning of Lagrange's Mean Value Theorem

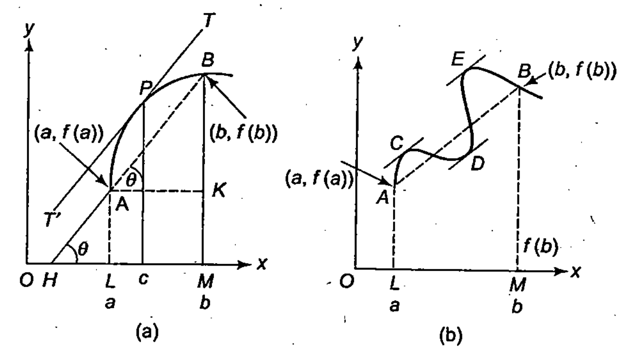


Fig. 5.20

Let $A(a, f(a))$ and $B(b, f(b))$ be two points on the curve $y = f(x)$. Then $OL = a$, $OM = b$, $AL = f(a)$, $BM = f(b)$.

Now, the slope of chord AB, $\tan\theta = \dfrac{BK}{AK} = \dfrac{f(b)-f(a)}{b-a} \qquad (1)$

By Lagrange's mean value theorem,

$\dfrac{f(b)-f(a)}{b-a} = f'(c) =$ slope of tangent at point $P(c, f(c))$

$\therefore$ from (1), $\tan\theta =$ slope of tangent at P

$\therefore$ slope of chord AB = slope of tangent at P

Hence, chord $AB \parallel$ tangent PT.

Thus geometrical meaning of the mean value theorem is as follows: If $y = f(x)$ is continuous and differentiable in (a, b), then there exists at least one point P on the curve in (a, b), where tangent will be parallel to chord AB. In Fig. 5.20(a) there is only one such point P where tangent is parallel to chord AB but in Fig. 5.20(b) there are more than one such points where tangents are parallel to chord AB.

Example 5.46 Consider the function $f(x) = 8x^2 - 7x + 5$ on the interval $[-6, 6]$. Find the value of c that satisfies the conclusion of Lagrange's mean value theorem.

Sol. $f'(c) = 16c - 7$

$$= \frac{f(6)-f(-6)}{12}$$

$$= \frac{(8\times36-7\times6+5)-(8\times36+7\times6+5)}{12} = -7$$

$\Rightarrow \quad c = 0$

Example 5.47 Let f be differentiable for all x. If $f(1) = -2$ and $f'(x) \geq 2$ for all $x \in [1, 6]$, then find the range of values of $f(6)$.

Sol. By Lagrange's mean value theorem, there exists $c \in (1, 6)$ such that

$$f'(c) = \frac{f(6)-f(1)}{6-1} \Rightarrow \frac{f(6)+2}{5} \geq 2$$

$(\because f'(x) \geq 2$ for all $x \in [1, 6])$

$\Rightarrow f(6)+2 \geq 10 \Rightarrow f(6) \geq 8$

Example 5.48 Let $f : [2, 7] \to [0, \infty)$ be a continuous and differentiable function. Then show that

$$(f(7)-f(2))\frac{(f(7))^2+(f(2))^2+f(2)f(7)}{3}$$
$$=5f^2(c)f'(c), \text{ where } c \in [2,7].$$

Sol. We have to prove that

$$(f(7)-f(2))\frac{(f(7))^2+(f(2))^2+f(2)f(7)}{3}$$
$$=5f^2(c)f'(c)$$

or $$\frac{(f(7))^3-(f(2))^3}{7-2}=3f^2(c)f'(c)$$

Then consider the function $g(x)=(f(x))^3$ which is continuous in [2, 7] and differentiable in (2, 7).
Then from Lagrange's mean value theorem there exists at least one $c \in [2, 7]$ such that

$$g'(c)=\frac{g(7)-g(2)}{7-2}$$

$$\Rightarrow \quad 3f^2(c)f'(c)=\frac{(f(7))^3-(f(2))^3}{7-2}$$

Example 5.49 Using Lagrange's mean value theorem prove that $|\cos a-\cos b| \le |a-b|$.

Sol. Consider $f(x)=\cos x$ in $[a, b]$ which is continuous and differentiable.
Hence, according to Lagrange's mean value theorem there exists at least one $c \in (a, b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

or $$-\sin c=\frac{\cos b-\cos a}{b-a}$$

$$\Rightarrow \left|\frac{\cos b-\cos a}{b-a}\right|=|-\sin c| \le 1$$

$$\Rightarrow |\cos b-\cos a| \le |a-b|$$

Example 5.50 Let $f(x)$ and $g(x)$ be differentiable functions in (a, b), continuous at a and b and $g(x) \ne 0$ in $[a, b]$.

Then prove that $$\frac{g(a)f(b)-f(a)g(b)}{g(c)f'(c)-f(c)g'(c)}$$

$$=\frac{(b-a)g(a)g(b)}{(g(c))^2} \text{ for at least one } c \in (a, b).$$

Sol. We have to prove $$\frac{g(a)f(b)-f(a)g(b)}{g(c)f'(c)-f(c)g'(c)}$$

$$=\frac{(b-a)g(a)g(b)}{(g(c))^2}$$

After rearranging,

$$\frac{\frac{f(b)}{g(b)}-\frac{f(a)}{g(a)}}{(b-a)}=\frac{g(c)f'(c)-f(c)g'(c)}{(g(c))^2}$$

Let $h(x)=\frac{f(x)}{g(x)}$

As $f(x)$ and $g(x)$ are differentiable functions in (a, b), $h(x)$ will also be differentiable in (a, b).
Further, h is continuous at a and b. So according to Lagrange's mean value theorem, there exists one $c \in (a, b)$ such that $h'(c)=\frac{h(b)-h(a)}{b-a}$, which proves the required result.

Example 5.51 Using mean value theorem, show that

$$\frac{\beta-\alpha}{1+\beta^2}<\tan^{-1}\beta-\tan^{-1}\alpha<\frac{\beta-\alpha}{1+\alpha^2}, \beta>\alpha>0.$$

Sol. Let $f(x)=\tan^{-1}x \Rightarrow f'(x)=\frac{1}{(1+x^2)}$

By mean value theorem for $f(x)$ in $[\alpha, \beta]$

$$\frac{f(\beta)-f(\alpha)}{\beta-\alpha}=f'(c)=\frac{1}{1+c^2}, \text{ where } a<c<b \quad (1)$$

$\because \alpha<c<\beta$

$$\Rightarrow \alpha^2<c^2<\beta^2 \text{ or } 1+\alpha^2<1+c^2<1+\beta^2$$

$$\Rightarrow \frac{1}{1+\alpha^2}>\frac{1}{1+c^2}>\frac{1}{1+\beta^2}$$

or $$\frac{1}{1+\beta^2}<f'(c)<\frac{1}{1+\alpha^2}$$

$$\Rightarrow \frac{1}{1+\beta^2}<\frac{f(\beta)-f(\alpha)}{\beta-\alpha}<\frac{1}{1+\alpha^2}$$

$$\Rightarrow \frac{(\beta-\alpha)}{1+\beta^2}<f(\beta)-f(\alpha)<\frac{(\beta-\alpha)}{(1+\alpha^2)}$$

$$\Rightarrow \frac{(\beta-\alpha)}{(1+\beta^2)}<\tan^{-1}\beta-\tan^{-1}\alpha<\frac{(\beta-\alpha)}{(1+\alpha^2)}$$

$(\because f(x)=\tan^{-1}x)$

Cauchy's Mean Value Theorem

Cauchy's mean value theorem, also known as the *extended mean value theorem*, is the more general form of the mean value theorem. It states that if both $f(t)$ and $g(t)$ are continuous functions on the closed interval $[a,b]$, differentiable on the open interval (a,b), and $g'(t)$ is not zero on that open interval, then there exists some c in (a,b), such that $\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$.

Proof:

The proof of Cauchy's mean value theorem is based on the same idea as the proof of the mean value theorem. First, we define a new function $h(t)$ and then we aim to transform this function so that it satisfies the conditions of Rolle's theorem.

Let $h(t) = f(t) - mg(t)$, where m is a constant. We choose m so that $h(a) = h(b) \Rightarrow m = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

Since h is continuous and $h(a) = h(b)$, by Rolle's theorem, there exists some c in (a, b) such that $h'(c) = 0$, i.e.,

$$h'(c) = 0 = f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) \text{ as required.}$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Example 5.52 Let $f(x)$ and $g(x)$ be two differentiable functions in R and $f(2) = 8$, $g(2) = 0$, $f(4) = 10$ and $g(4) = 8$, then prove that $g'(x) = 4f'(x)$ for at least one $x \in (2, 4)$.

Sol. Consider $h(x) = g(x) - 4f(x)$ in [2, 4]
also $h(2) = g(2) - 4f(2) = -32$, $h(4) = -32$
$\Rightarrow h'(x) = 0$ for at least one $x \in (2, 4)$ using Rolle's theorem.
Alternatively, using Cauchy's mean value theorem, there exists at least one $c \in (2, 4)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(4)-f(2)}{g(4)-g(2)} = \frac{10-8}{8-0} = \frac{1}{4}$$

$\Rightarrow 4f'(c) = g'(c)$

$\Rightarrow 4f'(x) = g'(x)$ (replacing c by x)

Example 5.53 Suppose α, β and θ are angles satisfying $0 < \alpha < \theta < \beta < \dfrac{\pi}{2}$, then prove that $\dfrac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = -\cot\theta$.

Sol. Let $f(x) = \sin x$ and $g(x) = \cos x$, then f and g are continuous and derivable.

Also, $\sin x \neq 0$ for any $x \in \left(0, \dfrac{\pi}{2}\right)$

So by Cauchy's mean value theorem, $\dfrac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} = \dfrac{f'(\theta)}{g'(\theta)}$

$$\Rightarrow \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta}$$

Concept Application Exercise 5.6

1. Find the condition if the equation $3x^2 + 4ax + b = 0$ has at least one root in (0, 1).
2. Find c of Lagrange's mean value theorem for the function $f(x) = 3x^2 + 5x + 7$ in the interval [1, 3].
3. Let $f(x)$ and $g(x)$ be differentiable for $0 \le x \le 2$ such that $f(0) = 2$, $g(0) = 1$ and $f(2) = 8$. Let there exist a real number c in [0, 2] such that $f'(c) = 3g'(c)$, then find the value of $g(2)$.
4. Prove that if $2a_0^2 < 15a$, all roots of $x^5 - a_0x^4 + 3ax^3 + bx^2 + cx + d = 0$ cannot be real. It is given that a_0, a, b, c, $d \in R$.
5. If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), then prove that there exists at least one $c \in (a, b)$ such that $\dfrac{f'(c)}{3c^2} = \dfrac{f(b)-f(a)}{b^3-a^3}$.
6. Using Lagrange's mean value theorem, prove that $\dfrac{b-a}{b} < \log\left(\dfrac{b}{a}\right) < \dfrac{b-a}{a}$, where $0 < a < b$.
7. Let $f(x)$ and $g(x)$ are two functions which are defined and differentiable for all $x \ge x_0$. If $f(x_0) = g(x_0)$ and $f'(x) > g'(x)$ for all $x > x_0$, then prove that $f(x) > g(x)$ for all $x > x_0$.
8. If $f(x)$ and $g(x)$ are continuous functions in $[a, b]$ and are differentiable in (a, b), then prove that there exists at least one $c(-(a-b))$ for which

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix} \text{ where } a < c < b.$$

EXERCISES

Subjective Type

Solutions on page 5.24

1. Prove that the equation of the normal to $x^{2/3} + y^{2/3} = a^{2/3}$ is $y\cos\theta - x\sin\theta = a\cos 2\theta$, where θ is the angle which the normal makes with the axis of x.
2. Show that the segment of the tangent to the curve $y = \dfrac{a}{2}\ln\left(\dfrac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}}\right) - \sqrt{a^2-x^2}$ contained between the y-axis and the point of tangency has a constant length.
3. If the tangent at (x_1, y_1) to the curve $x^3 + y^3 = a^3$ meets the curve again in (x_2, y_2), then prove that $\dfrac{x_2}{x_1} + \dfrac{y_2}{y_1} = -1$.
4. If the area of the triangle included between the axes and any tangent to the curve $x^n y = a^n$ is constant, then find the value of n.
5. Show the condition that the curves $ax^2 + by^2 = 1$ and $a'x^2 + b'y^2 = 1$ should intersect orthogonally is $\dfrac{1}{a} - \dfrac{1}{b} = \dfrac{1}{a'} - \dfrac{1}{b'}$.
6. Find the angle of intersection of curves, $y = [|\sin x| + |\cos x|]$ and $x^2 + y^2 = 5$, where [.] denotes the greatest integral function.
7. Tangents are drawn from the origin to curve $y = \sin x$. Prove that points of contact lie on $y^2 = \dfrac{x^2}{1+x^2}$.

8. Find the minimum value of

$$(x_1-x_2)^2+\left(\frac{x_1^2}{20}-\sqrt{(17-x_2)(x_2-13)}\right)^2,$$

where $x_1 \in R^+, x_2 \in (13, 17)$.

9. Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always 1/6th of the radius of the base. How fast does the height of the sand cone increase when the height is 4 cm?

10. Let $\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+\cdots+\frac{a_{n-1}}{2}+a_n=0$. Show that there exists at least real x between 0 and 1 such that $a_0x^n + a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n=0$.

11. Let a, b, c be non-zero real numbers such that

$$\int_0^1 (1+\cos^8 x)(ax^2+bx+c)dx$$

$$= \int_0^2 (1+\cos^8 x)(ax^2+bx+c)dx = 0,$$

then show that the equation $ax^2+bx+c=0$ will have one root between 0 and 1 and other root between 1 and 2.

12. If f is continuous and differentiable function and $f(0)=1$, $f(1) = 2$, then prove that there exists at least one $c \in [0, 1]$ for which $f'(c)\,(f(c))^{n-1} > \sqrt{2^{n-1}}$, where $n \in N$.

13. Let a, b, c be three real numbers such that $a<b<c, f(x)$ is continuous in $[a, c]$ and differentiable in (a, c). Also $f'(x)$ is strictly increasing in (a, c). Prove that $(b-c)f(a)+(c-a)f(b)+(a-b)f(c)<0$.

14. Prove that the portion of the tangent to the curve $\frac{x+\sqrt{a^2-y^2}}{a}=\log_e \frac{a+\sqrt{a^2-y^2}}{y}$ intercepted between the point of contact and the x-axis is constant.

15. Find the condition for the line $y=mx$ to cut at right angles the conic $ax^2+2hxy+by^2=1$.

16. Show that for the curve $by^2=(x+a)^3$, the square of the sub-tangent, varies as the sub-normal.

17. An aeroplane is flying horizontally at a height of $\frac{2}{3}$ km with a velocity of 15 km/h. Find the rate at which it is receding from a fixed point on the ground which it passed over 2 min ago.

18. Use the mean value theorem to prove $e^x \geq 1+x, \forall x \in R$,

Objective Type

Solutions on page 5.28

Each question has four choices a, b, c, and d, out of which *only one* is correct.

1. The number of tangents to the curve $x^{3/2}+y^{3/2}=2a^{3/2}$, $a>0$, which are equally inclined to the axes, is
   a. 2 b. 1
   c. 0 d. 4

2. The angle made by the tangent of the curve $x = a(t+\sin t \cos t)$; $y = a(1+\sin t)^2$ with the x-axis at any point on it is
   a. $\frac{1}{4}(\pi+2t)$ b. $\frac{1-\sin t}{\cos t}$
   c. $\frac{1}{4}(2t-\pi)$ d. $\frac{1+\sin t}{\cos 2t}$

3. If m is the slope of a tangent to the curve $e^y=1+x^2$, then
   a. $|m|>1$ b. $m>1$ c. $m>-1$ d. $|m|\leq 1$

4. If at each point of the curve $y=x^3-ax^2+x+1$, the tangent is inclined at an acute angle with the positive direction of the x-axis, then
   a. $a>0$ b. $a\leq\sqrt{3}$
   c. $-\sqrt{3}\leq a\leq\sqrt{3}$ d. None of these

5. The slope of the tangent to the curve $y=\sqrt{4-x^2}$ at the point, where the ordinate and the abscissa are equal, is
   a. -1 b. 1
   c. 0 d. None of these

6. The curve given by $x+y=e^{xy}$ has a tangent parallel to the y-axis at the point
   a. $(0,1)$ b. $(1,0)$
   c. $(1,1)$ d. None of these

7. If the line joining the points (0, 3) and (5, −2) is a tangent to the curve $y=\frac{c}{x+1}$, then the value of c is
   a. 1 b. -2
   c. 4 d. None of these

8. Let $f(x)=\begin{cases}-x^2, & \text{for } x<0\\ x^2+8, & \text{for } x\geq 0\end{cases}$. Then x-intercept of the line, that is, the tangent to the graph of $f(x)$ is
   a. zero b. -1 c. -2 d. -4

9. The distance between the origin and the tangent to the curve $y=e^{2x}+x^2$ drawn at the point $x=0$ is
   a. $\frac{1}{\sqrt{5}}$ b. $\frac{2}{\sqrt{5}}$
   c. $\frac{-1}{\sqrt{5}}$ d. $\frac{2}{\sqrt{3}}$

10. The point on the curve $3y=6x-5x^3$, the normal at which passes through the origin, is
    a. $(1, 1/3)$ b. $(1/3, 1)$
    c. $(2, -28/3)$ d. None of these

11. The normal to the curve $2x^2+y^2=12$ at the point (2, 2) cuts the curve again at
    a. $\left(-\frac{22}{9},-\frac{2}{9}\right)$ b. $\left(\frac{22}{9},\frac{2}{9}\right)$
    c. $(-2,-2)$ d. None of these

12. At what points of curve $y=\frac{2}{3}x^3+\frac{1}{2}x^2$, the tangent makes the equal angle with the axis?
    a. $\left(\frac{1}{2},\frac{5}{24}\right)$ and $\left(-1,-\frac{1}{6}\right)$ b. $\left(\frac{1}{2},\frac{4}{9}\right)$ and $(-1, 0)$
    c. $\left(\frac{1}{3},\frac{1}{7}\right)$ and $\left(-3,\frac{1}{2}\right)$ d. $\left(\frac{1}{3},\frac{4}{47}\right)$ and $\left(-1,-\frac{1}{3}\right)$

13. The equation of the tangent to the curve $y = be^{-x/a}$ at the point where it crosses the y-axis is

a. $\frac{x}{a} - \frac{y}{b} = 1$ b. $ax + by = 1$

c. $ax - by = 1$ d. $\frac{x}{a} + \frac{y}{b} = 1$

14. The angle of intersection of the normals at the point $\left(-\frac{5}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ of the curves $x^2 - y^2 = 8$ and $9x^2 + 25y^2 = 225$ is

a. 0 b. $\frac{\pi}{2}$ c. $\frac{\pi}{3}$ d. $\frac{\pi}{4}$

15. A function $y = f(x)$ has a second-order derivative $f''(x) = 6(x - 1)$. If its graph passes through the point (2, 1) and at that point tangent to the graph is $y = 3x - 5$, then the value of $f(0)$ is

a. 1 b. –1

c. 2 d. 0

16. If $x + 4y = 14$ is a normal to the curve $y^2 = \alpha x^3 - \beta$ at (2, 3), then the value of $\alpha + \beta$ is

a. 9 b. –5 c. 7 d. –7

17. The curve represented parametrically by the equations $x = 2 \ln \cot t + 1$ and $y = \tan t + \cot t$

a. tangent and normal intersect at the point (2, 1)

b. normal at $t = \pi/4$ is parallel to the y-axis

c. tangent at $t = \pi/4$ is parallel to the line $y = x$

d. tangent at $t = \pi/4$ is parallel to the x-axis

18. The abscissa of points P and Q on the curve $y = e^x + e^{-x}$ such that tangents at P and Q make 60° with the x-axis

a. $\ln\left(\frac{\sqrt{3}+\sqrt{7}}{7}\right)$ and $\ln\left(\frac{\sqrt{3}+\sqrt{5}}{2}\right)$

b. $\ln\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)$

c. $\ln\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)$ d. $\pm\ln\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)$

19. If a variable tangent to the curve $x^2y = c^3$ makes intercepts a, b on x- and y-axes, respectively, then the value of a^2b is

a. $27c^3$ b. $\frac{4}{27}c^3$ c. $\frac{27}{4}c^3$ d. $\frac{4}{9}c^3$

20. Let C be the curve $y = x^3$ (where x takes all real values). The tangent at A meets the curve again at B. If the gradient at B is K times the gradient at A, then K is equal to

a. 4 b. 2 c. –2 d. $\frac{1}{4}$

21. A curve is represented by the equations $x = \sec^2 t$ and $y = \cot t$, where t is a parameter. If the tangent at the point P on the curve, where $t = \pi/4$, meets the curve again at the point Q, then $|PQ|$ is equal to

a. $\frac{5\sqrt{3}}{2}$ b. $\frac{5\sqrt{5}}{2}$ c. $\frac{2\sqrt{5}}{3}$ d. $\frac{3\sqrt{5}}{2}$

22. The x-intercept of the tangent at any arbitrary point of the curve $\frac{a}{x^2} + \frac{b}{y^2} = 1$ is proportional to

a. square of the abscissa of the point of tangency

b. square root of the abscissa of the point of tangency

c. cube of the abscissa of the point of tangency

d. cube root of the abscissa of the point of tangency

23. At any point on the curve $2x^2y^2 - x^4 = c$, the mean proportional between the abscissa and the difference between the abscissa and the subnormal drawn to the curve at the same point is equal to

a. ordinate b. radius vector

c. x-intercept of tangent d. sub-tangent

24. If the length of sub-normal is equal to the length of sub-tangent at any point (3, 4) on the curve $y = f(x)$ and the tangent at (3, 4) to $y = f(x)$ meets the coordinate axes at A and B, then the maximum area of the triangle OAB, where O is origin, is

a. 45/2 b. 49/2 c. 25/2 d. 81/2

25. The number of points in the rectangle $\{(x, y) | -12 \le x \le 12$ and $-3 \le y \le 3\}$ which lie on the curve $y = x + \sin x$ and at which the tangent to the curve is parallel to the x-axis is

a. 0 b. 2 c. 4 d. 8

26. Tangent of acute angle between the curves $y = |x^2 - 1|$ and $y = \sqrt{7 - x^2}$ at their points of intersection is

a. $\frac{5\sqrt{3}}{2}$ b. $\frac{3\sqrt{5}}{2}$ c. $\frac{5\sqrt{3}}{4}$ d. $\frac{3\sqrt{5}}{4}$

27. The lines tangent to the curves $y^3 - x^2y + 5y - 2x = 0$ and $x^4 - x^3y^2 + 5x + 2y = 0$ at the origin intersect at an angle θ equal to

a. $\frac{\pi}{6}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{3}$ d. $\frac{\pi}{2}$

28. The two curves $x = y^2$, $xy = a^3$ cut orthogonally at a point, then a^2 is equal to

a. $\frac{1}{3}$ b. 3 c. 2 d. $\frac{1}{2}$

29. The curves $4x^2 + 9y^2 = 72$ and $x^2 - y^2 = 5$ at (3, 2)

a. touch each other b. cut orthogonally

c. intersect at 45° d. intersect at 60°

30. Let $f(1) = -2$ and $f'(x) \ge 4.2$ for $1 \le x \le 6$. The smallest possible value of $f(6)$ is

a. 9 b. 12 c. 15 d. 19

31. If $f(x) = x^3 + 7x - 1$, then $f(x)$ has a zero between $x = 0$ and $x = 1$. The theorem that best describes this is

a. Mean value theorem

b. Maximum-minimum value theorem

c. Intermediate value theorem

d. None of these

32. Consider the function $f(x) = \begin{cases} x\sin\frac{\pi}{x}, & \text{for } x > 0 \\ 0, & \text{for } x = 0 \end{cases}$ then the number of points in (0, 1) where the derivative $f'(x)$ vanishes is

a. 0 b. 1 c. 2 d. infinite

33. Let $f(x)$ and $g(x)$ be differentiable for $0 \le x \le 1$, such that $f(0)=0, g(0)=0, f(1)=6$. Let there exists a real number c in $(0, 1)$ such that $f'(c)=2g'(c)$, then the value of $g(1)$ must be
a. 1
b. 3
c. -2
d. -1

34. If $3(a+2c)=4(b+3d)$, then the equation $ax^3+bx^2+cx+d=0$ will have
a. no real solution
b. at least one real root in $(-1, 0)$
c. at least one real root in $(0, 1)$
d. none of these

35. If the function $f(x) = ax^3 + bx^2 + 11x - 6$ satisfies conditions of Rolle's theorem in $[1, 3]$ for $x = 2 + \frac{1}{\sqrt{3}}$, then value of a and b, respectively, are
a. $-3, 2$
b. $2, -4$
c. $1, 6$
d. None of these

36. A value of C for which the conclusion of mean value theorem holds for the function $f(x)=\log_e x$ on the interval $[1, 3]$ is
a. $\frac{1}{2}\log_e 3$
b. $\log_3 e$
c. $\log_e 3$
d. $2\log_3 e$

37. Let $f(x)$ be a twice differentiable function for all real values of x and satisfies $f(1)=1, f(2)=4, f(3)=9$. Then which of the following is definitely true?
a. $f''(x)=2$, for $\forall x \in (1, 3)$
b. $f''(x)=f'(x)=5$ for some $x \in (2, 3)$
c. $f''(x)=3\ \forall x \in (2, 3)$
d. $f''(x)=2$ for some $x \in (1, 3)$

38. The value of c in Lagrange's theorem for the function $f(x)=\log \sin x$ in the interval $[\pi/6, 5\pi/6]$
a. $\pi/4$
b. $\pi/2$
c. $2\pi/3$
d. None of these

39. In which of the following functions is Rolle's theorem applicable ?
a. $f(x)=\begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$ on $[0, 1]$
b. $f(x)=\begin{cases} \frac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$ on $[-\pi, 0]$
c. $f(x)=\frac{x^2-x-6}{x-1}$ on $[-2, 3]$
d. $f(x)=\begin{cases} \frac{x^3-2x^2-5x+6}{x-1}, & \text{if } x \ne 1, \text{ on } [-2, 3] \\ -6, & \text{if } x = 1 \end{cases}$

40. A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa is
a. $(2, 6)$
b. $(2, -6)$
c. $\left(\frac{9}{8}, -\frac{9}{2}\right)$
d. $\left(\frac{9}{8}, \frac{9}{2}\right)$

41. The rate of change of the volume of a sphere w.r.t. its surface area, when the radius is 2 cm, is
a. 1
b. 2
c. 3
d. 4

42. If there is an error of $k\%$ in measuring the edge of a cube, then the percent error in estimating its volume is
a. k
b. $3k$
c. $\frac{k}{3}$
d. None of these

43. A lamp of negligible height is placed on the ground ℓ_1 away from a wall. A man ℓ_2 m tall is walking at a speed of $\frac{\ell_1}{10}$ m/s from the lamp to the nearest point on the wall. When he is midway between the lamp and the wall, the rate of change in the length of this shadow on the wall is
a. $-\frac{5\ell_2}{2}$ m/s
b. $-\frac{2\ell_2}{5}$ m/s
c. $-\frac{\ell_2}{2}$ m/s
d. $-\frac{\ell_2}{5}$ m/s

44. A man is moving away from a tower 41.6 m high at a rate of 2 m/s. If the eye level of the man is 1.6 m above the ground, then the rate at which the angle of elevation of the top of the tower changes, when he is at a distance of 30 m from the foot of the tower, is
a. $-\frac{4}{125}$ radian/s
b. $-\frac{2}{25}$ radian/s
c. $-\frac{1}{625}$ radian/s
d. None of these

45. The co-ordinates of a point on the parabola $y^2=8x$ whose distance from the circle $x^2+(y+6)^2=1$ is minimum is
a. $(2, 4)$
b. $(2, -4)$
c. $(18, -12)$
d. $(8, 8)$

46. At the point $P(a, a^n)$ on the graph of $y=x^n$ $(n \in N)$ in the first quadrant, a normal is drawn. The normal intersects the y-axis at the point $(0, b)$. If $\lim_{a \to 0} b = \frac{1}{2}$, then n equals
a. 1
b. 3
c. 2
d. 4

47. Suppose that f is differentiable for all x and that $f'(x) \le 2$ for all x. If $f(1)=2$ and $f(4)=8$, then $f(2)$ has the value equal to
a. 3
b. 4
c. 6
d. 8

48. The radius of a right circular cylinder increases at the rate of 0.1 cm/min, and the height decreases at the rate of 0.2 cm/min. The rate of change of the volume of the cylinder, in cm^3/min, when the radius is 2 cm and the height is 3 cm is
a. $-2p$
b. $-\frac{8\pi}{5}$
c. $-\frac{3\pi}{5}$
d. $\frac{2\pi}{5}$

49. A cube of ice melts without changing its shape at the uniform rate of 4 cm^3/min. The rate of change of the surface area of the cube, in cm^2/min, when the volume of the cube is 125 cm^3, is
a. -4
b. $-16/5$
c. $-16/6$
d. $-8/15$

50. The tangent to the curve $y = e^{kx}$ at a point (0, 1) meets the x-axis at $(a, 0)$ where $a \in [-2, -1]$, then $k \in$
a. $[-1/2, 0]$
b. $[-1, -1/2]$
c. $[0, 1]$
d. $[1/2, 1]$

51. Let $f'(x) = e^{x^2}$ and $f(0) = 10$. If $A < f(1) < B$ can be concluded from the mean value theorem, then the largest value of $(A - B)$ equals
a. e
b. $1 - e$
c. $e - 1$
d. $1 + e$

52. If f be a continuous function on [0, 1], differentiable in (0, 1) such that $f(1) = 0$, then there exists some $c \in (0, 1)$ such that
a. $cf'(c) - f(c) = 0$
b. $f'(c) + cf(c) = 0$
c. $f'(c) - cf(c) = 0$
d. $cf'(c) + f(c) = 0$

53. Given $g(x) = \dfrac{x+2}{x-1}$ and the line $3x + y - 10 = 0$, then the line is
a. tangent to $g(x)$
b. normal to $g(x)$
c. chord of $g(x)$
d. none of these

54. Let f be a continuous, differentiable and bijective function. If the tangent to $y = f(x)$ at $x = a$ is also the normal to $y = f(x)$ at $x = b$, then there exists at least one $c \in (a, b)$ such that
a. $f'(c) = 0$
b. $f'(c) > 0$
c. $f'(c) < 0$
d. None of these

55. If $f(x)$ and $g(x)$ are differentiable functions for $0 \le x \le 1$ such that $f(0) = 10$, $g(0) = 2$, $f(1) = 2$, $g(1) = 4$, then in the interval (0, 1)
a. $f'(x) = 0$ for all x
b. $f'(x) + 4g'(x) = 0$ for at least one x
c. $f'(x) = 2g'(x)$ for at most one x
d. none of these

Multiple Correct Answers Type

Solutions on page 5.35

Each question has four choices a, b, c, and d, out of which *one* or *more* answers are correct.

1. Points on the curve $f(x) = \dfrac{x}{1-x^2}$ where the tangent is inclined at an angle of $\dfrac{\pi}{4}$ to the x-axis are
a. (0, 0)
b. $\left(\sqrt{3}, -\dfrac{\sqrt{3}}{2}\right)$
c. $\left(-2, \dfrac{2}{3}\right)$
d. $\left(-\sqrt{3}, \dfrac{\sqrt{3}}{2}\right)$

2. In the curve $y = ce^{x/a}$, the
a. sub-tangent is constant
b. sub-normal varies as the square of the ordinate
c. tangent at (x_1, y_1) on the curve intersects the x-axis at a distance of $(x_1 - a)$ from the origin
d. equation of the normal at the point where the curve cuts y-axis is $cy + ax = c^2$

3. Let $f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x$, where a_i's are real and $f(x) = 0$ has a positive root α_0. Then
a. $f'(x) = 0$ has a root α_1 such that $0 < \alpha_1 < \alpha_0$
b. $f'(x) = 0$ has at least one real root
c. $f''(x) = 0$ has at least one real root
d. None of these

4. Let the parabolas $y = x(c - x)$ and $y = x^2 + ax + b$ touch each other at the point (1, 0), then
a. $a + b + c = 0$
b. $a + b = 2$
c. $b - c = 1$
d. $a + c = -2$

5. Which of the following pair(s) of curves is/are orthogonal?
a. $y^2 = 4ax$; $y = e^{-x/2a}$
b. $y^2 = 4ax$; $x^2 = 4ay$ at (0, 0)
c. $xy = a^2$; $x^2 - y^2 = b^2$
d. $y = ax$; $x^2 + y^2 = c^2$

6. The co-ordinates of the point(s) on the graph of the function $f(x) = \dfrac{x^3}{3} - \dfrac{5x^2}{2} + 7x - 4$, where the tangent drawn cuts off intercepts from the co-ordinate axes which are equal in magnitude but opposite in sign, is
a. (2, 8/3)
b. (3, 7/2)
c. (1, 5/6)
d. None of these

7. The abscissa of a point on the curve $xy = (a + x)^2$, the normal which cuts off numerically equal intercepts from the coordinate axes, is
a. $-\dfrac{a}{\sqrt{2}}$
b. $\sqrt{2}a$
c. $\dfrac{a}{\sqrt{2}}$
d. $-\sqrt{2}a$

8. The angle formed by the positive y-axis and the tangent to $y = x^2 + 4x - 17$ at $(5/2, -3/4)$ is
a. $\tan^{-1}(9)$
b. $\dfrac{\pi}{2} - \tan^{-1}(9)$
c. $\dfrac{\pi}{2} + \tan^{-1}(9)$
d. None of these

9. If the tangent at any point $P(4m^2, 8m^3)$ of $x^3 - y^2 = 0$ is also a normal to the curve $x^3 - y^2 = 0$, then the value of m is
a. $m = \dfrac{\sqrt{2}}{3}$
b. $m = -\dfrac{\sqrt{2}}{3}$
c. $m = \dfrac{3}{\sqrt{2}}$
d. $m = -\dfrac{3}{\sqrt{2}}$

10. The angle between the tangents to the curves $y = x^2$ and $x = y^2$ at (1, 1) is
a. $\cos^{-1}\dfrac{4}{5}$
b. $\sin^{-1}\dfrac{3}{5}$
c. $\tan^{-1}\dfrac{3}{4}$
d. $\tan^{-1}\dfrac{1}{3}$

11. The angle between the tangents at any point P and the line joining P to the origin, where P is a point on the curve $\ln(x^2+y^2) = c\tan^{-1}\frac{y}{x}$, c is a constant, is

a. independent of x
b. independent of y
c. independent of x but dependent on y
d. independent of y but dependent on x

12. Given $f(x) = 4 - \left(\frac{1}{2} - x\right)^{2/3}$, $g(x) = \begin{cases} \frac{\tan[x]}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$,

$h(x) = \{x\}$, $k(x) = 5^{\log_2(x+3)}$
then in $[0, 1]$ Lagrange's mean value theorem is NOT applicable to the (where $[\cdot]$ and $\{\cdot\}$ represents greatest integer functions and fractional part functions, respectively)

a. f b. g c. k d. h

13. Which of the following is/are correct?

a. Between any two roots of $e^x\cos x = 1$, there exists at least one root of $\tan x = 1$.
b. Between any two roots of $e^x\sin x = 1$, there exists at least one root of $\tan x = -1$.
c. Between any two roots of $e^x\cos x = 1$, there exists at least one root of $e^x\sin x = 1$
d. Between any two roots of $e^x\sin x = 1$, there exists at least one root of $e^x\cos x = 1$.

14. Which of the following pair(s) of curves is/are orthogonal?

a. $y^2 = 4ax; y = e^{-x/2a}$
b. $y^2 = 4ax; x^2 = 4ay$ at $(0, 0)$
c. $xy = a^2; x^2 - y^2 = b^2$
d. $y = ax; x^2 + y^2 = c^2$

Reasoning Type

Solutions on page 5.37

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. if both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1
b. if both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1
c. if STATEMENT 1 is TRUE and STATEMENT 2 is FALSE
d. if STATEMENT 1 is FALSE and STATEMENT 2 is TRUE

1. **Statement 1:** Lagrange's mean value theorem is not applicable to $f(x) = |x-1|(x-1)$.
Statement 2: $|x-1|$ is not differentiable at $x = 1$.

2. **Statement 1:** If $27a + 9b + 3c + d = 0$, then the equation $f(x) = 4ax^3 + 3bx^2 + 2cx + d = 0$ has at least one real root lying between $(0, 3)$.
Statement 2: If $f(x)$ is continuous in $[a, b]$, derivable in (a, b) such that $f(a) = f(b)$, then at least one point $c \in (a, b)$ such that $f'(c) = 0$.

3. **Statement 1:** If both functions $f(t)$ and $g(t)$ are continuous on the closed interval $[a,b]$, differentiable on the open interval (a,b), and $g'(t)$ is not zero on that open interval, then there exists some c in (a,b), such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Statement 2: If $f(t)$ and $g(t)$ are continuous and differentiable in $[a, b]$, then there exists some c in $(a. b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ and $g'(c) = \frac{g(b)-g(a)}{b-a}$ from Lagrange's mean value theorem.

4. **Statement 1:** The maximum value of

$(\sqrt{-3+4x-x^2}+4)^2 + (x-5)^2$ (where $1 \le x \le 3$) is 36.

Statement 2: The maximum distance between the point $(5, -4)$ and the point on the circle $(x-2)^2 + y^2 = 1$ is 6.

5. **Statement 1:** If $g(x)$ is a differentiate function $g(2) \neq 0$, $g(-2) \neq 0$ and Rolle's theorem is not applicable to $f(x) = \frac{x^2-4}{g(x)}$ in $[-2, 2]$, then $g(x)$ has at least one root in $(-2, 2)$.
Statement 2: If $f(a) = f(b)$, then Rolle's theorem is applicable for $x \in (a, b)$.

6. **Statement 1:** The tangent at $x = 1$ to the curve $y = x^3 - x^2 - x + 2$ again meets the curve at $x = 0$.
Statement 2: When the equation of a tangent solved with the given curve, repeated roots are obtained at point of tangency.

7. Consider a curve $C: y = \cos^{-1}(2x-1)$ and a straight line $L: 2px - 4y + 2\pi - p = 0$
Statement 1: The set of values of 'p' for which the line L intersects the curve at three distinct points is $[-2\pi, -4)$
Statement 2: The line L is always passing through point of inflection of the curve C.

8. **Statement 1:** If $f(x)$ is differentiable in $[0, 1]$ such that $f(0) = f(1) = 0$, then for any $\lambda \in R$, there exists c such that $f'(c) = \lambda f(c)$, $0 < c < 1$.
Statement 2: If $g(x)$ is differentiable in $[0, 1]$, where $g(0) = g(1)$, then there exists c such that $g'(c) = 0$, $0 < c < 1$.

9. **Statement 1:** For the function $f(x) = x^2 + 3x + 2$, LMVT is applicable in $[1, 2]$ and the value of c is $3/2$ because
Statement 2: If LMVT is known to be applicable for any quadratic polynomial in $[a, b]$ then c of LMVT is $(a+b)/2$

10. Let $y = f(x)$ is a polynomial of degree odd (≥ 3) with real coefficients and (a, b) is any point
Statement 1: There always exists a line passing through (a, b) and touching the curve $y = f(x)$ at some point
Statement 2: A polynomial of degree odd with real coefficients have at least one real root

Linked Comprehension Type

Solutions on page 5.38

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1 – 3

Tangent at a point P_1 [other than $(0, 0)$] on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve again at P_3 and so on.

1. If P_1 has co-ordinates (1, 1), then the sum $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\frac{1}{x_n}$ is (where $x_1, x_2, \ldots$, are abscissas of $P_1, P_2, \ldots$, respectively)
   a. -2/3 b. 1/3 c. 1/2. d. 3/2

2. If P_1 has co-ordinates (1, 1), then the sum $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\frac{1}{y_n}$ is (where $y_1, y_2, \ldots$, are ordinates of $P_1, P_2, \ldots$, respectively)
   a. 1/8 b. 1/9 c. 8/9 d. 9/8

3. The ratio of area of $\Delta P_1P_2P_3$ to that of $\Delta P_2P_3P_4$ is
   a. 1/4 b. 1/2 c. 1/8 d. 1/16

For Problems 4 – 6

Consider the curve $x = 1 - 3t^2, y = t - 3t^3$. If a tangent at point $(1 - 3t^2, t - 3t^3)$ inclined at an angle θ to the positive x-axis and another tangent at point $P(-2, 2)$ cuts the curve again at Q.

4. The value of $\tan\theta + \sec\theta$ is equal to
   a. 3t b. t c. $t - t^2$ d. $t^2 - 2t$

5. The point Q will be
   a. $(1, -2)$ b. $\left(-\frac{1}{3}, -\frac{2}{3}\right)$
   c. $(-2, 1)$ d. None of these

6. The angle between the tangents at P and Q will be
   a. $\frac{\pi}{4}$ b. $\frac{\pi}{6}$
   c. $\frac{\pi}{2}$ d. $\frac{\pi}{3}$

For Problems 7 – 8

A spherical balloon is being inflated so that its volume increases uniformly at the rate of 40 cm^3/min.

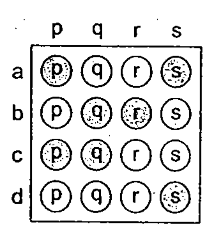
7. At $r = 8$, its surface area increases at the rate
   a. 8 cm^2/min b. 10 cm^2/min
   c. 20 cm^2/min d. None of these

8. When $r = 8$, then the increase in radius in the next 1/2 min is
   a. 0.025 cm b. 0.050 cm
   c. 0.075 cm d. 0.01 cm

Matrix-Match Type

Solutions on page 5.39

Each question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct match are a – p, a – s, b – r, c – p, c – q and d – s, then the correctly bubbled 4 × 4 matrix should be as follows:



1.

Column I	Column II
a. The sides of a triangle vary slightly in such a way that its circum-radius remains constant, if $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} + 1 = \lvert m\rvert$, then the values of m is	p. 1
b. The length of sub-tangent to the curve $x^2y^2 = 16$ at the point (–2, 2) is $\lvert k\rvert$, then the value of k is	q. –1
c. The curve $y = 2e^{2x}$ intersects the y-axis at an angle $\cot^{-1}\lvert(8n-4)/3\rvert$, then the value of n is	r. 2
d. The area of a triangle formed by normal at the point (1, 0) on the curve $x = e^{\sin y}$ with axes is $\lvert 2t+1\rvert/6$ sq. units, then the value of t is	s. –2

2.

Column I	Column II
a. A circular plate is expanded by heat from radius 6 cm to 6.06 cm. Approximate increase in the area is	p. 5
b. If an edge of a cube increases by 2%, then the percentage increase in the volume is	q. 0.72π
c. If the rate of decrease of $\frac{x^2}{2} - 2x + 5$ is thrice the rate of decrease of x, then x is equal to (rate of decrease is non-zero)	r. 6
d. The rate of increase in the area of an equailateral triangle of side 30 cm, when each side increases at the rate of 0.1 cm/s is	s. $\frac{3\sqrt{3}}{2}$

3.

Column I: Curves	Column II: Angle between the curves
a. $y^2 = 4x$ and $x^2 = 4y$	p. 90°
b. $2y^2 = x^3$ and $y^2 = 32x$	q. any one of $\tan^{-1}\frac{3}{4}$ or $\tan^{-1}(16^{\frac{1}{3}})$
c. $xy = a^2$ and $x^2 + y^2 = 2a^2$	r. 0°
d. $y^2 = x$ and $x^3 + y^3 = 3xy$ at other than origin	s. $\tan^{-1}\frac{1}{2}$

Integer Type

Solutions on page 5.40

1. There is a point (p, q) on the graph of $f(x) = x^2$ and a point (r, s) on the graph of $g(x) = \dfrac{-8}{x}$, where $p > 0$ and $r > 0$. If the line through (p, q) and (r, s) is also tangent to both the curves at these points, respectively, then the value of $p + r$ is
2. A curve is defined parametrically by the equations $x = t^2$ and $y = t^3$. A variable pair of perpendicular lines through the origin 'O' meet the curve at P and Q. If the locus of the point of intersection of the tangents at P and Q is $ay^2 = bx - 1$, then the value of $(a + b)$ is
3. If d is the minimum distance between the curves $f(x) = e^x$ and $g(x) = \log_e x$, then the value of d^6 is
4. Let $f(x)$ be a non-constant thrice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(6-x)$ and $f'(0) = 0 = f'(2) = f'(5)$. If n is the minimum number or roots of $(f''(x)^2) + f'(x)f'''(x) = 0$ in the interval $[0, 6]$, then the value of $n/2$ is
5. At the point $P(a, a^n)$ on the graph of $y = x^n$ $(n \in N)$ in the first quadrant a normal is drawn. The normal intersects the y-axis at the point $(0, b)$. If $\lim\limits_{a \to 0} b = \dfrac{1}{2}$, then n equals
6. A curve is given by the equations $x = \sec^2 \theta$, $y = \cot \theta$. If the tangent at P where $\theta = \pi/4$ meets the curve again at Q, then $[PQ]$ is, where $[\cdot]$ represents the greatest integer function,
7. Water is dropped at the rate of $2 m^3/s$ into a cone of semi-vertical angle $45°$. If the rate at which periphery of water surface changes when the height of the water in the cone is 2 m is d, then the value of $5d$ is
8. If the slope of line through the origin which is tangent to the curve $y = x^3 + x + 16$ is m, then the value of $m - 4$ is
9. Let $y = f(x)$ be drawn with $f(0) = 2$ and for each real number the tangent to $y = f(x)$ at $(a, f(a))$, has x intercept $(a - 2)$. If $f(x)$ is of the form of $k e^{px}$, then $\left(\dfrac{k}{p}\right)$ has the value equal to
10. Suppose a, b, c are such that the curve $y = ax^2 + bx + c$ is tangent to $y = 3x - 3$ at $(1, 0)$ and is also tangent to $y = x + 1$ at $(3, 4)$, then the value of $(2a - b - 4c)$ equals
11. Let C be a curve defined by $y = e^{a+bx^2}$. The curve C passes through the point $P(1, 1)$ and the slope of the tangent at P is (-2). Then the value of $2a - 3b$ is
12. If the curve C in the xy plane has the equation $x^2 + xy + y^2 = 1$, then the fourth power of the greatest distance of a point on C from the origin, is

Archives

Solutions on page 5.42

Subjective

1. For all $x \in [0, 1]$, let the second derivative $f''(x)$ of a function $f(x)$ exist and satisfy $|f''(x)| < 1$. If $f(0) = f(1)$, then show that $|f'(x)| < 1$ for all x in $[0, 1]$. **(IIT-JEE, 1981)**
2. If $f(x)$ and $g(x)$ are differentiable functions for $0 \le x \le 1$ such that $f(0) = 2, g(0) = 0, f(1) = 6; g(1) = 2$, then show that there exists c satisfying $0 < c < 1$ and $f'(c) = 2g'(c)$. **(IIT-JEE, 1982)**
3. Find the shortest distance of the point $(0, c)$ from the parabola $y = x^2$, where $0 \le c \le 5$. **(IIT-JEE, 1982)**
4. Find all the tangents to the curve $y = \cos(x + y)$, where $-2\pi \le x \le 2\pi$, that are parallel to the line $x + 2y = 0$. **(IIT-JEE, 1985)**
5. Find the equation of the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at $x = 0$. **(IIT-JEE, 1993)**
6. The curve $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at $P(-2, 0)$ and cuts the y-axis at a point Q where its gradient is 3. Find a, b, c. **(IIT-JEE, 1994)**
7. If the function $f: [0, 4] \to R$ is differentiable, then show that for $a, b \in (0, 4)$, $(f(4))^2 - (f(0))^2 = 8f'(a)f(b)$ and $\int_0^4 f(t)\,dt = 2[\alpha f(\alpha^2) + \beta f(\beta^2)]\ \forall\ 0 < \alpha, \beta < 2$. **(IIT-JEE, 2003)**
8. Using the Rolle's theorem, prove that there is at least one root in $(45^{1/100}, 46)$ of the equation $P(x) = 51x^{101} - 2323(x)^{100} - 45x + 1035 = 0$. **(IIT-JEE, 2004)**
9. If $|f(x_1) - f(x_2)| < (x_1 - x_2)^2$, for all $x_1, x_2 \in R$. Find the equation of tangent to the curve $y = f(x)$ at the point $(1, 2)$. **(IIT-JEE, 2005)**
10. For a twice differentiable function $f(x)$, $g(x)$ is defined as $g(x) = f'(x)^2 + f''(x)f(x)$ on $[a, e]$. If for $a < b < c < d < e$, $f(a) = 0, f(b) = 2, f(c) = -1, f(d) = 2, f(e) = 0$, then find the minimum number of zero of $g(x)$. **(IIT-JEE, 2006)**

Objectives

Fill in the blanks

1. Let C be the curve $y^3 - 3xy + 2 = 0$. If H is the set of points on the curve C where the tangent is horizontal and V is the set of points on the curve C where the tangent is vertical, then $H =$ ________ and $V =$ ________ **(IIT-JEE, 1994)**

Multiple choice questions with one correct answer

1. If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has
   a. at least one root in $[0, 1]$
   b. one root in $[2, 3]$ and the other in $[-2, -1]$
   c. imaginary roots
   d. none of these **(IIT-JEE, 1983)**
2. The normal to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any point θ is such that
   a. it makes a constant angle with the x-axis
   b. it passes through the origin
   c. it is at a constant distance from the origin
   d. none of these **(IIT-JEE, 1983)**
3. The slope of the tangent to the curve $y = f(x)$ at $[x, f(x)]$ is $2x + 1$. If the curve passes through the point $(1, 2)$, then the area bounded by the curve, the x-axis and the line $x = 1$ is
   a. $\dfrac{5}{6}$ b. $\dfrac{6}{5}$ c. $\dfrac{1}{6}$ d. 6 **(IIT-JEE, 1995)**

4. If the normal to the curve $y=f(x)$ at the point (3, 4) makes an angle $\frac{3\pi}{4}$ with the positive x-axis, then $f'(3)$ is equal to

a. -1 b. $-\frac{3}{4}$ c. $\frac{4}{5}$ d. 1

(IIT-JEE, 2000)

5. The triangle formed by the tangent to the curve $f(x)=x^2+bx-b$ at the point (1, 1) and the co-ordinate axes lies in the first quadrant. If its area is 2, then the value of b is

a. -1 b. 3
c. -3 d. 1 **(IIT-JEE, 2001)**

6. The point(s) on the curve $y^3+3x^2=12y$, where the tangent is vertical, is (are)

a. $\left(\pm\frac{4}{\sqrt{3}},-2\right)$ b. $\left(\pm\sqrt{\frac{11}{3}},1\right)$

c. $(0,0)$ d. $\left(\pm\frac{4}{\sqrt{3}},2\right)$

(IIT-JEE, 2002)

7. In [0, 1], Lagrange's mean value theorem is NOT applicable to

a. $f(x)=\begin{cases}\frac{1}{2}-x, & x<\frac{1}{2}\\ \left(\frac{1}{2}-x\right)^2, & x\ge\frac{1}{2}\end{cases}$

b. $f(x)=\begin{cases}\frac{\sin x}{x}, & x\ne 0\\ 1, & x=0\end{cases}$

c. $f(x)=x|x|$

d. $f(x)=|x|$ **(IIT-JEE, 2003)**

8. If $f(x)=x^{\alpha}\log x$ and $f(0)=0$, then the value of α for which Rolle's theorem can be applied in [0, 1]

a. -2 b. -1
c. 0 d. 1/2 **(IIT-JEE, 2004)**

9. If $P(x)$ is a polynomial of degree less than or equal to 2 and S is the set of all such polynomials so that $P(0)=0$, $P(1)=1$ and $P'(x)>0\ \forall\in[0,1]$, then

a. $S=\phi$
b. $S=ax+(1-a)x^2\ \forall\ a\in(0,2)$
c. $S=ax+(1-a)x^2\ \forall\ a\in(0,\infty)$
d. $S=ax+(1-a)x^2\ \forall\ a\in(0,1)$ **(IIT-JEE, 2005)**

10. The tangent to the curve $y=e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c-1, e^{c-1})$ and $(c+1, e^{c+1})$

a. on the left of $x=c$ b. on the right of $x=c$
c. at no point d. at all point

(IIT-JEE, 2007)

Multiple choice question with one or more than one correct answer

1. If the line $ax+by+c=0$ is a normal to the curve $xy=1$, then

a. $a>0, b>0$ b. $a>0, b<0$
c. $a<0, b>0$ d. $a<0, b<0$
e. None of these **(IIT-JEE, 1986)**

2. Which one of the following curves cut the parabola $y^2=4ax$ at right angles?

a. $x^2+y^2=a^2$ b. $y=e^{-x/2a}$
c. $y=ax$ d. $x^2=4ay$

(IIT-JEE, 1994)

Linked comprehension type

Read the passage given below and answer the questions that follows **(IIT-JEE, 2007)**

If a continuous function f, defined on the real line R, assumes positive and negative values in R, then the equation $f(x)=0$ has a root in R. For example, if it is known that a continuous function f on R is positive at some point and its minimum value is negative, then the equation $f(x)=0$ has a root in R.

Consider $f(x)=ke^x-x$ for all real x, where k is a real constant.

1. The line $y=x$ meets $y=ke^x$ for $k\le 0$ at

a. no point b. one point
c. two points d. more than two points

2. The positive value of k for which $ke^x-x=0$ has only one root

a. $\frac{1}{e}$ b. 1 c. e d. $\log_e 2$

3. For $k>0$, the set of the values of k for which $ke^x-x=0$ has two distinct roots is

a. $\left(0,\frac{1}{e}\right)$ b. $\left(\frac{1}{e},1\right)$ c. $\left(\frac{1}{e},\infty\right)$ d. $(0,1)$

ANSWERS AND SOLUTIONS

Subjective Type

1. We have $x^{2/3}+y^{2/3}=a^{2/3}$. (1)

Differentiating, we get $dy/dx=-y^{1/3}/x^{1/3}$

$\therefore$ Slope of the normal $=-dx/dy=x^{1/3}/y^{1/3}=\tan\theta$

or $\frac{x^{1/3}}{\sin\theta}=\frac{y^{1/3}}{\cos\theta}=\frac{\sqrt{(x^{1/3})^2+(y^{1/3})^2}}{\sqrt{\sin^2\theta+\cos^2\theta}}$

$=\sqrt{(a^{2/3})}=a^{1/3}$, using (1).

$\therefore x=a\sin^3\theta, y=a\cos^3\theta$

$\therefore$ equation of the normal whose slope is $\tan\theta$ is

$y-a\cos^3\theta=\tan\theta(x-a\sin^3\theta)$

$\Rightarrow y\cos\theta-a\cos^4\theta=x\sin\theta-a\sin^4\theta$

$\Rightarrow y\cos\theta-x\sin\theta=a(\cos^4\theta-\sin^4\theta)$

$=a(\cos^2\theta+\sin^2\theta)(\cos^2\theta-\sin^2\theta)$

$=a\cos 2\theta$

2. Given $y=\frac{a}{2}\ln\left(\frac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}}\right)-\sqrt{a^2-x^2}$

Let $x = a\sin\phi$ (1)

$\therefore y=\frac{a}{2}\ln\left(\frac{1+\cos\phi}{1-\cos\phi}\right)-a\cos\phi=-a\ln\tan(\phi/2)-a\cos\phi$

$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{d\phi}\right)}{\left(\frac{dx}{d\phi}\right)}=\frac{-a\operatorname{cosec}\phi+a\sin\phi}{a\cos\phi}=-\cot\phi$

Equation of the tangent at $P(x_1, y_1)$ is

$y-y_1=-\cot\phi(x-x_1)$

Point on y-axis is $Q(0, y_1+x_1\cot\phi)$ (from (1))

$\therefore PQ=\sqrt{x_1^2+x_1^2\cot^2\phi}$

$= x\operatorname{cosec}\phi = a =$ constant.

3. Slope of the tangent at $(x_1, y_1) = -\frac{x_1^2}{y_1^2}$

The tangent cuts the curve again at (x_2, y_2)

$\therefore$ Slope of the tangent $=\frac{y_2-y_1}{x_2-x_1}$

$\Rightarrow -\frac{x_1^2}{y_1^2}=\frac{y_2-y_1}{x_2-x_1}$

Also, $x_1^3+y_1^3=a^3$ and $x_2^3+y_2^3=a^3$

$\therefore x_1^3+y_1^3=x_2^3+y_2^3$

$\frac{y_2^3-y_1^3}{x_1^3-x_2^3}=1$

$\Rightarrow \frac{y_2-y_1}{x_1-x_2}=\frac{x_1^2+x_2^2+x_1x_2}{y_1^2+y_2^2+y_1y_2}$

$\Rightarrow \frac{x_1^2}{y_1^2}=\frac{x_1^2+x_2^2+x_1x_2}{y_1^2+y_2^2+y_1y_2}$

$\Rightarrow x_1^2y_1^2+x_1^2y_2^2+x_1^2y_1y_2=y_1^2x_1^2+x_2^2y_1^2+y_1^2x_1x_2$

$\Rightarrow x_2^2y_1^2-y_2^2x_1^2=x_1y_1[x_1y_2-x_2y_1]$

$\Rightarrow x_2y_1+y_2x_1=-x_1y_1$

$\Rightarrow \frac{x_2}{x_1}+\frac{y_2}{y_1}=-1$

4. Let $P(x_1, y_1)$ be any point on the curve $x^ny=a^n$. Then,

$x_1^ny_1=a^n$ (1)

Now, $x^ny=a^n \Rightarrow nx^{n-1}y+x^n\frac{dy}{dx}=0$ (differentiate w.r.t. x)

$\Rightarrow \frac{dy}{dx}=-n\frac{y}{x}\Rightarrow\left(\frac{dy}{dx}\right)_{(x_1, y_1)}=\frac{-ny_1}{x_1}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)}=-n\frac{a^n}{x_1^{n+1}}$ [Using (1)]

The equation of the tangent at $P(x_1, y_1)$ is

$y-y_1=-\frac{na^n}{x_1^{n+1}}(x-x_1)$

This meets the co-ordinate axes at $A\left(\frac{x_1^{n+1}y_1}{na^n}+x_1, 0\right)$ and $B\left(0, y_1+\frac{na^n}{x_1^n}\right)$

$\therefore$ area of $\Delta AOB=\frac{1}{2}(OA\times OB)$

$=\frac{1}{2}\left(\frac{x_1^{n+1}y_1}{na^n}+x_1\right)\left(y_1+\frac{na^n}{x_1^n}\right)$

$=\frac{1}{2}\left(\frac{x_1}{n}+x_1\right)\left(\frac{a^n}{x_1^n}+\frac{na^n}{x_1^n}\right)$ [Using (1)]

$=\frac{1}{2}\frac{(n+1)^2}{n}a^nx_1^{1-n}$

For the area to be a constant, we must have $1-n=0$, i.e., $n=1$.

5. The given curves are

$ax^2+by^2=1$ (1)

$a'x^2+b'y^2=1$ (2)

Differentiating (1), $\frac{dy}{dx}=-\frac{ax}{by}=m_1$ (say)

Differentiating (2), $\frac{dy}{dx}=-\frac{a'x}{b'y}=m_2$ (say)

If the curves (1) and (2) intersect at $P(x_1, y_1)$, then at this point P,

$m_1=-\frac{ax_1}{by_1}$ and $m_2=-\frac{a'x_1}{b'y_1}$

If the curves (1) and (2) intersect orthogonally, at P, then

$m_1m_2=-1\Rightarrow\frac{aa'x_1^2}{bb'y_1^2}=-1$ (3)

Since point $P(x_1, y_1)$ lie on both (1) and (2),

$\therefore ax_1^2+by_1^2=1$ and $a'x_1^2+b'y_1^2=1$

Subtracting, we get $(a-a')x_1^2+(b-b')y_1^2=0$

$\Rightarrow \frac{x_1^2}{y_1^2}=-\frac{b-b'}{a-a'}$

Substituting in equation (3), we get

$\left(\frac{aa'}{bb'}\right)\left(-\frac{b-b'}{a-a'}\right)=-1$

$\Rightarrow \frac{b-b'}{bb'}=\frac{a-a'}{aa'}\Rightarrow\frac{1}{b'}-\frac{1}{b}=\frac{1}{a'}-\frac{1}{a}$

$\Rightarrow \frac{1}{a}-\frac{1}{b}=\frac{1}{a'}-\frac{1}{b'}$

6. We know that,

$1\le|\sin x|+|\cos x|\le\sqrt{2}$

$\Rightarrow y=[|\sin x|+|\cos x|]=1$

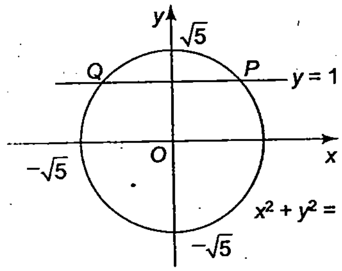


Fig. 5.21

Let P and Q be the points of intersection of given curves. Clearly the given curves meet at points where $y = 1$, so we get

$x^2 + 1 = 5 \Rightarrow x = \pm 2$

Now, $P(2, 1)$ and $Q(-2, 1)$

Differentiating $x^2 + y^2 = 5$ w.r.t. x, we get $2x + 2y\dfrac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(2,1)} = -2 \text{ and } \left(\frac{dy}{dx}\right)_{(-2,1)} = 2$$

Clearly, the slope of line $y = 1$ is zero and the slopes of the tangents at P and Q are (-2) and (2), respectively.

Thus, the angle of intersection is $\tan^{-1}(2)$.

7. Let $P(h, k)$ be a point of contact of tangents from the origin $(0, 0)$ on the curve $y = \sin x$

Since P lies on the curve $\Rightarrow k = \sin h$ (1)

Also $\dfrac{dy}{dx} = \cos x \Rightarrow \left(\dfrac{dy}{dx}\right)_{(h, k)} = \cos h$ (2)

Slope of the line joining $O(0, 0)$ and $P(h, k)$ is $\dfrac{k}{h}$

Given that $\cos h = \dfrac{k}{h}$ (3)

Squaring and adding (1) and (3), $k^2 + \dfrac{k^2}{h^2} = 1$

$$\Rightarrow h^2k^2 + k^2 = h^2 \Rightarrow k^2 = \frac{h^2}{1+h^2} \Rightarrow y^2 = \frac{x^2}{1+x^2}$$

8. The given expression resembles with $(x_1 - x_2)^2 + (y_1 - y_2)^2$,

where $y_1 = \dfrac{x_1^2}{20}$ and $y_2 = \sqrt{(17 - x_2)(x_2 - 13)}$.

Thus, we can think about two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lying on the curves $x^2 = 20y$ and $(x - 15)^2 + y^2 = 4$, respectively. Let D be the distance between P_1 and P_2, then the given expression simply represents D^2.

Now, as per the requirements, we have to locate the points on these curves (in the first quadrant) such that the distance between them is minimum.

Since the shortest distance between two curves always occurs along the common normal, it implies that we have to locate a point $P(x_1, y_1)$ on the parabola $x^2 = 20y$ such the normal drawn to the parabola at this point passes through $(15,0)$.

Now, the equation of the normal to the parabola at (x_1, y_1) is $\left(y - \dfrac{x_1^2}{20}\right) = -\dfrac{10}{x_1}(x - x_1)$. It should pass through $(15, 0)$.

$\Rightarrow x_1^3 + 200x_1 - 3000 = 0 \Rightarrow x_1 = 10 \Rightarrow y_1 = 5$

$\Rightarrow D = \sqrt{(10-15)^2 + 5^2} - 2 = (5\sqrt{2} - 2)$.

$\Rightarrow$ The minimum value of the given expression is $(5\sqrt{2} - 2)^2$.

9. Let h be the height and r the radius of the cone, then

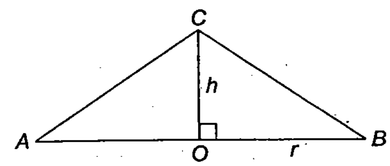


Fig. 5.22

$h = \dfrac{1}{6}r$ (given)

$\Rightarrow r = 6h$ (1)

$\Rightarrow$ Volume $V = \dfrac{1}{3}\pi r^2 h$

$= \dfrac{1}{3}\pi(6h)^2 h = 12\pi h^3$ (From (1))

$\Rightarrow \dfrac{dV}{dt} = 36\pi h^2 \dfrac{dh}{dt}$

$\Rightarrow 12 = 36\pi h^2 \dfrac{dh}{dt}$ $\left(\because \dfrac{dV}{dt} = 12 \text{ cm}^3/\text{s}\right)$

$\Rightarrow \dfrac{dh}{dt} = \dfrac{1}{3\pi h^2}$

When $h = 4$ cm, then $\dfrac{dh}{dt} = \dfrac{1}{3\pi(4)^2} = \dfrac{1}{48\pi}$ cm/s.

Hence, the rate, at which the height of the sand cone increases when the height is 4 cm, is $\dfrac{1}{48\pi}$ cm/s.

10. Let $f'(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$

Integrating both sides, we get

$$\Rightarrow f(x) = \frac{a_0x^{n+1}}{(n+1)} + \frac{a_1x^n}{n} + \frac{a_2x^{n-1}}{(n-1)} + \cdots + \frac{a_{n-1}x^2}{2} + a_nx + d$$

$\Rightarrow f(0) = d$

and $f(1) = \dfrac{a_0}{n+1} + \dfrac{a_1}{n} + \dfrac{a_2}{n-1} + \cdots + \dfrac{a_{n-1}}{2} + a_n + d = 0 + d = d$ (given)

$\Rightarrow f(0) = f(1)$

Now, since $f(x)$ is a polynomial, it is continuous and differentiable for all x. Consequently, $f(x)$ is continuous in the closed interval $[0, 1]$ and differentiable in the open interval $(0, 1)$.

Thus, all the three conditions of Rolle's theorem are satisfied. Hence, there is at least one value of x in the open interval $(0, 1)$ where $f'(x) = 0$, i.e., $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$.

11. Let $f(x) = \int (1 + \cos^8 x)(ax^2 + bx + c)dx$

$\therefore f'(x) = (1 + \cos^8 x)(ax^2 + bx + c)$ (1)

From the given conditions

$f(1) - f(0) = 0 \quad \Rightarrow f(0) = f(1)$ (2)

and $f(2) - f(0) = 0 \quad \Rightarrow f(0) = f(2)$ (3)

From (2) and (3), we get $f(0) = f(1) = f(2)$

By Rolle's theorem for $f(x)$ in $[0, 1]$: $f'(\alpha) = 0$, at least one α such that $0 < \alpha < 1$

By Rolle's theorem for $f(x)$ in $[1, 2]$: $f'(\beta) = 0$, at least one β such that $1 < \beta < 2$

Now from (1), $f'(\alpha) = 0 \Rightarrow (1 + \cos^8 \alpha)(a\alpha^2 + b\alpha + c) = 0$

$(\because 1 + \cos^8 \alpha \neq 0)$

$\Rightarrow a\alpha^2 + b\alpha + c = 0,$

i.e., α is a root of the equation $ax^2 + bx + c = 0$.

Similarly, β is a root of the equation $ax^2 + bx + c = 0$.

But equation $ax^2 + bx + c = 0$ being a quadratic equation cannot have more than two roots.

Hence, equation $ax^2 + bx + c = 0$ has one root α between 0 and 1, and the other root β between 1 and 2.

12. Let $g(x) = (f(x))^n$. Given $f(x)$ is continuous and differential, then $g(x)$ is also continuous and differentiable.

Then from Lagrange's mean value theorem, there exists at least one $c \in (0, 1)$ for which

$$g'(c) = nf'(c)f(c)^{n-1} = \frac{(f(1))^n - (f(0))^n}{1-0} = 2^n - 1$$

$$\Rightarrow nf'(c)(f(c))^{n-1} = \frac{2^n - 1}{2-1} = (1 + 2 + 2^2 + \cdots + 2^{n-1})$$

$$\Rightarrow f'(c)(f(c))^{n-1} = \frac{1 + 2 + 2^2 + \cdots + 2^{n-1}}{n} > (1.2...2^{n-1})^{\frac{1}{n}}$$

(as A.M. > G.M.)

$$\Rightarrow f'(c)(f(c))^{n-1} > \sqrt{2^{n-1}}.$$

13. By Lagrange's mean value theorem in $[a, b]$ for f,

$$\frac{f(b) - f(a)}{b - a} = f'(u), \text{ where } a < u < b$$

and applying Lagrange's mean value theorem in $[b, c]$,

$$\frac{f(c) - f(b)}{c - b} = f'(v), \text{ where } b < v < c$$

Since $f'(x)$ is strictly increasing

$\Rightarrow f'(u) < f'(v)$

$$\Rightarrow \frac{f(b) - f(a)}{b - a} < \frac{f(c) - f(b)}{c - b}$$

$\Rightarrow f(b)(c - b + b - a) - f(a)(c - b) - f(c)(b - a) < 0$

$\Rightarrow (b - c)f(a) + (c - a)f(b) + (a - b)f(c) < 0$

14. Substituting $y = a \sin \theta$ (1)

$$\Rightarrow \frac{x + a\cos\theta}{a} = \log_e \frac{a + a\cos\theta}{a\sin\theta} = \log_e \frac{1 + \cos\theta}{\sin\theta}$$

$= \log_e(\operatorname{cosec} \theta + \cot \theta)$

$$\Rightarrow \frac{x}{a} = \log_e(\operatorname{cosec} \theta + \cot \theta) - \cos \theta$$

$$\Rightarrow \frac{1}{a}\frac{dx}{d\theta} = \frac{-\operatorname{cosec}\theta(\operatorname{cosec}\theta + \cot\theta)}{\operatorname{cosec}\theta + \cot\theta} = + \sin \theta$$

$$= -\operatorname{cosec} \theta + \sin \theta = \sin \theta - \frac{1}{\sin\theta}$$

$$= -\frac{\cos^2\theta}{\sin\theta}$$

$$\therefore \frac{dx}{d\theta} = -\frac{a\cos^2\theta}{\sin\theta} \quad (1)$$

also $$\frac{dy}{d\theta} = a\cos\theta \quad (2)$$

$$\therefore \frac{dy}{dx} = \frac{a\cos\theta\sin\theta}{-a\cos^2\theta} = -\tan\theta$$

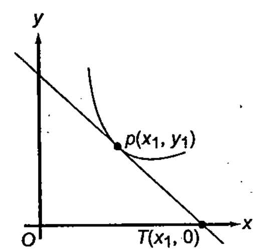


Fig. 5.23

Equation of tangent at point $P(x_1, y_1)$ is

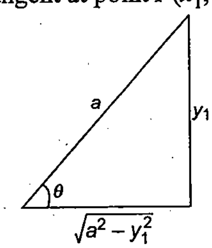


Fig. 5.24

$y - y_1 = -\tan\theta(x - x_1)$

or $$y - y_1 = -\frac{y_1}{\sqrt{a^2 - y_1^2}}(x - x_1)$$

$y = 0 \quad \Rightarrow \quad x = x_1 + \sqrt{a^2 - y_1^2}$

$PT^2 = a^2 - y^2 + y^2$

$\Rightarrow PT = a =$ constant.

15.

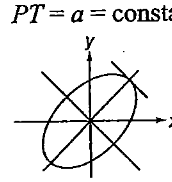


Fig. 5.25

$ax^2 + 2hxy + by^2 = 1$ (1)

$$\Rightarrow 2ax + 2h\left[x\frac{dy}{dx} + y\right] + 2by\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{ax + hy}{hx + by}$$

Now line $y = mx$ and curve intersect at right angle.

$$\Rightarrow m\left(\frac{ax+hy}{hx+by}\right)=1$$

Put $y=mx$

We have $m\left[\frac{ax+h\cdot mx}{hx+b\cdot mx}\right]=1$

$\Rightarrow m(a+hm)=h+bm$

$\Rightarrow m^2h+(a-b)m-h=0$

16. We have to prove that

$$\left(\frac{y}{(dy/dx)}\right)^2=k\cdot y\frac{dy}{dx}$$

or $y=k\left(\frac{dy}{dx}\right)^3$

Differentiating $by^2=(x+a)^3$ w.r.t. x, we have

$$2by\frac{dy}{dx}=3(x+a)^2$$

$$\Rightarrow \frac{dy}{dx}=\frac{3(x+a)^2}{2by}$$

$$\Rightarrow \frac{y}{(dy/dx)^3}=\frac{y\ (8b^3y^3)}{27(x+a)^6}=\frac{8b^3y^4}{27b^2y^4}=\frac{8b}{27}$$

Hence proved.

17.

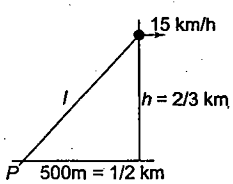


Fig. 5.26

$l^2=h^2+x^2$

$$2l\frac{dl}{dt}=0+2x\frac{dx}{dt}$$

$$\Rightarrow \frac{dl}{dt}=\frac{x}{l}\cdot\frac{dx}{dt}$$

where $x=\frac{1}{2}$ km, $h=\frac{2}{3}$ km

then $l=\frac{1}{4}+\frac{4}{9}=\frac{5}{6}$ km

$$\therefore \frac{dl}{dt}=\frac{1}{2}\cdot\frac{6}{5}\cdot 15=9 \text{ km/h}$$

18. Consider the function $f(x)=e^x-1$ in $[0,x]$ $\forall$, where $x>0$. Therefore, f is continuous and differentiable. Hence using LMVT $\exists$ some $c\in(0,x)$

$$f'(c)=\frac{(e^x-1)-0}{x}=\frac{e^x-1}{x}\left[f'(c)=\frac{f(x)-f(0)}{x-0}\right]$$

but $f'(c)=e^c$;

Hence, $\frac{e^x-1}{x}=e^c>1$, for $x>0$

$\therefore e^x-1>x$

$\therefore e^x>x+1$ for $x>0$ (1)

Again consider the function

$f(x)=e^x-1$ in $[x,0]$ where $x<0$

Using LMVT $\exists$ some $c\in(x,0)$ such that

$$f'(c)=\frac{0-(e^x-1)}{-x}=\frac{1-e^x}{-x}=\frac{e^x-1}{x}$$

But $f'(c)=e^c$. Hence $\frac{e^x-1}{x}=e^c<1$ for $c<0$

Hence $\frac{(e^x-1)}{x}<1$ for $x<0$

$\Rightarrow (e^x-1)>x$ (as x is –ve) (2)

From (1) and (2)

$e^x>x+1$ for $x\neq 0$

$\therefore$ for $x=0$ equality holds

$\therefore e^x\geq x+1$ for $x\in R$

Objective Type

1.b. Given curve is $x^{3/2}+y^{3/2}=2a^{3/2}$ (1)

$\therefore \frac{3}{2}\sqrt{x}+\frac{3}{2}\sqrt{y}\frac{dy}{dx}=0$ (Differentiate w.r.t. x)

or $\frac{dy}{dx}=-\frac{\sqrt{x}}{\sqrt{y}}$

Since the tangent is equally inclined to the axes

$\therefore \frac{dy}{dx}=\pm 1$

$\therefore -\frac{\sqrt{x}}{\sqrt{y}}=\pm 1\Rightarrow -\frac{\sqrt{x}}{\sqrt{y}}=-1$ $[\because \sqrt{x}>0, \sqrt{y}>0]$

$\Rightarrow \sqrt{x}=\sqrt{y}$

Putting $\sqrt{y}=\sqrt{x}$ in (1), we get

$2x^{3/2}=2a^{3/2}\Rightarrow x^3=a^3$

$\therefore x=a$ and so $y=a$.

2.a. $\frac{dx}{dt}=a+\frac{a}{2}2\cos 2t=a[1+\cos 2t]=2a\cos^2 t$

and $\frac{dy}{dt}=2a(1+\sin t)\cos t$

$$\Rightarrow \frac{dy}{dx}=\frac{2a(1+\sin t)\cos t}{2a\cos^2 t}=\frac{(1+\sin t)}{\cos t}$$

Then, the slope of the tangent

$$\tan\theta=\frac{(\cos(t/2)+\sin(t/2))^2}{\cos^2(t/2)-\sin^2(t/2)}$$

$$= \frac{1+\tan\frac{t}{2}}{1-\tan\frac{t}{2}} = \tan\left(\frac{\pi}{4}+\frac{t}{2}\right)$$

$$\Rightarrow \theta = \frac{\pi+2t}{4}$$

3. d. Differentiating w.r.t. x, we get $e^y \frac{dy}{dx} = 2x$

$$\Rightarrow \frac{dy}{dx} = \frac{2x}{1+x^2} \qquad (\because e^y = 1+x^2)$$

$$\Rightarrow m = \frac{2x}{1+x^2} \quad \text{or} \quad |m| = \frac{2|x|}{1+|x|^2}$$

But $1+|x|^2-2|x| = (1-|x|)^2 \ge 0$

$\Rightarrow 1+|x|^2 \ge 2|x|$,

$\therefore |m| \le 1$

4. c. $\frac{dy}{dx} = 3x^2 - 2ax + 1$

Given that $\frac{dy}{dx} \ge 0$

$\Rightarrow 3x^2 - 2ax + 1 \ge 0$ for all x.

$\Rightarrow D \le 0$ or $4a^2 - 12 \le 0$

$\Rightarrow -\sqrt{3} \le a \le \sqrt{3}$

5. a. Here $y > 0$. Putting $y = x$ in $y = \sqrt{4-x^2}$, we get $x = \sqrt{2}, -\sqrt{2}$.

So, the point is $(\sqrt{2}, \sqrt{2})$.

Differentiating $y^2 + x^2 = 4$ w.r.t. x,

$$2y\frac{dy}{dx} + 2x = 0 \text{ or } \frac{dy}{dx} = -\frac{x}{y}$$

$\Rightarrow$ at $(\sqrt{2}, \sqrt{2})$, $\frac{dy}{dx} = -1$

6. b. Differentiating w.r.t. x, we get $1 + \frac{dy}{dx} = e^{xy}\left(y + x\frac{dy}{dx}\right)$ or

$$\frac{dy}{dx} = \frac{ye^{xy}-1}{1-xe^{xy}}$$

$$\frac{dy}{dx} = \infty \Rightarrow 1 - xe^{xy} = 0$$

This holds for $x = 1, y = 0$.

7. c. The equation of the line is $y - 3 = \frac{3+2}{0-5}(x-0)$, i.e.,

$x + y - 3 = 0$

$$y = \frac{c}{x+1} \Rightarrow \frac{dy}{dx} = \frac{-c}{(x+1)^2}$$

Let the line touches the curve at (α, β).

$$\Rightarrow \alpha + \beta - 3 = 0, \left(\frac{dy}{dx}\right)_{\alpha,\beta} = \frac{-c}{(\alpha+1)^2} = -1 \text{ and } \beta = \frac{c}{\alpha+1}$$

$$\Rightarrow \frac{c}{(c/\beta)^2} = 1 \text{ or } \beta^2 = c \text{ or } (3-\alpha)^2 = c = (\alpha+1)^2$$

$\Rightarrow 3 - \alpha = \pm(\alpha+1)$ or $3 - \alpha = \alpha + 1$

$\Rightarrow \alpha = 1$. So, $c = (1+1)^2 = 4$

8. b.

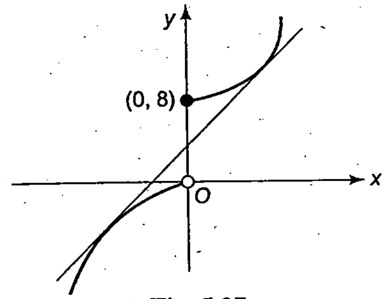


Fig. 5.27

Let $y = mx + c$ be a tangent to $f(x)$

$y = x^2 + 8$ for $x \ge 0$

$mx + c = x^2 + 8$

$x^2 - mx + 8 - c = 0$ (for the line to be tangent $D = 0$)

$\therefore m^2 = 4(8-c)$ (1)

Again $y = -x^2$, for $x < 0$

$mx + c = -x^2$

$x^2 + mx + c = 0$

$D = 0 \Rightarrow m^2 = 4c$ (2)

From (1) and (2), we get

$c = 4, m = 4$

$\therefore y = 4x + 4$

Put $y = 0 \Rightarrow x = -1$.

9. a. Putting $x = 0$ in the given curve, we obtain $y = 1$.

So, the given point is (0, 1)

$$\text{Now, } y = e^{2x} + x^2 \Rightarrow \frac{dy}{dx} = 2e^{2x} + 2x \Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)} = 2$$

The equation of the tangent at (0, 1) is

$y - 1 = 2(x - 0) \Rightarrow 2x - y + 1 = 0$ (1)

Required distance = length of the $\perp$ from (0, 0) on (1)

$$= \frac{1}{\sqrt{5}}.$$

10. a. Let the required point be (x_1, y_1)

Now, $3y = 6x - 5x^3$

$$\Rightarrow 3\frac{dy}{dx} = 6 - 15x^2$$

$$\Rightarrow \frac{dy}{dx} = 2 - 5x^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2 - 5x_1^2$$

The equation of the normal at (x_1, y_1) is

$$y - y_1 = \frac{-1}{2-5x_1^2}(x - x_1)$$

If it passes through the origin, then

$$0-y_1=\frac{-1}{2-5x_1^2}(0-x_1)$$

$$\Rightarrow \quad y_1=\frac{-x_1}{2-5x_1^2} \quad (1)$$

Since (x_1, y_1) lies on the given curve.

Therefore, $3y_1=6x_1-5x_1^3$ (2)

Solving equations (1) and (2), we obtain $x_1=1$ and $y_1=1/3$

Hence, the required point is (1, 1/3).

11. a. $2x^2+y^2=12 \Rightarrow \frac{dy}{dx}=-\frac{2x}{y}$.

Slope of normal at point $A(2, 2)$ is $\frac{1}{2}$

Also point $B\left(-\frac{22}{9},-\frac{2}{9}\right)$ lies on the curve and slope of AB is $=\frac{2-(-2/9)}{2-(-22/9)}=\frac{1}{2}$

Hence the normal meets the curve again at point $\left(-\frac{22}{9},-\frac{2}{9}\right)$

12. a. $y=\frac{2}{3}x^3+\frac{1}{2}x^2$

$\therefore \frac{dy}{dx}=\frac{2}{3}3x^2+\frac{1}{2}2x=2x^2+x$

Since the tangent makes equal angles with the axes.

$\Rightarrow \frac{dy}{dx}=\pm 1$

$\Rightarrow 2x^2+x=\pm 1$

$\Rightarrow 2x^2+x-1=0$ ($2x^2+x+1=0$ has no real roots)

$\Rightarrow (2x-1)(x+1)=0$

$\Rightarrow x=\frac{1}{2}$ or $x=-1$

13. d. $y=b\,e^{-x/a}$ meets the y-axis at $(0, b)$

Again $\frac{dy}{dx}=b\,e^{-x/a}\left(-\frac{1}{a}\right)$

At $(0, b)$, $\frac{dy}{dx}=be^0\left(-\frac{1}{a}\right)=-\frac{b}{a}$

$\therefore$ required tangent is $y-b=-\frac{b}{a}(x-0)$ or $\frac{x}{a}+\frac{y}{b}=1$.

14. b. $x^2-y^2=8 \Rightarrow \frac{dy}{dx}=\frac{x}{y} \Rightarrow -\frac{1}{dy/dx}=-\frac{y}{x}$

At the point $\left(-\frac{5}{\sqrt{2}},\frac{3}{\sqrt{2}}\right)$, $-\frac{1}{dy/dx}=\frac{-3/\sqrt{2}}{-5/\sqrt{2}}=\frac{3}{5}$

Also $9x^2+25y^2=225$

$\Rightarrow 18x+50y\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=-\frac{9x}{25y} \Rightarrow -\frac{dx}{dy}=\frac{25y}{9x}$

At the point $\left(-\frac{5}{\sqrt{2}},\frac{3}{\sqrt{2}}\right)$,

$$-\frac{dx}{dy}=\frac{25\times 3/\sqrt{2}}{9(-5/\sqrt{2})}=-\frac{15}{9}=-\frac{5}{3}$$

Since the product of the slopes $=-1$. Therefore, the normals cut orthogonally, i.e., the required angle is equal to $\frac{\pi}{2}$.

15. b. We have $f''(x)=6(x-1)$

Integrating, we get $f'(x)=3(x-1)^2+c$ (1)

At (2, 1), $y=3x-5$ is tangent to $y=f(x)$

$\therefore \quad f'(2)=3$

From equation (1), $3=3(2-1)^2+c \Rightarrow 3=3+c \Rightarrow c=0$

$\therefore \quad f'(x)=3(x-1)^2$

Integrating, we get $f(x)=(x-1)^3+c'$

Since the curve passes through (2, 1)

$\therefore \quad 1=(2-1)^3+c' \Rightarrow c'=0$

$\therefore \quad f(x)=(x-1)^3$.

$\therefore \quad f(0)=-1$

16. a. $y^2=\alpha x^3-\beta \Rightarrow \frac{dy}{dx}=\frac{3\alpha x^2}{2y}$

$\Rightarrow$ Slope of the normal at (2, 3) is

$$\left(-\frac{dx}{dy}\right)_{(2,3)}=-\frac{2\times 3}{3\,\alpha(2)^2}=-\frac{1}{2\alpha}=-\frac{1}{4}$$

$\Rightarrow \alpha=2$

Also, (2, 3) lies on the curve

$\Rightarrow 9=8\alpha-\beta \Rightarrow \beta=16-9=7 \Rightarrow \alpha+\beta=9$.

17. d. $x=2\ln\cot t+1, y=\tan t+\cot t$

Slope of the tangent

$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}}=\left(\frac{\sec^2 t-\operatorname{cosec}^2 t}{-\frac{2}{\cot t}\operatorname{cosec}^2 t}\right)_{t=\frac{\pi}{4}}=0$$

18. b. $y=e^x+e^{-x} \Rightarrow \frac{dy}{dx}=e^x-e^{-x}=\tan\theta$, where θ is the angle of the tangent with the x-axis

For $\theta=60°$, we have $\tan 60°=e^x-e^{-x}$

$\Rightarrow e^{2x}-\sqrt{3}\,e^x-1=0$

$\Rightarrow e^x=\frac{\sqrt{3}\pm\sqrt{7}}{2} \Rightarrow x=\log_e\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)$

19. c. $x^2y=c^3$

Differentiating w.r.t. x, we have

$x^2\frac{dy}{dx}+2xy=0 \Rightarrow \frac{dy}{dx}=-\frac{2y}{x}$

Equation of the tangent at (h, k) $y-k=-\frac{2k}{h}(x-h)$

$y=0$ gives $x=\frac{3h}{2}=a$, and $x=0$ gives $y=3k=b$

Now, $a^2b = \frac{9h^2}{4}\cdot 3k = \frac{27}{4}h^2k = \frac{27}{4}c^3$

20. a.

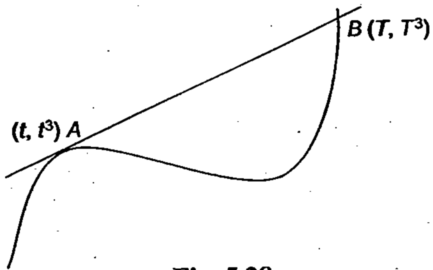


Fig. 5.28

$$\frac{dy}{dx} = 3x^2 = 3t^2 \text{ at } A$$

$$\therefore \quad 3t^2 = \frac{T^3 - t^3}{T - t} = T^2 + Tt + t^2$$

$\Rightarrow T^2 + Tt - 2t^2 = 0$

$\Rightarrow (T - t)(T + 2t) = 0 \Rightarrow T = t$ or $T = -2t$

($T = t$ is not possible)

Now, $m_A = 3t^2$ and $m_B = 3T^2$

$\Rightarrow \frac{m_B}{m_A} = \frac{T^2}{t^2} = \frac{4t^2}{t^2}$ (using $T = -2t$)

21. d.

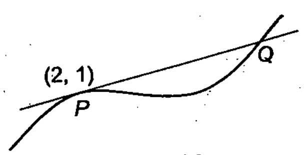


Fig. 5.29

Eliminating t gives $y^2(x-1) = 1$

Equation of the tangent at $P(2, 1)$ is $x + 2y = 4$

Solving with curve $x = 5$ and $y = -1/2$

$\Rightarrow Q(5, -1/2) \Rightarrow PQ = \frac{3\sqrt{5}}{2}$

22. c. $\frac{a}{x^2} + \frac{b}{y^2} = 1 \Rightarrow ay^2 + bx^2 = x^2y^2$ (1)

$$-\frac{2a}{x^3} - \frac{2b}{y^3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{ay^3}{bx^3}$$

Equation of the tangent at (h, k) is $y - k = -\frac{ak^3}{bh^3}(x - h)$

for x-intercept, put $y = 0$

$$\Rightarrow x = \frac{bh^3}{ak^2} + h \Rightarrow x = h\left[\frac{bh^2 + ak^2}{ak^2}\right] = h\left[\frac{h^2k^2}{ak^2}\right] = \frac{x^3}{a}$$

$\Rightarrow$ x-intercept is proportional to the cube of abscissa.

23. a.

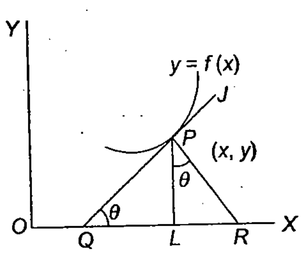


Fig. 5.30

Given curve is $2x^2y^2 - x^4 = c$ (1)

Sub-normal at $P(x, y) = y\frac{dy}{dx}$ (2)

From (1), we get $2\left(x^2 2y\frac{dy}{dx} + 2xy^2\right) - 4x^3 = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{x(x^2 - y^2)}{x^2y} \quad (3)$$

Now, $x(x - yy') = x^2 - xy\frac{dy}{dx}$

$= x^2 - (x^2 - y^2)$ [from (3)]

$= y^2$

$\Rightarrow$ Mean proportion $= \sqrt{x(x - yy')} = y$

24. b. Length of sub-normal = length of sub-tangent $\Rightarrow \frac{dy}{dx} = \pm 1$

If $\frac{dy}{dx} = 1$, equation of the tangent $y - 4 = x - 3$

$\Rightarrow y - x = 1$, area of $\Delta OAB = \frac{1}{2}\times 1 \times 1 = \frac{1}{2}$

If $\frac{dy}{dx} = -1$, equation of the tangent is $y - 4 = -x + 3$

$\Rightarrow y + x = 7$, area $= \frac{1}{2}\times 7 \times 7 = \frac{49}{2}$

25. a. $y = x + \sin x \Rightarrow$ If $\frac{dy}{dx} = 1 + \cos x = 0$, then $\cos x = -1$

$\Rightarrow x = \pm\pi, \pm 3\pi \ldots.$

Also $y = \pm\pi, \pm 3\pi \ldots.$

But for the given constraint on x and y, no such y exists.

Hence, no such tangent exists.

26. c. Solving $y = |x^2 - 1|$ and $y = \sqrt{7 - x^2}$

we have $|x^2 - 1| = \sqrt{7 - x^2}$

$\Rightarrow x^4 - 2x^2 + 1 = 7 - x^2$

$\Rightarrow x^4 - x^2 - 6 = 0$

$\Rightarrow (x^2 - 3)(x^2 + 2) = 0$

$\Rightarrow x = \pm\sqrt{3}$

Points of intersection of the curves $y = |x^2 - 1|$ and $y = \sqrt{7 - x^2}$ are $(\pm\sqrt{3}, 2)$.

Since both the curves are symmetrical about the y-axis, points of intersection are also symmetrical.

Now, $y = x^2 - 1 \Rightarrow \frac{dy}{dx} = 2x$

$$\Rightarrow m_1 = \left.\frac{dy}{dx}\right|_{(\sqrt{3}, 2)} = 2\sqrt{3}$$

and $y = \sqrt{7 - x^2} \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

$$\Rightarrow m_2 = \left.\frac{dy}{dx}\right|_{(\sqrt{3}, 2)} = -\frac{\sqrt{3}}{2} \Rightarrow \tan\theta = \left|\frac{5\sqrt{3}}{4}\right|$$

27. d. Differentiating $y^3 - x^2y + 5y - 2x = 0$ w.r.t. x, we get

$3y^2y' - 2xy - x^2y' + 5y' - 2 = 0$

$\Rightarrow \; y' = \dfrac{2xy+2}{3y^2 - x^2 + 5} \quad \Rightarrow y'_{(0,0)} = 2/5$

Differentiating $x^4 - x^3y^2 + 5x + 2y = 0$ w.r.t. x,

we have $4x^3 - 3x^2y^2 - 2x^3yy' + 5 + 2y' = 0$

$\Rightarrow \quad y' = \dfrac{3x^2y^2 - 4x^3 - 5}{2 - 2x^3y} \quad \Rightarrow y'_{(0,0)} = -5/2.$

Thus, both the curves intersect at right angle.

28. d. Solving the curves, we get point of intersection (a^2, a)

For $x = y^2$, $\dfrac{dy}{dx} = \dfrac{1}{2y}$

At (a^2, a), $\dfrac{dy}{dx} = \dfrac{1}{2a}$

For $xy = a^3$, $\dfrac{dy}{dx} = -\dfrac{y}{x}$

At (a^2, a), $\dfrac{dy}{dx} = -\dfrac{a}{a^2} = -\dfrac{1}{a}$

Since the curves cut orthogonally.

$\therefore \; \dfrac{1}{2a} \times -\dfrac{1}{a} = -1 \Rightarrow 2a^2 = 1 \Rightarrow a^2 = \dfrac{1}{2}$

29. b. $4x^2 + 9y^2 = 72$

Differentiating w.r.t. x, we have

$\Rightarrow 8x + 18y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{4}{9}\dfrac{x}{y}$

At $(3, 2)$, $\dfrac{dy}{dx} = -\dfrac{4}{9} \times \dfrac{3}{2} = -\dfrac{2}{3}$

Also $x^2 - y^2 = 5 \Rightarrow \dfrac{dy}{dx} = \dfrac{x}{y}$. At $(3, 2)$, $\dfrac{dy}{dx} = \dfrac{3}{2}$

$\therefore$ the curves cut orthogonally.

30. d. Using Lagrange's mean value theorem, for some $c \in (1, 6)$

such that $f'(c) = \dfrac{f(6) - f(1)}{5} = \dfrac{f(6) + 2}{5} \geq 4.2$

$\Rightarrow f(6) + 2 \geq 21$

$\Rightarrow f(6) \geq 19$

31. c. $f(0) = -1$; $f(1) = 7$. So $f(0)$ and $f(1)$ have opposite sign.

32. d. $f(x)$ vanishes at points where

$\sin\dfrac{\pi}{x} = 0$, i.e., $\dfrac{\pi}{x} = k\pi$, $k = 1, 2, 3, 4, \ldots$.

Hence $x = \dfrac{1}{k}$.

Also $f'(x) = \sin\dfrac{\pi}{x} - \dfrac{\pi}{x}\cos\dfrac{\pi}{x}$, if $x \neq 0$.

Since the function has a derivative at any interior point of the interval (0, 1), also continuous in [0,1] and $f(0) = f(1)$. Hence, Rolle's theorem is applicable to any one of the

interval $\left[\dfrac{1}{2}, 1\right], \left[\dfrac{1}{3}, \dfrac{1}{2}\right], \ldots, \left[\dfrac{1}{k+1}, \dfrac{1}{k}\right]$.

Hence, there exists at least one c in each of these intervals where $f'(c) = 0 \Rightarrow$ infinite points.

33. b. Applying Rolle's theorem to $F(x) = f(x) - 2g(x)$, we get

$F(0) = 0$,

$F(1) = f(1) - 2g(1)$

$\Rightarrow \quad 0 = 6 - 2g(1)$

$\Rightarrow \; g(1) = 3.$

34. b. Let $f(x) = \dfrac{ax^4}{4} + \dfrac{bx^3}{3} + \dfrac{cx^2}{2} + dx$,

which is continuous and differentiable.

$f(0) = 0, f(-1) = \dfrac{a}{4} - \dfrac{b}{3} + \dfrac{c}{2} - d$

$= \dfrac{1}{4}(a + 2c) - \dfrac{1}{3}(b + 3d) = 0$

So, according to Rolle's theorem, there exists at least one root of $f'(x) = 0$ in $(-1, 0)$.

35. d. $f(x) = ax^3 + bx^2 + 11x - 6$

satisfies conditions of Rolle's theorem in [1, 3]

$\Rightarrow \; f(1) = f(3)$

$\Rightarrow \; a + b + 11 - 6 = 27a + 9b + 33 - 6$

$\Rightarrow \; 13a + 4b = -11 \qquad (1)$

and $f'(x) = 3ax^2 + 2bx + 11$

$\Rightarrow \; f'\left(2 + \dfrac{1}{\sqrt{3}}\right) = 3a\left(2 + \dfrac{1}{\sqrt{3}}\right)^2 + 2b\left(2 + \dfrac{1}{\sqrt{3}}\right) + 11 = 0$

$\Rightarrow \; 3a\left(4 + \dfrac{1}{3} + \dfrac{4}{\sqrt{3}}\right) + 2b\left(2 + \dfrac{1}{\sqrt{3}}\right) + 11 = 0 \qquad (2)$

From equations (1) and (2), we get $a = 1$, $b = -6$

36. d. Here, $f(x) = \log_e x$

$\therefore f'(c) = \dfrac{f(b) - f(a)}{b - a} \Rightarrow \dfrac{1}{c} = \dfrac{\log_e 3 - \log_e 1}{3 - 1}$

$\Rightarrow \dfrac{1}{c} = \dfrac{1}{2}\log_e 3 \Rightarrow c = 2\log_3 e.$

37. d. Let $g(x) = f(x) - x^2$. We have $g(1) = 0, g(2) = 0, g(3) = 0$

$[\because f(1) = 1, f(2) = 4, f(3) = 9]$

From Rolle's theorem on $g(x)$, $g'(x) = 0$ for at least $x \in (1, 2)$. Let $g'(c_1) = 0$ where $c_1 \in (1, 2)$.

Similarly, $g(x) = 0$ for at least one $x \in (2, 3)$. Let $g'(c_2) = 0$ where $c_2 \in (2, 3)$

$\therefore g'(c_1) = g'(c_2) = 0$

By Rolle's theorem, at least one $x \in (c_1, c_2)$ such that $g''(x) = 0 \Rightarrow f''(x) = 2$ for some $x \in (1, 3)$.

38. b. $f\left(\dfrac{5\pi}{6}\right) = \log\sin\left(\dfrac{5\pi}{6}\right) = \log\sin\dfrac{\pi}{6} = \log\dfrac{1}{2} = -\log 2$,

$f\left(\dfrac{\pi}{6}\right) = \log\sin\dfrac{\pi}{6} = -\log 2$.

$f'(c) = \dfrac{1}{\sin x}\cos x = \cot x$

By Lagrange's mean value theorem,

$\dfrac{f(5\pi/6) - f(\pi/6)}{(5\pi/6) - (\pi/6)} = \cot c$

$\Rightarrow \cot c = 0 \Rightarrow c = \dfrac{\pi}{2}$

Thus, $c = \dfrac{\pi}{2} \in (\pi/6, 5\pi/6)$.

39.d. **a.** Discontinuous at $x=1 \Rightarrow$ not applicable.
b. $F(x)$ is not continuous (jump discontinuity) at $x=0$.
c. Discontinuity(missing point) at $x=1 \Rightarrow$ not applicable.
d. Notice that $x^3-2x^2-5x+6=(x-1)(x^2-x-6)$.
Hence, $f(x)=x^2-x-6$ if $x \neq 1$ and $f(1)=-6$.
$\Rightarrow f$ is continuous at $x=1$. So $f(x)=x^2-x-6$ is continuous in the interval $[-2, 3]$.
Also, note that $f(-2)=f(3)=0$. Hence, Rolle's theorem applies $f'(x)=2x-1$.
Setting $f'(x)=0$, we obtain $x=1/2$ which lies between -2 and 3.

40. d. We have $y^2=18x$ (1)

$$\therefore 2y\frac{dy}{dx}=18 \Rightarrow \frac{dy}{dx}=\frac{9}{y}.$$

Given that $\frac{dy}{dx}=2 \Rightarrow \frac{9}{y}=2 \Rightarrow y=\frac{9}{2}$

Putting in (1), we get $\frac{81}{4}=18x \Rightarrow x=\frac{9}{8}$.

Hence, the point is $\left(\frac{9}{8},\frac{9}{2}\right)$.

41. a. $V=\frac{4}{3}\pi r^3, S=4\pi r^2$

$$\frac{dV}{dr}=4\pi r^2, \frac{dS}{dr}=8\pi r$$

$$\Rightarrow \frac{dV}{dS}=\frac{dV/dr}{dS/dr}=\frac{4\pi r^2}{8\pi r}=\frac{r}{2}$$

when $r=2$, $\frac{dV}{dS}=\frac{2}{2}=1$

42. b. $V=x^3$ and the percent error in measuring $x=\frac{dx}{x}\times 100=k$

The percent error in measuring volume $=\frac{dV}{V}\times 100$

Now, $\frac{dV}{dx}=3x^2$

$$\Rightarrow dV=3x^2dx \Rightarrow \frac{dV}{V}=\frac{3x^2dx}{x^3}=3\frac{dx}{x}$$

$$\therefore \frac{dV}{V}\times 100=3\frac{dx}{x}\times 100=3k$$

43. b.

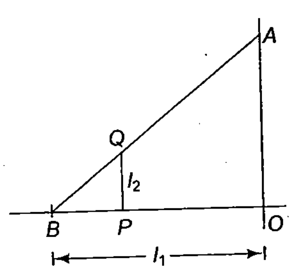


Fig. 5.31

Let $BP=x$. From similar triangle property, we get $\frac{AO}{l_1}=\frac{l_2}{x}$

$$\Rightarrow AO=\frac{l_1l_2}{x} \Rightarrow \frac{d(AO)}{dt}=\frac{-l_1l_2}{x^2}\frac{dx}{dt},\text{ when}$$

$$x=\frac{l_1}{2}, \frac{d(AO)}{dt}=-\frac{2l_2}{5}\text{m/s}.$$

44. a. Let CD be the position of man at any time t. Let $BD=x$, then $EC=x$. Let $\angle ACE=\theta$

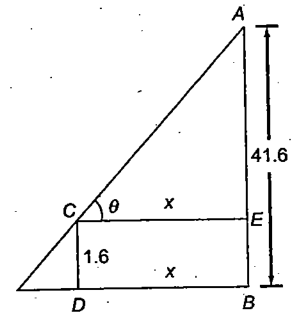


Fig. 5.32

Given, $AB=41.6$ m, $CD=1.6$ m and $\frac{dx}{dt}=2$ m/s.

$AE=AB-EB=AB-CD=41.6-1.6=40$ m

We have to find $\frac{d\theta}{dt}$ when $x=30$ m

From ΔAEC, $\tan\theta=\frac{AE}{EC}=\frac{40}{x}$ (1)

Differentiating w.r.t. to t, $\sec^2\theta\frac{d\theta}{dt}=\frac{-40}{x^2}\frac{dx}{dt}$

$$\Rightarrow \sec^2\theta\frac{d\theta}{dt}=\frac{-40}{x^2}\times 2$$

$$\Rightarrow \frac{d\theta}{dt}=\frac{-80}{x^2}\cos^2\theta=-\frac{80}{x^2}\frac{x^2}{x^2+40^2}$$

$$\left[\because \cos\theta=\frac{x}{\sqrt{x^2+40^2}}\right]$$

$$\Rightarrow \frac{d\theta}{dt}=-\frac{80}{x^2+40^2} \quad (2)$$

when $x=30$ m, $\frac{d\theta}{dt}=-\frac{80}{30^2+40^2}=-\frac{4}{125}$ radian/s.

45. b. Any point on the parabola $y^2=8x$ ($4a=8$ or $a=2$) is $(at^2, 2at)$ or $(2t^2, 4t)$.
For its minimum distance from the circle means its distance from the centre $(0,-6)$ of the circle.
Let D be the distance, then

$$z=D^2=(2t^2)^2+(4t+6)^2=4(t^4+4t^2+12t+9)$$

$$\therefore \frac{dz}{dt}=4(4t^3+8t+12)=0$$

$$\Rightarrow 16(t^3+2t+3)=0$$

$$\Rightarrow 16(t+1)(t^2-t+3)=0$$

$$\Rightarrow t=-1$$

$$\frac{d^2z}{dt^2} = 16(3t^2+2) = +\text{ve, hence minimum.}$$

$\therefore$ point is $(2, -4)$.

46. c. $y = x^n$

$$\frac{dy}{dx} = n x^{n-1} = na^{n-1}$$

Slope of the normal $= -\dfrac{1}{na^{n-1}}$

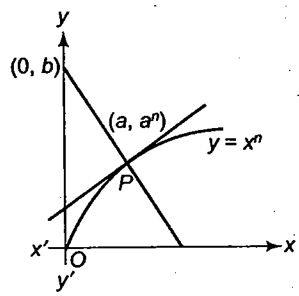


Fig. 5.33

Equation of the normal $y - a^n = -\dfrac{1}{na^{n-1}}(x-a)$

put $x = 0$ to get y-intercept

$y = a^n + \dfrac{1}{na^{n-2}}$; hence, $b = a^n + \dfrac{1}{na^{n-2}}$

$$\lim_{a\to 0} b = \begin{cases} 0, & \text{if } n<2 \\ \dfrac{1}{2}, & \text{if } n=2 \\ \infty, & \text{if } n>2 \end{cases}$$

47. b. Using Lagrange's mean value theorem for f in $[1, 2]$

for $c \in (1, 2)$, $\dfrac{f(2)-f(1)}{2-1} = f'(c) \le 2$

$\Rightarrow f(2) - f(1) \le 2$

$\Rightarrow f(2) \le 4$ (1)

again using Lagrange's mean value theorem in $[2, 4]$

for $d \in (1, 2)$, $\dfrac{f(4)-f(2)}{4-2} = f'(d) \le 2$

$\Rightarrow f(4) - f(2) \le 4$

$\Rightarrow 8 - f(2) \le 4$

$\Rightarrow f(2) \ge 4$ (2)

from (1) and (2), $f(2) = 4$.

48. d. Given, $V = \pi r^2 h$

Differentiating both sides, we get

$$\frac{dV}{dt} = \pi\left(r^2\frac{dh}{dt} + 2r\frac{dr}{dt}h\right) = \pi r\left(r\frac{dh}{dt} + 2h\frac{dr}{dt}\right)$$

$$\frac{dr}{dt} = \frac{1}{10} \text{ and } \frac{dh}{dt} = -\frac{2}{10}$$

$$\frac{dV}{dt} = \pi r\left(r\left(-\frac{2}{10}\right) + 2h\left(\frac{1}{10}\right)\right) = \frac{\pi r}{5}(-r+h)$$

Thus, when $r = 2$ and $h = 3$,

$$\frac{dV}{dt} = \frac{\pi(2)}{5}(-2+3) = \frac{2\pi}{5}$$

49. b. $\dfrac{dV}{dt} = -4 \text{ cm}^3/\text{min}$; $\dfrac{dS}{dt} = ?$ when $V = 125 \text{ cm}^2$

$$V = x^3;\ S = 6x^2 \Rightarrow \frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

$$-4 = 3x^2\frac{dx}{dt} \quad (1)$$

Also $\dfrac{dS}{dt} = 12x\dfrac{dx}{dt}$

$\dfrac{dS}{dt} = -\dfrac{16}{x}$; when $V = 125 = x^3 \Rightarrow x = 5$

$$\Rightarrow \left(\frac{dS}{dt}\right)_{x=5} = -\frac{16}{5} \text{ cm}^2/\text{min}.$$

50. d. $\dfrac{dy}{dx} = ke^{kx} = k$ at $(0, 1)$. Equation of the tangent is $y - 1 = kx$.

Point of intersection with x-axis is $x = -\dfrac{1}{k}$, where

$$-2 \le -\frac{1}{k} \le -1 \Rightarrow k \in \left[\frac{1}{2}, 1\right].$$

51. b. Applying LMVT in $[0, 1]$ to the function $y = f(x)$, we get

$f'(c) = \dfrac{f(1)-f(0)}{1-0}$, for some $c \in (0, 1)$

$\Rightarrow e^{c^2} = \dfrac{f(1)-f(0)}{1}$

$\Rightarrow f(1) - 10 = e^{c^2}$ for some $c \in (0, 1)$

but $1 < e^{c^2} < e$ in $(0, 1)$

$\Rightarrow 1 < f(1) - 10 < e$

$\Rightarrow 11 < f(1) < 10 + e$

$\Rightarrow A = 11, B = 10 + e$

$\Rightarrow A - B = 1 - \text{e}$

52. d. Consider a function $g(x) = x f(x)$

Since $f(x)$ is continuous, $g(x)$ is also continuous in $[0, 1]$ and differentiable in $(0, 1)$

As $f(1) = 0$

$\therefore g(0) = 0 = g(1)$

Hence Rolle's theorem is applicable for $g(x)$.

Therefore, there exists atleast one $c \in (0, 1)$ such that $g'(c) = 0$

$\Rightarrow xf'(x) + f(x) = 0$

$\Rightarrow cf'(c) + f(c) = 0$

53. a. $g(x) = \dfrac{x+2}{x-1}$

$$\Rightarrow g'(x) = \frac{-3}{(x-1)^2}$$

Slope of given line $=-3 \Rightarrow = \dfrac{-3}{(x-1)^2}=-3$

$\Rightarrow x=2$, also $g(2)=4$

(2, 4) also lies on given line.

Hence the given line is tangent to the curve.

54. a. Since the same line is tangent at one point $x=a$ and normal at other point $x=b$

$\Rightarrow$ Tangent at $x=b$ will be perpendicular to tangent at $x=a$

$\Rightarrow$ Slope of tangent changes from positive to negative or negative to positive. Therefore, it takes the value zero somewhere. Thus, there exists a point $c \in (a, b)$ where $f'(c)=0$

55. b. We know that there exists at least one x in (0, 1) for which

$$\frac{f(1)-f(0)}{g(1)-g(0)}=\frac{f'(x)}{g'(x)}$$

or $\dfrac{2-10}{4-2}=\dfrac{f'(x)}{g'(x)}$ or $f'(x)+4g'(x)=0$ for at least one x in (0,1)

Multiple Correct Answers Type

1. a, b, d.

$$f(x)=\frac{x}{1-x^2}$$

$$\therefore f'(x)=\frac{1+x^2}{\left(1-x^2\right)^2}=1, \text{ i.e., } x=0, -\sqrt{3}, \sqrt{3}$$

$\Rightarrow$ The points are (0, 0), $\left(\pm\sqrt{3}, \mp\dfrac{\sqrt{3}}{2}\right)$

2. a, b, c, d.

We have $y=ce^{x/a}$

$$\Rightarrow \frac{dy}{dx}=\frac{c}{a}e^{x/a} \Rightarrow \frac{dy}{dx}=\frac{1}{a}y$$

$$\Rightarrow \frac{y}{dy/dx}=a=\text{const.}$$

$\Rightarrow$ sub-tangent = const.

$\Rightarrow$ Length of the sub-normal $=y\dfrac{dy}{dx}=y\dfrac{y}{a}=\dfrac{y^2}{a} \propto$ (square of the ordinate)

Equation of the tangent at (x_1, y_1) is $y-y_1=\dfrac{y_1}{a}(x-x_1)$

This meets the x-axis at a point given by

$$-y_1=\frac{y_1}{a}(x-x_1) \Rightarrow x=x_1-a$$

The curve meets the y-axis at $(0, c)$.

$$\therefore \left(\frac{dy}{dx}\right)_{(0,c)}=c/a$$

So, the equation of the normal at $(0, c)$ is

$$y-c=-\frac{1}{c/a}(x-0) \Rightarrow ax+cy=c^2$$

3. a, b, c.

Clearly, $f(0)=0$. So, $f(x)=0$ has two real roots $0, \alpha_0(>0)$. Therefore, $f'(x)=0$ has a real root α_1 lying between 0 and α_0. So, $0<\alpha_1<\alpha_0$.

Again, $f'(x)=0$ is a fourth-degree equation. As imaginary roots occur in conjugate pairs, $f'(x)=0$ will have another real root α_2. Therefore, $f''(x)=0$ will have a real root lying between α_1 and α_2. As $f(x)=0$ is an equation of the fifth degree, it will have at least three real roots and so $f'(x)$ will have at least two real roots.

4. a, c, d.

$y=x(c-x)$ (1)

$y=x^2+ax+b$ (2)

Slope of (1) curve $=c-2x$

And at (1, 0), $c-2=m_1$ (say)

Slope of (2) curve $=2x+a$

at (1, 0), $2+a=m_1$ (say)

Curves are touching at (1, 0)

$\Rightarrow m_1=m_2$

$\Rightarrow 2+a=c-2$ (3)

Also (1, 0) lies on both the curves

$\Rightarrow 0=c-1$ and $0=1+a+b$ (4)

Solving (3) and (4), we get

$a=-3, b=2, c=1$

5. a, b, c, d.

a. $y^2=4ax \Rightarrow m_1=y'=\dfrac{2a}{y}$

$y=e^{-x}/2a \Rightarrow m_2=y'=-\dfrac{1}{2a}e^{-x/2a}=-\dfrac{1}{2a}y$

$m_1m_2=-1$. Hence, orthogonal.

b. $y^2=4ax$

$\Rightarrow y'=\dfrac{4a}{2y_1}=\dfrac{2a}{y_1}$, not defined at (0, 0)

$x^2=4ay$

$\Rightarrow y'=\dfrac{2x_1}{4a}=\dfrac{x_1}{2a}=0$ at (0, 0)

$\therefore$ The two curves are orthogonal at (0, 0).

c. $xy=a^2, x^2-y^2=b^2$

$m_1m_2=-\dfrac{a^2}{x_1y_1}=-\dfrac{a^2}{a^2}=-1 \Rightarrow$ orthogonal.

d. $y=ax, \Rightarrow y'=a$

$x^2+y^2=c^2 \Rightarrow y'=-\dfrac{x_1}{y_1}$

$m_1m_2=-\dfrac{ax_1}{y_1}=-\dfrac{y_1}{y_1}=-1 \Rightarrow$ orthogonal.

6. a, b. Since the intercepts are equal in magnitude but opposite in sign

$$\Rightarrow \left.\frac{dy}{dx}\right|_P=1$$

now $\frac{dy}{dx}=x^2-5x+7=1$

$\Rightarrow x^2-5x+6=0$

$\Rightarrow x=2$ or 3.

7. a, c. $xy=(a+x)^2$

$\Rightarrow y+xy'=2(a+x)$

Now $y'=\pm 1$

$\Rightarrow y\pm x=2(a+x)$

$\frac{(a+x)^2}{x}\pm x=2(a+x)$

$\Rightarrow \pm x=2(a+x)-\frac{(a+x)^2}{x}$

$\Rightarrow \pm x^2=(a+x)(x-a)$

$\Rightarrow \pm x^2=x^2-a^2$

$\Rightarrow 2x^2=a^2 \Rightarrow x=\pm\frac{a}{\sqrt{2}}$

8. b, c. $y=x^2+4x-17 \Rightarrow \frac{dy}{dx}=2(x+2) \Rightarrow \left(\frac{dy}{dx}\right)_{x=\frac{5}{2}}=9$

$\Rightarrow \tan\theta=9$, where θ is the angle with positive direction of x-axis.

$\Rightarrow$ Angle with y-axis is $\frac{\pi}{2}\pm\theta=\frac{\pi}{2}\pm\tan^{-1}9$.

9. a, b.

$x^3-y^2=0$ (1)

$\Rightarrow 2y\times\frac{dy}{dx}=3x^2$

Slope of the tangent at $P=\left.\frac{dy}{dx}\right|_P=\left.\frac{3x^2}{2y}\right|_{(4m^2,8m^3)}=3m$

$\therefore$ Equation of the tangent at P is

$y-8m^3=3m(x-4m^2)$ or $y=3mx-4m^3$ (2)

It cuts the curve again at point Q. Solving (1) and (2), we get $x=4m^2, m^2$

Put $x=m^2$ in equation (2)

$\Rightarrow y=3m(m^2)-4m^3=-m^3$ $\therefore Q$ is $(m^2,-m^3)$

Slope of the tangent at $Q=\left.\frac{dy}{dx}\right|_{(m^2,-m^3)}=\frac{3(m^4)}{2\times(-m^3)}=\frac{-3}{2}m$

Slope of the normal at $Q=\frac{1}{(-3/2)m}=\frac{2}{3m}$.

Since tangent at P is normal at Q $\Rightarrow \frac{2}{3m}=3m$

$\Rightarrow 9m^2=2$

10. a, b, c.

$y=x^2 \Rightarrow \frac{dy}{dx}=2x=2$ at (1, 1)

$\Rightarrow \frac{dy}{dx}=\frac{1}{2\sqrt{x}}=\frac{1}{2}$ at (1, 1)

$\Rightarrow \tan\theta=\frac{2-\frac{1}{2}}{1+2\left(-\frac{1}{2}\right)}=\frac{\frac{3}{2}}{1+1}=\frac{3}{4}$

$\Rightarrow \theta=\tan^{-1}\frac{3}{4}=\cos^{-1}\frac{4}{5}=\sin^{-1}\frac{3}{5}$

11. a, b.

Let $P(x, y)$ be a point on the curve $\ln(x^2+y^2)=c\tan^{-1}\frac{y}{x}$

Differentiating both sides with respect to x, we get

$\frac{2x+2yy'}{(x^2+y^2)}=\frac{c(xy'-y)}{(x^2+y^2)} \Rightarrow y'=\frac{2x+cy}{cx-2y}=m_1$ (say)

Slope of $OP=\frac{y}{x}=m_2$ (say) (where O is origin)

Let the angle between the tangents at P and OP be θ

$\Rightarrow \tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{\frac{2x+cy}{cx-2y}-\frac{y}{x}}{1+\frac{2xy+cy^2}{cx^2-2xy}}\right|=\frac{2}{c}$

$\Rightarrow \theta=\tan^{-1}\left(\frac{2}{c}\right)$ which is independent of x and y.

12. a, b, d.

f is not differentiable at $x=\frac{1}{2}$

g is not continuous in [0, 1] at $x=0$

h is not continuous in [0, 1] at $x=1$

$k(x)=(x+3)^{\ln_2 5}=(x+3)p$, where $2<p<3$, which is continuous and differentiable.

13. a, b, c

a. Let $f(x)=e^x\cos x-1$

$\Rightarrow f'(x)=e^x(\cos x-\sin x)=0$

$\Rightarrow \tan x=1$, which has a root between two roots of $f(x)=0$

b. Let $f(x)=e^x\sin x-1$,

$f'(x)=e^x(\sin x+\cos x)=0$

$\Rightarrow \tan x=-1$, which has a root between two roots of $f(x)=0$

c. Let $f(x)=e^{-x}-\cos x$,

$f'(x)=-e^{-x}+\sin x=0$

$\Rightarrow e^{-x}=\sin x$, which has a root between two roots of $f(x)=0$

14. a, b, c, d.

a. $y^2=4ax$ and $y=e^{-x/2a}$

$y'=\frac{2a}{y}$ and $y'=-\frac{1}{2a}e^{-x/2a}=-\frac{1}{2a}y$

Let the intersection point be (x_1, y_1)

$y'=\frac{2a}{y_1}$ and $y'=-\frac{1}{2a}y_1$

$m_1m_2 = -1$. Hence orthogonal

b. $y^2 = 4ax$ and $x^2 = 4ay$

$$y' = \frac{4a}{2y_1} = \frac{2a}{y_1}, \text{ not defined at } x = 0$$

$$y' = \frac{2x_1}{4a} = \frac{x_1}{2a}\ 0 \text{ at } x = 0$$

$\therefore$ The two curves are orthogonal at (0, 0)

c. $xy = a^2$ and $x^2 - y^2 = b^2$

$$m_1m_2 = -\frac{a^2}{x_1y_1} = -\frac{a^2}{a^2} = -1 \text{ orthogonal}$$

d. $y = ax$ and $x^2 + y^2 = c^2$

$$y' = a \text{ and } y' = -\frac{x_1}{y_1}$$

$$m_1m_2 = -\frac{ax_1}{y_1} = -\frac{y_1}{x_1} = -1 \text{ orthogonal}$$

Reasoning Type

1. d. Though $|x-1|$ is non-differentiable at $x = 1$, $(x-1)|x-1|$ is differentiable at $x = 1$, for which Lagrange's mean value theorem is applicable.

2. a. Consider $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$
$\Rightarrow f(x) = ax^4 + bx^3 + cx^2 + dx + e$
$f(0) = e$ and $f(3) = 81a + 27b + 9c + 3d + e$
$= 3(27a + 9b + 3c + d) + e = e$
Hence, Rolle's theorem is applicable for $f(x)$,
$\Rightarrow$ there exists at least one c in (a, b) such that $f'(c) = 0$.

3. c. Statement 1 is correct as it is the statement of Cauchy's mean value theorem. Statement 2 is false as it is necessary that c in both $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ and $g'(c) = \dfrac{g(b)-g(a)}{b-a}$ is same.

4. a. Let $y = \sqrt{-3+4x-x^2}$
$\Rightarrow x^2 + y^2 - 4x + 3 = 0$ or point (x, y) lies on this circle.
Then, the given expression is $(y+4)^2 + (x-5)^2$, which is the square of distance between point $P(5, -4)$ and any point on the circle $x^2 + y^2 - 4x + 3 = 0$ which has centre $C(2, 0)$ and radius 1.
Now $CP = 5$, then the maximum distance between the point P and any point on the circles is 6.
$\Rightarrow$ Maximum value of $(\sqrt{-3+4x-x^2} + 4)^2 + (x-5)^2$ is 36.

5. c. Statement 1 is correct as $f(-2) = f(2) = 0$ and Rolle's theorem is not applicable, then it implies that either $f(x)$ is discontinuous or $f'(x)$ does not exist at at least one point in $(-1, 1)$. Since it is given that $g(x)$ is differentiable, $g(x) = 0$ has at least one value of x in $(-1, 1)$.
Statement 2 is false as $f(x)$ must be differentiable in (a, b) is not given.

6. d. when $x = 1, y = 1$
$$\frac{dy}{dx} = 3x^2 - 2x - 1 \Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = 0$$
$\Rightarrow$ Equation of the tangent is $y = 1$.
Solving with the curve, $x^3 - x^2 - x + 2 = 1$
$\Rightarrow x^3 - x^2 - x + 1 = 0 \Rightarrow x = -1, 1$ (1 is repeated root)
$\therefore$ the tangent meets the curve again at $x = -1$
$\therefore$ statement 1 is false and statement 2 is true.

7. b. Point of inflection of the curve is $\left(\frac{1}{2}, \frac{\pi}{2}\right)$ and this satisfies the line L

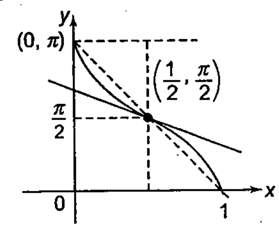


Fig. 5.34

Slope of the tangent to the curve C at $\left(\frac{1}{2}, \frac{\pi}{2}\right)$

$$\frac{dy}{dx} = \frac{-2}{\sqrt{1-(2x-1)^2}} = \frac{-1}{\sqrt{x-x^2}} = -(x-x^2)^{-1/2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}\,\frac{(1-2x)}{(x-x^2)^{3/2}} = 0;\ x = \frac{1}{2}$$

$$\left.\frac{dy}{dx}\right|_{x=1/2} = -2$$

As the slope decreases from -2, line cuts the curve at three distinct points and the minimum slope of the line when it intersects the curve at three distinct points is

$$\frac{\pi - \frac{\pi}{2}}{0 - \frac{1}{2}} = -\pi$$

$$\therefore \frac{p}{2} \in [-\pi, -2) \Rightarrow p \in [-2\pi, -4)$$

8. a. Consider, $F(x) = e^{-\lambda x} f(x), \lambda \in R$
$F(0) = f(0) = 0$
$F(1) = e^{-1} f(1) = 0$
$\therefore$ By Rolle's theorm, $F'(c) = 0$
$F'(x) = e^{-\lambda x}(f'(x) - \lambda f(x))$
$F'(c) = 0 \Rightarrow e^{-\lambda c}(f'(c) - \lambda f(c)) = 0$
$\Rightarrow f'(c) = \lambda f(c), 0 < c < 1$.

9. a. Verify by taking $f(x) = lx^2 + mx + n$ in $[a, b]$

10. a. Equation of a tangent at (h, k) on $y = f(x)$ is
$y - k = f'(h)(x - h)$ (1)
Suppose (1) passes through (a, b)
$b - k = f'(h)[a - h]$ must hold good for some (h, k)
Now $hf'(h) - f(h) - af'(h) + b = 0$ represents an equation of degree odd in h.
$\therefore \exists$ some 'h' for which LHS vanishes

Linked Comprehension Type

For Problems 1–3

1.a, 2.c, 3.d.

Sol.

1. a. Let $P_1(t_1, t_1^3)$ is a point on the curve $y = x^3$

$$\therefore \left.\frac{dy}{dx}\right|_{(t_1, t_1^3)} = 3t_1^2$$

Tangent at P_1 is $y - t_1^3 = 3t_1^2 (x - t_1)$ (1)

The intersection of (1) and $y = x^3$

$\Rightarrow x^3 - t_1^3 = 3t_1^2(x - t_1)$

$\Rightarrow (x - t_1)(x^2 + xt_1 + t_1^2) - 3t_1^2(x - t_1) = 0$

$\Rightarrow (x - t_1)^2(x + 2t_1) = 0$

If $P_2(t_2, t_2^3)$, then

$(t_2 - t_1)^2 (t_2 + 2t_1) = 0$

$\therefore t_2 = -2t_1 \ (t_2 \neq t_1)$.

Similarly, the tangent at P_2 will meet the curve at the point $P_3(t_3\ t_3^3)$ when $t_3 = -2t_2 = 4t_1$ and so on.

The abscissae of $P_1, P_2, \ldots, P_n$ are

$t_1, -2t_1, 4t_1, \ldots; (-2)^{n-1} t_1$ in G.P.

$$\therefore \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \cdots = -2 \ (r \text{ say})$$

$\therefore t_2 = t_1 r, t_3 = t_2 r$ and $t_4 = t_3 r$

If $x_1 = 1$, then $x_2 = -2, t_3 = 4, \ldots$

Then $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{x_n}$ = sum of infinite G.P. with common ratio (–1/2) with first term 1

$$= \frac{1}{1 - \left(-\frac{1}{2}\right)} = \frac{2}{3}$$

2. c. Then $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{y_n}$ = sum of infinite G.P. with common ratio (–1/8) with first term 1

$$= \frac{1}{1 - \left(-\frac{1}{8}\right)} = \frac{8}{9}.$$

3. d. $\therefore$ Area of $\Delta P_2P_3P_4 = \frac{1}{2}\begin{vmatrix} t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \\ t_4 & t_4^3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} rt_1 & r^3t_1^3 & 1 \\ rt_2 & r^3t_2^3 & 1 \\ rt_3 & r^3t_3^3 & 1 \end{vmatrix}$

$= r^4$ (Area of $(\Delta P_1P_2P_3)$

$$\therefore \frac{\text{Area of } (\Delta P_1P_2P_3)}{\text{Area of } (\Delta P_2P_3P_4)} = \frac{1}{r^4} = \frac{1}{(-2)^4} = \frac{1}{16}$$

For Problems 4–6

4. a, 5.b, 6.c.

Sol.

4. a. $\frac{dy}{dx} = \frac{1 - 9t^2}{-6t} = \tan\theta$

$\Rightarrow 9t^2 - 6\tan\theta.\, t - 1 = 0$

$\Rightarrow 3t = \tan\theta \pm \sec\theta$

$\Rightarrow \tan\theta + \sec\theta = 3t.$

5. b. $P(-2, 2) \Rightarrow t = -1 \Rightarrow \left.\frac{dy}{dx}\right|_{t=-1} = -\frac{4}{3}$

Equation of the tangent $y - 2 = -\frac{4}{3}(x + 2)$

$\Rightarrow t - 3t^3 - 2 = -\frac{4}{3}(1 - 3t^2 + 2)$

$\Rightarrow 9t^3 + 12t^2 - 3t - 6 = 0$

$\Rightarrow 3t^3 + 4t^2 - t - 2 = 0$

$\Rightarrow (3t^2 + t - 2)(t + 1) = 0$

$\Rightarrow (3t - 2)(t + 1)^2 = 0$

$\Rightarrow t = \frac{2}{3}$

$\Rightarrow Q = \left(-\frac{1}{3}, -\frac{2}{3}\right)$

6. c. $\left.\frac{dy}{dx}\right|_{t=2/3} = \frac{3}{4}$

$m_{po}m_Q = -1 \Rightarrow$ angle $90°$

For Problems 7–8

7.b, 8.a.

Sol. Let V be the volume and r the radius of the balloon at any time, then

$$V = \left(\frac{4}{3}\right)\pi r^3$$

$$\Rightarrow \frac{dV}{dt} = \left(\frac{4}{3}\right)(3\pi r^2)\frac{dr}{dt}$$

$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 40 \text{ (given)}$$

$$\Rightarrow \frac{dr}{dt} = \frac{10}{\pi r^2} \quad (1)$$

Now let S be the surface area of the balloon when its radius is r, then $S = 4\pi r^2$

$$\Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt} \quad (2)$$

From (1) and (2), $\frac{dS}{dt} = 8\pi r \frac{10}{\pi r^2} = \frac{80}{r}$

When $r = 8$, the rate of increase of $S = \frac{80}{8} = 10 \text{ cm}^2/\text{min}.$

$\Rightarrow$ Increase of S in $\frac{1}{2}$ minute $= 10 \times \left(\frac{1}{2}\right) = 5 \text{ cm}^2/\text{min}.$

If r_1 is the radius of the balloon after (1/2) min, then $4\pi r_1^2 = 4\pi(8)^2 + 5$

or $r_1^2 - 8^2 = \frac{5}{4\pi} = 0.397$ nearly or $r_1^2 = 64.397$ or $r_1 = 8.025$ nearly.

$\Rightarrow$ Required increase in the radius $= r_1 - 8 = 8.025 - 8 = 0.025$ cm.

Matrix-Match Type

1. a → p, q; b → r, s; c → r, q; d → p, s;

a. Given $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ (say)

$\therefore da = 2R\cos A\ dA$

$db = 2R\cos B\ dB$

$dc = 2R\cos C\ dC$

$$\therefore \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R(dA + dB + dC) \quad (1)$$

Also $A + B + C = \pi$ So, $dA + dB + dC = 0$ (2)

From equations (1) and (2), we get

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} + 1 = 1$$

$\Rightarrow m = \pm 1$

b. $x^2y^2 = 16 \Rightarrow xy = \pm 4$ (1)

$$L_{ST} = \left|\frac{y}{dy/dx}\right|$$

Differentiating (1) w.r.t. x, we get $y + xy' = 0 \Rightarrow y' = \frac{-y}{x}$

$$L_{ST} = \left|\frac{y}{y/x}\right| = |x| \Rightarrow L_{ST} = 2$$

$\Rightarrow k = \pm 2.$

c. $y = 2e^{2x}$ intersects y-axis at (0, 2)

$$\frac{dy}{dx} = 4e^{2x} \therefore \left.\frac{dy}{dx}\right|_{\text{at } x=0} = 4$$

$\therefore$ Angle of intersection with y-axis $= \frac{\pi}{2} - \tan^{-1} 4 = \cot^{-1} 4$

$\Rightarrow n = 2$ or -1

d. $\frac{dy}{dx} = e^{\sin y}\cos y$: slope of the normal at (1, 0) $= -1$

equation of the normal is $x + y = 1$

Area $= \frac{1}{2}$

$\Rightarrow t = 1, -2.$

2. a → q; b → r; c → p; d → s;

a. $r = 6$ cm $\delta r = 0.06$

$A = \pi r^2 \Rightarrow \delta A = 2\pi r\delta r = 2\pi(6)(0.06) = 0.72\pi.$

b. $V = x^3$, $\delta V = 3x^2\,\delta x$

$$\frac{\delta V}{V} \times 100 = 3\frac{\delta x}{x} \times 100 = 3 \times 2 = 6.$$

c. $(x-2)\frac{dx}{dt} = 3\frac{dx}{dt}$

$\Rightarrow x = 5$

d. $A = \frac{\sqrt{3}}{4}x^2$

$$\Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{2}\left(x\frac{dx}{dt}\right) = \frac{\sqrt{3}}{2} \times 30 \times \frac{1}{10} = \frac{3\sqrt{3}}{2}.$$

3. a → p, q; b → p, s; c → r; d → q;

a. $y^2 = 4x$ and $x^2 = 4y$ intersect at point (0, 0) and (4, 4)

$C_1 : y^2 = 4x$ $\quad$ $C_2 : x^2 = 4y$

$\frac{dy}{dx} = \frac{2}{y}$ $\quad$ $\frac{dy}{dx} = \frac{x}{2}$

$\left.\frac{dy}{dx}\right|_{0,0} = \infty$ $\quad$ $\left.\frac{dy}{dx}\right|_{0,0} = 0$

Hence, $\tan\theta = 90°$ at point (0, 0)

$\left.\frac{dy}{dx}\right|_{(4,4)} = \frac{1}{2}$ $\quad$ $\left.\frac{dy}{dx}\right|_{(4,4)} = 2$

$$\tan\theta = \left|\frac{2 - \frac{1}{2}}{1 + 2 \times \frac{1}{2}}\right| = \frac{3}{4}$$

b. Solving I: $2y^2 = x^3$ and II: $y^2 = 32x$, we get (0, 0), (8, 16) and (8, – 16)

at (0, 0) $\left.\frac{dy}{dx}\right|_{(0,0)} = 0$ for I

at (0, 0) $\left.\frac{dy}{dx}\right|_{(0,0)} = \infty$ for II

Hence, angle = 90°

now $\left.\frac{dy}{dx}\right|_{(8,16)} = \frac{3x^2}{4y} = \frac{3}{4}\frac{64}{16} = 3$ for I

$\left.\frac{dy}{dx}\right|_{(8,16)} = \frac{32}{2y} = \frac{16}{16} = 1$ for II

$\therefore \tan\theta = \frac{3-1}{1+3} = \frac{2}{4} = \frac{1}{2}$

$\Rightarrow$ angle between the two curves at the origin is 90°.

c. The two curves are

$xy = a^2$ (1)

$x^2 + y^2 = 2a^2$ (2)

Solving (1) and (2), the points of intersection are (a, a) and $(-a, -a)$

Differentiating (1), $dy/dx = -y/x = m_1$ (say)
Differentiating (2), $dy/dx = -x/y = m_2$ (say)
At both points, $m_1 = -1 = m_2$
Hence, the two curves touch each other.

d. $y^2 = x, x^3 + y^3 = 3xy$

For the 1st curve, $2y\dfrac{dy}{dx} = 1 \Rightarrow \left.\dfrac{dy}{dx}\right|_P = \dfrac{1}{2y_1}$

Again for the 2nd curve, $\left.\dfrac{dy}{dx}\right|_P = \dfrac{y_1 - x_1^2}{y_1^2 - x_1}$

solving $y^2 = x$ and $x^3 + y^3 = 3xy$;
$y^6 + y^3 = 3y^3 \Rightarrow y^3 + 1 = 3 \Rightarrow y^3 = 2$
$\therefore y_1 = 2^{1/3}$ and $x_1 = 2^{2/3}$.

Now $m_1 = \dfrac{1}{2\times 2^{\frac{1}{3}}} = \dfrac{1}{2^{\frac{4}{3}}}$; $m_2 = \dfrac{2^{\frac{1}{3}} - 2^{\frac{4}{3}}}{2^{\frac{2}{3}} - 2^{\frac{2}{3}}} \to \infty$

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{1 - \frac{m_1}{m_2}}{\frac{1}{m_2} + m_1}\right| = \left|\frac{1}{m_1}\right| = 2^{\frac{4}{3}} = 16^{\frac{1}{3}}$$

$\therefore \theta = \tan^{-1}(16^{\frac{1}{3}})$

Integer Type

1. (5) $y = x^2$ and $y = -\dfrac{8}{x}$; $q = p^2$ and $s = -\dfrac{8}{r}$ (1)

Equating $\dfrac{dy}{dx}$ at A and B, we get

$2p = \dfrac{8}{r^2}$ (1)

$\Rightarrow pr^2 = 4$

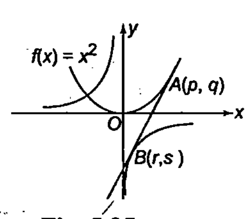


Fig. 5.35

Now $m_{AB} = \dfrac{q-s}{p-r} \Rightarrow 2p = \dfrac{p^2 + \frac{8}{r}}{p-r}$

$\Rightarrow p^2 = 2pr + \dfrac{8}{r} \Rightarrow p^2 = \dfrac{16}{r}$

$\Rightarrow \dfrac{16}{r^4} = \dfrac{16}{r} = \Rightarrow r = 1\ (r \neq 0) \Rightarrow p = 4$

$\therefore r = 1, p = 1$
Hence $p + r = 5$

2. (7) $x = t^2; y = t^3$

$\dfrac{dx}{dt} = 2t; \dfrac{dy}{dt} = 3t^2$

$\dfrac{dy}{dx} = \dfrac{3t}{2}$

$y - t^3 = \dfrac{3t}{2}(x - t^2)$

$2k - 2t^3 = 3th - 3t^3$

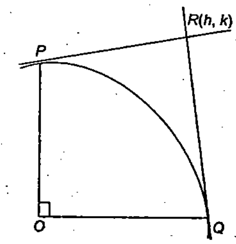


Fig. 5.36

$\therefore t^3 - 3th + 2k = 0$ (1)

$t_1t_2t_3 = -2k$ (put $t_1t_2 = -1$); hence $t_3 = 2k$
Product of roots
Now t_3 must satisfy equation (1)
$\Rightarrow (2k)^3 - 3(2k)h + 2k = 0$
$\Rightarrow 4y^2 - 3x + 1 = 0$ or $4y^2 = 3x - 1$
$\Rightarrow a + b = 7$

3. (8)

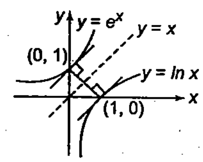


Fig. 5.37

Since the graphs of $y = e^x$ and $y = \log_e x$ are symmetrical about the line $y = x$, minimum distance is the distance along the common normal to both the curves, i.e., $y = x$ must be parallel to the tangent as both the curves are inverse of each other.

$\left.\dfrac{dy}{dx}\right|_{x_1} = e^{x_1} = 1$

$\Rightarrow x_1 = 0$ and $y_1 = 1$
$\Rightarrow A \equiv (0, 1)$ and $B \equiv (1, 0)$
$\Rightarrow AB = \sqrt{2}$

4. (6) $f(x) = f(6-x)$ (1)

On differentiating (1) w.r.t. x, we get
$f'(x) = -f'(6-x)$ (2)
Putting $x = 0, 2, 3, 5$ in (2), we get
$f'(0) = -f'(6) = 0$
Similarly $f'(2) = -f'(4) = 0$
$f'(3) = 0$
$f'(5) = -f'(1) = 0$
$\therefore f'(0) = 0 = f'(2) = f'(3) = f'(5) = f'(1) = f'(4) = f'(6)$
$\therefore f'(x) = 0$ has minimum 7 roots in [0, 6]
Now, consider a function $y = f'(x)$
As $f'(x)$ satisfy Rolle's theorem in intervals [0, 1], [1, 2], [2, 3], [3, 4], [4, 5] and [5, 6] respectively.
So, by Rolle's theorem, the equation $f''(x) = 0$ has minimum 6 roots.
Now $g(x) = (f''(x))^2 + f'(x)f'''(x) = = h'(x)$, where $h(x) = f'(x)f''(x)$
Clearly $h(x) = 0$ has minimum 13 roots in [0, 6]
Hence again by Rolle's theorem, $g(x) = h'(x)$ has minimum 12 zeroes in [0, 6].

5. (2) $y = x^n$

$\dfrac{dy}{dx} = n\,x^{n-1} = na^{n-1}$

Slope of normal $= -\frac{1}{na^{n-1}}$

Equation of normal $y - a^n = -\frac{1}{na^{n-1}}(x-a)$

put $x = 0$ to get y-intercept

$y = a^n + \frac{1}{na^{n-2}}$; Hence $b = a^n + \frac{1}{na^{n-2}}$

$$\lim_{a\to 0} b = \begin{cases} 0 & \text{if } n<2 \\ \frac{1}{2} & \text{if } n=2 \\ \infty & \text{if } n>2 \end{cases}$$

6. (3) $\frac{dy}{dx} = \frac{y}{x} = -\frac{1}{2}\cot^3\theta = -\frac{1}{2}$ at $\theta = \frac{\pi}{4}$

Also the point P for $\theta = \pi/4$ is (2, 1)

Equation of tangent is $y - 1 = -\frac{1}{2}(x-2)$

or $x + 2y - 4 = 0$ (1)

This meets the curve whose Cartesian equation on eliminating θ by $\sec^2\theta - \tan^2\theta = 1$ is

$y^2 = \frac{1}{x-1}$ (2)

Solving (1) and (2), we get $y = 1, -\frac{1}{2}$

$\therefore x = 2, 5$

Hence P is (2, 1) as given and Q is $\left(5, -\frac{1}{2}\right)$

$\therefore PQ = \sqrt{\frac{45}{4}} = \frac{3\sqrt{5}}{2}$

7. (5)

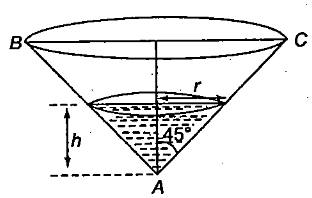


Fig. 5.38

We have

$\frac{dV}{dt} = 2 \Rightarrow \frac{d}{dt}\left(\frac{1}{3}\pi r^3\right) = 2$ [Here $r = h$, as $\theta = 45°$]

$\Rightarrow \pi r^2 \frac{dr}{dt} = 2 \Rightarrow \frac{dr}{dt} = \frac{2}{\pi r^2}$ (1)

Now, perimeter $= 2\pi r = p$(let)

$\Rightarrow \frac{d}{dt}(2\pi r) = 2\pi\left(\frac{2}{\pi r^2}\right) = \frac{4}{r^2}$ (2) (Using equation (1))

When $h = 2$ m $\Rightarrow r = 2$ m

Hence $\frac{dp}{dt} = \frac{4}{4} = 1$ m/s.

8. (9) $y = x^3 + x + 16$

$\left(\frac{dy}{dx}\right)_{x_1,y_1} = 3x_1^2 + 1$

$\therefore 3x_1^2 + 1 = \frac{y_1}{x_1}$

$\Rightarrow 3x_1^3 + x_1 = x_1^3 + x_1 + 16$

$\Rightarrow 2x_1^3 = 16$

$\Rightarrow x_1 = 2 \Rightarrow y_1 = 26$

$\therefore m = 13$.

9. (4) We have $f(0) = 2$

Now $y - f(a) = f'(a)[x-a]$

For x intercept $y = 0$, so

$x = a - \frac{f(a)}{f'(a)} = a - 2 \Rightarrow \frac{f(a)}{f'(a)} = 2$

$\Rightarrow \frac{f'(a)}{f(a)} = \frac{1}{2}$

$\therefore$ On integrating both sides w.r.t. a, we get

$\ln f(a) = \frac{a}{2} + C$

$f(a) = Ce^{a/2}$

$f(x) = Ce^{x/2}$

$f(0) = C \Rightarrow C = 2$

$\therefore f(x) = 2e^{x/2}$

Hence $k = 2$, $p = \frac{1}{2} \Rightarrow \frac{k}{p} = 4$

10. (9) $y = ax^2 + bx + c$; $\frac{dy}{dx} = 2ax + b$

When $x = 1, y = 0 \Rightarrow a + b + c = 0$ (1)

$\left.\frac{dy}{dx}\right|_{x=1} = 3$ and $\left.\frac{dy}{dx}\right|_{x=3} = 1$

$2a + b = 3$ (2)

$6a + b = 1$ (3)

Solving (1), (2) and (3)

$a = -\frac{1}{2}; b = 4, c = -\frac{7}{2}$

$\therefore 2a - b - 4c = -1 - 4 + 14 = 9$

11. (5) $y = e^{a+bx^2}$, passes through (1, 1)

$\Rightarrow 1 = e^{a+b}$

$\Rightarrow a + b = 0$

also $\left(\frac{dy}{dx}\right)_{(1,\,1)} = -2$

$\Rightarrow e^{a+bx^2} \cdot 2bx = -2$

$\Rightarrow e^{a+b} \cdot 2b(1) = -2$

$\Rightarrow b = -1$ and $a = 1$

$\Rightarrow 2a - 3b = 5$

12. (4) Let $x=r\cos\theta, y=r\sin\theta$
$\Rightarrow r^2(1+\cos\theta\sin\theta)=1$
$\Rightarrow r^2=\dfrac{2}{2+\sin 2\theta}$
$\Rightarrow r^2_{max}=\dfrac{2}{1}$

Archives

Subjective

1. $\because f''(x)$ exists for all x in [0, 1]
$\therefore f(x)$ and $f'(x)$ are differentiable and continuous in [0, 1]
Now $f(x)$ is continuous in [0, 1] and differentiable in (0, 1) and $f(0)=f(1)$
$\therefore$ By Rolle's theorem, there is at least one c such that $f'(c)=0$, where $0<c<1$
Case I: Let $x=c$, then
$f'(x)=f'(c)=0 \Rightarrow |f'(x)|=|0|=0<1$
Case II: Let $x>c$. By Lagrange's mean value theorem for $f'(x)$ in $[c, x]$
$$\frac{f'(x)-f'(c)}{x-c}=f''(\alpha) \text{ for at least one } \alpha, c<\alpha<x$$
or $f'(x)=(x-c)f''(\alpha)$ $\quad [\because f'(c)=0]$
or $|f'(x)|=|x-c||f''(\alpha)|$
But $x\in[0,1], c\in(0,1)$
$\Rightarrow |x-c|<1-0 \Rightarrow |x-c|<1$ and given $|f''(x)|\le 1$, $\forall x\in[0,1]$
$\therefore |f''(\alpha)|\le 1$
$\therefore |f'(x)|<1.1$ $\quad (\because |f'(x)|=|x-c||f''(\alpha)|)$
or $|f'(x)|<1\ \forall\ x\in[0,1]$.
Case III: Let $x<c$, then
$$\frac{f'(c)-f'(x)}{c-x}=f''(\alpha) \Rightarrow |-f'(x)|=|c-x|\,|f''(\alpha)|$$
$\Rightarrow |f'(x)|<1\times 1 \Rightarrow |f'(x)|<1$
Combining all cases, we get $|f'(x)|<1\ \forall\ x\in[0,1]$.

2. Given that $f(x)$ and $g(x)$ are differentiable for $x\in[0,1]$ such that
$f(0)=2; f(1)=6$
$g(0)=0; g(1)=2$
Consider $h(x)=f(x)-2g(x)$
Then $h(x)$ is continuous on [0, 1] and differentiable on (0, 1)
Also $h(0)=f(0)-2g(0)=2-2\times 0=2$
$h(1)=f(1)-2g(1)=6-2\times 2=2$
$\therefore h(0)=h(1)$
$\therefore$ All the conditions of Rolle's theorem are satisfied for $h(x)$ on [0, 1]
$\therefore$ there exists at least one $c\in(0,1)$ such that $h'(c)=0$.
$\Rightarrow f'(c)-2g'(c)=0 \Rightarrow f'(c)=2g'(c)$.

3. $(0, c)\ y=x^2, 0\le c\le 5$
Any point on the parabola is (x, x^2)
Distance between (x, x^2) and $(0, c)$ is $D=\sqrt{x^2+\left(x^2-c\right)^2}$
To find D_{min} we consider $D^2=x^4-(2c-1)x^2+c^2$
$=\left(x^2-\dfrac{2c-1}{2}\right)^2+c-\dfrac{1}{4}$ which is minimum when
$x^2-\dfrac{2c-1}{2}=0$
$\Rightarrow D_{min}=\sqrt{c-\dfrac{1}{4}}$

4. Slope of the given line is $-1/2$, (1)
$\Rightarrow$ Slope of the tangent $=\left(\dfrac{dy}{dx}\right)=-1/2$
The equation of given curve $y=\cos(x+y)$
Differentiating the curve w.r.t. x, we get
$$\frac{dy}{dx}=-\sin(x+y)\left\{1+\frac{dy}{dx}\right\} \Rightarrow \frac{dy}{dx}=-\frac{\sin(x+y)}{1+\sin(x+y)}$$
= slope of tangent (2)
From (1) and (2), $\dfrac{-\sin(x+y)}{1+\sin(x+y)}=-\dfrac{1}{2}$
$\Rightarrow \sin(x+y)=1$ (3)
$\Rightarrow \cos(x+y)=0$
From the given curve $y=\cos(x+y) \Rightarrow y=0$ (4)
and $\sin(x)=1$ [using (3) and (4)]
$\Rightarrow x=\dfrac{\pi}{2}, -\dfrac{3\pi}{2}$
$\Rightarrow$ The points are $P(\pi/2, 0), Q(-3\pi/2, 0)$.
Tangent at P is $x+2y=\dfrac{\pi}{2}$ and tangent at Q is
$x+2y=-\dfrac{3\pi}{2}$

5. At $x=0, y=1$.
Hence, the point at which normal is drawn is $P(0, 1)$
Differentiating the given equation w.r.t. x, we have
$$(1+x)^y\left\{\log(1+x)\frac{dy}{dx}+\frac{y}{1+x}\right\}$$
$$-\frac{dy}{dx}+\frac{1}{\sqrt{1-\sin^4 x}}\ 2\sin x\cos x=0$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)}=\frac{(1+0)^1\times\dfrac{1}{1+0}-\dfrac{2\sin 0}{\sqrt{1-\sin^2 0}}}{1-(1+0)^1\log 1}=1$$
$\Rightarrow$ Slope of the normal $=-1$
$\Rightarrow$ Equation of the normal having slope -1 at point $P(0, 1)$ is given by $y-1=-(x-0) \Rightarrow x+y=1$.

6. Since the curve $y=ax^3+bx^2+cx+5$ touches x-axis at $P(-2, 0)$, then x-axis is the tangent at $(-2, 0)$. The curve meets y-axis in $(0, 5)$. We have

$\frac{dy}{dx} = 3ax^2 + 2bx + c$

$\Rightarrow \left.\frac{dy}{dx}\right|_{(0,5)} = 0 + 0 + c = 3$ (given)

$\Rightarrow c = 3$ (1)

and $\left.\frac{dy}{dx}\right|_{(-2,0)} = 0$

$\Rightarrow 12a - 4b + c = 0$

$\Rightarrow 12a - 4b + 3 = 0$ [From (1)] (2)

and (– 2, 0) lies on the curve, then

$0 = -8a + 4b - 2c + 5$

$\Rightarrow 0 = -8a + 4b - 1$ ($\because c = 3$)

$\Rightarrow 8a - 4b + 1 = 0$ (3)

From (2) and (3), we get $a = -\frac{1}{2}$, $b = -\frac{3}{4}$

Hence, $a = -\frac{1}{2}$, $b = -\frac{3}{4}$ and $c = 3$.

7. (i) Given that $f(x)$ is a differentiable function on [0, 4]

$\therefore$ It will be continuous on [0, 4]

$\therefore$ By Lagrange's mean value theorem, we get

$\frac{f(4) - f(0)}{4 - 0} = f'(a)$, for $a \in (0, 4)$ (1)

Again since f is continuous on [0, 4] by intermediate mean value theorem, we get

$\frac{f(4) + f(0)}{2} = f(b)$ for $b \in (0, 4)$ (2)

Multiplying (1) and (2), we get

$\frac{[f(4)]^2 - [f(0)]^2}{8} = f'(a) f(b)$; $a, b \in (0, 4)$

or $[f(4)]^2 - [f(0)]^2 = 8 f'(a) f(b)$

(ii) To prove $\int_0^4 f(t)\, dt = 2\left[\alpha f(\alpha^2) + \beta f(\beta^2)\right]$,

$\forall\, 0 < \alpha, \beta < 2$

Let $I = \int_0^4 f(t)\, dt$

Let $t = u^2 \Rightarrow dt = 2u\, du$

$\therefore I = \int_0^2 f(u^2) 2u du$ (3)

Consider $F(x) = \int_0^x f(u^2)\, 2u\, du$

Then clearly $F(x)$ is differentiable and hence continuous on [0, 2].

By Lagrange's mean value theorem, we get some, $\mu \in (0, 2)$

such that $F'(\mu) = \frac{F(2) - F(0)}{2 - 0}$

$\Rightarrow f(\mu^2) 2\mu = \frac{\int_0^2 f(u^2)\, 2u\, du}{2}$ (4)

Again by intermediate mean values theorem, there exist at least one α, β such that $0 < \alpha < \mu < \beta < 2$

$\Rightarrow F'(\mu) = \frac{F'(\alpha) + F'(\beta)}{2}$

[as f is continuous on [0, 2] $\Rightarrow F$ is continuous on [0, 2]]

$\Rightarrow f(\mu^2) 2\mu = \frac{f(\alpha^2)\, 2\alpha + f(\beta^2)\, 2\beta}{2}$

$\Rightarrow f(\mu^2) 2\mu = \alpha f(\alpha^2) + \beta f(\beta^2)$ (5)

From (4) and (5), we get

$\int_0^2 f(u^2)\, 2u\, du = 2[\alpha f(\alpha^2) + \beta f(\beta^2)]$, where $0 < \alpha, \beta < 2$

$\Rightarrow \int_0^4 f(t)\, dt = 2[\alpha f(\alpha^2) + \beta f(\beta^2)]$, $0 < \alpha, \beta < 2$.

8. Given that

$P(x) = 51x^{101} - 2323x^{100} - 45x + 1035$

To show that at least one root of $P(x)$ lies in $(45^{1/100}, 46)$, using Rolle's theorem, we consider anti-derivative of $P(x)$,

i.e., consider $F(x) = \frac{x^{102}}{2} - \frac{2323x^{101}}{101} - \frac{45x^2}{2} + 1035\, x + c$

Then being a polynomial function $F(x)$ is continuous and differentiable.

Now, $F(45^{1/100}) = \frac{(45)^{\frac{102}{100}}}{2} - \frac{2323(45)^{\frac{101}{100}}}{101} - \frac{45\,(45)^{\frac{2}{100}}}{2} + 1035(45)^{\frac{1}{100}} + c$

$= \frac{45}{2}(45)^{\frac{2}{100}} - 23 \times 45(45)^{\frac{1}{100}} - \frac{45\,(45)^{\frac{2}{100}}}{2} + 1035(45)^{\frac{1}{100}} + c$

$= c$

and $F(46) = \frac{(46)^{102}}{2} - \frac{2323(46)^{101}}{101} - \frac{45(46)^2}{2} + 1035\,(46) + c$

$= 23(46)^{101} - 23(46)^{101} - 23 \times 45 \times 46 + 1035 \times 46 = c$

$\therefore F\left(45^{\frac{1}{100}}\right) = F(46) = c$

$\therefore$ Rolle's theorem is applicable.

Hence, there must exist at least one root of $F'(x) = 0$

i.e., $P(x) = 0$ in the interval $\left(45^{1/100}, 46\right)$.

9. Given that $|f(x_1) - f(x_2)| < (x_1 - x_2)^2$, $\forall\, x_1, x_2 \in R$

Let $x_1 = x + h$ and $x_2 = x$, then we get

$|f(x + h) - f(x)| < h^2$

$\Rightarrow |f(x+h)-f(x)| < |h|^2$

$\Rightarrow \left|\dfrac{f(x+h)-f(x)}{h}\right| < |h|$

Taking limit as $h \to 0$ on both sides, we get

$\lim\limits_{h\to 0}\left|\dfrac{f(x+h)-f(x)}{h}\right| < 0$

$\Rightarrow |f'(x)| < 0$

$\Rightarrow f'(x) = 0$

$\Rightarrow f(x)$ is a constant function.

Let $f(x) = k$, i.e., $y = k$

As $f(x)$ passes through $(1, 2) \Rightarrow y = 2$

$\therefore$ equation of the tangent at $(1, 2)$ is $y - 2 = 0\,(x - 1)$, i.e., $y = 2$.

10. $g(x) = (f'(x))^2 + f''(x)f(x) = \dfrac{d}{dx}\,(f(x)f'(x))$

Let $h(x) = f(x)f'(x)$

Since $f(a) = 0, f(b) = 2, f(c) = -1, f(d) = 2, f(e) = 0$

$f(x) = 0$ has four roots, namely, a, α, β, e where $b < \alpha < c$ and $c < \beta < d$.

And $f'(x) = 0$ at three points k_1, k_2, k_3 where $a < k_1 < \alpha$, $\alpha < k_2 < \beta$, $\beta < k_3 < c$

[$\because$ Between any two roots of a polynomial function $f(x) = 0$ there lies at least one root of $f'(x) = 0$]

$\therefore$ There are at least 7 roots of $f(x)f'(x) = 0$

$\Rightarrow$ There are at least 6 roots of $\dfrac{d}{dx}\,(f(x)f'(x)) = 0$

i.e., of $g(x) = 0$.

Objective

Fill in the blanks

1. The given curve is $C: y^3 - 3xy + 2 = 0$

Differentiating w.r.t. x, we get $3y^2\dfrac{dy}{dx} - 3x\dfrac{dy}{dx} - 3y = 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y}{-x + y^2}$

$\therefore$ Slope of the tangent to C at point (x_1, y_1) is $\dfrac{dy}{dx} = \dfrac{y_1}{-x_1 + y_1^2}$

For horizontal tangent $\dfrac{dy}{dx} = 0 \Rightarrow y_1 = 0$

For $y_1 = 0$ in C, we get no value of x_1

$\therefore$ there is no point on C at which tangent is horizontal

$\therefore H = \phi$.

For vertical tangent $\dfrac{dy}{dx} \to \infty$

$\Rightarrow -x_1 + y_1^2 = 0$

$\Rightarrow x_1 = y_1^2$

From C, $y_1^3 - 3y_1^3 + 2 = 0 \Rightarrow y_1^3 = 1 \Rightarrow y_1 = 1 \Rightarrow x_1 = 1$

Therefore, there is only one point $(1, 1)$ at which vertical tangent can be drawn

$\therefore V = \{(1, 1)\}$.

Multiple choice questions with one correct answer

1. a. Consider the function $f(x) = ax^3 + bx^2 + cx + d$ on $[0, 1]$ then being a polynomial, it is continuous on $[0, 1]$ and differentiable on $(0, 1)$ and $f(0) = f(1) = d$.

$f(0) = d, f(1) = a + b + c + d = d$ [as given $a + b + c = 0$]

$\therefore$ By Rolle's theorem, there exists at least one $x \in (0, 1)$ such that $f'(x) = 0$

$\Rightarrow 3ax^2 + 2bx + c = 0$

Thus, equation $3ax^2 + 2bx + c = 0$ has at least one root in $[0, 1]$.

2. c. $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$

$$\frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta \qquad (1)$$

$$\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta) = a\theta\sin\theta \qquad (2)$$

$\Rightarrow \dfrac{dy}{dx} = \tan\theta$ (slope of the tangent)

$\Rightarrow$ Slope of the normal $= -\cot\theta$

$\therefore$ Equation of the normal is

$$y - a(\sin\theta - \theta\cos\theta) = -\frac{\cos\theta}{\sin\theta}\,(x - a(\cos\theta + \theta\sin\theta))$$

$\Rightarrow y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta$
$= -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$

$\Rightarrow x\cos\theta + y\sin\theta = a$

As θ varies, inclination is not constant. Therefore, (a) is not correct.

Clearly, it does not pass through $(0, 0)$

Its distance from the origin $= \left|\dfrac{a}{\sqrt{\cos^2\theta + \sin^2\theta}}\right| = a$,

which is a constant.

3. a. Slope of the tangent at $(x, f(x))$ is $2x + 1$

$\Rightarrow f'(x) = 2x + 1$

$\Rightarrow f(x) = x^2 + x + c$

Also the curve passes through $(1, 2)$. Therefore, $f(1) = 2$

$\Rightarrow 2 = 1 + 1 + c \Rightarrow c = 0 \Rightarrow f(x) = x^2 + x$

$\Rightarrow$ Required area $= \int_0^1 (x^2 + x)\,dx$

$$= \left(\frac{x^3}{3} + \frac{x^2}{2}\right)_0^1 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

4. d. Slope of the tangent $y = f(x)$ is $\dfrac{dy}{dx} = f'(x)_{(3,4)}$

Therefore, slope of the normal $= -\dfrac{1}{f'(x)_{(3,4)}}$

$$= -\frac{1}{f'(3)}$$

$$= \tan\left(\frac{3\pi}{4}\right) \text{ (given)}$$

$\Rightarrow f'(3) = 1$.

5. c. $y=x^2+bx-b \Rightarrow \frac{dy}{dx}=2x+b$

$\Rightarrow$ Equation of the tangent at (1, 1) is

$y-1=(2+b)(x-1)$

$\Rightarrow (b+2)x-y=b+1$

$x\text{-intercept} = \frac{b+1}{b+2} = OA$

and y-intercept $= -(b+1) = OB$

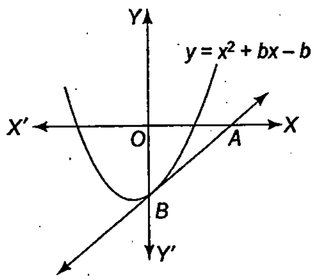


Fig. 5.39

Given area of triangle OAB is $=2$

$\Rightarrow \frac{1}{2} OA \times OB = 2$

$\Rightarrow \frac{1}{2}\left(\frac{b+1}{b+2}\right)[-(b+1)]=2$

$\Rightarrow b^2+2b+1=-4(b+2)$

$\Rightarrow b^2+6b+9=0$

$\Rightarrow (b+3)^2=0 \Rightarrow b=-3.$

6. d. The given curve is $y^3+3x^2=12y$

$\Rightarrow 3y^2\frac{dy}{dx}+6x=12\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}=\frac{2x}{4-y^2}$

For vertical tangents $\frac{dy}{dx}\to\infty$

$\Rightarrow 4-y^2=0$

$\Rightarrow y=\pm 2.$

For $y=2$, $x^2=\frac{24-8}{3}=\frac{16}{3} \Rightarrow x=\pm\frac{4}{\sqrt{3}}$

For $y=-2$, $x^2=\frac{-24+8}{3}=-$ve (not possible)

Hence, the required points are $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$.

7. a. It can be easily seen that functions in options (b), (c) and (d) are continuous on [0, 1] and differentiable in (0, 1)

In (a), $f(x)=\begin{cases}\left(\frac{1}{2}-x\right), x<1/2\\ \left(\frac{1}{2}-x\right)^2, x\ge 1/2\end{cases}$ which is continuous at

$x=1/2$

Also $f'(x)=\begin{cases}-1, & x<1/2\\ -2\left(\frac{1}{2}-x\right), & x>1/2\end{cases}$

Here $f'\left(\frac{1^-}{2}\right)=-1$ and $f'(1/2^+)=-2\left(\frac{1}{2}-\frac{1}{2}\right)=0$

Since $f'\left(\frac{1^-}{2}\right)\ne f'(1/2^+)$, f is not differentiable at $\frac{1}{2}\in(0,1)$

$\therefore$ Lagrange's mean value theorem is not applicable for this function in [0, 1].

8. d For Rolle's theorem in $[a, b]$

$f(a)=f(b)$, In $[0,1] \Rightarrow f(0)=f(1)=0$

$\because$ The function has to be continuous in [0, 1]

$\Rightarrow f(0)=\lim_{x\to 0^+} f(x)=0$

$\Rightarrow \lim_{x\to 0} x^\alpha \log x=0$

$\Rightarrow \lim_{x\to 0}\frac{\log x}{x^{-\alpha}}=0$

Applying L' Hopital rule, we get

$\Rightarrow \lim_{x\to 0}\frac{1/x}{-\alpha x^{-\alpha-1}}=0 \Rightarrow \frac{-x^\alpha}{\alpha}=0 \Rightarrow \alpha>0$

$\Rightarrow$ d is the correct option.

9. b. Let the polynomial be $P(x)=ax^2+bx+c$

Given $P(0)=0$ and $P(1)=1$

$\Rightarrow c=0$ and $a+b=1 \Rightarrow a=1-b$

$\therefore P(x)=(1-b)x^2+bx$

$\Rightarrow P'(x)=2(1-b)x+b$

Given $P'(x)>0,\ \forall\ x\in(0,1)$

$\Rightarrow 2(1-b)x+b>0,\ \forall\ x\in(0,1)$

Now when $x=0$, $b>0$ and when $x=1$, $b<2$

$\Rightarrow 0<b<2$

$\therefore S=\{(1-a)x^2+ax, a\in(0,2)\}.$

10. a. Slope of the tangent to $y=e^x$ at (c, e^c) is given by

$m_1=\left(\frac{dy}{dx}\right)_{(c,e^c)}=e^c$

Also the slope of the line joining the points $(c-1, e^{c-1})$ and $(c+1, e^{c+1})$

$m_2=\frac{e^{c+1}-e^{c-1}}{(c+1)-(c-1)}$

$=\frac{e^{c+1}-e^{c-1}}{2}$

$=e^c\left(\frac{e-e^{-1}}{2}\right)$

we observe $m_2 > m_1$

$\Rightarrow$ tangent to the curve $y = e^x$ will intersect the given line to the left of the line $x = c$ as shown in Fig. 5.40.

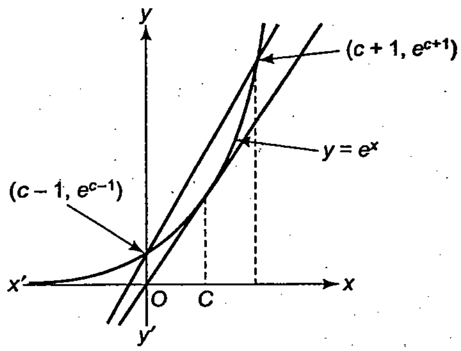


Fig. 5.40

Multiple choice question with one or more than one correct answer

1. b, c. Let the line $ax + by + c = 0$ be normal to the curve $xy = 1$.
Differentiate the curve $xy = 1$ w.r.t. x, we get

$$y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{y_1}{x_1}$$

$\therefore$ Slope of the normal $= \dfrac{x_1}{y_1}$

Slope of the given line $= \dfrac{-a}{b}$

Given that $\dfrac{x_1}{y_1} = \dfrac{-a}{b}$ (1)

Also (x_1, y_1) lies on the given curve $\Rightarrow x_1y_1 = 1$ (2)

From (1) and (2), we can conclude that a and b must have opposite sign.

2. b, d.

For $y^2 = 4ax$, y-axis is tangent at (0, 0), while for $x^2 = 4ay$, x-axis is tangent at (0, 0).

Thus the two curves cut each other at right angles.

$\therefore$ Also for $y^2 = 4ax$, $\dfrac{dy}{dx} = \dfrac{2a}{y} = m_1$

For $y = e^{-x/2a}$, $\dfrac{dy}{dx} = \dfrac{-1}{2a}e^{-x/2a} = \dfrac{-y}{2a} = m_2$

$\Rightarrow m_1m_2 = -1$

$\Rightarrow y^2 = 4ax$ and $y = e^{-x/2a}$ intersect at right angle.

Linked comprehension type

1. b. For $k = 0$, line $y = x$ meets $y = 0$, i.e., x-axis only at one point.
For $k < 0$, $y = ke^x$ meets $y = x$ only once as shown in Fig. 5.41.

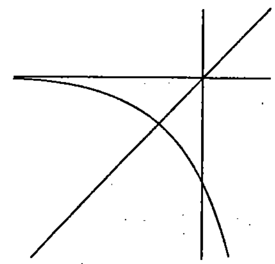


Fig. 5.41

2. a. Let $f(x) = ke^x - x$
Now for $f(x) = 0$ to have only one root means the line $y = x$ must be tangential to the curve $y = ke^x$.
Let it be so at (x_1, y_1), then

$$\left(\frac{dy}{dx}\right)_{x_1} = ke^{x_1} = 1$$

$$\Rightarrow e^{x_1} = \frac{1}{k} \text{ also } y_1 = ke^{x_1} \text{ and } y_1 = x_1$$

$\Rightarrow x_1 = 1 \Rightarrow 1 = ke \Rightarrow k = 1/e.$

3. a. $\because$ For $y = x$ to be the tangent to the curve $y = ke^x$, $k = 1/e$
$\therefore$ For $y = ke^x$ to meet $y = x$ at two points, we should have

$$k < \frac{1}{e} \Rightarrow k \in \left(0, \frac{1}{e}\right) \text{ as } k > 0.$$