Rational and Irrational Numbers

- The numbers, which can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$, are called rational numbers. Rational numbers can be positive as well as negative. Rational numbers include all integers and fractions.

  For example

  $$-\frac{2}{7}, \frac{41}{366}, 2 = \frac{2}{1}, \text{ etc.}$$

- To find rational numbers between any two given rational numbers, firstly we have to make their denominators same and then find the respective rational numbers.

  Example:

  Find some rational numbers between $\frac{1}{6}$ and $\frac{7}{8}$.

  Solution:

  The L.C.M. of 6 and 8 is 24.

  Now, we can write

$$\frac{1}{6}=\frac{1\times 4}{6\times 4}=\frac{4}{24}$$
$$\frac{7}{8}=\frac{7\times 3}{8\times 3}=\frac{21}{24}$$

Therefore, some of the rational numbers between $\frac{4}{24}\left(\frac{1}{6}\right)$ and $\frac{21}{24}\left(\frac{7}{8}\right)$ are

$$\frac{5}{24},\frac{6}{24},\frac{7}{24},\frac{8}{24},\frac{9}{24},\frac{10}{24},\frac{11}{24},\frac{12}{24},\frac{13}{24},\frac{14}{24},\frac{15}{24},\frac{16}{24},\frac{17}{24},\frac{18}{24},\frac{19}{24},\frac{20}{24}$$

- Natural numbers are a collection of all positive numbers starting from 1.
- Whole numbers are a collection of all natural numbers including 0.
- Integers are the set of numbers comprising of all the natural numbers 1, 2, 3 ... and their negatives −1, −2, −3 ..., and the number 0.
- Rational numbers are the numbers that can be written in $\frac{p}{q}$ form, where p and q are integers and $q \neq 0$
- **Closure property**
    - Whole numbers are closed under addition and multiplication. However, they are **not** closed under subtraction and division.
    - Integers are also closed under addition, subtraction and multiplication. However, they are **not** closed under division.
    - Rational numbers:
        1. Rational numbers are closed under addition.

        Example: $\frac{2}{5}+\frac{3}{2}=\frac{19}{10}$ is a rational number.

        2. Rational numbers are closed under subtraction.

        Example: $\frac{1}{5}-\frac{3}{4}=\frac{-11}{20}$ is rational number.

        3. Rational numbers are closed under multiplication.

        Example: $\frac{2}{3}\times\left(\frac{-3}{5}\right)=\frac{-2}{5}$ is a rational number.

4. Rational numbers are **not** closed under division.

Example: $2 \div 0$ is not defined.

- Decimal expansion of a rational number can be of two types:

  (i) Terminating

  (ii) Non-terminating and repetitive

  In order to find decimal expansion of rational numbers we use long division method.

  For example, to find the decimal expansion of $\frac{1237}{25}$.

  We perform the long division of 1237 by 25.

$$
\begin{array}{r|l}
 & 49.48 \\ \hline
25 & 1237.00 \\
 & \underline{100} \\
 & 237 \\
 & \underline{225} \\
 & 120 \\
 & \underline{100} \\
 & 200 \\
 & \underline{200} \\
 & \underline{0}
\end{array}
$$

  Hence, the decimal expansion of $\frac{1237}{25}$ is 49.48. Since the remainder is obtained as zero, the decimal number is terminating.

- Every number of the form $\sqrt{p}$, where p is a prime number is called an irrational number. For example, $\sqrt{3}$, $\sqrt{11}$, $\sqrt{12}$ etc.

Theorem: If a prime number p divides a^2, then p divides a, where a is a positive integer.

Example:
Prove that $\sqrt{7}$ is an irrational number.

Solution:
If possible, suppose $\sqrt{7}$ is a rational number.

Then, $\sqrt{7} = \frac{p}{q}$, where p, q are integers, $q \neq 0$.

If HCF $(p, q) \neq 1$, then by dividing p and q by HCF(p, q), $\sqrt{7}$ can be reduced as

$\sqrt{7} = \frac{a}{b}$ where HCF $(a, b) = 1$ … (1)

$\Rightarrow \sqrt{7}b = a$

$\Rightarrow 7b^2 = a^2$

$\Rightarrow a^2$ is divisible by 7
$\Rightarrow a$ is divisible by 7 … (2)
$\Rightarrow a = 7c$, where c is an integer
$\therefore \sqrt{7}c = b$

$\Rightarrow 7b^2 = 49c^2$

$\Rightarrow b^2 = 7c^2$

$\Rightarrow b^2$ is divisible by 7
$\Rightarrow b$ is divisible by 7 … (3)
From (2) and (3), 7 is a common factor of a and b. which contradicts (1)

$\therefore \sqrt{7}$ is an irrational number.

Example:
Show that $\sqrt{12} - 6$ is an irrational number.

Solution:
If possible, suppose $\sqrt{12} - 6$ is a rational number.

Then $\sqrt{12} - 6 = \frac{p}{q}$ for some integers p, q (q ^{1} 0)

Now,

$$\sqrt{12} - 6 = \frac{p}{q}$$
$$\Rightarrow 2\sqrt{3} = \frac{p}{q} + 6$$
$$\Rightarrow \sqrt{3} = \frac{1}{2}\left(\frac{p}{q} + 6\right)$$

As p, q, 6 and 2 are integers, $\frac{1}{2}\left(\frac{p}{q} + 6\right)$ is rational number, so is $\sqrt{3}$.
This conclusion contradicts the fact that $\sqrt{3}$ is irrational.
Thus, $\sqrt{12} - 6$ is an irrational number.

- **Operation on irrational numbers:**
    - **Like terms:** The terms or numbers whose irrational parts are the same are known as like terms. We can add or subtract like irrational numbers only.
    - **Unlike terms:** The terms or numbers whose irrational parts are not the same are known as unlike terms.

We can perform addition, subtraction, multiplication and division involving irrational numbers.

Note:

(1) The sum or difference of a rational and an irrational number is always irrational.

(2) The product or quotient of a non-zero rational number and an irrational number is always irrational.

Example:

(1) $\left(\mathbf{2\sqrt{3} + \sqrt{2}}\right) + \left(\mathbf{3\sqrt{3} - 5\sqrt{2}}\right)$

$= \left(2\sqrt{3} + 3\sqrt{3}\right) + \left(\sqrt{2} - 5\sqrt{2}\right)$ (Collecting like terms)
$= \left(2 + 3\right)\sqrt{3} + \left(1 - 5\right)\sqrt{2}$
$= 5\sqrt{3} - 4\sqrt{2}$

(2) $\left(\mathbf{5\sqrt{7} - 3\sqrt{2}}\right) - \left(\mathbf{7\sqrt{7} + 3\sqrt{2}}\right)$

$= 5\sqrt{7} - 3\sqrt{2} - 7\sqrt{7} - 3\sqrt{2}$
$= 5\sqrt{7} - 7\sqrt{7} - 3\sqrt{2} - 3\sqrt{2}$
$= (5-7)\sqrt{7} - (3+3)\sqrt{2}$ (Collecting like terms)
$= -2\sqrt{7} - 6\sqrt{2}$

(3) $\mathbf{\left(4\sqrt{5} + 3\sqrt{2}\right) \times \sqrt{2}}$

$= 4\sqrt{5} \times \sqrt{2} + 3\sqrt{2} \times \sqrt{2}$
$= 4\sqrt{10} + 3 \times 2 \qquad \left(\sqrt{2} \times \sqrt{2} = 2\right)$
$= 4\sqrt{10} + 6$

(4) $\mathbf{5\sqrt{6} \div \sqrt{12}}$

$= 5\sqrt{6} \times \frac{1}{\sqrt{12}}$
$= \frac{5 \times \sqrt{2} \times \sqrt{3}}{2 \times \sqrt{3}}$
$= \frac{5}{2}\sqrt{2}$

• If $\sqrt[a]{x}$ is an irrational number such that x is a positive rational number and a ($a \neq 1$) is a natural number, then $\sqrt[a]{x}$ is known as a **surd**. Here, $\sqrt{\ }$ is **radical** sign, a is **order** of the surd and x is **radicand**.

For example, $\sqrt[3]{10}$ is a surd of order 3.

• A surd whose order is 2 is called **quadratic surd**.

• **Rules for surds:**

If $x, y \in$ Q, $x, y > 0$ and $a, b, c \in$ N, then

(1) $\sqrt[a]{x} = (x)^{\frac{1}{a}}$

(2) $\left(\sqrt[a]{x}\right)^a = x$

(3) $\sqrt[a]{x}.\sqrt[a]{y} = \sqrt[a]{xy}$

(4) $\frac{\sqrt[a]{x}}{\sqrt[a]{y}} = \sqrt[a]{\frac{x}{y}}$

(5) $\sqrt[a]{\sqrt[b]{x}} = \sqrt[b]{\sqrt[a]{x}} = \sqrt[ab]{x}$

(6) $\sqrt[a]{x^b} = \sqrt[ax]{x^{bx}}$

(7) $\sqrt[a]{x^b} = \left(\sqrt[a]{x}\right)^b$

For example, $\sqrt[4]{\sqrt{8}} = \sqrt[4\times 2]{8} = \sqrt[8]{8}$ $\left[\sqrt[a]{\sqrt[b]{x}} = \sqrt[ab]{x}\right]$

Forms of surds

Pure form: A surd of the form $k\sqrt[a]{x}$ where $k \in$ Q such that $k = \pm 1$. For example, $\sqrt[3]{7}, -\sqrt{11}$ are pure surds.

Mixed form: A surd of the form $k\sqrt[a]{x}$ where $k \in$ Q such that $k \neq 0$ and $k \neq \pm 1$. For example, $3\sqrt[3]{5}, -4\sqrt{16}$ are mixed surds.

• **Conversion of mixed surds into pure surds:**

For example, $-5\sqrt{3} = -\sqrt{25}\sqrt{3} = -\sqrt{25\times 3} = -\sqrt{75}$

• **Conversion of pure surds into mixed surds:**

For example, $\sqrt{27} = \sqrt{9\times 3} = \sqrt{9}\times\sqrt{3} = \sqrt{3^2}\times\sqrt{3} = 3\sqrt{3}$

• In cases, where the radicand is a prime number or it has the factors whose roots are irrational, it is not possible to express pure surd as mixed surd.

For example, $\sqrt{11}, \sqrt{21}$ etc.

• If a rational number is obtained after multiplying two surds then each surd is called the **rationalizing factor** of the other.

For example, $\sqrt{3}\times\sqrt{27} = \sqrt{3\times 27} = \sqrt{81} = \sqrt{9^2} = 9$

So, $\sqrt{3}$ and $\sqrt{27}$ are rationalizing factors of each other.

- **Rationalization of denominators:**
    - The denominator of $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{x}+\sqrt{y}}$ can be rationalized by multiplying both the numerator and the denominator by $\sqrt{x}-\sqrt{y}$, where *a*, *b*, *x* and *y* are integers.
    - The denominator of $\frac{\sqrt{a}+\sqrt{b}}{c+\sqrt{d}}$ can be rationalized by multiplying both the numerator and the denominator by $c-\sqrt{d}$, where *a*, *b*, *c* and *d* are integers.

Note: $\sqrt{x}-\sqrt{y}$ and $c-\sqrt{d}$ are the conjugates of $\sqrt{x}+\sqrt{y}$ and $c+\sqrt{d}$ respectively.

Example: Rationalize $\frac{2\sqrt{2}}{\sqrt{5}+\sqrt{3}}$

Solution:

$$\frac{2\sqrt{2}}{\sqrt{5}+\sqrt{3}}$$

$$=\frac{2\sqrt{2}}{\sqrt{5}+\sqrt{3}}\times\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}$$

$$=\frac{2\sqrt{2\times5}-2\sqrt{2\times3}}{\left(\sqrt{5}\right)^2-\left(\sqrt{3}\right)^2}\qquad\left[(a+b)(a-b)=a^2-b^2\right]$$

$$=\frac{2\sqrt{10}-2\sqrt{6}}{5-3}$$

$$=\frac{2\left(\sqrt{10}-\sqrt{6}\right)}{2}$$

$$=\sqrt{10}-\sqrt{6}$$