CHAPTER

3

Parabola

- Introduction
- Standard Equation of Parabola
- General Equation of a Parabola

INTRODUCTION

Let l be a fixed vertical line and m be another line intersecting it at a fixed point V and inclined to it at an angle α. Suppose we rotate the line m around the line l in such a way that the angle α remains constant. Then the surface generated is a double-napped right circular hollow cone herein after referred as cone extending indefinitely far in both directions. The point V is called the *vertex*; the line l is the *axis* of the cone. The rotating line m is called a *generator* of the cone. The *vertex* separates the cone into two parts called *nappes*.

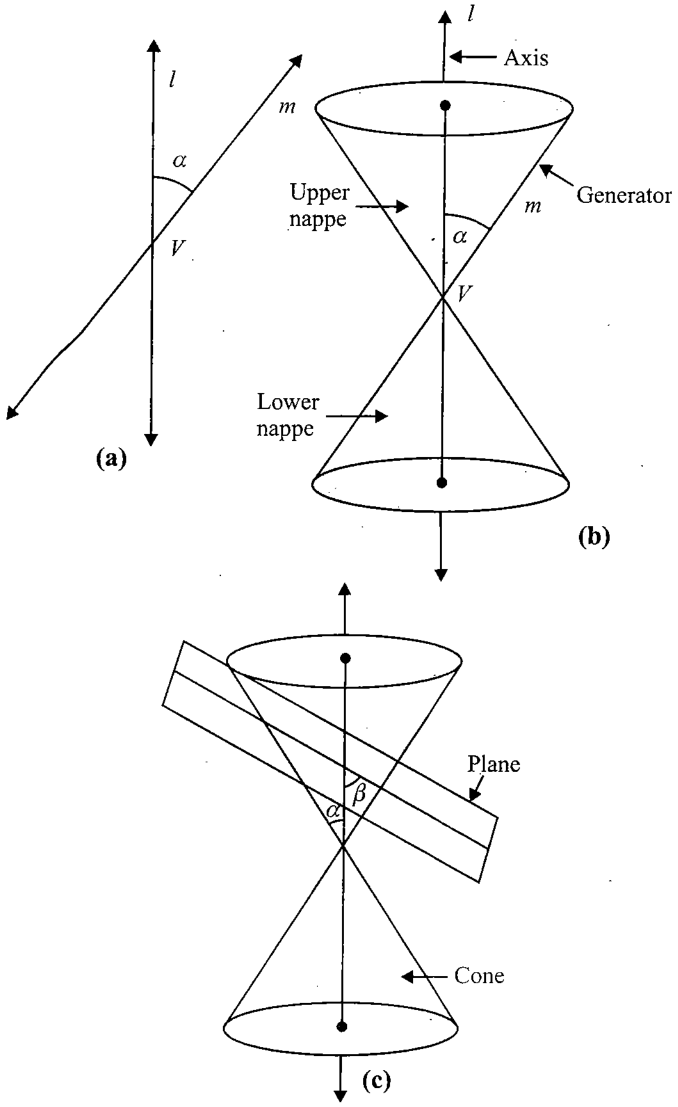


Fig. 3.1

If we take the intersection of a plane with a cone, the section so obtained is called a *conic section*. Thus, conic sections are the curves obtained by intersecting a right circular cone by a plane. We obtain different kinds of conic sections depending on the position of the intersecting plane with respect to the cone and the angle made by it with the vertical axis of the cone. Let β be the angle made by the intersecting plane with the vertical axis of the cone. The intersection of the plane with the cone can take place either at the vertex of the cone or at any other part of the nappe either below or above the vertex.

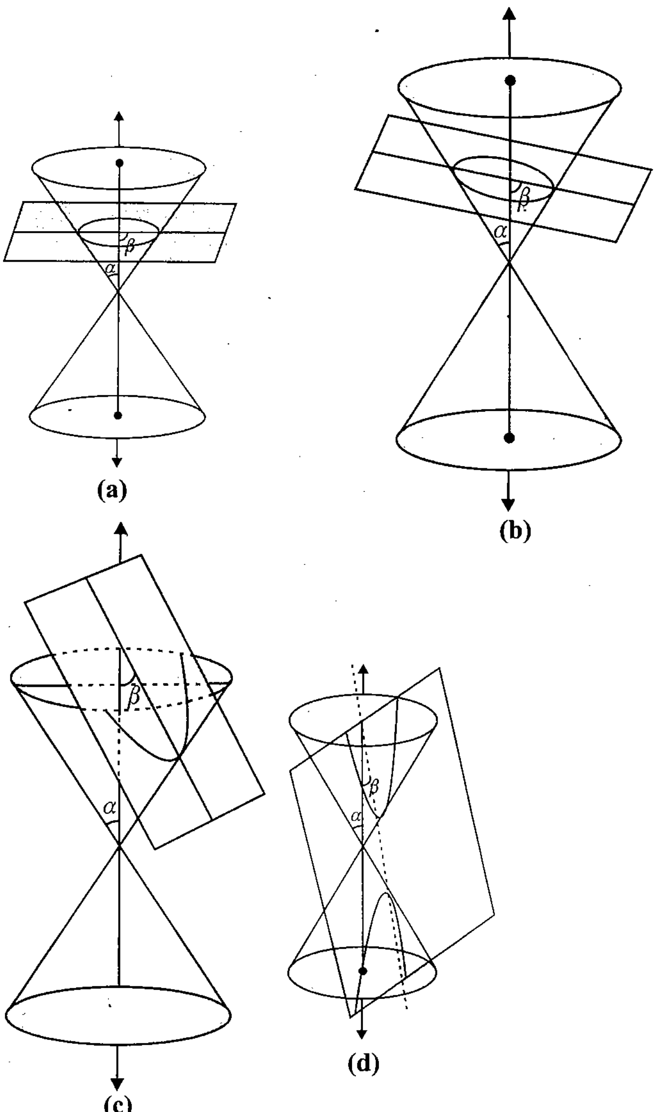


Fig. 3.2

Circle, Ellipse, Parabola and Hyperbola

When the plane cuts the nappe (other than the vertex) of the cone, we have the following situations:

a. When $\beta = 90°$, the section is a *circle*.

b. When $\alpha < \beta < 90°$, the section is an *ellipse*.

c. When $\beta = \alpha$, the section is a *parabola* (in each of the above three situations, the plane cuts entirely across one nappe of the cone).

d. When $0 \le \beta < \alpha$; the plane cuts through both the nappes and the curves of intersection is a *hyperbola*.

Conic Section as a Locus of a Point

The locus of a point which moves in a plane such that the ratio of its distance from a fixed point to its perpendicular distance from a fixed straight line not passing through given fixed point is always constant, is known as a conic section or conic.

The fixed point is called the *focus* of the conic and fixed line is called the *directrix* of the conic.

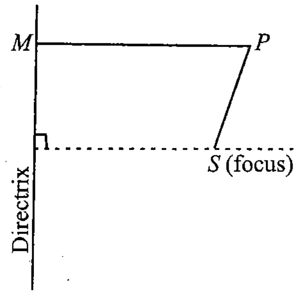


Fig. 3.3

Also this constant ratio is called the eccentricity of the conic and is denoted by e.

If $e = 1$, the conic is called parabola.
If $e < 1$, the conic is called ellipse.
If $e > 1$, the conic is called hyperbola.
If $e = 0$, the conic is called circle.
If $e = \infty$, the conic is called pair of straight lines.
In Fig. 3.3 $\frac{SP}{PM}$ = constant = e or $SP = ePM$

Equation of Conic Section

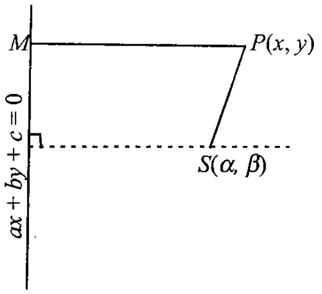


Fig. 3.4

If the focus is (α, β) and the directrix is $ax + by + c = 0$ then the equation of the conic section whose eccentricity = e is

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = e\frac{|ax + by + c|}{\sqrt{(a^2 + b^2)}}$$

$$\text{or} \quad (x-\alpha)^2 + (y-\beta)^2 = e^2\frac{(ax + by + c)^2}{(a^2 + b^2)}$$

Important Terms

Axis: The straight line passing through the focus and perpendicular to the directrix is called the axis of the conic section.

Vertex: The points of intersection of the conic section and the axis is (are) called vertex (vertices) of the conic section.

Focal chord: Any chord passing through the focus is called focal chord of the conic section.

Double ordinate: A straight line that is drawn perpendicular to the axis and terminates at both ends of the curve is a double ordinate of the conic section.

Latus rectum: The double ordinate passing through the focus is called the latus rectum of the conic section.

Centre: The point that bisects every chord of the conic passing through it is called the centre of the conic section.

Note:
Parabola has no centre, but circle, ellipse, hyperbola have centre.

STANDARD EQUATION OF PARABOLA

Consider the focus of the parabola as $S(a, 0)$ and directrix be $x + a = 0$, and axis as x-axis.

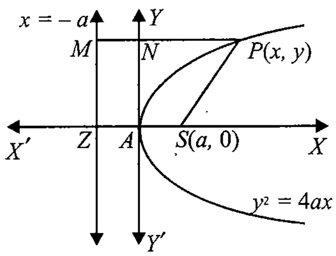


Fig. 3.5

Now according to the definition of the parabola, for any point on the parabola, we must have

$$SP = PM$$

$$\Rightarrow \quad \sqrt{(x-a)^2 + (y-0)^2} = PN + NM = x + a$$

$$\Rightarrow \quad (x-a)^2 + y^2 = (a+x)^2$$

$$\Rightarrow \quad y^2 = (a+x)^2 - (x-a)^2$$

$$\Rightarrow \quad y^2 = 4ax$$

Vertex:	(0, 0)
Tangent at vertex:	$x = 0$
Equation of latus rectum:	$x = a$
Extremities of latus rectum:	$(a, 2a), (a, -2a)$
Length of latus rectum:	$4a$
Focal distance (SP):	$SP = PM = x + a$
Parametric form:	$x = at^2$ and $y = 2at$, where t is parameter

Other Standard Forms of Parabola

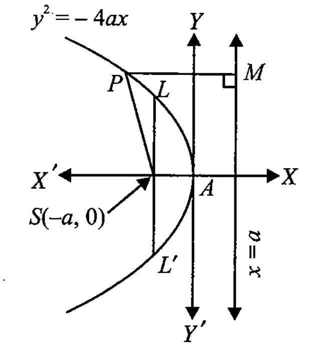


Fig. 3.6

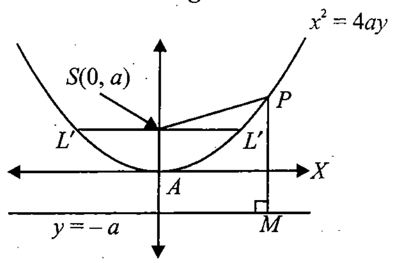


Fig. 3.7

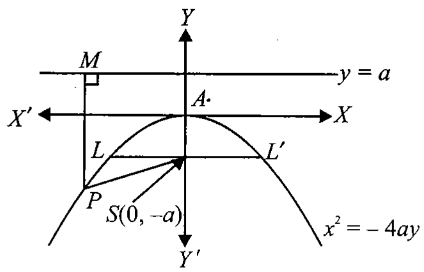


Fig. 3.8

Equation of curve:	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Vertex:	(0, 0)	(0, 0)	(0, 0)
Focus:	$(-a, 0)$	$(0, a)$	$(0, -a)$
Directrix:	$x - a = 0$	$y + a = 0$	$y - a = 0$
Axis:	$y = 0$	$x = 0$	$x = 0$
Tangent at vertex:	$x = 0$	$y = 0$	$y = 0$
Parametric form:	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$

If focus and vertex of the parabola are $(p, 0)$ and $(q, 0)$, then its equation is

$y^2 = 4(q - p)(x - p)$(when $p < q$)

or $y^2 = -4(p - q)(x - p)$ (where $q < p$)

If focus and vertex of the parabola are $(p, 0)$ and $(q, 0)$, then its equation is

$x^2 = 4(q - p)(y - p)$ (when $q > p$)

or $x^2 = -4(p - q)(x - p)$ (when $q < p$)

Example 3.1 **Find the equation of a parabola**

i. having its vertex at A (1, 0) and focus at S (3, 0)

ii. having its focus at S (2, 0) and one extremity of its latus rectum as (2, 2)

iii. having focus at (0, − 3) and its directrix is $y = 3$

Sol. i. Clearly the axis of the parabola is x-axis. Corresponding value of $a = 3 - 1 = 2$. Thus equation of the parabola is $y^2 = 8(x - 1)$.

ii. Clearly the other extremity of latus rectum is $(2, -2)$. Its axis is x-axis. Corresponding value of $a = \frac{2-0}{2} = 1$. Hence, its vertex is (1, 0) or (3, 0). Thus its equation is $y^2 = 4(x - 1)$ or $y^2 = -4(x - 3)$.

iii.

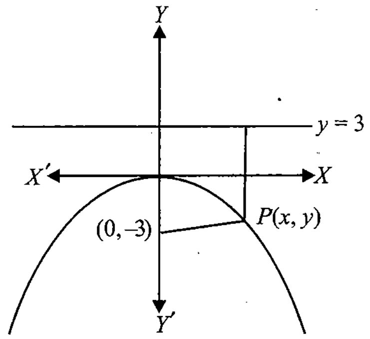


Fig. 3.9

Let $P(x, y)$ be any point on the parabola.

Then by definition $\sqrt{(x-0)^2 + (y+3)^2} = y - 3$

$\Rightarrow \quad x^2 = -12y.$

Example 3.2 **A beam is supported at its ends by two supports which are 12 m apart. Since the load is concentrated at its centre, there is a deflection of 3 cm at the centre and the deflected beam is in the shape of a parabola. How far from the centre is the deflection 1 cm?**

Sol.

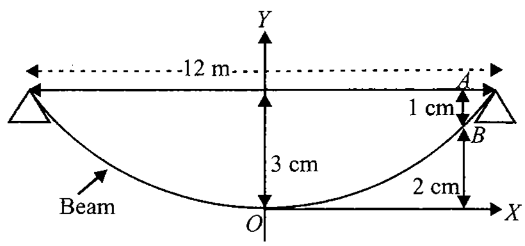


Fig. 3.10

Let the vertex be at the lowest point and the axis vertical. Let the coordinate axis be chosen as shown in Fig. 3.10.

The equation of the parabola takes the form $x^2 = 4ay$. Since it passes through $\left(6, \frac{3}{100}\right)$, we have $6^2 = 4a \times \frac{3}{100}$, i.e., $a = 300$ m.

Let AB be the deflection of the beam which is $\frac{1}{100}$ m. Coordinates of B are $\left(x, \frac{2}{100}\right)$.

Therefore, $\quad x^2 = 4 \times 300 \times \frac{2}{100} = 24.$

$\Rightarrow \quad x = 2\sqrt{6}$ m

Example 3.3 **If the two ends of the latus rectum are given. How many parabolas can be drawn?**

Sol.

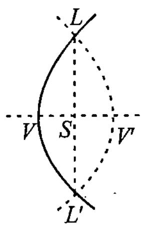


Fig. 3.11

L, L' are the ends of the latus rectum. S bisects LL'. VSV' is the perpendicular bisector of LL', where $VS = \frac{1}{4} LL' = V'S$. Clearly, two parabolas are possible.

Example 3.4 **Find the coordinates of a point on the parabola $y^2 = 8x$ whose focal distance is 4.**

Sol. If the coordinates of a point on the parabola $y^2 = 4ax$ are $P(x, y)$, then its focal distance is $SP = x + a$.

Here, $\quad a = 2$ and $SP = 4$.

$\therefore \quad 4 = x + 2$

$\Rightarrow \quad x = 2$

$\Rightarrow \quad y^2 = 8 \times 2$

$\Rightarrow \quad y = \pm 4$

Thus, the coordinates of the required point are $(2, \pm 4)$.

Example 3.5 **If the vertex of a parabola is the point $(-3, 0)$ and the directrix is the line $x + 5 = 0$, then find its equation.**

Sol. Since the line passing through the focus and perpendicular to the directirx is x-axis, therefore axis of the required parabola is x-axis.

Let the coordinates of the focus be $S(a, 0)$.

Since the vertex is the midpoint of the line joining the focus and the point $(-5, 0)$ where the directrix $x + 5 = 0$ meets the axis.

Therefore,

$$-3 = \frac{a-5}{2}$$

$\Rightarrow \quad a = -1$

Thus, the coordinates of the focus are $(-1, 0)$.

Let $P(x, y)$ be a point on the parabola.

Then by definition

$$\sqrt{(x+1)^2 + y^2} = (x+5)^2$$

$\Rightarrow \quad y^2 = 8(x + 3)$

Example 3.6 **The chord AB of the parabola $y^2 = 4ax$ cuts the axis of the parabola at C. If $A = (at_1^2, 2at_1)$, $B = (at_2^2, 2at_2)$ and $AC{:}AB = 1{:}3$, then prove that $t_2 + 2t_1 = 0$.**

Sol.

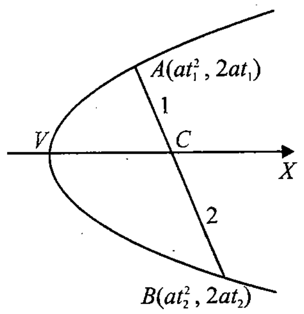


Fig. 3.12

$$\frac{AC}{AB} = \frac{1}{3} \Rightarrow \frac{AC}{BC} = \frac{1}{2}$$

Here, $\quad C = \left(\frac{2at_1^2 + at_2^2}{3}, \frac{4at_1 + 2at_2}{3}\right)$

It lies on $\quad y = 0$

$\therefore \quad \frac{4at_1 + 2at_2}{3} = 0 \Rightarrow t_2 + 2t_1 = 0$

Example 3.7 **M is the foot of the perpendicular from a point P on a parabola $y^2 = 4ax$ to its directrix and SPM is an equilateral triangle, where S is the focus. Then find SP.**

Sol.

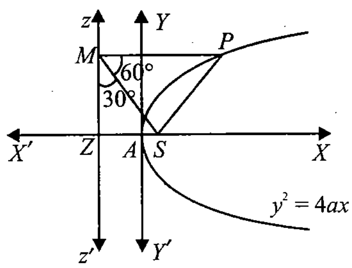


Fig. 3.13

From the definition of the parabola, we have

$$SP = PM$$

SPM is an equilateral triangle.

Therefore,

$$SP = PM = SM$$

$\Rightarrow \quad \angle PMS = 60°$

$\Rightarrow \quad \angle SMZ = 30°$

In ΔSMZ, we have

$$\sin 30° = \frac{SZ}{SM}$$

$\Rightarrow \qquad \frac{1}{2} = \frac{2a}{SM}$

$\Rightarrow \qquad SM = 4a$

Hence, $\qquad SP = SM = 4a$

Example 3.8 **An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, such that one vertex of this triangle coincides with the vertex of the parabola. Then find the side length of this triangle.**

Sol.

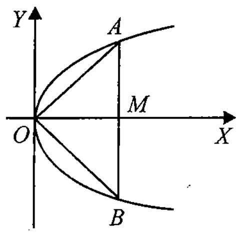


Fig. 3.14

If triangle OAB is equilateral then $OA = OB = AB = p$.

Thus AB will be a double ordinate of the parabola.

Thus $\qquad \angle AOM = \angle MOB = \frac{\pi}{6}$

$\Rightarrow \qquad OM = p \cos \frac{\pi}{6}$ and $AM = p \sin \frac{\pi}{6}$

Then A has coordinates $\left(\frac{\sqrt{3}p}{2}, \frac{p}{2}\right)$

A lies on the parabola, then $\frac{p^2}{4} = 4a\frac{\sqrt{3}p}{2}$

$\Rightarrow \qquad p = 8\sqrt{3}a$

Example 3.9 **A quadrilateral is inscribed in a parabola $y^2 = 4ax$ and three of its sides pass through fixed points on the axis. Show that the fourth side also passes through fixed point on the axis of the parabola.**

Sol.

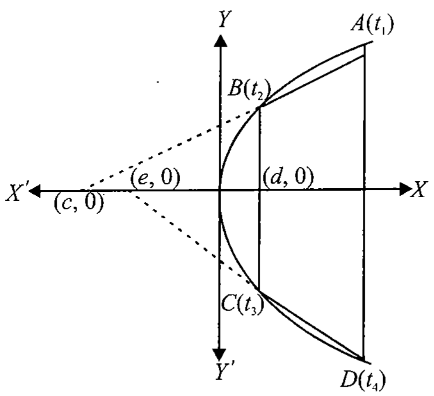


Fig. 3.15

Equation of chord AB is

$$2x - (t_1 + t_2)y + 2at_1t_2 = 0$$

It passes through the point $(c, 0)$,

then given $\qquad t_1t_2 = -\frac{c}{a} \qquad$ (i)

Similarly for chord BC, $t_2t_3 = -\frac{d}{a} \qquad$ (ii)

And for chord CD, $\qquad t_3t_4 = -\frac{e}{a} \qquad$ (iii)

Multiplying (i) and (iii), we get

$$t_1t_2t_3t_4 = +\frac{ec}{a^2}$$

$\Rightarrow \qquad t_1t_4\left(-\frac{d}{a}\right) = \frac{ec}{a^2}$

$\Rightarrow \qquad t_1t_4 = -\frac{ec}{ad}$

Hence, chord AD passes through the fixed point.

Example 3.10 **Find the locus of middle points of chords of a parabola $y^2 = 4ax$ which subtend a right angle at the vertex of the parabola.**

Sol. Here, $h = \frac{at_1^2 + at_2^2}{2}$, $k = \frac{2at_1 + 2at_2}{2} = a(t_1 + t_2)$

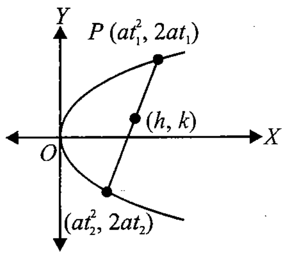


Fig. 3.16

Also, $\frac{2at_1 - 0}{at_1^2 - 0} \times \frac{2at_2 - 0}{at_2^2 - 0} = -1$

$\Rightarrow \qquad t_1t_2 = -4$

Now, $\qquad \frac{2h}{a} = t_1^2 + t_2^2 = (t_1 + t_2)^2 - 2t_1t_2$

$$= \left(\frac{k}{a}\right)^2 - 2(-4)$$

Therefore, the locus of (h, k) is

$$\frac{2x}{a} = \frac{y^2}{a^2} + 8$$

which is a parabola.

Example 3.11 **Let there be two parabolas $y^2 = 4ax$ and $y^2 = -4bx$ (where $a \neq b$ and $a, b > 0$). Then find the locus of**

the middle points of the intercepts between the parabolas made on the lines parallel to the common axis.

Sol. Let the line parallel to common axis be $y = h$.

Then coordinates of A and B are $\left(\frac{h^2}{4a}, h\right)$ and $\left(-\frac{h^2}{4b}, h\right)$, respectively.

If $P(\alpha, \beta)$ is a midpoint of AB, then $\alpha = \frac{1}{2}\left(\frac{h^2}{4a} - \frac{h^2}{4b}\right)$ and $\beta = h$

$$\therefore \quad 2\alpha = \frac{\beta^2}{4}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Hence, locus of P is

$$2x = \frac{y^2}{4}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Example 3.12 **A right-angled triangle ABC is inscribed in parabola $y^2 = 4x$, where A is vertex of parabola and $\angle BAC = \pi/2$. If $AB = \sqrt{5}$, then find the area of ΔABC.**

Sol.

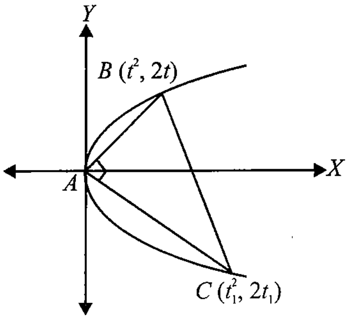


Fig. 3.17

From the figure, $AB = \sqrt{5}$

$\Rightarrow \quad t^4 + 4t^2 = 5$

$\Rightarrow \quad (t^2 - 1)(t^2 + 4) = 0$

$\Rightarrow \quad t = \pm 1$

$\Rightarrow \quad B \equiv (1, 2)$

Also $m_{AB} \times m_{AC} = -1$

$\therefore \; \frac{2}{t} \cdot \frac{2}{t_1} = -1$

$\therefore \; t_1 = -4$

$\Rightarrow C$ has coordinates $(16, -8)$

Now $\quad AC = \sqrt{256 + 64} = \sqrt{320}$

Then, the area of $\angle ABC$ is $\frac{1}{2}\sqrt{5}\,\sqrt{320} = \frac{1}{2}\sqrt{1600} = 20$.

Example 3.13 **AP is perpendicular to PB, where A is vertex of parabola $y^2 = 4x$ and P on the parabola. B is on the axis of parabola. Then find the locus of centroid of ΔPAB.**

Sol.

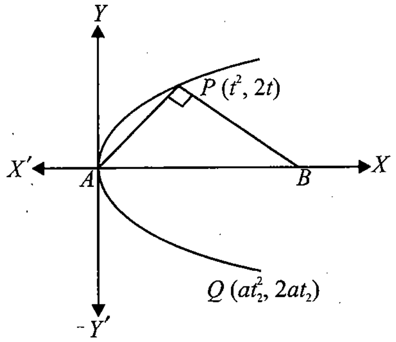


Fig. 3.18

Let P be $(t^2, 2t)$

Slope of $AP = \frac{2}{t}$

$\Rightarrow$ Slope of $BP = -\frac{t}{2}$

$\Rightarrow$ Equation of line BP is $y - 2t = -\frac{t}{2}(x - t^2)$

$\Rightarrow$ Point B is $(t^2 + 4, 0)$

Now let centroid of ΔPAB is (h, k)

$$\Rightarrow \quad h = \frac{t^2 + t^2 + 4}{3} \text{ and } k = \frac{2t}{3}$$

$$\Rightarrow \quad 3h - 4 = 2\left(\frac{3k}{2}\right)^2$$

Hence, the required locus is

$$3x - 4 = \frac{9y^2}{2}$$

which is a parabola.

Position of a Point With Respect to a Parabola $y^2 = 4ax$

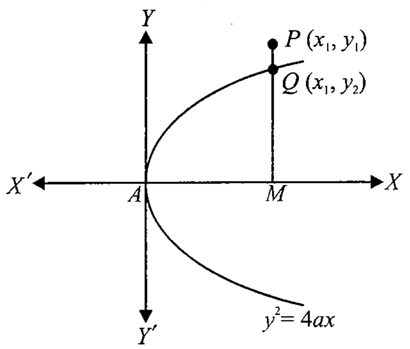


Fig. 3.19

Let $P(x_1, y_1)$ be a point in the plane (in 1st or 4th quadrant) From P draw $PM \perp AX$, meeting the parabola $y^2 = 4ax$ at Q. Then the coordinates of Q be (x_1, y_2). Since Q lies on the parabola, we have $y_2^2 = 4ax_1$

Now, point (x_1, y_1) will be outside, on or inside the parabola $y^2 = 4ax$ according to

$$PM >, = \text{ or } < QM$$

$$\Rightarrow \quad PM^2 >, = \text{ or } < QM^2$$

$$\Rightarrow \quad y_1^2 >, = \text{ or } < y_2^2$$

$$\Rightarrow \quad y_1^2 >, = \text{ or } < 4ax_1$$

For P lying in 3rd or 4th quadrant, $y_1^2 - 4ax_1 > 0$ ($\because x_1 < 0$)

Example 3.14 **The equation of a parabola is $y^2 = 4x$. P (1, 3) and Q (1, 1) are two points in the xy plane. Then, for the parabola**

a. **P and Q are exterior points**
b. **P is an interior point while Q is an exterior point**
c. **P and Q are interior points**
d. **P is an exterior point while Q is an interior point**

Sol. d. Here, $S \equiv y^2 - 4x = 0$

$$S(1, 3) \equiv 3^2 - 4 \cdot 1 > 0$$

$\Rightarrow$ $P(1, 3)$ is an exterior point.

$$S(1, 1) = 1^2 - 4 \cdot 1 < 0$$

$\Rightarrow$ $Q(1, 1)$ is an interior point.

Example 3.15 **The point $(a, 2a)$ is an interior point of the region bounded by the parabola $y^2 = 16x$ and the double ordinate through the focus. Then find the values of a.**

Sol.

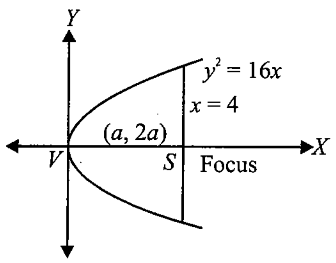


Fig. 3.20

$(a, 2a)$ is an interior point of $y^2 - 16x = 0$ if

$$(2a)^2 - 16a < 0, \text{ i.e., } a^2 - 4a < 0$$

$V(0, 0)$ and $(a, 2a)$ are on the same side of $x - 4 = 0$.

So, $a - 4 < 0$, i.e., $a < 4$.

Now,

$$a^2 - 4a < 0$$

$$\Rightarrow \quad 0 < a < 4$$

Equation of Parabola When Vertex is (h, k) and Axis is Parallel to x-Axis

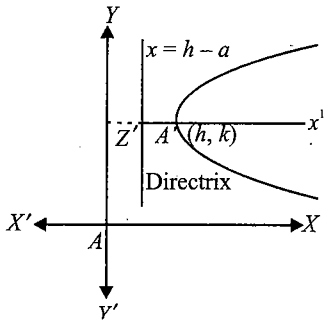


Fig. 3.21

The parabola

$$y^2 = 4ax \quad \text{(i)}$$

can be written as

$$(y - 0)^2 = 4a(x - 0)$$

The vertex of this parabola is A (0, 0).

Now when origin is shifted to A' (h, k) without changing the direction of axes, its equation becomes

$$(y - k)^2 = 4a(x - h) \quad \text{(ii)}$$

This is called general form of the parabola Eq. (i) and axis $A'X' \parallel AX$ with its vertex at A' (h, k). Its focus is at $(a + h, k)$ and length of latus rectum $= 4a$.

The equation of the directrix is

$$x = h - a$$

or $$x + a - h = 0$$

Equation of Parabola When Vertex is (h, k) and Axis is Parallel to y-Axis

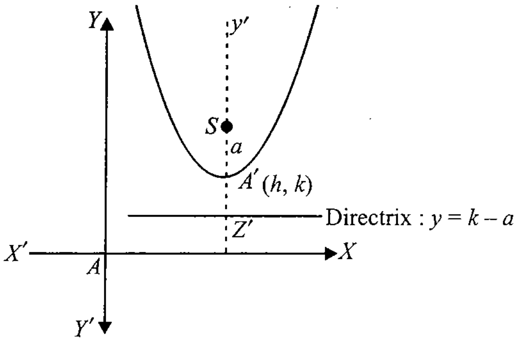


Fig. 3.22

The equation of parabola with vertex (h, k) is

$$(x - h)^2 = 4a(y - k)$$

Its focus is at $(h, a + k)$ and length of latus rectum $= 4a$.

The equation of the directrix is

$$y = k - a$$

or $$y + a - k = 0$$

Parabolic Curve

The equations $y = Ax^2 + Bx + C$ and $x = Ay^2 + By + C$ $(A \neq 0)$ represent parabola and are called parabolic curve.

Now,

$$y = Ax^2 + Bx + C$$
$$= A\left\{x^2 + \frac{B}{A}x + \frac{C}{A}\right\}$$
$$= A\left\{\left(x + \frac{B}{2A}\right)^2 - \frac{B^2}{4A^2} + \frac{C}{A}\right\}$$
$$= A\left\{\left(x + \frac{B}{2A}\right)^2 - \frac{(B^2 - 4AC)}{4A^2}\right\}$$

or
$$\left(x + \frac{B}{2A}\right)^2 = \frac{1}{A}\left(y + \frac{B^2 - 4AC}{4A}\right)$$

Comparing it with $(x - h)^2 = 4a(y - k)$ it represents a parabola with vertex at $(h, k) \equiv \left(-\frac{B}{2A}, -\frac{B^2 - 4AC}{4A}\right)$, axis parallel to y-axis, latus rectum $= \frac{1}{|A|}$ and the curve opening upwards and downwards depending upon the sign of A (for $A > 0$ curve opens upward, for $A < 0$ curve opens downward).

Similarly, $x = Ay^2 + By + C$ can be simplified to

$$\left(y + \frac{B}{2A}\right)^2 = \frac{1}{A}\left(x + \frac{B^2 - 4AC}{4A}\right)$$

Comparing it with $(y - k)^2 = 4a(x - h)$ it represents a parabola with vertex at $(h, k) \equiv \left(-\frac{B^2 - 4AC}{4A}, -\frac{B}{2A}\right)$ axis parallel to x-axis and latus rectum $= \frac{1}{|A|}$ and the curve opening left and right according to $A < 0$ and $A > 0$ respectively.

Note:
Parametric form of the parabola $(y - k)^2 = 4a(x - h)$ is $x = h + at^2$ and $y = k + 2at$.

Example 3.16 **$y^2 + 2y - x + 5 = 0$ represents a parabola. Find its vertex, equation of axis, equation of latus rectum, coordinates of the focus, equation of the directrix, extremities of the latus rectum and the length of the latus rectum.**

Sol.

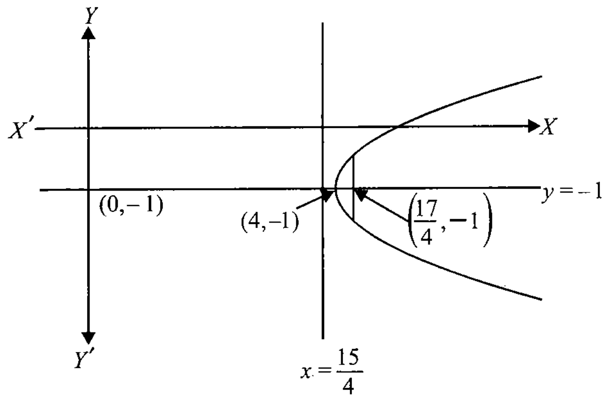


Fig. 3.23

$$y^2 + 2y - x + 5 = 0$$
$$\Rightarrow \quad y^2 + 2y + 1 = x - 4$$
$$\Rightarrow \quad (y + 1)^2 = (x - 4)$$

Comparing this equation with $(y - k)^2 = 4a(x - h)$ we have $a = \frac{1}{4}$.

Vertex: $(4, -1)$.

Equation of axis: $y + 1 = 0$

Equation of latus rectum: $x - 4 = \frac{1}{4}$ or $x = \frac{17}{4}$

Focus: Intersection point of axis and latus rectum is $\left(\frac{17}{4}, -1\right)$.

Directrix: $x - 4 = -\frac{1}{4}$ or $4x - 15 = 0$

Extremities of its latus rectum: These points lie at distance $2a$ from the focus on the latus rectum line which are $\left(\frac{17}{4}, -1 + \frac{1}{2}\right)$ and $\left(\frac{17}{4}, -1 - \frac{1}{2}\right)$, or $\left(\frac{17}{4}, \frac{-1}{2}\right)$ and $\left(\frac{17}{4}, \frac{-3}{2}\right)$

Length of its latus rectum: 1 unit.

Example 3.17 **Find the equation of parabola which has axis parallel to y-axis and which passes through points (0, 2), (– 1, 0) and (1, 6).**

Sol. General equation of such parabola is

$$y = Ax^2 + Bx + C$$

It passes through points (0, 2), (– 1, 0) and (1, 6). Then we have,

$$C = 2 \qquad \text{(i)}$$
$$A - B + C = 0 \qquad \text{(ii)}$$
$$A + B + C = 6 \qquad \text{(iii)}$$

Solving (i), (ii) and (iii) we get $C = 2$, $A = 1$ and $B = 3$.

Hence equation of parabola is

$$y = x^2 + 3x + 2$$

Example 3.18 **Prove that the locus of centre of circle which touches given circle externally and given line is parabola.**

Sol.

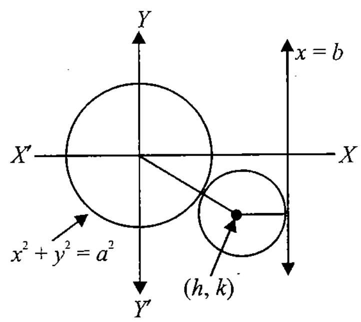


Fig. 3.24

Let given circle be $x^2 + y^2 = a^2$ and given line be $x = b$.

From the diagram radius of variable circle is $b - h$.

If it touches $x^2 + y^2 = a^2$,

then

$$a + (b - h) = \sqrt{h^2 + k^2}$$

$$\Rightarrow (a + b)^2 - 2(a + b)h + h^2 = h^2 + k^2$$

$$\Rightarrow \quad y^2 = (a + b)^2 - 2(a + b)x$$

which is equation of parabola.

Example 3.19 **If on a given base BC a triangle be described such that the sum of the tangents of the base angles is m, then prove that locus of opposite vertex A is parabola.**

Sol.

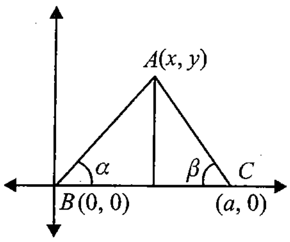


Fig. 3.25

Let the given points B and C are (0, 0) and $(a, 0)$.

According to the given condition

$\tan\alpha + \tan\beta = m$ (where m is constant)

$$\Rightarrow \quad \frac{y}{x} + \frac{y}{a - x} = m$$

$$\Rightarrow \quad y\frac{a}{x(a - x)} = m$$

$$\Rightarrow \quad ay = mx(a - x)$$

which is equation of parabola.

Example 3.20 **The parametric equation of a parabola is $x = t^2 + 1, y = 2t + 1$. Then find the equation of directrix.**

Sol. Eliminating t, we have

$$x = \left(\frac{y - 1}{2}\right)^2 + 1$$

or

$$(y - 1)^2 = 4(x - 1)$$

Putting $y - 1 = Y,\ x - 1 = X$, the equation becomes

$$Y^2 = 4X$$

So, the equation of the directrix is

$$X + 1 = 0 \Rightarrow x = 0$$

Example 3.21 **Find the points on the parabola $y^2 - 2y - 4x = 0$ whose focal length is 6.**

Sol. $y^2 - 2y - 4x = 0$

or

$$(y - 1)^2 = 4x + 1$$

or

$$(y - 1)^2 = 4\left(x + \frac{1}{4}\right)$$

Vertex of parabola is $\left(-\frac{1}{4}, 1\right)$. Corresponding parabola with vertex at origin is $y^2 = 4x$.

For point on this parabola having focal length 6, we have

$$6 = a + x = 1 + x$$

or

$$\therefore \quad x = 5$$

Hence points on this parabola are $(5, \pm 2\sqrt{5})$.

Hence corresponding points on the parabola

$$(y - 1)^2 = 4\left(x + \frac{1}{4}\right) \text{ are } \left(5 - \frac{1}{4}, 1 \pm 2\sqrt{5}\right)$$

$$\equiv \left(\frac{19}{4}, 1 \pm 2\sqrt{5}\right)$$

Example 3.22 **If the length of chord of circle $x^2 + y^2 = 4$ and $y^2 = 4(x - h)$ is maximum, then find the value of h.**

Sol.

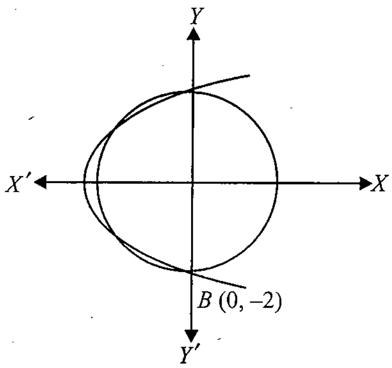


Fig. 3.26

Obviously, maximum lengths of chord occur when the parabola passes through (0, 2) and (0, –2).

Hence, from $y^2 = 4(x - h)$ we have,

$$4 = 4(0 - h)$$

$$\Rightarrow \quad h = -1$$

Concept Application Exercise 3.1

1. If the focus and vertex of a parabola are the points (0, 2) and (0, 4), respectively, then find its equation.
2. Find the coordinates of any point on the parabola whose focus is (0, 1) and the directrix is $x + 2 = 0$.
3. Find the length of the common chord of the parabola $y^2 = 4(x + 3)$ and the circle $x^2 + y^2 + 4x = 0$.

4. Find the vertex of the parabola $x^2 = 2(2x + y)$.
5. Find the equation of the directrix of the parabola $x^2 - 4x - 3y + 10 = 0$.
6. The vertex of a parabola is (2, 2) and the coordinates of its two extremities of latus rectum are (–2, 0) and (6, 0). Then find the equation of the parabola.
7. Find the length of latus rectum of parabola whose focus is at (2, 3) and directrix is the line $x - 4y + 3 = 0$.
8. Find the angle made by a double ordinate of length $8a$ at the vertex of the parabola $y^2 = 4ax$.
9. LOL' and MOM' are two chords of parabola $y^2 = 4ax$ with vertex A passing through a point O on its axis. Prove that the radical axis of the circles described on LL' and MM' as diameters passes though the vertex of the parabola.
10. If (a, b) is the midpoint of a chord passing through the vertex of the parabola $y^2 = 4(x + 1)$, then prove that $2(a + 1) = b^2$.
11. Find the range of values of λ for which the point $(\lambda, -1)$ is exterior to both the parabolas $y^2 = |x|$.
12. Prove that the locus of a point, which moves so that its distance from a fixed line is equal to the length of the tangent drawn from it to a given circle, is a parabola.
13. Prove that the locus of the centre of a circle, which intercepts a chord of given length $2a$ on the axis of x and passes through a given point on the axis of y distant b from the origin, is parabola.

GENERAL EQUATION OF A PARABOLA

Let $S(a, b)$ be the focus and $lx + my + n = 0$ is the equation of the directrix.

Let $P(x, y)$ be any point on the parabola.

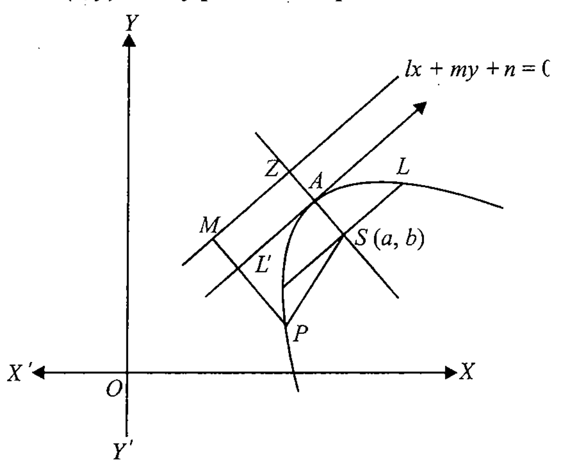


Fig. 3.27

By definition,

$$SP = PM$$

$$\Rightarrow \quad \sqrt{(x-a)^2 + (y-b)^2} = \frac{|lx + my + n|}{\sqrt{(l^2 + m^2)}}$$

$$\Rightarrow \quad (x-a)^2 + (y-b)^2 = \frac{(lx + my + n)^2}{(l^2 + m^2)}$$

$\Rightarrow m^2x^2 + l^2y^2 - 2lmxy + x$ term $+ y$ term + constant = 0

This is of the form $(mx - ly)^2 + 2gx + 2fy + c = 0$

This equation is the general equation of parabola.

Note:

Second degree terms in the general equation of a parabola forms of a perfect square.

Recognition of Conics

The equation of conics represented by the general equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{(i)}$$

can be recognized easily by the condition given in the tabular form. For this, first we have to find $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$. When $\Delta \neq 0$, Eq. (i) represents the non-degenerate conic whose nature is given in the following table:

Condition	Nature of conics
$\Delta \neq 0, h = 0, a = b$	A circle
$\Delta \neq 0, ab - h^2 = 0$ ($\because$ 2nd degree terms form a perfect square)	A parabola
$\Delta \neq 0, ab - h^2 > 0$	An ellipse or empty set
$\Delta \neq 0, ab - h^2 < 0$	A hyperbola
$\Delta \neq 0, ab - h^2 < 0$ and $a + b = 0$	A rectangular hyperbola

Example 3.23 **Find the value of λ if equation $9x^2 + 4y^2 + 2\lambda xy + 4x - 2y + 3 = 0$ represents parabola.**

Sol. Comparing this equation with $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ we have $a = 9, b = 4, c = 3, h = \lambda, g = 2, f = -1$.

If the equation $9x^2 + 4y^2 + 2\lambda xy + 4x - 2y + 3 = 0$ represents the parabola then its second degree terms must form the perfect square.

$\Rightarrow \lambda^2 = 36$ (using $h^2 - ab = 0$)

$\Rightarrow \lambda = \pm 6$

Also for these values of λ, $\Delta \neq 0$.

Example 3.24 **Find the value of λ if the equation $(x-1)^2 + (y-2)^2 = \lambda(x+y+3)^2$ represents a parabola. Also, find its focus, the equation of its directrix, the equation of its axis, the coordinates of its vertex, the equation of its latus rectum, length of the latus rectum and the extremities of the latus rectum.**

Sol. $(x-1)^2 + (y-2)^2 = \lambda(x+y+3)^2 = 2\lambda\left(\frac{x+y+3}{\sqrt{2}}\right)^2$

$\therefore \quad \sqrt{(x-1)^2+(y-2)^2} = \sqrt{2\lambda}\,\frac{|x+y+3|}{\sqrt{2}}$

This represents parabola if $\sqrt{2\lambda} = 1$

$\Rightarrow \quad \lambda = \frac{1}{2}$

Its focus is (1, 2). Its directirx is

$$x + y + 3 = 0.$$

Axis of parabola is a line through focus (1, 2) and perpendicular to the directrix $x + y + 3 = 0$

Hence, its axis is the line $y = x + 1$.

Axis $y = x + 1$ and the directrix $x + y + 3 = 0$ meet at (–2, –1), thus its vertex is $\left(\frac{1-2}{2}, \frac{2-1}{2}\right)$, i.e., $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Its latus rectum will be in the form of $x + y + b = 0$,

which passes through focus (1, 2). Hence its equation is $x + y - 3 = 0$

Distance between focus and directrix is $\frac{|1+2+3|}{\sqrt{2}} = \frac{6}{\sqrt{2}}$.

Thus length of its latus rectum is $\frac{12}{\sqrt{2}}$, i.e., $6\sqrt{2}$ units.

If (x_1, y_1) is the extremity of its latus rectum, then

$$\frac{x_1 - 1}{-\frac{1}{\sqrt{2}}} = \frac{y_1 - 2}{\frac{1}{\sqrt{2}}} = \pm\frac{6}{\sqrt{2}}$$

$\Rightarrow (x_1, y_1)$ is (–2, 5) or (4, –1)

Example 3.25 **Find the equation to the parabola whose focus is S (–1, 1) and directrix is $4x + 3y - 24 = 0$. Also find its axis, the vertex, the length and the equation of the latus rectum.**

Sol. Let $P(x, y)$ be any point on the parabola. Since the distance of P from the focus is equal to its distance from the directrix, i.e., $PS = PQ$ or $PS^2 = PQ^2$

or $\quad (x+1)^2 + (y-1)^2 = \left[\frac{(4x+3y-24)}{5}\right]^2$

i.e., $9x^2 + 16y^2 - 24xy + 242x + 94y - 526 = 0 \quad$ (i)

This is the required equation of the parabola.

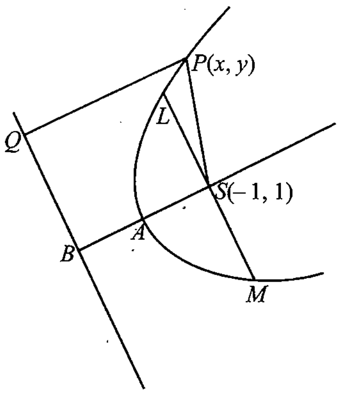


Fig. 3.28

The axis is a line through S (–1, 1) and ⊥ to the directrix $4x + 3y - 24 = 0$. Thus the equation of the axis is

$$3(x+1) - 4(y-1) = 0 \text{ or } 3x - 4y + 7 = 0 \quad \text{(ii)}$$

The axis and the directrix intersect at B. Solving them, we get B (3, 4).

The vertex A is the midpoint of S (–1, 1) and B (3, 4)

Thus vertex A is $\left(1, \frac{5}{2}\right)$ (iii)

Also $\quad AS = \sqrt{\left[2^2 + \left(\frac{3}{2}\right)^2\right]} = \frac{5}{2}$

Hence length of the latus rectum = $4AS = 10$ (iv)

Now, latus rectum is a straight line through the focus S and parallel to the directrix.

Hence its equation is $4x + 3y + 1 = 0$ (v)

Properties of Focal Chord

Any point on the parabola $y^2 = 4ax$ can be taken as $(at^2, 2at)$, where t is parameter and $t \in R$. Any line passing through the focus of the parabola is called focal chord of the parabola.

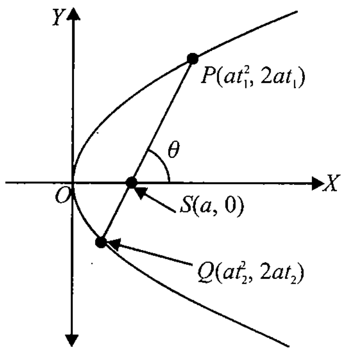


Fig. 3.29

1. If the chord joining $P \equiv (at_1^2, 2at_1)$ and $Q \equiv (at_2^2, 2at_2)$ is the focal chord then $t_1t_2 = -1$.

   Proof:

$$P \equiv (at_1^2, 2at_1) \text{ and } Q \equiv (at_2^2, 2at_2)$$

Since PQ passes through the focus $S(a, 0)$

$\therefore$ Q, S, P are collinear

$\therefore$ Slope of PS = Slope of QS

$$\Rightarrow \quad \frac{2at_1 - 0}{at_1^2 - a} = \frac{0 - 2at_2}{a - at_2^2}$$

$$\Rightarrow \quad \frac{2t_1}{t_1^2 - 1} = \frac{2t_2}{t_2^2 - 1}$$

$$\Rightarrow \quad t_1(t_2^2 - 1) = t_2(t_1^2 - 1)$$

$$\Rightarrow \quad t_1t_2(t_2 - t_1) + (t_2 - t_1) = 0$$

$$t_2 - t_1 \neq 0 \therefore t_1t_2 + 1 = 0$$

$$\therefore \quad t_1t_2 = -1 \text{ or } t_2 = -\frac{1}{t_1}, \qquad \text{(i)}$$

which is the required relation.

Note:

If one extremity of a focal chord is $(at_1^2, 2at_1)$ *then the other extremity* $(at_2^2, 2at_2)$ *becomes* $\left(\frac{a}{t_1^2}, -\frac{2a}{t_1}\right)$.

2. If point P is $(at^2, 2at)$, then length of focal chord PQ is $a\left(t + \frac{1}{t}\right)^2$.

Proof:

$$PQ = SP + SQ = a + at^2 + a + \frac{a}{t^2}$$
$$= a\left(t + \frac{1}{t}\right)^2$$

3. The length of the focal chord which makes an angle θ with positive direction of x-axis is $4a \operatorname{cosec}^2\theta$.

Proof:

$$PQ = a\left(t + \frac{1}{t}\right)^2$$

Now slope of $\quad PQ = \dfrac{2}{t - \frac{1}{t}} = \tan\theta$

$$\Rightarrow \quad 2\cot\theta = t - \frac{1}{t}$$

$$\Rightarrow \quad PQ = a\left(t + \frac{1}{t}\right)^2 = a\left[\left(t - \frac{1}{t}\right)^2 + 4\right]$$

$$= a[4\cot^2\theta + 4]$$
$$= 4a \operatorname{cosec}^2\theta$$

From this we can conclude that the minimum length of focal chord is $4a$, which is the length of latus rectum.

4. Semi-latus rectum is harmonic mean of SP and SQ, where P and Q are extremities of focal chord.

Proof:

PQ is focal chord.

$$SP = a + at^2 \text{ and } SQ = a + \frac{a}{t^2}$$

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a + at^2} + \frac{1}{a + a/t^2} = \frac{1}{a}$$

$$\Rightarrow \quad 2a = \frac{2SP \times SQ}{SP + SQ}$$

5. Circle described on the focal length as diameter touches the tangent at vertex.

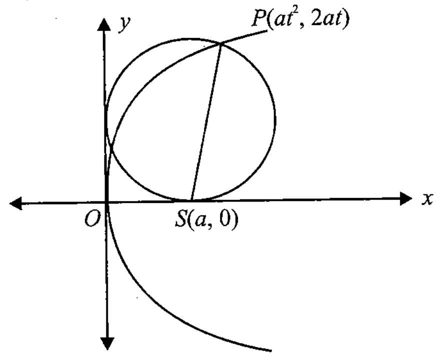


Fig. 3.30

Proof:

Equation of the circle described on SP as diameter is

$(x - at^2)(x - a) + (y - 2at)(y - 0) = 0$

Solving it with y-axis, $x = 0$, we have

$(0 - at^2)(0 - a) + (y - 2at)(y - 0) = 0$

or $\quad y^2 - 2aty + a^2t^2 = 0$ which has equal roots.

Hence, y-axis touches the circle.

Also point of contact is $(0, at)$.

6. Circle described on the focal chord as diameter touches directrix.

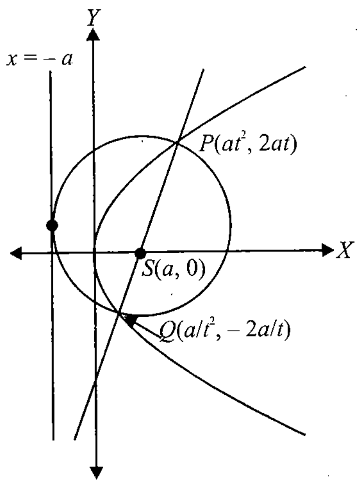


Fig. 3.31

Proof:

Equation of the circle described on PQ as diameter is

$$(x - at^2)\left(x - \frac{a}{t^2}\right) + (y - 2at)\left(y + \frac{2a}{t}\right) = 0$$

Solving it with $x = -a$, we have

$$(-a - at^2)\left(-a - \frac{a}{t^2}\right) + (y - 2at)\left(y + \frac{2a}{t}\right) = 0$$

or $y^2 - 2a\left(t - \frac{1}{t}\right)y + a^2\left(t - \frac{1}{t}\right) = 0$, which is the perfect square.

Hence, $x = -a$ touches the circle.

Example 3.26 **If the length of a focal chord of the parabola $y^2 = 4ax$ at a distance b from the vertex is c. Then prove that $b^2c = 4a^3$.**

Sol.

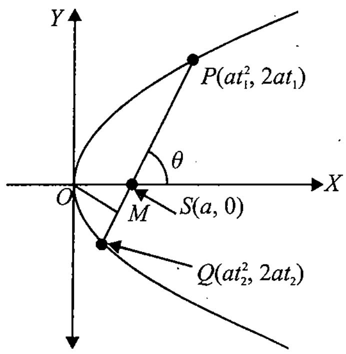


Fig. 3.32

In the figure, $OM = b$ = distance of focal chord from vertex.

Now let focal chord makes an angle θ with positive x- axis.

Then its length, $PQ = 4a \operatorname{cosec}^2\theta$

Now in right-angled triangle OMS, $\sin\theta = \frac{OM}{OS} = \frac{b}{a}$

$\Rightarrow$ $PQ = 4a\left(\frac{a}{b}\right)^2 = \frac{4a^3}{b^2}$, i.e., $b^2c = 4a^3$

Example 3.27 **If (2, –8) is at an end of a focal chord of the parabola $y^2 = 32x$, then find the other end of the chord.**

Sol. For $y^2 = 32x$, $a = 8$.

Then any point on the parabola is $(8t^2, 16t)$.

Comparing this with point P (2, –8), we get $t = -\frac{1}{2}$.

Now if PQ is the focal chord and point Q is $(8t_1^2, 16t_1)$, then $t_1 = -\frac{1}{t} = 2$.

Hence, point Q has coordinates (32, 32).

Example 3.28 **Circles are drawn with diameter being any focal chord of the parabola $y^2 - 4x - y - 4 = 0$ will always touch a fixed line, find its equation.**

Sol. $y^2 - 4x - y - 4 = 0$

$\Rightarrow$ $y^2 - y + \frac{1}{4} = 4x + \frac{17}{4}$

$\Rightarrow$ $\left(y - \frac{1}{2}\right)^2 = 4\left(x + \frac{17}{16}\right)$ circle drawn with diameter being any focal chord of the parabola always touches the directrix of the parabola.

Thus circle will touch the line $x + \frac{17}{16} = -1$, i.e., $16x + 33 = 0$.

Example 3.29 **If AB is a focal chord of $x^2 - 2x + y - 2 = 0$ whose focus is 'S'. If $AS = l_1$ then find BS.**

Sol. $x^2 - 2x + y - 2 = 0$

$\Rightarrow$ $x^2 - 2x + 1 = 3 - y$

$\Rightarrow$ $(x - 1)^2 = -(y - 3)$.

Length of its latus rectum is 1 unit.

Since AS, $\frac{1}{2}$, BS are in H.P., thus

$$\frac{1}{2} = \frac{2AS \times BS}{AS + BS}$$

$\Rightarrow$ $BS = \frac{l_1}{(4l_1 - 1)}$

Concept Application Exercise 3.2

1. Circle drawn having its diameter equal to focal distance of any point lying on the parabola $x^2 - 4x + 6y + 10 = 0$ will touch a fixed line, find its equation.
2. A circle is drawn to pass through the extremities of the latus rectum of the parabola $y^2 = 8x$. It is given that this circle also touches the directrix of the parabola. Find the radius of this circle.
3. If a focal chord of $y^2 = 4ax$ makes an angle $\alpha \in \left[0, \frac{\pi}{4}\right]$ with the positive direction of x-axis, then find the minimum length of this focal chord.
4. If the line passing through the focus S of the parabola $y = ax^2 + bx + c$ meets the parabola at P and Q and if $SP = 4$ and $SQ = 6$ then find the value of a.
5. The coordinates of the ends of a focal chord of a parabola $y^2 = 4ax$ are (x_1, y_1) and (x_2, y_2), then find the value of $x_1x_2 + y_1y_2$.

Intersection of a Line and a Parabola

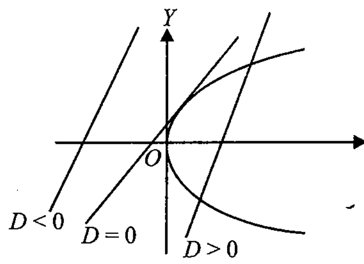


Fig. 3.33

Let the parabola be $y^2 = 4ax$ (i)

and the given line be $y = mx + c$ (ii)

Eliminating y from (i) and (ii), then $(mx + c)^2 = 4ax$

or $m^2x^2 + 2x(mc - 2a) + c^2 = 0$ (iii)

This equation, being quadratic in x, gives two values of x and shows that every straight line will cut the parabola in two points may be real, coincident or imaginary according as discriminant of Eq. (iii) is greater, equal or less than 0.

That is, $4(mc - 2a)^2 - 4m^2c^2 >, =, < 0$

or $4a^2 - 4amc >, =, < 0$ in equation (ii),

or $a >, =, < mc$ (iv)

Equation of Tangent

Equation of Tangent at Point $P(x_1, y_1)$ to Parabola $y^2 = 4ax$

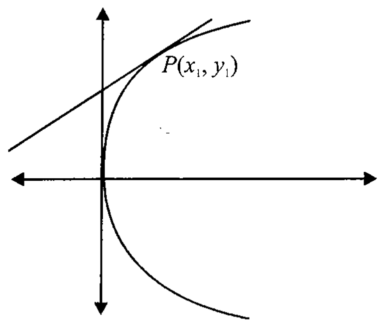


Fig. 3.34

Differentiating $y^2 = 4ax$ with respect to x, we have

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y}$$

then equation of tangent at point P is

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$

or $$yy_1 - 2ax = y_1^2 - 2ax_1$$

$$\Rightarrow \quad yy_1 - 2ax - 2ax_1 = y_1^2 - 4ax_1$$

(adding $-2ax_1$ both sides)

$$\Rightarrow \quad yy_1 - 2a(x + x_1) = 0 \quad \text{(i)}$$

(since point (x_1, y_1) lies on the parabola)

Hence, equation of tangent at point $P(x_1, y_1)$ is given by

$$T = 0$$

where T is an expression which we get by replacing y^2 by yy_1, and $2x$ by $x + x_1$.

Equation of Tangent at Point $P(t)$ or $P(at^2, 2at)$

In Eq. (i) replace y_1 by $2at$ and x_1 by at^2

we have $2aty = 2a(x + at^2)$ or $ty = x + at^2$ (ii)

Equation of Tangent if Slope of Tangent is m

In Eq. (ii), slope of tangent $m = \frac{1}{t}$

In Eq. (ii) replacing t by $\frac{1}{m}$ we have $y = mx + \frac{a}{m}$ which is equation of tangent in terms of slope.

This is tangent at point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

If line $y = mx + c$ touches parabola $y^2 = 4ax$ we must have $c = \frac{a}{m}$ (comparing equation with $y = mx + \frac{a}{m}$)

Note:

- *Equation of tangent to the parabola $(y - k)^2 = 4a(x - h)$ having slope m is $y - k = m(x - h) + \frac{a}{m}$.*

  Equation of tangent at point $p(t)$ on different parabolas:

-

Equations of parabola	Parametric co-ordinates t	Tangent at $P(t)$
$y^2 = 4ax$	$(at^2, 2at)$	$ty = x + at^2$
$y^2 = -4ax$	$(-at^2, 2at)$	$ty = -x + at^2$
$x^2 = 4ay$	$(2at, at^2)$	$tx = y + at^2$
$x^2 = -4ay$	$(2at, -at^2)$	$tx = -y + at^2$

-

Equation of parabola	Point of contact in terms of slope (m)	Equation of tangent in terms of slope (m)	Condition of tangency for line $y = mx + c$
$y^2 = 4ax$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx + \frac{a}{m}$	$c = \frac{a}{m}$
$y^2 = -4ax$	$\left(-\frac{a}{m^2}, -\frac{2a}{m}\right)$	$y = mx - \frac{a}{m}$	$c = -\frac{a}{m}$

$x^2 = 4ay$	$(2am, am^2)$	$y = mx - am^2$	$c = -am^2$
$x^2 = -4ay$	$(-2am, -am^2)$	$y = mx + am^2$	$c = am^2$

Pair of Tangents from Point (x_1, y_1)

Let $T(h, k)$ be any point on the pair of tangents PQ or PR drawn from any external point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$.

Equation of PT is

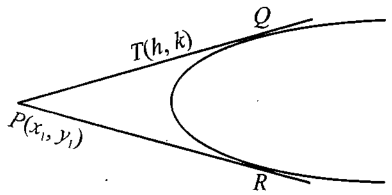


Fig. 3.35

$$y - y_1 = \frac{k - y_1}{h - x_1}(x - x_1)$$

or
$$y = \left(\frac{k - y_1}{h - x_1}\right)x + \left(\frac{hy_1 - kx_1}{h - x_1}\right)$$

which is tangent to the parabola $y^2 = 4ax$

$\therefore$
$$c = \frac{a}{m}$$

or
$$cm = a$$

or
$$\left(\frac{hy_1 - kx_1}{h - x_1}\right)\left(\frac{k - y_1}{h - x_1}\right) = a$$

or
$$(k - y_1)(hy_1 - kx_1) = a(h - x_1)^2$$

$\therefore$ Locus of (h, k), equation of pair of tangents is

$$(y - y_1)(xy_1 - x_1y) = a(x - x_1)^2$$

or
$$(y^2 - 4ax)(y_1^2 - 4ax_1) = \{(yy_1 - 2a(x + x_1)\}^2$$

or
$$SS_1 = T^2,$$

where $S = y^2 - 4ax$, $S_1 = y_1^2 - 4ax_1$ and $T = yy_1 - 2a(x + x_1)$

Example 3.30 **Find equation of the tangent to the parabola $y^2 = 8x$ having slope 2 and also find the point of contact.**

Sol. Equation of the tangent to $y^2 = 4ax$ having slope m is $y = mx + \frac{a}{m}$.

Hence, for the given parabola, equation of the tangent is $y = 2x + \frac{2}{2}$ or $y = 2x + 1$ and point of contact is

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right) \equiv \left(\frac{2}{2^2}, \frac{2(2)}{2}\right) \equiv \left(\frac{1}{2}, 2\right).$$

Example 3.31 **Find the equations of the tangents of the parabola $y^2 = 12x$, which passes through the point (2, 5).**

Sol. Equation of the given parabola is $y^2 = 12x$ (i)

Comparing with $y^2 = 4ax$, we get $a = 3$.

$\therefore$ Let the equation of the tangent from (2, 5)

i.e.,
$$y = mx + \frac{3}{m} \qquad \text{(ii)}$$

Eq. (ii) passes through the point (2, 5), then

$$5 = 2m + \frac{3}{m}$$

$\Rightarrow$
$$2m^2 - 5m + 3 = 0$$

$\Rightarrow$
$$m = 1, \frac{3}{2}$$

Therefore, From Eq. (ii), the equations of the required tangents are

$$y = x + 3 \text{ and } 2y = 3x + 4.$$

Example 3.32 **If the line $y = 3x + c$ touches the parabola $y^2 = 12x$ at point P, then find the equation of the tangent at point Q where PQ is a focal chord.**

Sol. Line $y = 3x + c$ touches $y^2 = 12x$ then we must have $c = \frac{a}{m}$ or $c = \frac{3}{3} = 1$ and the point of contact is

$$P\left(\frac{a}{m^2}, \frac{2a}{m}\right) \equiv P\left(\frac{3}{3^2}, \frac{2(3)}{3}\right) \equiv P\left(\frac{1}{3}, 2\right)$$

Comparing this point to $(at^2, 2at)$, we have $2at = 2$

$\Rightarrow$
$$t = \frac{1}{3}$$

Hence, point P has parameter $\frac{1}{3}$, then point Q has parameter -3.

Now tangent at point Q is $(-3)y = x + 3(-3)^2$

or
$$x + 3y - 27 = 0.$$

Example 3.33 **Find the equation of tangent to parabola $y = x^2 - 2x + 3$ at point (2, 3).**

Sol. Since the equation of parabola is not in the standard form, we use calculus method to find the equation of tangent.

$y = x^2 - 2x + 3$

Differentiating, w.r.t. x, we have $\frac{dy}{dx} = 2x - 2$

We want to find tangent at point (2, 3)

Then $\left(\frac{dy}{dx}\right)_{(2,3)} = 2(2) - 2 = 2$

Hence using point slope form equation of tangent is
$y - 3 = 2(x - 2)$ or $y = 2x - 1$

Example 3.34 **Find the equation of tangent to parabola $x = y^2 + 3y + 2$ having slope 1.**

Sol. $x = y^2 + 3y + 2$

Differentiating both sides w.r.t. x, we have

$$1 = 2y\frac{dy}{dx} + 3\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2y+3}$$

Now slope of tangent is 1

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2y+3} = 1$$

$\Rightarrow$ $y = -1$, which is the ordinate of the point on the curve where slope of tangent is 1.

Putting $y = -1$, in equation of parabola we get $x = 0$

Hence using point-slope form we have $y - (-1) = 1(x - 0)$ or $x - y - 1 = 0$

Example 3.35 **Find the equation of tangents drawn to parabola $y = x^2 - 3x + 2$ from the point (1, – 1).**

Sol. Tangents are drawn to the parabola from the point (1, – 1).

Now equation of line from (1, – 1) having slope m is $y - (-1) = m(x - 1)$

or $mx - y - m - 1 = 0$ or $y = mx - m - 1$

Since this line touches the parabola, when we solve line and parabola and the resulting quadratic will have equal roots

Solving we have $mx - m - 1 = x^2 - 3x + 2$

$x^2 - (3 + m)x + 3 + m = 0$

This equation has equal roots

$$\Rightarrow (m + 3)^2 - 4(m + 3) = 0$$

$$\Rightarrow m = -3 \text{ or } m = 1$$

Hence equation of tangents are $y + 1 = -3(x - 1)$ and $y + 1 = x - 1$

or $3x + y - 2 = 0$ and $x - y - 2 = 0$

Example 3.36 **Find the shortest distance between the line $y = x - 2$ and the parabola $y = x^2 + 3x + 2$.**

Sol. Let $P(x_1, y_1)$ be a point closest to the line $y = x - 2$

then $\left.\frac{dy}{dx}\right|_{(x_1, y_1)}$ = slope of given line

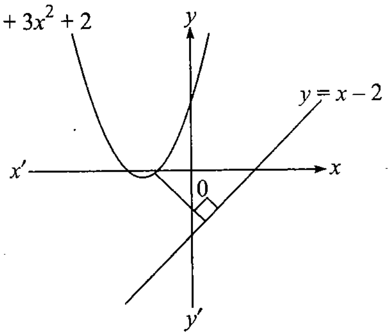


Fig. 3.36

$2x_1 + 3 = 1 \Rightarrow x_1 = -1 \Rightarrow y_1 = 0$

Hence point (– 1, 0) is the closest and its perpendicular distance from the line $y = x - 2$ will be the shortest distance $\Rightarrow$ Shortest distance = $\frac{3}{\sqrt{2}}$

Example 3.37 **Find the equation of common tangent of $y^2 = 4ax$ and $x^2 = 4ay$.**

Sol.

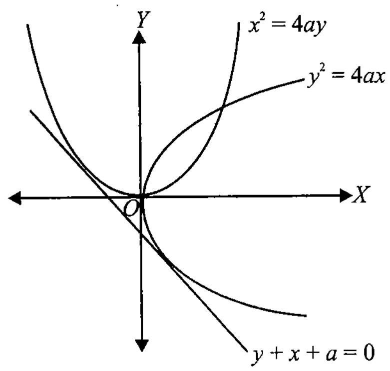


Fig. 3.37

Equation of the tangent to $y^2 = 4ax$ having slope m is $y = mx + \frac{a}{m}$.

It will touch $x^2 = 4ay$, if $x^2 = 4a\left(mx + \frac{a}{m}\right)$ has equal roots.

Thus, $\quad 16a^2m^2 = -16\frac{a^2}{m}$ ($\because D = 0$)

$\Rightarrow \quad m = -1.$

Thus, common tangent is $y + x + a = 0$.

Example 3.38 **A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$. Then find one of the points of contact.**

Sol. Equation of the tangent to the parabola

$y^2 = 4ax$ at $(at^2, 2at)$ is

$ty = x + at^2$

Here $a = 2$, so the equation of the tangent at $(2t^2, 4t)$ to the parabola

$y^2 = 8x$ is $ty = x + 2t^2$ (i)

Slope of Eq. (i) is $\frac{1}{t}$ and that of given line is 3.

$$\Rightarrow \quad \frac{\frac{1}{t}-3}{1+\frac{1}{t}\times 3} = \pm \tan 45° = \pm 1$$

$$\Rightarrow \quad t = -\frac{1}{2} \text{ or } 2$$

For $t = -\frac{1}{2}$, tangent is

$$\left(-\frac{1}{2}\right)y = x + 2\left(\frac{1}{4}\right),$$

i.e., $2x + y + 1 = 0$ at point of contact $\left(\frac{1}{2}, -2\right)$.

For $t = 2$, tangent is $2y = x + 8$, at point of contact (8, 8).

Example 3.39 **Show that $x \cos \alpha + y \sin \alpha = p$ touches the parabola $y^2 = 4ax$ if $p \cos \alpha + a \sin^2 \alpha = 0$ and that the point of contact is $(a \tan^2 \alpha, -2a \tan \alpha)$.**

Sol. The given line is $x \cos \alpha + y \sin \alpha = p$

or $\quad y = -x \cot \alpha + p \text{ coses } \alpha$

$\therefore \quad m = -\cot \alpha$ and $c = p \operatorname{cosec} \alpha$

since the given line touches the parabola

$\therefore \quad c = \frac{a}{m}$ or $cm = a$

$\Rightarrow \quad (p \operatorname{cosec} \alpha)(-\cot \alpha) = a$

and the point of contact is

$$\left(\frac{a}{\cot^2 \alpha}, -\frac{2a}{\cot \alpha}\right) \equiv (a \tan^2 \alpha, -2a \tan \alpha)$$

Example 3.40 **The tangents to a parabola $y^2 = 4ax$ at the vertex V and any point P meet at Q. If S be the focus, then prove that SP, SQ, SV are in G.P.**

Sol. Let the parabola be $y^2 = 4ax$.

Q is the intersection of the lines $x = 0$ and tangent at point $P(at^2, 2at)$, $ty = x + at^2$.

Solving these, we get $Q = (0, at)$. Also $S = (a, 0)$.

Now focal length $\quad SP = a + at^2$

$$SQ^2 = a^2 + a^2t^2 = a^2(t^2 + 1)$$

and $\quad SV = a$

$\therefore \quad SQ^2 = SP \times SV$

$\Rightarrow SP, SQ, SV$ are in G.P.

Example 3.41 **Parabola $y^2 = 4x$ and the circle having its centre at (6, 5) intersect at right angle. Then find the possible points of intersection of these curves.**

Sol. Let the possible point be $(t^2, 2t)$. Equation of the tangent at this point is $yt = x + t^2$.

It must pass through centre of the circle (6, 5).

$$\Rightarrow \quad t^2 - 5t + 6 = 0$$

$$\Rightarrow \quad t = 2, 3$$

$\Rightarrow$ Possible points are (4, 4), (9, 6).

Example 3.42 **If a tangent to the parabola $y^2 = 4ax$ meets the x-axis at T and intersect tangent at vertex A at P, and the rectangle $TAPQ$ be completed. Then find the locus of point Q.**

Sol.

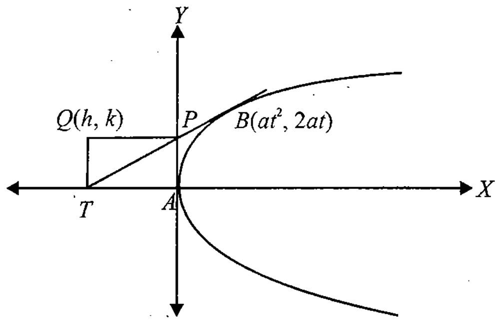


Fig. 3.38

The tangent at any point $B(at^2, 2at)$ to the parabola is

$$ty = x + at^2. \quad \text{(i)}$$

Since tangent at the vertex A is y-axis, so T and P are $(-at^2, 0)$ and $(0, at)$, respectively. Clearly A is (0, 0).

If Q be (h, k), then $h = AT = -at^2$ and $k = AP = at$

Eliminating t, we get $k^2 + ah = 0$.

Hence, the locus of Q is $y^2 + ax = 0$,

which is a parabola.

Example 3.43 **Two tangents are drawn from the point (–2, –1) to the parabola $y^2 = 4x$. If α is the angle between these tangents, then find the value of $\tan \alpha$.**

Sol. Here $a = 1$. Any tangent having slope m is $y = mx + \frac{1}{m}$.

It passes through (– 2, –1)

$$\Rightarrow \quad 2m^2 - m - 1 = 0$$

$$\Rightarrow \quad m = 1, -\frac{1}{2}$$

$$\Rightarrow \quad \tan \alpha = \frac{1+\frac{1}{2}}{1-\frac{1}{2}} = 3$$

Example 3.44 **If two tangents drawn from the point (α, β) to the parabola $y^2 = 4x$ be such that the slope of one tangent is double of the other, then prove that $\alpha = \frac{2}{9}\beta^2$.**

Sol. Any tangent to the parabola $y^2 = 4x$ having slope m is

$$y = mx + \frac{1}{m}$$

It passes through (α, β), therefore,

$$\beta = m\alpha + \frac{1}{m}$$

or $\quad \alpha m^2 - \beta m + 1 = 0$

According to the question, it has roots m_1, $2m_1$

Now, $\quad m_1 + 2m_1 = \frac{\beta}{\alpha}$ and $m_1 \cdot 2m_1 = \frac{1}{\alpha}$

$$\Rightarrow \quad 2\left(\frac{\beta}{3\alpha}\right)^2 = \frac{1}{\alpha} \Rightarrow \alpha = \frac{2}{9}\beta^2$$

Example 3.45 **A pair of tangents are drawn to parabola $y^2 = 4ax$ which are equally inclined to a straight line $y = mx + c$, whose inclination to the axis is α. Prove that the locus of their point of intersection is the straight line $y = (x - a)\tan 2\alpha$.**

Sol.

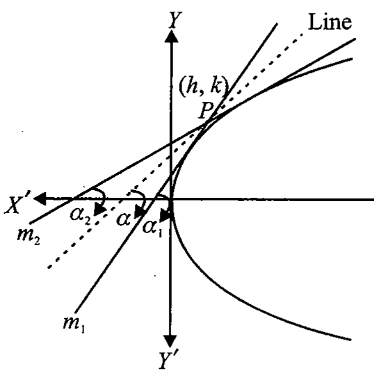


Fig. 3.39

We have $\quad \theta = \alpha_1 - \alpha = \alpha - \alpha_2$

$\therefore \quad 2\alpha = \alpha_1 + \alpha_2$

$$\Rightarrow \quad \tan 2\alpha = \tan(\alpha_1 + \alpha_2) = \frac{m_1 + m_2}{1 - m_1 m_2}$$

but (h, k) lies on $\quad y = mx + \frac{a}{m}$

$\Rightarrow \quad m^2 h - km + a = 0$

Hence, $\quad m_1 + m_2 = \frac{k}{h}$ and $m_1 m_2 = \frac{a}{h}$;

$$\tan 2\alpha = \frac{\frac{k}{h}}{1 - \frac{a}{h}} = \frac{k}{h - a}$$

$$\Rightarrow \quad y = (x - a)\tan 2\alpha$$

Example 3.46 **Tangents are drawn from the point (–1, 2) on the parabola $y^2 = 4x$. Find the length that these tangents will intercept on the line $x = 2$.**

Sol.

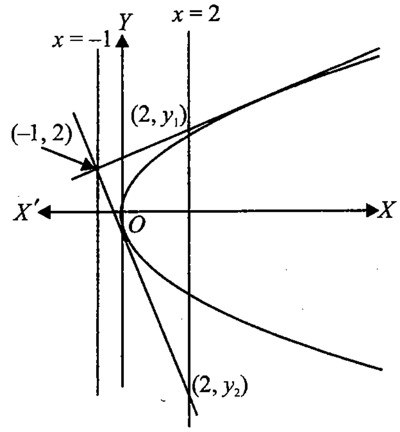


Fig. 3.40

Equation of the pair of tangents from point (x_1, y_1) is $SS_1 = T^2$

or $\quad (y^2 - 4x)(y_1^2 - 4x_1) = (yy_1 - 2(x + x_1))^2$

Then equation of the pair of tangents from (–1, 2) is

$$(y^2 - 4x)(4 + 4) = [2y - 2(x - 1)]^2$$
$$= 4(y - x + 1)^2$$

or $\quad 2(y^2 - 4x) = (y - x + 1)^2$

Solving with the line $\quad x = 2$, we get

$$2(y^2 - 8) = (y - 1)^2$$

or $\quad y^2 + 2y - 17 = 0$

where $\quad y_1 + y_2 = -2$ and $y_1 y_2 = -17$

Now $\quad |y_1 - y_2|^2 = (y_1 + y_2)^2 - 4y_1 y_2$

$$= 4 - 4(-17) = 72$$

$\therefore \quad |y_1 - y_2| = \sqrt{72} = 6\sqrt{2}$

Properties of Tangents

Point of Intersection of Tangents at Any Two Points on the Parabola

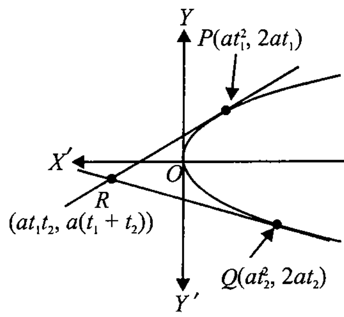


Fig. 3.41

Let the equation of the parabola be $y^2 = 4ax$

The two points on the parabola are $P \equiv (at_1^2, 2at_1)$ and $Q \equiv (at_2^2, 2at_2)$

Equation of the tangents at P and Q are

$$t_1 y = x + at_1^2 \qquad \text{(i)}$$

and

$$t_2 y = x + at_2^2 \qquad \text{(ii)}$$

Solving these equations, we get

$$x = at_1t_2,\ y = a(t_1 + t_2).$$

Thus, the co-ordinates of the point of intersection of tangents at $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are $(at_1t_2, a(t_1 + t_2))$.

Note:

- *The geometric mean of the x-coordinates of P and Q (i.e., $\sqrt{at_1^2 \times at_2^2} = at_1t_2$) is x-coordinates of the point of intersection of tangents at P and Q on the parabola.*
- *The arithmetic mean of the y-coordinates of P and Q $\left(i.e., \dfrac{2at_1 + 2at_2}{2} = a(t_1 + t_2)\right)$ is the y-coordinate of the point of intersection of tangents at P and Q on the parabola.*

Locus of Foot of Perpendicular From Focus Upon Any Tangent is Tangent at Vertex

Equation of the tangent to parabola $y^2 = 4ax$ at point $P(t)$ is $ty = x + at^2$

It meets y-axis at $Q(0, at)$

Now $$m_{SQ} = \frac{at - 0}{0 - a} = -t$$

Slope of the tangent PQ is $\frac{1}{t}$. Hence, SQ is perpendicular to PQ.

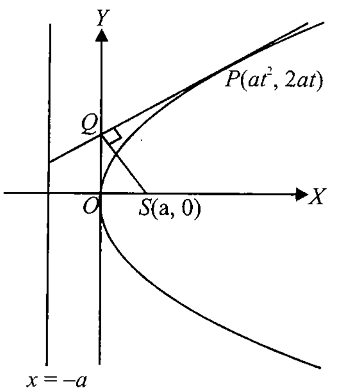


Fig. 3.42

Length of Tangent Between the Point of Contact P(t) and the Point Where it Meets the Directrix Q Subtends Right Angle at Focus

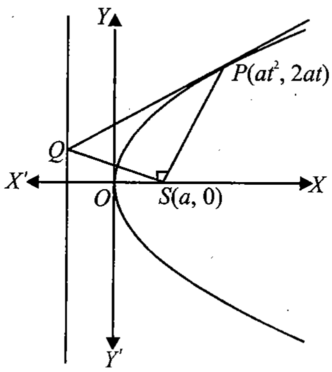


Fig. 3.43

Equation of the tangent to parabola $y^2 = 4ax$ at point $P(t)$ is $ty = x + at^2$

It meets $$x = -a \text{ at } \left(-a, \frac{at^2 - a}{t}\right)$$

Now the slope of $$SP = m_{SP} = \frac{2at - 0}{at^2 - a} = \frac{2t}{t^2 - 1}$$

Slope of $$SQ = m_{SQ} \frac{\frac{at^2 - a}{t} - 0}{-a - a} = \frac{1 - t^2}{2t}$$

Hence, SP is perpendicular to SQ.

Tangents at Extremities of Focal Chord are Perpendicular and Intersect on Directrix

Slope of the tangent at point $P(t)$ is $\frac{1}{t}$

If PQ is focal chord then point Q has parameter $-\frac{1}{t}$

Then slope of the tangent at point Q is $-t$

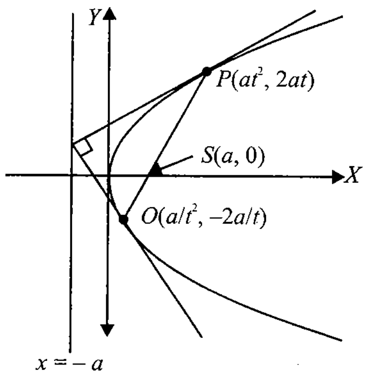


Fig. 3.44

Hence, the tangents are perpendicular.

Moreover, the point of intersection of the tangents at point $P(t_1)$ and $Q(t_2)$ is

$$(at_1t_2, a(t_1+t_2)) \equiv \left(a\frac{1}{t}(-t), a\left(t-\frac{1}{t}\right)\right) \equiv \left(-a, a\left(t-\frac{1}{t}\right)\right)$$

Thus, tangents intersect on the directix.

Example 3.47 **Find the points of contact Q and R of a tangent from the point $P(2, 3)$ on the parabola $y^2 = 4x$.**

Sol. $\left.\begin{array}{l} t_1t_2 = 2 \\ t_1+t_2 = 3 \end{array}\right] \Rightarrow t_1 = 1 \text{ and } t_2 = 2$

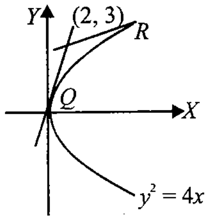


Fig. 3.45

Hence, points $(t_1^2, 2t_1)$ and $(t_2^2, 2t_2)$. i.e., (1, 2) and (4, 4).

Example 3.48 **Two straight lines $(y - b) = m_1(x + a)$ and $(y - b) = m_2(x + a)$ are the tangents of $y^2 = 4ax$. Prove $m_1m_2 = -1$.**

Sol. Clearly both the lines pass through $(-a, b)$, which is a point lying on the directrix of the parabola.

Thus, $m_1m_2 = -1$.

Because tangents drawn from any point on the directrix are always mutually perpendicular.

Example 3.49 **Mutually perpendicular tangents TA and TB are drawn to $y^2 = 4ax$, then find the minimum length of AB.**

Sol. Chord of contact of mutually perpendicular tangents is always a focal chord. Thus, minimum length of AB is $4a$.

Example 3.50 **Tangents PA and PB are drawn from point P on the directrix of the parabola $(x - 2)^2 + (y - 3)^2 = \frac{(5x - 12y + 3)^2}{169}$. Find the least radius of the circumcircle of the triangle PAB.**

Sol. Tangents from any point on the directrix are perpendicular and the corresponding chord of contact is the focal chord which is the diameter of the circumcircle of the triangle PAB. The least value of the diameter is the latus rectum.

For the given parabola, focus is (2, 3) and directrix is $5x - 12y + 3 = 0$

Hence, latus rectum $= \frac{|10 - 36 + 3|}{13} = \frac{23}{13}$.

Example 3.51 **Tangents are drawn to the parabola $(x - 3)^2 + (x + 4)^2 = \frac{(3x - 4y - 6)^2}{25}$ at the extremities of the chord $2x - 3y - 18 = 0$. Find the angle between the tangents.**

Sol. The given chord $2x - 3y - 18 = 0$ satisfies the point $(3, -4)$ which is focus of the given parabola. Hence, it is focal chord and tangents at extremities are perpendicular.

Locus of Point of Intersection of Tangent under Different Conditions

Tangents to the parabola $y^2 = 4ax$ at points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are $t_1y = x + at_1^2$ and $t_2y = x + at_2^2$, respectively. These tangents intersect at point $R(at_1t_2, a(t_1 + t_2))$. If we want to find the locus of point R under some conditions then let point R has coordinates (h, k). We have $h = at_1t_2$ and $k = a(t_1 + t_2)$.

Example 3.52 **Find the locus of the point of intersection of tangents to the parabola $y^2 = 4ax$.**

- **a. which are inclined at an angle θ to each other**
- **b. which intercept constant length c on the tangent at the vertex**
- **c. such that area of ΔABR is constant c, where A and B are the point of intersection of tangents with y-axis and R is a point of intersection of tangents.**

Sol.

a. Given that $\tan\theta = \left|\frac{\frac{1}{t_1} - \frac{1}{t_1}}{1 + \frac{1}{t_1}\frac{1}{t_2}}\right| = \left|\frac{t_1 - t_2}{1 + t_1t_2}\right|$

$\Rightarrow \tan^2\theta\,(1 + t_1t_2)^2 = (t_1 - t_2)^2$

$\Rightarrow \tan^2\theta\,(1 + t_1t_2)^2 = (t_1 + t_2)^2 + 4t_1t_2$

$\Rightarrow \tan^2\theta\left(1 + \frac{h}{a}\right)^2 = \left(\frac{k}{a}\right)^2 + 4\frac{h}{a}$

$\Rightarrow \tan^2\theta\,(x + a)^2 = y^2 + 4ax$

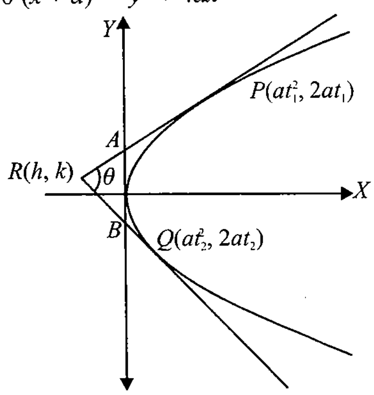


Fig. 3.46

b. Points A and B have coordinates $(0, at_1)$ and $(0, at_2)$

Given $\qquad AB = c$ or $|at_1 - at_2| = c$

$\Rightarrow \quad a^2[(t_1 + t_2)^2 - 4t_1t_2] = c^2$

$\Rightarrow \quad a^2\left[\left(\frac{k}{a}\right)^2 - 4\frac{h}{a}\right] = c^2$

$\Rightarrow \quad y^2 - 4ax = c^2$ which is a parabola.

c. Points A and B have coordinates $(0, at_1)$ and $(0, at_2)$

Given area of triangle $ABR = c$

$\Rightarrow \quad \frac{1}{2} AB \times RM = c$

$\Rightarrow \quad \frac{1}{2}|at_1 - at_2|(at_1t_2) = c^2$

$\Rightarrow a^4[(t_1 + t_2)^2 - 4t_1t_2](t_1t_2)^2 = 4c^2$

$\Rightarrow \quad a^4\left[\left(\frac{k}{a}\right)^2 - 4\frac{h}{a}\right]\left(\frac{h}{a}\right)^2 = 4c^2$

$\Rightarrow \quad (y^2 - 4ax)x^2 = 4c^2$

Concept Application Exercise 3.3

1. Find the point on the curve $y^2 = ax$ the tangent at which makes an angle of 45° with x-axis.
2. Find the equation of the straight lines touching both $x^2 + y^2 = 2a^2$ and $y^2 = 8ax$.
3. Find the angle at which the parabolas $y^2 = 4x$ and $x^2 = 32y$ intersect.
4. How many distinct real tangents that can be drawn from (0, –2) to the parabola $y^2 = 4x$?
5. The tangents to the parabola $y^2 = 4x$ at the points (1, 2) and (4, 4) meet on which of the following lines?

   a. $x = 3$ **b.** $x + y = 4$

   c. $y = 3$ **d.** $y = 4$
6. If the tangents at the points P and Q on the parabola $y^2 = 4ax$ meet at T and S is its focus, then prove that SP, ST and SQ are in G.P.
7. If the line $x + y = a$ touches the parabola $y = x - x^2$, then find the value of a.
8. From an external point P, pair of tangent are drawn to the parabola $y^2 = 4x$. If θ_1 and θ_2 are the inclinations of these tangents with x-axis such that $\theta_1 + \theta_2 = \frac{\pi}{4}$, then find the locus of P.
9. Find the angle between the tangents drawn from (1, 3) to the parabola $y^2 = 4x$.
10. Find the slopes of the tangents to the parabola $y^2 = 8x$ which are normal to the circle $x^2 + y^2 + 6x + 8y - 24 = 0$.
11. Find the angle between the tangents drawn to $y^2 = 4x$, where it is intersected by the line $y = x - 1$.
12. Find the locus of the point of intersection of the perpendicular tangents of the curve $y^2 + 4y - 6x - 2 = 0$.
13. From a point P on directrix, tangents PA and PB are drawn to the parabola $y^2 = 16x$. Find the minimum radius of the circle circumscribing ΔPAB.
14. Find the angle between the tangents drawn from the origin to the parabolas $y^2 = 4a(x - a)$.
15. Find the locus of point from which the two tangents drawn to a parabola $y^2 = 4ax$ are such that slope of one is thrice of the other.

Equation of Normal

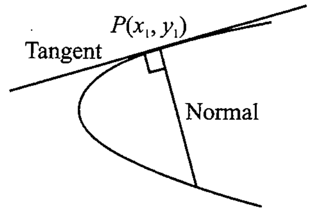


Fig. 3.47

Differentiating $y^2 = 4ax$ with respect to x, we have $\frac{dy}{dx} = \frac{2a}{y}$

The slope of the tangent at $(x_1, y_1) = \frac{2a}{y_1}$

Since the normal at (x_1, y_1) is perpendicular to the tangent at (x_1, y_1)

$\therefore$ Slope of normal at $(x_1, y_1) = -\frac{y_1}{2a}$

Hence, the equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \qquad \text{(i)}$$

Parametric Form

Replacing x_1 by at^2 and y_1 by $2at$, then Eq. (i) becomes $y - 2at = -t(x - at^2)$ or $y = -tx + 2at + at^3$

The equations of normals of all standard parabolas are as follows:

Equations of parabola	Parametric coordinates t	Normals at t
$y^2 = 4ax$	$(at^2, 2at)$	$y + tx = 2at + at^3$
$y^2 = -4ax$	$(-at^2, 2at)$	$y - tx = 2at + at^3$
$x^2 = 4ay$	$(2at, at^2)$	$x + ty = 2at + at^3$
$x^2 = -4ay$	$(2at, -at^2)$	$x - ty = 2at + at^3$

Slope Form

The equation of normal to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is

$$y = -tx + 2at + at^3 \quad \text{(i)}$$

Since m is the slope of the normal, then $m = -t$

Then equation of normal is

$$y = mx - 2am - am^3. \quad \text{(ii)}$$

Thus $y = mx - 2am - am^3$ is a normal to the parabola $y^2 = 4ax$ where m is the slope of the normal.

The coordinates of the foot of normal are $(am^2, -2am)$.

Comparing (ii) with $y = mx + c$

$\therefore \quad c = -2am - am^3$

which is the condition when $y = mx + c$ is the normal of $y^2 = 4ax$.

Equation of normal for all parabolas in terms of m.

Equations of parabolas	**Point of contact in terms of slope (m)**	**Equations of normals in terms of slope (m)**	**Condition for line $y = mx + c$ is normal**
$y^2 = 4ax$	$(am^2, -2am)$	$y = mx - 2am - am^3$	$c = -2am - am^3$
$y^2 = -4ax$	$(-am^2, 2am)$	$y = mx + 2am + am^3$	$c = 2am + am^3$
$x^2 = 4ay$	$\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$	$y = mx + 2a + \frac{a}{m^2}$	$c = 2a + \frac{a}{m^2}$
$x^2 = -4ay$	$\left(\frac{2a}{m}, -\frac{a}{m^2}\right)$	$y = mx - 2a - \frac{a}{m^2}$	$c = -2a - \frac{a}{m^2}$

Example 3.53 **Three normals to $y^2 = 4x$ pass through the point (15, 12). Show that one of the normals is given by $y = x - 3$ and find the equations of the others.**

Sol. Equation of the normal to $y^2 = 4x$ having slope m is $y = mx - 2m - m^3$.

If it passes through the point (15, 12), then

$$12 = 15m - 2m - m^3$$

$\Rightarrow \quad m^3 - 13m + 12 = 0$

$\Rightarrow (m - 1)(m - 3)(m + 4) = 0$

$\Rightarrow \quad m = 1, 3, -4$

Taking $m = 1$, the equation of normal is $y = x - 3$, which is one of the normals.

Taking $m = 3$, and -4, the equations of other two normals are $y = 3x - 33$ and $y + 4x = 72$.

Example 3.54 **Find the equations of normals to the parabola $y^2 = 4ax$ at the ends of the latus rectum.**

Sol. Differentiating $y^2 = 4ax$ w.r.t. x, we have $\frac{dy}{dx} = \frac{2a}{y}$

Hence, slope of normal at point $P(a, 2a)$ is $-\frac{2a}{2a} = -1$

Slope of normal at point $Q(a, -2a)$ is $-\frac{-2a}{2a} = 1$

Hence, equation of normal at point P and Q are $x + y - 3a = 0$ and $x - y - 3a = 0$.

Example 3.55 **If $y = x + 2$ is normal to parabola $y^2 = 4ax$, then find the value of a.**

Sol. Normal to parabola $y^2 = 4ax$ having slope m is $y = mx - 2am - am^3$

Given normal is $y = x + 2 \Rightarrow m = 1$ and $-2am - am^3 = 2$

$\Rightarrow \quad -2a(1) - a(1)^3 = 1$

$\Rightarrow \quad a = -1/3$

Example 3.56 **Find the equation of normal to the parabola $y = x^2 - x - 1$ which has equal intercept on axis.**

Also find the point where this normal meets the curve again.

Sol. Normal has equal intercept on axis, then its slope is -1.

Now differentiating $y = x^2 - x - 1$ w.r.t. x both sides

we have $\frac{dy}{dx} = 2x - 1$, which is the slope of the tangent to the parabola at any point on the parabola.

Now Slope of normal to curve at any point is

$$m = -\frac{dx}{dy} = \frac{1}{1-2x}$$

Then we want slope of normal as $-1 \Rightarrow \frac{1}{1-2x} = -1$

$\Rightarrow x = 1$

$\Rightarrow y = -1$ (from $y = x^2 - x - 1$)

Hence from point – slope form equation of normal is

$y - (-1) = -1(x - 1)$ or $x + y = 0$

Solving this equation of normal with the equation of parabola.

$-x = x^2 - x - 1$ or $x^2 = 1$ or $x = \pm 1$

Hence normal meets parabola again at point whose abscissa is -1, for which ordinate is 1

Thus normal meets parabola again at $(1, -1)$.

Example 3.57 **Find the minimum distance between the curves $y^2 = 4x$ and $x^2 + y^2 - 12x + 31 = 0$.**

Sol. Centre and radius of the given circle is $P(6, 0)$ and $\sqrt{5}$, respectively.

Now the shortest distance always occurs along common normal.

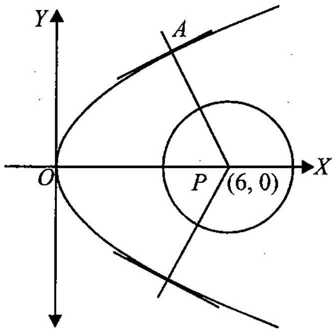


Fig. 3.48

Differentiating $y^2 = 4x$ with respect to x, we get

$$\frac{dy}{dx} = \frac{2}{y}$$

Then the slope of normal at point $A\left(y_1^2/4, y_1\right)$ is $\frac{y_1}{2}$.

Also from definition, the slope of AP is given by

$$\frac{y_1 - 0}{\frac{y_1^2}{4} - 6} = -\frac{y_1}{2}$$

$\Rightarrow$ $y_1 = 0$ or $y_1 = \pm 4$

Hence, the points are $O(0, 0)$, $A(4, 4)$, $C(4, -4)$.

The shortest distance is $AP - \sqrt{5} = \sqrt{20} - \sqrt{5} = \sqrt{5}$.

Example 3.58 **Prove that the length of the intercept on the normal at the point $P(at^2, 2at)$ of a parabola $y^2 = 4ax$ made by the circle described on the line joining the focus and P as diameter is $a\sqrt{1+t^2}$.**

Sol.

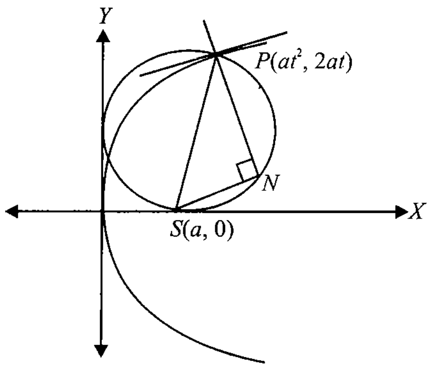


Fig. 3.49

Equation of normal at $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ is

$$y = -tx + 2at + at^3 \quad \text{(i)}$$

Let this normal meets the circle on SP as diameter in N, then $\angle SNP = \frac{\pi}{2}$ (angle in a semicircle)

$\therefore$ $PN^2 = SP^2 - SN^2$

SN is perpendicular to the normal

Now $SP = a + at^2$

and $SN = \frac{|at - 2at - at^3|}{\sqrt{1+t^2}}$

$= at\sqrt{1+t^2}$

$\therefore$ $PN^2 = a^2(1+t^2)^2 - a^2t^2(1+t^2)$

$= a^2(1+t^2)[1 + t^2 - t^2]$

$= a^2(1+t^2)$

$\therefore$ $PN = a\sqrt{1+t^2}$

Properties of Normal

1. Normal other than axis of parabola never passes through the focus.

   Proof:

   Let normal at $P(am^2, -2am)$, $y = mx - 2am - am^3$ passes through the focus $(a, 0)$.

   Then $0 = am - 2am - am^3$

   $\Rightarrow$ $m^2 + 1 = 0$, which is not possible.

   Hence, proved.

2. Point of intersection of normal at point $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$:

   Solving normal at point P, $y = -t_1x + 2at_1 + at_1^3$ and normal at point Q, $y = -t_2x + 2at_2 + at_2^3$,

   we have point of intersection, which is $[2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)]$.

3. Normal at the point $P(t_1)$ meets the curve again at point $Q(t_2)$ such that

   $$t_2 = -t_1 - \frac{2}{t_1}$$

   Proof:

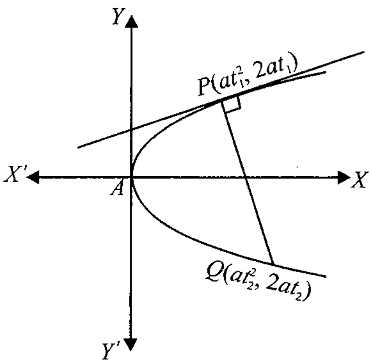


Fig. 3.50

For the parabola be $y^2 = 4ax$, the equation of normal at $P(at_1^2, 2at_1)$ is

$$y = -t_1x + 2at_1 + at_1^3 \quad \text{(i)}$$

Since it meets the parabola again at Q $(at_2^2, 2at_2)$, Eq. (i) passes through $Q(at_2^2, 2at_2)$.

$\therefore \quad 2at_2 = -at_1t_2^2 + 2at_1 + at_1^3$

$\Rightarrow 2a(t_2 - t_1) + at_1(t_2^2 - t_1^2) = 0$

$\Rightarrow a(t_2 - t_1)[2 + t_1(t_2 + t_1)] = 0$

$\because \quad a(t_2 - t_1) \neq 0 \quad (\because t_1 \text{ and } t_2 \text{ are different})$

$\therefore \quad 2 + t_1(t_2 + t_1) = 0$

$\therefore \quad t_2 = -t_1 - \frac{2}{t_1}$

Example 3.59 **In the parabola $y^2 = 4ax$, the tangent at P whose abscissa is equal to the latus rectum meets its axis at T and normal at P cuts the curve again at Q. Show that PT: $PQ = 4 : 5$.**

Sol. Let P be $(at^2, 2at)$. Since $at^2 = 4a$, $t = \pm 2$. Consider $t = 2$, and $P(4a, 4a)$. Tangent at P is $2y = x + 4a$ which meets x-axis at $T(-4a, 0)$.

If coordinates of Q are $(at_1^2, 2at_1)$ then $t_1 = -t - \frac{2}{t} = -3$

so Q is $(9a, -6a)$

$\therefore \quad (PQ)^2 = 125a^2$ and $(PT)^2 = 80a^2$

or $\quad PT{:}PQ = 4{:}5$

Example 3.60 **If the normal to the parabola $y^2 = 4ax$ at point t_1 cuts the parabola again at point t_2, then prove that $t_2^2 \geq 8$.**

Sol. A normal at point t_1 cuts the parabola again at t_2, then

$$t_2 = -t_1 - \frac{2}{t_1}$$

$\Rightarrow \quad t_1^2 + t_1t_2 + 2 = 0$

Since, t_1 is real, discriminant ≥ 0

$\Rightarrow \quad t_2^2 - 8 \geq 0$

$\Rightarrow \quad t_2^2 \geq 8$

Example 3.61 **Find the length of normal chord which subtends an angle of 90° at the vertex of the parabola $y^2 = 4x$.**

Sol.

$$t_1 = -t - \frac{2}{t} \text{ also, } tt_1 = -4$$
$(\because OP \perp OQ)$

$\Rightarrow \quad t_1 = -\frac{4}{t}$

$\Rightarrow \quad -\frac{4}{t} = -t - \frac{2}{t}$

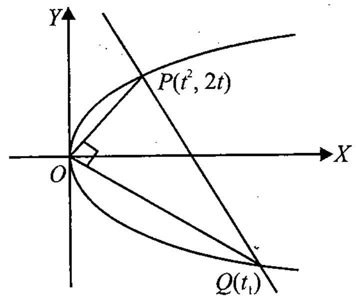


Fig. 3.51

$\Rightarrow \quad \frac{2}{t} = t$

$\Rightarrow \quad t = \sqrt{2}$

$\Rightarrow \quad t_1 = -2\sqrt{2}$

$\Rightarrow \quad Q \equiv (8, -4\sqrt{2}), P \equiv (1, 2)$

$\Rightarrow \quad PQ = \sqrt{7^2 + (2 + 4\sqrt{2})^2} = \sqrt{85 + 16\sqrt{2}}$

Example 3.62 **Find the locus of the point of intersection of the normals at the end of the focal chord of the parabola $y^2 = 4ax$.**

Sol. Point of intersection of normal at point $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ is

$$R \equiv [2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)] = (h, k)$$

Also PQ is focal chord, then $t_1t_2 = -1$

Then $\quad k = a(t_1 + t_2) \quad$ (i)

and

$$h = 2a + a[(t_1 + t_2)^2 - t_1t_2]$$
$$= 2a + a[(t_1 + t_2)^2 + 1] \quad \text{(ii)}$$

Eliminating $t_1 + t_2$ from Eqs. (i) and (ii), we have

$$h = 2a + a\left(\frac{k^2}{a^2} + 1\right)$$

$\Rightarrow \quad y^2 = a(x - 3a)$

Example 3.63 **Find the locus of the point of intersection of two normals to a parabola which are at right angles to one another.**

Sol. The equation of the normal to the parabola $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$

It passes through the point (h, k) if

$$k = mh - 2am - am^3$$

$\Rightarrow \quad am^3 + m(2a - h) + k = 0 \quad$ (i)

Let the roots of the above equation be m_1, m_2 and m_3.

Let the perpendicular normals correspond to the values of m_1 and m_2 so that $m_1m_2 = -1$.

From Eq. (i), $m_1m_2m_3 = -\frac{k}{a}$

Since $m_1m_2 = -1, m_3 = \frac{k}{a}$

Since m_3 is a root of Eq. (i), we have

$$a\left(\frac{k}{a}\right)^3 + \frac{k}{a}(2a - h) + k = 0$$

$$\Rightarrow \quad k^2 + a(2a - h) + a^2 = 0$$

$$\Rightarrow \quad k^2 = a(h - 3a)$$

Hence, the locus of (h, k) is

$$y^2 = a(x - 3a)$$

Example 3.64 Prove that the locus of the point of intersection of the normals at the ends of a system of parallel chords of a parabola is a straight line which is a normal to the curve.

Sol. Consider a chord PQ joining points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on parabola $y^2 = 4ax$.

If slope of PQ is m, then we have

$$m = 2a(t_2 - t_1)/a(t_2^2 - t_1^2)$$

$$= 2/(t_1 + t_2) \quad \text{(i)}$$

Normals at P and Q are

$$y + t_1x = 2at_1 + at_1^3$$

and $$y + t_2x = 2at_2 + at_2^3$$

Let the normal meet at $A(x_1, y_1)$, then $x_1 = 2a + a(t_1^2 + t_2^2 + t_1t_2) = 2a + a[(t_1 + t_2)^2 - t_1t_2]$

and $$y_1 = -at_1t_2(t_1 + t_2) \quad \text{(ii)}$$

Using Eq. (i) and (ii), we get $x_1 - 2a = a\left[4/m^2 + y_1m/2a\right]$

The locus of $A(x_1, y_1)$ is $\frac{1}{2}my = x - 2a - \frac{4a}{m^2}$

i.e., $$y - \left(\frac{2}{m}\right)x = -\frac{4a}{m} - \frac{8a}{m^3} \quad \text{(ii)}$$

Putting $-\frac{2}{m} = t$, locus of Eq. (ii) can be expressed as $y + tx = 2at + at^3$, which is normal to the parabola.

Co-normal Points

Let $P(h, k)$ be any given point and $y^2 = 4ax$ be a parabola.

The equation of any normal to $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$

If it passes through (h, k) then

$$k = mh - 2am - am^3$$

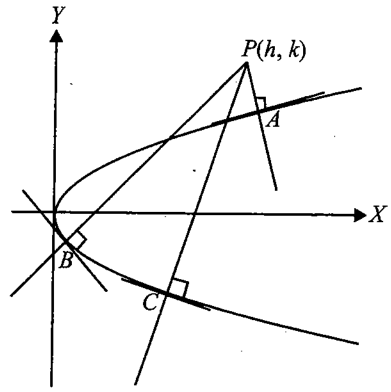


Fig. 3.52

$$\Rightarrow \quad am^3 + m(2a - h) + k = 0 \quad \text{(i)}$$

This is a cubic equation in m. So, it has three roots, say m_1, m_2 and m_3.

$$\therefore \quad m_1 + m_2 + m_3 = 0,$$

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{(2a - h)}{a} \quad \text{(ii)}$$

Hence, for any given point $P(h, k)$, Eq. (i) has maximum three real roots. Corresponding to each of these three roots, we have one normal passing through $P(h, k)$.

Hence, in total, we have maximum three normals PA, PB and PC drawn through P to the parabola. Points A, B, C in which the three normals from $P(h, k)$ meet the parabola are called co-normal points.

Note:

1. *The algebraic sum of ordinates of the feet of three normals drawn to a parabola from a given point is 0.*

   Proof:

   Let the ordinates of A, B, C be y_1, y_2, y_3 respectively. Then $y_1 = -2am_1, y_2 = -2am_2$ and $y_3 = -2am_3$.

   Therefore, algebraic sum of these ordinates is

   $y_1 + y_2 + y_3$

   $$= -2am_1 - 2am_2 - 2am_3$$

   $$= -2a(m_1 + m_2 + m_3)$$

   $$= -2a \times 0 \quad \text{\{from Eq. (i)\}}$$

   $$\Rightarrow y_1 + y_2 + y_3 = 0$$

2. *Centroid of the triangle formed by the feet of the three normals lies on the axis of the parabola.*

   Proof:

   If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be vertices of ΔABC, then its centroid is

   $$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right) = \left(\frac{x_1 + x_2 + x_3}{3}, 0\right)$$

   Since $y_1 + y_2 + y_3 = 0$.

Therefore, the centroid lies on the x-axis OX, which is the axis of the parabola also.

3. *If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) be real, then $h > 2a$.*

Proof:

When normals are real, then all the three roots of Eq. (i) are real and in that case

$m_1^2 + m_2^2 + m_3^2 > 0$ (*for any value of* m_1, m_2, m_3)

$\Rightarrow (m_1 + m_2 + m_3)^2 - 2(m_1m_2 + m_2m_3 + m_3m_1) > 0$

$\Rightarrow (0)^2 - \dfrac{2(2a-h)}{a} > 0$

$\Rightarrow h - 2a > 0$

$\Rightarrow h > 2a$

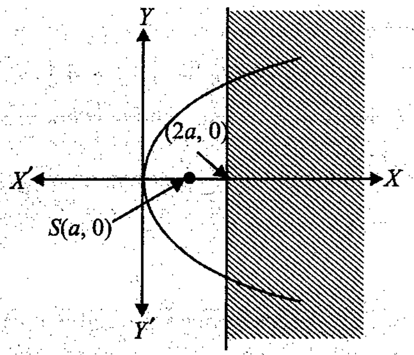


Fig. 3.53

As shown in Fig. 3.53, point (h, k) must lie in the shaded region, from which we can draw three normals to the parabola $y^2 = 4ax$.

Example 3.65 **Find the number of distinct normals that can be drawn from (–2, 1) to the parabola $y^2 - 4x - 2y - 3 = 0$.**

Sol.

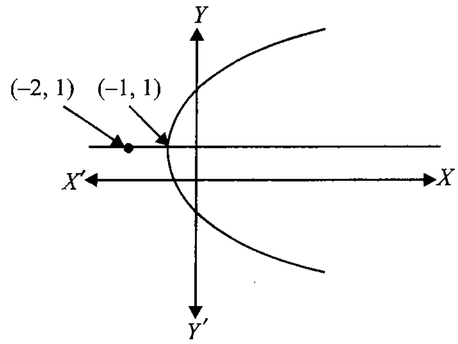


Fig. 3.54

Here, $y^2 - 2y + 1 = 4(x + 1)$

or

$(y - 1)^2 = 4(x + 1)$

So, the axis is $y - 1 = 0$.

Also (–2, 1) lies on the axis, and it is exterior to the parabola because $1^2 - 4(-2) - 2(1) - 3 > 0$.

Hence, only one normal is possible.

Example 3.66 **If two of the three feet of normals drawn from a point to the parabola $y^2 = 4x$ be (1, 2) and (1, –2), then find the third foot.**

Sol. The sum of the ordinates of the feet $= y_1 + y_2 + y_3 = 0$.

$\therefore \quad 2 + (-2) + y_3 = 0$

$\therefore \quad y_3 = 0$

$\therefore$ third foot is (0, 0)

Example 3.67 **If three distinct normals can be drawn to the parabola $y^2 - 2y = 4x - 9$ from the point $(2a, b)$, then find the range of the value of a.**

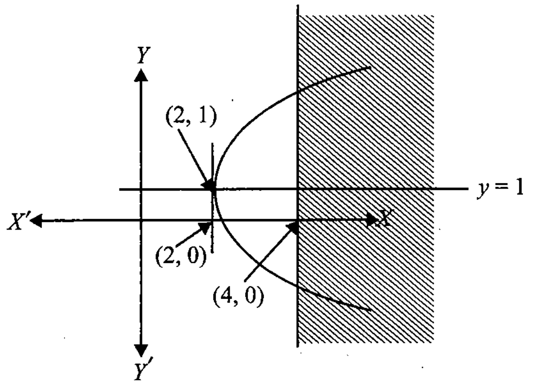


Fig. 3.55

Sol. Given parabola is $(y - 1)^2 = 4(x - 2)$. Hence, vertex is (2, 1).

Focus is (2, 2). Hence, all the points $(2a, b)$ must lie in the shaded region

$\Rightarrow \quad 2a > 4$

$\Rightarrow \quad a > 2$

Example 3.68 **If (h, k) is a point on the axis of the parabola $2(x-1)^2 + 2(y-1)^2 = (x + y + 2)^2$ from where three distinct normals may be drawn, then prove that $h > 2$.**

Sol.

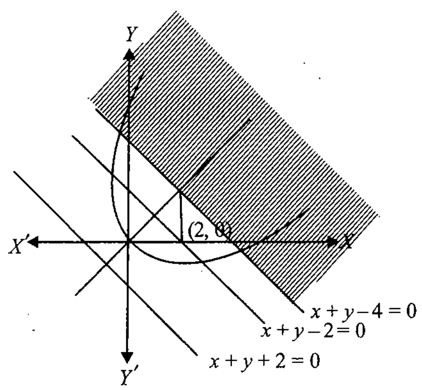


Fig. 3.56

We have, $2(x-1)^2+2(y-1)^2=(x+y+2)^2$

$$\Rightarrow \sqrt{(x-1)^2+(y-1)^2} = \left|\frac{x+y+2}{\sqrt{1+1}}\right|$$

Focus is (1, 1) and directrix is $x+y+2=0$

Equation of latus rectum is $x+y-2=0$

Then the required points lie above the line $x+y-4=0$ which intersect the line $x=y$ (axis of parabola) at (2, 2).

Hence, $h>2$

Example 3.69 **If the normals from any point to the parabola $y^2=4x$ cut the line $x=2$ in points whose ordinates are in A.P., then prove that slopes of tangents at the co-normal points are in G.P.**

Sol. Equation of the normal to the parabola $y^2=4x$ is given by

$$y=-tx+2t+t^3 \quad \text{(i)}$$

Since it intersects $x=2$, we get $y=t^3$

Let the three ordinates be t_1^3, t_2^3, t_3^3 are in A.P.

$$\Rightarrow 2t_2^3=t_1^3+t_3^3$$
$$=(t_1+t_3)^3-3t_1t_3(t_1+t_3) \quad \text{(ii)}$$

Now $t_1+t_2+t_3=0$

$\Rightarrow t_1+t_3=-t_2$

Hence, Eq. (ii) reduces to

$$2t_2^3=(-t_2)^3-3t_1t_3(-t_2)$$
$$=-t_2^3+3t_1t_2t_3$$

$$\Rightarrow 3t_2^3=3t_1t_2t_3 \Rightarrow t_2^2=t_1t_3$$

$\Rightarrow t_1, t_2, t_3$ are in G.P.

Hence, slopes of tangents $\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}$ are in G.P.

Circle through the Co-normal Points

To find the equation of the circle passing through the three (co-normal) points on the parabola, normal at which pass through a given point (h, k)

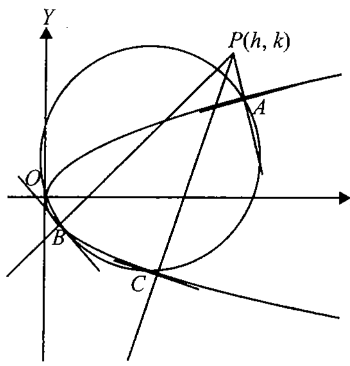


Fig. 3.57

Let $A(am_1^2, -2am_1)$ $B(am_2^2, -2am_2)$ and $C(am_3^2, -2am_3)$ be the three points on the parabola $y^2=4ax$. Since the point of intersection of normals at these points is (h, k)

$$\therefore \quad am^3+(2a-h)m+k=0 \quad \text{(i)}$$

$$\Rightarrow \quad m_1+m_2+m_3=0 \quad \text{(ii)}$$

$$m_1m_2+m_2m_3+m_3m_1=\frac{(2a-h)}{a} \quad \text{(iii)}$$

and

$$m_1m_2m_3=-\frac{k}{a} \quad \text{(iv)}$$

Let the equation of the circle through A, B and C be

$$x^2+y^2+2gx+2fy+c=0 \quad \text{(v)}$$

If the point $(am^2, -2am)$ lies on it, then

$$(am^2)^2+(-2am)^2+2g(am^2)+2f(-2am)+c=0$$

or $a^2m^4+(4a^2+2ag)m^2-4afm+c=0$ (vi)

This equation has four roots m_1, m_2, m_3 and m_4 such that the circle passes through the points $A\left(am_1^2, -2am_1\right)$, $B\left(am_2^2, -2am_2\right)$, $C\left(am_3^2, -2am_3\right)$ and $D\left(am_4^2, -2am_4\right)$.

$$\therefore \quad m_1+m_2+m_3+m_4=0 \quad \text{(vii)}$$

$$0+m_4=0 \quad \text{\{From Eq. (i)\}}$$

$$\therefore \quad m_4=0$$

$$\Rightarrow \quad \left(am_4^2, -2am_4\right)=(0, 0)$$

Thus, the circle passes through the vertex of the parabola $y^2=4ax$.

$$\therefore \quad c=0$$

From Eq. (vi), $a^2m^4+(4a^2+2ag)m^2-4afm=0$

$\Rightarrow am^3+(4a+2g)m-4f=0$ (viii)

Now, Eqs. (i) and (viii) are identical

$$\therefore \quad 1=\frac{4a+2g}{2a-h}=-\frac{4f}{k}$$

$$\therefore \quad 2g=-(2a+h),\ 2f=-k/2$$

$\therefore$ The equation of the required circle is

$$x^2+y^2-(2a+h)x-\frac{k}{2}y=0$$

Example 3.70 **A circle and a parabola $y^2=4ax$ intersect at four points. Show that the algebraic sum of the ordinates of the four points is zero. Also show that the line joining one pair of these four points are equally inclined to the axis.**

Sol.

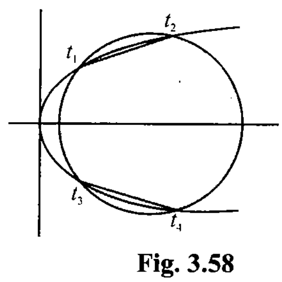


Fig. 3.58

$x^2 + y^2 + 2gx + 2fy + c = 0$

Solving it with $x = at^2, y = 2at$

$a^4t^4 + 4a^2t^2 + 2gat^2 + 4aft + c = 0$

$a^4t^4 + 2a(2a + g)t^2 + 4aft + c = 0$

Hence, $t_1 + t_2 + t_3 + t_4 = 0$ (i)

or $2a(t_1 + t_2 + t_3 + t_4) = 0$. Hence, proved

Slope of line joining

$t_1, t_2 = \frac{2}{t_1 + t_2} = -\frac{2}{t_3 + t_4}$

$= m_1$ [using Eq. (i)]

Slope of line joining $t_3, t_4 = \frac{2}{t_3 + t_4} = m_2$

Hence, $m_1 + m_2 = 0$

Reflection Property of Parabola

The tangent at any point P to a parabola bisects the angle between the focal chord through P and the perpendicular from P to the directrix.

Let the tangent at $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ meets the axis of the parabola, i.e., x-axis or $y = 0$ at T.

The equation of tangent to the parabola $y^2 = 4ax$ at $P(at^2, 2at)$ is $ty = x + at^2$.

For co-ordinate of T solve it with $y = 0$

$\therefore \quad T(-at^2, 0)$

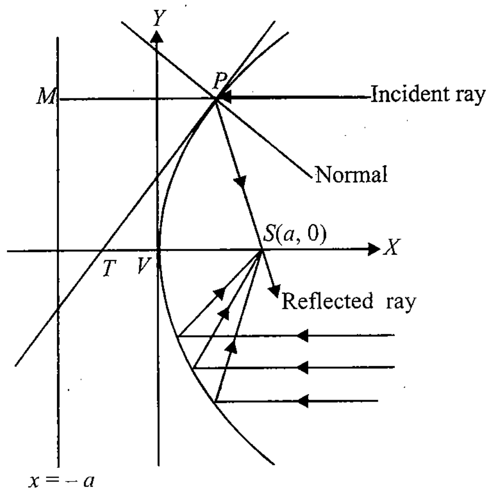


Fig. 3.59

$\therefore \quad ST = SV + VT = a + at^2$

$= a(1 + t^2)$

Also, $SP = PM = a + at^2 = a(1 + t^2)$

$\therefore \quad SP = ST$

i.e. $\angle STP = \angle SPT$

But $\angle STP = \angle MPT$ (alternate angles)

$\therefore \quad \angle SPT = \angle MPT$

Thus, if any light ray is sent along a line parallel to the axis of the parabola then the reflected ray passes through the focus, as the normal bisects the angle between the incident ray and reflected ray.

Example 3.71 **A ray of light moving parallel to the x-axis gets reflected form a parabolic mirror whose equation is $(y - 2)^2 = 4(x + 1)$. Find the point on the axis of parabola through which the ray must pass after reflection.**

Sol. The equation of the axis of the parabola is $y - 2 = 0$ which is parallel to the x-axis. We know that any ray parallel to the axis of a parabola passes through the focus after reflection. Hence, it passes through focus (0, 2).

Example 3.72 **If incident from point $(-1, 2)$ parallel to the axis of the parabola $y^2 = 4x$ strikes the parabola, then find the equation of reflected ray.**

Sol. Incident ray as shown in the figure strikes the parabola at $P(1, 2)$.

Reflected ray passes through the focus

Hence, equation of reflected ray is $x = 1$.

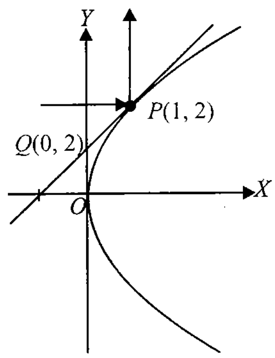


Fig. 3.60

Concept Application Exercise 3.4

1. Prove that the chord $y - x\sqrt{2} + 4a\sqrt{2} = 0$ is a normal chord of the parabola $y^2 = 4ax$. Also find the point on the parabola when the given chord is normal to the parabola.
2. If $y = 2x + 3$ is a tangent to the parabola $y^2 = 24x$, then find its distance from the parallel normal.
3. Find the point where the line $x + y = 6$ is a normal to the parabola $y^2 = 8x$.

4. Find the locus of the midpoints of the portion of the normal to the parabola $y^2 = 4ax$ intercepted between the curve and the axis.
5. Find the angle at which normal at point $P(at^2, 2at)$ to the parabola meets the parabola again at point Q.
6. If tangents are drawn to $y^2 = 4ax$ from any point P on the parabola $y^2 = a(x + b)$, then show that the normals drawn at their point for contact meet on a fixed line.
7. If normal to parabola $y^2 - 4ax = 0$ at α point intersect the parabola again such that sum of ordinates of these two points is 3, then show that the semi-latus rectum is equal to -1.5α.
8. If the parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ have a common normal other than x-axis (a, b, c being distinct positive real numbers), then prove that $\frac{b}{a-c} > 2$.
9. If a normal chord subtends a right angle at the vertex of the parabola $y^2 = 4ax$, then find its inclination to the axis.

Chord of Contact

Let PQ and PR be tangents to the parabola $y^2 = 4ax$ drawn from any external point $P(h, k)$, then QR is called chord of contact of the parabola $y^2 = 4ax$.

Let $\quad Q \equiv (x_1, y_1)$ and $R \equiv (x_2, y_2)$

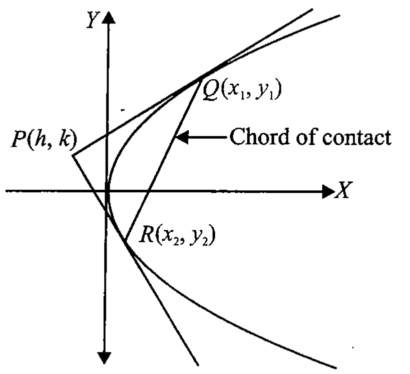


Fig. 3.61

Equation of the tangent PQ is

$$yy_1 = 2a(x + x_1) \quad \text{(i)}$$

and equation of the tangent PR is

$$yy_2 = 2a(x + x_2) \quad \text{(ii)}$$

Since Eqs. (i) and (ii) pass through (h, k)

$$\therefore \quad ky_1 = 2a(h + x_1) \quad \text{(iii)}$$

$$\text{and} \quad ky_2 = 2a(h + x_2) \quad \text{(iv)}$$

Hence, it is clear that $Q(x_1, y_1)$ and $R(x_2, y_2)$ lie on $yk = 2a(x + h)$ which is chord of contact of QR.

Example 3.73 **Tangents are drawn to porabola $y^2 = 4ax$ at point where the line $lx + my + n = 0$ meets this parabola. Find the point of intersection of these tangents.**

Sol. Let the tangent intersect at $P(h, k)$, then $lx + my + n = 0$ will be the chord of contact of 'P'. That means $lx + my + n = 0$ and $yk - 2ax - 2ah = 0$ will represent the same line. Thus,

$$\frac{k}{m} = \frac{-2a}{l} = \frac{-2ah}{n}$$

$$\Rightarrow \quad h = \frac{n}{l}, k = -\frac{2am}{l}$$

Example 3.74 **If the chord of contact of tangents from a point P to the parabola $y^2 = 4ax$ touches the parabola $x^2 = 4by$, then find the locus of P.**

Sol. Chord of contact of parabola $y^2 = 4ax$ w.r.t. point $P(x_1, y_1)$ is

$$yy_1 = 2a(x + x_1) \quad \text{(i)}$$

This line touches the parabola $x^2 = 4by$

Solving Eq. (i) with parabola, we have

$$x^2 = 4b[(2a/y_1)(x + x_1)]$$

or $\quad y_1x^2 - 8abx - 8abx_1 = 0$

According to the question, this equation must have equal roots.

$$\Rightarrow \quad D = 0$$

$$\Rightarrow \quad 64a^2b^2 + 32abx_1y_1 = 0$$

$$\Rightarrow \quad x_1y_1 = -2ab \text{ or } xy = -2ab$$

which is a rectangular hyperbola.

Example 3.75 **Tangents are drawn from any point on the line $x + 4a = 0$ to the parabola $y^2 = 4ax$. Then find the angle subtended by the chord of contact at the vertex.**

Sol. Let $P(at_1^2, 2at_1)$ and $Q(at_1^2, 2at_2)$ be the points of contact for tangents drawn from any point on the line $x + 4a = 0$. Their point of intersection will be on this line.

$$\therefore \quad at_1t_2 + 4a = 0$$

$$\text{or} \quad t_1t_2 = -4$$

This is also the condition for chord PQ to subtend a right angle at the vertex.

Equation of Chord Whose Midpoint is (x_1, y_1)

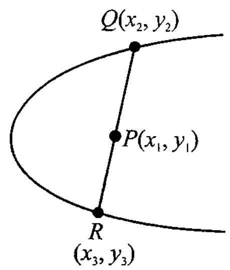


Fig. 3.62

Equation of the parabola is

$$y^2 = 4ax \quad \text{(i)}$$

Let QR be the chord of the parabola whose midpoint is $P(x_1, y_1)$.

Since Q and R lie on parabola (i)

$\therefore \quad y_2^2 = 4ax_2$ and $y_3^2 = 4ax_3$

$\therefore \quad y_3^2 - y_2^2 = 4a(x_3 - x_2)$

or $\quad \dfrac{y_3 - y_2}{x_3 - x_2} = \dfrac{4a}{y_3 + y_2} = \dfrac{4a}{2y_1}$

($\because P(x_1, y_1)$ is midpoint of QR)

$\therefore \quad \dfrac{y_3 - y_2}{x_3 - x_2} = \dfrac{2a}{y_1}$ = slope of QR

$\Rightarrow$ Equation of QR is

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$

$\Rightarrow \quad yy_1 - y_1^2 = 2ax - 2ax_1$

$\Rightarrow \quad yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$

(Subtracting $2ax_1$ from both sides)

$\Rightarrow \quad T = S_1$

where $\quad T = yy_1 - 2a(x + x_1)$

and $\quad S_1 = y_1^2 - 4ax_1$.

Example 3.76 **Find the locus of midpoint of chords of the parabola $y^2 = 4ax$ that pass through the point $(3a, a)$.**

Sol. Let the midpoint of chord be $P(h, k)$, then it's equation is

$$T = S_1$$

i.e., $\quad yk - 2a(x + h) = k^2 - 4ah$

It must pass through $(3a, a)$, hence

$$ak - 2a(3a + h) = k^2 - 4ah$$

Thus, locus of 'P' is

$$y^2 - 2ax - ay + 6a^2 = 0.$$

Example 3.77 **If the tangent at the point $P(2, 4)$ to the parabola $y^2 = 8x$ meets the parabola $y^2 = 8x + 5$ at Q and R, then find the midpoint of chord, QR.**

Sol. The equation of the tangent to $y^2 = 8x$ at $P(2, 4)$ is

$$4y = 4(x + 2) \text{ or } x - y + 2 = 0 \quad \text{(i)}$$

Let (x_1, y_1) be the midpoint of chord QR. Then, equation of QR is

$$yy_1 - 4(x + x_1) - 5 = y_1^2 - 8x_1 - 5$$

$$4x - yy_1 - 4x_1 + y_1^2 = 0 \quad \text{(ii)}$$

Clearly, Eqs. (i) and (ii) represent the same line. So,

$$\frac{4}{1} = -\frac{y_1}{-1} = \frac{-4x_1 + y_1^2}{2}$$

$\Rightarrow \quad y_1 = 4$

and $\quad 8 = -4x_1 + y_1^2$

$\Rightarrow \quad y_1 = 4$ and $x_1 = 2$

Example 3.78 **Find the locus of midpoint of normal chord of parabola $y^2 = 4ax$.**

Sol. Normal at point $P(t)$ meets the parabola again at point $Q(t_1 = -t - 2/t)$.

Let its midpoint be $R(h, k)$, then we have

$$h = \frac{at^2 + at_1^2}{2}$$

and

$$k = \frac{2at + 2at_1}{2} = a\left(t - t - \frac{2}{t}\right) = -\frac{2a}{t}$$

$\Rightarrow \quad t = -\dfrac{2a}{k}$

$\Rightarrow \quad h = \dfrac{at^2 + at_1^2}{2} = \dfrac{at^2 + a\left(-t - \frac{2}{t}\right)^2}{2}$

$\Rightarrow \quad \dfrac{2h}{a} = \left(-\dfrac{2a}{k}\right)^2 + a\left(\dfrac{2a}{k} + \dfrac{k}{a}\right)^2$

$\Rightarrow \quad \dfrac{2x}{a} = \left(\dfrac{2a}{y}\right)^2 + a\left(\dfrac{2a}{y} + \dfrac{y}{a}\right)^2$

which is required locus.

Concept Application Exercise 3.5

1. TP and TQ are tangents to the parabola, $y^2 = 4ax$ at P and Q. If the chord PQ passes through the fixed point $(-a, b)$, then find the locus of T.
2. If the distance of the point $(\alpha, 2)$ from its chord of contact w.r.t. parabola $y^2 = 4x$ is 4, then find the value of α.
3. Find the locus of the middle points of the focal chord of the parabola $y^2 = 4ax$.
4. From a variable point on the tangent at the vertex of a parabola $y^2 = 4ax$, a perpendicular is drawn to its chord of contact. Show that these variable perpendicular lines pass through a fixed point on the axis of the parabola.

EXERCISES

Subjective Type

Solutions on page 3.47

1. Prove that line joining the orthocentre to the centroid of a triangle formed by the focal chord of a parabola and tangents drawn at its extremities is parallel to the axis of the parabola.
2. A line AB makes intercepts of length a and b on the coordinate axes. Find the equation of the parabola passing through A, B and the origin, if AB is the shortest focal chord of the parabola.
3. From a point on the circle $x^2 + y^2 = a^2$, two tangents are drawn to the circle $x^2 + y^2 = b^2$ $(a > b)$. If chord of contact touches a variable circle passing through origin, show that locus of the centre of the variable circle is always a parabola.
4. Show that the common tangents to the parabola $y^2 = 4x$ and the circle $x^2 + y^2 + 2x = 0$ form an equilateral triangle.
5. The vertices A, B and C of a variable right triangle lie on a parabola $y^2 = 4x$. If the vertex B containing the right angle always remains at the point (1, 2), then find the locus of the centroid of the triangle ABC.
6. If a leaf of a book be folded so that one corner moves along an opposite side, then prove that the line of crease will always touch parabola.
7. A parabola of latus rectum l touches a fixed equal parabola. The axes of two parabolas are parallel. Then find the locus of the vertex of the moving parabola.
8. A variable parabola touches the x and the y-axis at (1, 0) and (0, 1). Then find the locus of the focus of the parabola.
9. Let N by the foot of perpendicular to the x-axis from point p on the parabola $y^2 = 4ax$. A straight line is drawn parallel to the axis which bisects PN and cuts the curve at Q; if NO meets the tangent at the vertex at a point T, then prove that $AT = \frac{2}{3} PN$.
10. Two lines are drawn at right angles, one being a tangent to $y^2 = 4ax$ and the other to $x^2 = 4by$. Then find the locus of their point of intersection.
11. Find the area of the trapezium whose vertices lie on the parabola $y^2 = 4x$ and its diagonals pass through (1, 0) and having length $\frac{25}{4}$ unit each.
12. Find the range of parameter a for which a unique circle will pass through the points of intersection of the hyperbola $x^2 - y^2 = a^2$ and the parabola $y = x^2$. Also find the equation of the circle.
13. Find the radius of the largest circle, which passes through the focus of the parabola $y^2 = 4(x + y)$ and also contained in it.
14. A tangent is drawn to the parabola $y^2 = 4ax$ at P such that it cuts the y-axis at Q. A line perpendicular to this tangent is drawn through Q which cuts the axis of the parabola at R. If the rectangle $PQRS$ is completed, then find the locus of S.
15. Tangents are drawn to the parabola at three distinct points. Prove that these tangent lines always make a triangle and that the locus of the orthocenter of the triangle is the directrix of the parabola.
16. A series of chords are drawn so that their projections on the straight line, which is inclined at an angle α to the axis, are of constant length c. Prove that the locus of their middle point is the curve
$(y^2 - 4ax)(y \cos \alpha + 2a \sin \alpha)^2 + a^2c^2 = 0.$
17. A parabola is drawn touching the axis of x at the origin and having its vertex at a given distance k form this axis. Prove that the axis of the parabola is a tangent to the parabola $x^2 = -8k(y - 2k)$.
18. Prove that for a suitable point P on the axis of the parabola, a chord AB through the point P can be drawn such that $\left[\left(\frac{1}{AP^2}\right) + \left(\frac{1}{BP^2}\right)\right]$ is the same for all positions of the chord.

Objective Type

Solutions on page 3.52

Each question has four choices a, b, c and d, out of which *only one* answer is correct. Find the correct answer.

1. The equation of the parabola whose vertex and focus lie on the axis of x at distances a and a_1 from the origin respectively is
   a. $y^2 = 4(a_1 - a)x$
   b. $y^2 = 4(a_1 - a)(x - a)$
   c. $y^2 = 4(a_1 - a)(x - a_1)$
   d. none of these
2. The vertex of a parabola is the point (a, b) and latus rectum is of length l. If the axis of the parabola is along the positive direction of y-axis, then its equation is
   a. $(x + a)^2 = \frac{l}{2}(2y - 2b)$
   b. $(x - a)^2 = \frac{l}{2}(2y - 2b)$
   c. $(x + a)^2 = \frac{l}{4}(2y - 2b)$
   d. $(x - a)^2 = \frac{l}{8}(2y - 2b)$
3. Which one of the following equations represented parametrically equation to a parabolic curve?

a. $x = 3 \cos t; y = 4 \sin t$

b. $x^2 - 2 = 2 \cos t; y = 4 \cos^2 \frac{t}{2}$

c. $\sqrt{x} = \tan t;\ \sqrt{y} = \sec t$

d. $x = \sqrt{1 - \sin t}; y = \sin \frac{t}{2} + \cos \frac{t}{2}$

4. The equation of the parabola whose focus is the point (0, 0) and the tangent at the vertex is $x - y + 1 = 0$ is

a. $x^2 + y^2 - 2xy - 4x - 4y - 4 = 0$

b. $x^2 + y^2 - 2xy + 4x - 4y - 4 = 0$

c. $x^2 + y^2 + 2xy - 4x + 4y - 4 = 0$

d. $x^2 + y^2 + 2xy - 4x - 4y + 4 = 0$

5. The curve represented by the equation $\sqrt{px} + \sqrt{qy} = 1$, where $p, q \in R, p, q > 0$ is

a. a circle b. a parabola

c. an ellipse d. a hyperbola

6. If parabola $y^2 = \lambda x$ and $25[(x - 3)^2 + (y + 2)^2] = (3x - 4y - 2)^2$ are equal, then value of λ is

a. 9 b. 3 c. 7 d. 6

7. The length of the latus rectum of the parabola whose focus is $\left(\frac{u^2}{2g} \sin 2\alpha, -\frac{u^2}{2g} \cos 2\alpha\right)$ and directrix is $y = \frac{u^2}{2g}$ is

a. $\frac{u^2}{g} \cos^2 \alpha$ b. $\frac{u^2}{g} \cos 2\alpha$

c. $\frac{2u^2}{g} \cos 2\alpha$ d. $\frac{2u^2}{g} \cos^2 \alpha$

8. If the segment intercepted by the parabola $y = 4ax$ with the line $lx + my + n = 0$ subtends a right angle at the vertex, then

a. $4al + n = 0$ b. $4al + 4am + n = 0$

c. $4am + n = 0$ d. $al + n = 0$

9. The graph of the curve $x^2 + y^2 - 2xy - 8x - 8y + 32 = 0$ falls wholly in the

a. first quadrant b. second quadrant

c. third quadrant d. none of these

10. A point $P(x, y)$ moves in xy plane such that $x = a \cos^2 \theta$ and $y = 2a \sin \theta$, where θ is a parameter. The locus of the point P is

a. circle

b. ellipse

c. unbounded parabola

d. part of the parabola

11. Locus of the point $\sqrt{3h}, \sqrt{3k + 2}$ if it lies on the line $x - y - 1 = 0$ is a

a. straight line b. circle

c. parabola d. none of these

12. A water jet from a fountain reaches its maximum height of 4 m at a distance 0.5 m from the vertical passing through the point O of water outlet. The height of the jet above the horizontal OX at a distance of 0.75 m from the point O is

a. 5 m b. 6 m c. 3 m d. 7 m

13. Vertex of the parabola whose parametric equation is $x = t^2 - t + 1, y = t^2 + t + 1; t \in R$, is

a. (1, 1) b. (2, 2)

c. $\left(\frac{1}{2}, \frac{1}{2}\right)$ d. (3, 3)

14. The ratio in which the line segment joining the points (4, –6) and (3, 1) is divided by the parabola $y^2 = 4x$ is

a. $\frac{-20 \pm \sqrt{155}}{11} : 1$

b. $\frac{-2 \pm 2\sqrt{155}}{11} : 2$

c. $-20 \pm 2\sqrt{155} : 11$

d. $-20 \pm \sqrt{155} : 11$

15. If (a, b) is the midpoint of a chord passing through the vertex of the parabola $y^2 = 4x$, then

a. $a = 2b$ b. $2a = b$ c. $a^2 = 2b$ d. $2a = b^2$

16. A set of parallel chords of the parabola $y^2 = 4ax$ have their midpoints on

a. any straight line through the vertex

b. any straight line through the focus

c. a straight line parallel to the axis

d. another parabola

17. A line L passing through the focus of the parabola $y^2 = 4(x - 1)$ intersects the parabola in two distinct points. If 'm' be the slope of the line L then

a. $-1 < m < 1$ b. $m < -1$ or $m > 1$

c. $m \in R$ d. none of these

18. If PSQ is the focal chord of the parabola $y^2 = 8x$ such that $SP = 6$. Then the length of SQ is

a. 6 b. 4

c. 3 d. none of these

19. The circle $x^2 + y^2 + 2\lambda x = 0, \lambda \in R$, touches the parabola $y^2 = 4x$ externally. Then

a. $\lambda > 0$ b. $\lambda < 0$

c. $\lambda > 1$ d. none of these

20. If y_1, y_2 and y_3 are the ordinates of the vertices of a triangle inscribed in the parabola $y^2 = 4ax$, then its area is

a. $\frac{1}{2a}(y_1-y_2)(y_2-y_3)(y_3-y_1)$

b. $\frac{1}{4a}(y_1-y_2)(y_2-y_3)(y_3-y_1)$

c. $\frac{1}{8a}(y_1-y_2)(y_2-y_3)(y_3-y_1)$

d. none of these

21. If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, then

a. $d^2 + (2b + 3c)^2 = 0$

b. $d^2 + (3b + 2c)^2 = 0$

c. $d^2 + (2b - 3c)^2 = 0$

d. none of these

22. Let P be the point (1, 0) and Q a point on the locus $y^2 = 8x$. The locus of the midpoint of PQ is

a. $y^2 + 4x + 2 = 0$ b. $y^2 - 4x + 2 = 0$

c. $x^2 - 4y + 2 = 0$ d. $x^2 + 4y + 2 = 0$

23. The locus of the vertex of the family of parabolas $y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$ is

a. $xy = \frac{105}{64}$ b. $xy = \frac{3}{4}$

c. $xy = \frac{35}{16}$ d. $xy = \frac{64}{105}$

24. A circle touches the x-axis and also touches the circle with centre (0, 3) and radius 2. The locus of the centre of the circle is

a. a circle b. an ellipse

c. a parabola d. a hyperbola

25. Parabolas $y^2 = 4a(x - c_1)$ and $x^2 = 4a(y - c_2)$, where c_1 and c_2 are variable, are such that they touch each other. Locus of their point of contact is

a. $xy = 2a^2$ b. $xy = 4a^2$

c. $xy = a^2$ d. none of these

26. The locus of a point on the variable parabola $y^2 = 4ax$, whose distance from focus is always equal to k, is equal to: (a is parameter)

a. $4x^2 + y^2 - 4kx = 0$

b. $x^2 + y^2 - 4kx = 0$

c. $2x^2 + 4y^2 - 8kx = 0$

d. $4x^2 - y^2 + 4kx = 0$

27. If the line $y - \sqrt{3}x + 3 = 0$ cuts the parabola $y^2 = x + 2$ at P and Q, then $AP \cdot AQ$ is equal to [where $A \equiv (\sqrt{3}, 0)$]

a. $\frac{2(\sqrt{3}+2)}{3}$ b. $\frac{4\sqrt{3}}{2}$

c. $\frac{4(2-\sqrt{2})}{3}$ d. $\frac{4(\sqrt{3}+2)}{3}$

28. A line is drawn form A (– 2, 0) to intersect the curve $y^2 = 4x$ in P and Q in the first quadrant such that $\frac{1}{AP} + \frac{1}{AQ} < \frac{1}{4}$, then slope of the line is always

a. $> \sqrt{3}$ b. $< \frac{1}{\sqrt{3}}$ c. $> \sqrt{2}$ d. $> \frac{1}{\sqrt{3}}$

29. Let $y = f(x)$ be a parabola, having its axis parallel to y-axis, which is touched by the line $y = x$ at $x = 1$, then

a. $2f(0) = 1 - f'(0)$ b. $f(0) + f'(0) + f''(0) = 1$

c. $f'(1) = 1$ d. $f'(0) = f'(1)$

30. An equilateral triangle SAB is inscribed in the parabola $y^2 = 4ax$ having it's focus at 'S'. If chord AB lies towards the left of S, then side length of this triangle is

a. $2a(2 - \sqrt{3})$ b. $4a(2 - \sqrt{3})$

c. $a(2 - \sqrt{3})$ d. $8a(2 - \sqrt{3})$

31. Let S be the focus of $y^2 = 4x$ and a point P is moving on the curve such that its abscissa is increasing at the rate of 4 units/sec, then the rate of increase of projection of SP on $x + y = 1$ when P is at (4, 4) is

a. $\sqrt{2}$ b. -1

c. $-\sqrt{2}$ d. $-\frac{3}{\sqrt{2}}$

32. Two parabolas have the same focus. If their directrices are the x-axis and the y-axis, respectively, then the slope of their common chord is

a. ± 1 b. $\frac{4}{3}$

c. $\frac{3}{4}$ d. none of these

33. C is the centre of the circle with centre (0, 1) and radius unity. P is the parabola $y = ax^2$. The set of values of 'a' for which they meet at a point other than the origin, is

a. $a > 0$ b. $a \in \left(0, \frac{1}{2}\right)$

c. $\left(\frac{1}{4}, \frac{1}{2}\right)$ d. $\left(\frac{1}{2}, \infty\right)$

34. The length of the chord of the parabola $y^2 = x$ which is bisected at the point (2, 1) is

a. $2\sqrt{3}$ b. $4\sqrt{3}$ c. $3\sqrt{2}$ d. $2\sqrt{5}$

35. The circle $x^2 + y^2 = 5$ meets the parabola $y^2 = 4x$ at P and Q. Then the length PQ is equal to

a. 2 b. $2\sqrt{2}$

c. 4 d. none of these

36. The triangle PQR of area 'A' is inscribed in the parabola $y^2 = 4ax$ such that the vertex P lies at the vertex of the parabola and the base QR is a focal chord. The modules of the difference of the ordinates of the points Q and R is

a. $\frac{A}{2a}$ b. $\frac{A}{a}$ c. $\frac{2A}{a}$ d. $\frac{4A}{a}$

37. If A_1B_1 and A_2B_2 are two focal chords of the parabola $y^2 = 4ax$, then the chords A_1A_2 and B_1B_2 intersect on

a. directrix b. axis
c. tangent at vertex d. none of these

38. If a line $y = 3x + 1$ cuts the parabola $x^2 - 4x - 4y + 20 = 0$ at A and B, then the tangent of the angle subtended by line segment AB at origin is

a. $\frac{8\sqrt{3}}{205}$ b. $\frac{8\sqrt{3}}{209}$
c. $\frac{8\sqrt{3}}{215}$ d. none of these

39. $P(x, y)$ is a variable point on the parabola $y^2 = 4ax$ and $Q(x + c, y + c)$ is another variable point, where 'c' is a constant. The locus of the midpoint of PQ is

a. parabola b. ellipse
c. hyperbola d. circle

40. If a and c are the lengths of segments of any focal chord of the parabola $y^2 = 2bx$ $(b > 0)$, then the roots of the equation $ax^2 + bx + c = 0$ are

a. real and distinct b. real and equal
c. imaginary d. none of these

41. AB is a chord of the parabola $y^2 = 4ax$ with vertex A. BC is drawn is drawn perpendicular to AB meeting the axis at C. The projection of BC on the axis of the parabola is

a. a b. $2a$ c. $4a$ d. $8a$

42. Set of values of α for which the point $(\alpha, 1)$ lies inside the curves c_1:$x^2 + y^2 - 4 = 0$ and c_2:$y^2 = 4x$ is

a. $|\alpha| < \sqrt{3}$ b. $|\alpha| < 2$
c. $\frac{1}{4} < \alpha < \sqrt{3}$ d. none of these

43. If P be a point on the parabola $y^2 = 3(2x - 3)$ and M is the foot perpendicular drawn from P on the directrix of the parabola, then length of each side of an equilateral triangle SMP, where S is focus of the parabola is

a. 2 b. 4 c. 6 d. 8

44. If $y = mx + c$ touches the parabola $y^2 = 4a(x + a)$, then

a. $c = \frac{a}{m}$ b. $c = am + \frac{a}{m}$
c. $c = a + \frac{a}{m}$ d. none of these

45. The angle between the tangents to the parabola $y^2 = 4ax$ at the points where it intersects with the line $x - y - a = 0$ is

a. $\frac{\pi}{3}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{6}$ d. $\frac{\pi}{2}$

46. The area of the triangle formed by the tangent and the normal to the parabola $y^2 = 4ax$, both drawn at the same end of the latus rectum, and the axis of the parabola is

a. $2\sqrt{2}a^2$ b. $2a^2$
c. $4a^2$ d. none of these

47. Double ordinate AB of the parabola $y^2 = 4ax$ subtends an angle $\pi/2$ at the focus of the parabola, then tangents drawn to parabola at A and B will intersect at

a. $(-4a, 0)$ b. $(-2a, 0)$
c. $(-3a, 0)$ d. none of these

48. $y = x + 2$ is any tangent to the parabola $y^2 = 8x$. The point P on this tangent is such that the other tangent from it which is perpendicular to it is

a. $(2, 4)$ b. $(-2, 0)$
c. $(-1, 1)$ d. $(2, 0)$

49. The tangent and normal at the point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ meet the x-axis in T and G, respectively, then the angle at which the tangent at P to the parabola is inclined to the tangent at P to the circle through P, T, G is

a. $\tan^{-1}(t^2)$ b. $\cot^{-1}(t^2)$ c. $\tan^{-1}(t)$ d. $\cot^{-1}(t)$

50. Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) is

a. $\frac{\pi}{2}$ b. $\frac{\pi}{3}$ c. $\frac{\pi}{6}$ d. $\frac{\pi}{4}$

51. Radius of the circle that passes through origin and touches the parabola $y^2 = 4ax$ at the point $(a, 2a)$ is

a. $\frac{5}{\sqrt{2}}a$ b. $2\sqrt{2}\,a$ c. $\sqrt{\frac{5}{2}}\,a$ d. $\frac{3}{\sqrt{2}}a$

52. If the line $x + y = 1$ touches the parabola $y^2 - y + x = 0$, then the coordinates of the point of contact are

a. $(1, 1)$ b. $\left(\frac{1}{2}, \frac{1}{2}\right)$ c. $(0, 1)$ d. $(1, 0)$

53. Two straight lines are perpendicular to each other. One of them touches the parabola $y^2 = 4a(x + a)$ and the other touches $y^2 = 4b(x + b)$. Their point of intersection lies on the line

a. $x - a + b = 0$ b. $x + a - b = 0$
c. $x + a + b = 0$ d. $x - a - b = 0$

54. The mirror image of the parabola $y^2 = 4x$ in the tangent to the parabola at the point $(1, 2)$ is

a. $(x-1)^2 = 4(y+1)$
b. $(x+1)^2 = 4(y+1)$
c. $(x+1)^2 = 4(y-1)$
d. $(x-1)^2 = 4(y-1)$

55. Consider the parabola $y^2 = 4x$. $A \equiv (4, -4)$ and $B \equiv (9, 6)$ be two fixed points on the parabola. Let 'C' be a moving point on the parabola between A and B such that the area of the triangle ABC is maximum, then coordinate of 'C' is

a. $\left(\frac{1}{4}, 1\right)$ b. $(4, 4)$
c. $(3, 2\sqrt{3})$ d. $(3, -2\sqrt{3})$

56. $\min\left[(x_1 - x_2)^2 + \left(5 + \sqrt{1 - x_1^2} - \sqrt{4x_2}\right)^2\right] \forall x_1, x_2 \in R$ is

a. $4\sqrt{5} + 1$ b. $4\sqrt{5} - 1$
c. $\sqrt{5} + 1$ d. $\sqrt{5} - 1$

57. Two mutually perpendicular tangent of the parabola $y^2 = 4ax$ meet the axis in P_1 and P_2. If S is the focus of the parabola, then $\frac{1}{(SP_1)} + \frac{1}{(SP_2)}$ is equal to

a. $\frac{4}{a}$ b. $\frac{2}{a}$ c. $\frac{1}{a}$ d. $\frac{1}{4a}$

58. A tangent is drawn to the parabola $y^2 = 4ax$ at the point 'P' whose abscissa lies in the interval (1, 4). The maximum possible area of the triangle formed by the tangent at 'P' ordinates of the point 'P' and the x-axis is equal to

a. 8 b. 16 c. 24 d. 32

59. A parabola $y = ax^2 + bx + c$ crosses the x-axis at $(\alpha, 0)$ $(\beta, 0)$ both to the right of the origin. A circle also passes through these two points. The length of a tangent from the origin to the circle is

a. $\sqrt{\frac{bc}{a}}$ b. ac^2 c. $\frac{b}{a}$ d. $\sqrt{\frac{c}{a}}$

60. The straight line joining any point P on the parabola $y^2 = 4ax$ to the vertex and perpendicular from the focus to the tangent at P, intersect at R, then the equation of the locus of R is

a. $x^2 + 2y^2 - ax = 0$
b. $2x^2 + y^2 - 2ax = 0$
c. $2x^2 + 2y^2 - ay = 0$
d. $2x^2 + y^2 - 2ay = 0$

61. Through the vertex O of the parabola $y^2 = 4ax$, two chords OP and OQ are drawn and the circles on OP and OQ as diameters intersect in R. If θ_1, θ_2 and ϕ are the angles made with axis by the tangents at P and Q on the parabola and by OR, then the value of, $\cot\theta_1 + \cot\theta_2$

a. $-2\tan\phi$ b. $-2\tan(\pi - \phi)$
c. 0 d. $2\cot\phi$

62. A line of slope $\lambda (0 < \lambda < 1)$ touches parabola $y + 3x^2 = 0$ at P. If S is the focus and M is the foot of the perpendicular of directrix from P, then $\tan\angle MPS$ equals

a. 2λ b. $\frac{2\lambda}{-1 + \lambda^2}$
c. $\frac{1 - \lambda^2}{1 + \lambda^2}$ d. none of these

63. If $y = 2x - 3$ is a tangent to the parabola $y^2 = 4a\left(x - \frac{1}{3}\right)$, then '$a$' is equal to

a. $\frac{22}{3}$ b. -1 c. $\frac{14}{3}$ d. $\frac{-14}{3}$

64. AB is a double ordinate of the parabola $y^2 = 4ax$. Tangents drawn to parabola at A and B meet y-axis at A_1 and B_1, respectively. If the area of trapezium AA_1B_1B is equal to $12a^2$, then angle subtended by A_1B_1 at the focus of the parabola is equal to

a. $2\tan^{-1}(3)$ b. $\tan^{-1}(3)$
c. $2\tan^{-1}(2)$ d. $\tan^{-1}(2)$

65. If $y + 3 = m_1(x + 2)$ and $y + 3 = m_2(x + 2)$ are two tangents to the parabola $y^2 = 8x$, then

a. $m_1 + m_2 = 0$ b. $m_1m_2 = -1$
c. $m_1m_2 = 1$ d. none

66. The tangent at any point P on the parabola $y^2 = 4ax$ intersects the y-axis at Q. The tangent to the circum circle of triangle PQS (S is the focus) at Q is

a. a line parallel to x-axis
b. y-axis
c. a line parallel to y-axis
d. none of these

67. If $y = m_1x + c$ and $y = m_2x + c$ are two tangents to the parabola $y^2 + 4a(x + a) = 0$, then

a. $m_1 + m_2 = 0$ b. $1 + m_1 + m_2 = 0$
c. $m_1m_2 - 1 = 0$ d. $1 + m_1m_2 = 0$

68. If the parabola $y = ax^2 - 6x + b$ passes through (0, 2) and has its tangent at $x = \frac{3}{2}$ parallel to the x-axis then

a. $a = 2, b = -2$ b. $a = 2, b = 2$
c. $a = -2, b = 2$ d. $a = -2, b = -2$

69. If the angle between the tangents from the point $(\lambda, 1)$ to the parabola $y^2 = 16x$ be $\frac{\pi}{2}$ then λ is

a. 4 b. -4 c. -1 d. 2

70. Minimum area of circle which touches the parabolas $y = x^2 + 1$ and $y^2 = x - 1$ is

a. $\frac{9\pi}{16}$ sq. unit b. $\frac{9\pi}{32}$ sq. unit

c. $\frac{9\pi}{8}$ sq. unit d. $\frac{9\pi}{4}$ sq. unit

71. If the locus of middle of point of contact of tangent drawn to the parabola $y^2 = 8x$ and foot of perpendicular drawn from its focus to the tangents is a conic then length of latus rectum of this conic is

a. $\frac{9}{4}$ b. 9 c. 18 d. $\frac{9}{2}$

72. If d is the distance between parallel tangents with positive slope to $y^2 = 4x$ and $x^2 + y^2 - 2x + 4y - 11 = 0$, then

a. $10 < d < 2$ b. $4 < d < 6$

c. $d < 4$ d. none of these

73. If bisector of the angle APB, where PA and PB are the tangents to the parabola $y^2 = 4ax$, is equally inclined to the coordinate axes, then the point P lies on

a. tangent at vertex of the parabola

b. directrix of the parabola

c. circle with centre at the origin and radius a

d. the line of latus rectum

74. The locus of the centre of a circle which cuts orthogonally the parabola $y^2 = 4x$ at (1, 2) will pass though points

a. (3, 4) b. (4, 3) c. (5, 3) d. (2, 4)

75. From a point $A(t)$ on the parabola $y^2 = 4ax$, a focal chord and a tangent is drawn. Two circles are drawn in which one circle is drawn taking focal chord AB as diameter and other is drawn by taking intercept of tangent between point A and point P on the directrix, as diameter. Then the common chord of the circles is

a. line joining focus and P

b. line joining focus and A

c. tangent to the parabola at point A

d. none of these

76. The normal at the point $P(ap^2, 2ap)$ meets the parabola $y^2 = 4ax$ again at $Q(aq^2, 2aq)$ such that the lines joining the origin to P and Q are at right angle. Then

a. $p^2 = 2$ b. $q^2 = 2$ c. $p = 2q$ d. $q = 2p$

77. The set of points on the axis of the parabola $y^2 = 4x + 8$ from which the three normals to the parabola are all real and different is

a. $\{(k, 0) \mid k \le -2\}$ b. $\{(k, 0) \mid k > -2\}$

c. $\{(0, k) \mid k > -2\}$ d. none of these

78. Tangents and normal drawn to parabola $y^2 = 4ax$ at point $P(at^2, 2at)$, $t \neq 0$, meet the x-axis at points T and N, respectively. If 'S' is the focus of the parabola, then

a. $SP = ST \neq SN$ b. $SP \neq ST = SN$

c. $SP = ST = SN$ d. $SP \neq ST \neq SN$

79. Locus of the midpoint of any normal chords of $y^2 = 4ax$ is

a. $x = a\left(\frac{4a^2}{y^2} - 2 + \frac{y^2}{2a^2}\right)$

b. $x = a\left(\frac{4a^2}{y^2} + 2 + \frac{y^2}{2a^2}\right)$

c. $x = a\left(\frac{4a^2}{y^2} - 2 - \frac{y^2}{2a^2}\right)$

d. $x = a\left(\frac{4a^2}{y^2} + 2 - \frac{y^2}{2a^2}\right)$

80. Normals AO, AA_1, AA_2 are drawn to parabola $y^2 = 8x$ from the point $A(h, 0)$. If triangle OA_1A_2 is equilateral, then possible values of 'h' is

a. 26 b. 24

c. 28 d. none of these

81. If $2x + y + \lambda = 0$ is a normal to the parabola $y^2 = -8x$, then λ is

a. 12 b. -12 c. 24 d. -24

82. At what point on the parabola $y^2 = 4x$ the normal makes equal angle with axes?

a. (4, 4) b. (9, 6) c. (4, -4) d. (1, ± 2)

83. If the normals to the parabola $y^2 = 4ax$ at three points P, Q and R meet at A and S be the focus, then $SP.SQ.SR$ is equal to

a. a^2SA b. SA^3

c. aSA^2 d. none of these

84. If the normals to the parabola $y^2 = 4ax$ at the ends of the latus rectum meet the parabola at Q, Q', then QQ' is

a. $10a$ b. $4a$ c. $20a$ d. $12a$

85. From a point ($\sin\theta$, $\cos\theta$) if three normals can be drawn to the parabola $y^2 = 4ax$, then the value of 'a' is

a. $\left(\frac{1}{2}, 1\right)$ b. $\left[-\frac{1}{2}, 0\right)$

c. $\left[\frac{1}{2}, 1\right]$ d. $\left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$

86. If the normals at points 't_1' and 't_2' meet on the parabola, then

a. $t_1t_2 = -1$ b. $t_2 = -t_1 - \frac{2}{t_1}$

c. $t_1t_2 = 2$ d. none of these

87. Length of the normal chord of the parabola $y^2 = 4x$ which makes an angle of $\frac{\pi}{4}$ with the axis of x is

a. 8 b. $8\sqrt{2}$ c. 4 d. $4\sqrt{2}$

88. If the normal to a parabola $y^2 = 4ax$ at P meets the curve again in Q and if PQ and the normal at Q makes angle α and β, respectively, with the x-axis then $\tan\alpha(\tan\alpha + \tan\beta)$ has the value equal to

a. 0 **b.** -2 **c.** $-\frac{1}{2}$ **d.** -1

89. If two normals to a parabola $y^2 = 4ax$ intersect at right angles, then the chord joining their feet passes through a fixed point whose co-ordinates are

a. $(-2a, 0)$ **b.** $(a, 0)$ **c.** $(2a, 0)$ **d.** none

90. PQ is a normal chord of the parabola $y^2 = 4ax$ at P, A being the vertex of the parabola. Through P a line is drawn parallel to AQ meeting the x-axis in R. Then line length of AR is

a. equal to the length of the latus rectum

b. equal to the focal distance of the point P

c. equal to twice the focal distance of the point P

d. equal to the distance of the point P from the directrix

91. Tangent and normal drawn to parabola at $A(at^2, 2at)$, $t \neq 0$ meet the x-axis at point B and D respectively. If the rectangle $ABCD$ is completed, then locus of 'C' is

a. $y = 2a$ **b.** $y + 2a = c$

b. $x = 2a$ **d.** $x + 2a = 0$

92. An equation for the line that passes through $(10, -1)$ and is perpendicular to $y = \frac{x^2}{4} - 2$ is

a. $4x + y = 39$ **b.** $2x + y = 19$

c. $x + y = 9$ **d.** $x + 2y = 8$

93. P, Q, R are the feet of the normals drawn to a parabola $(y-3)^2 = 8(x-2)$. A circle cuts the above parabola in points P, Q, R and S. Then this circle always passes through the point

a. $(2, 3)$ **b.** $(3, 2)$ **c.** $(0, 3)$ **d.** $(2, 0)$

94. Normals at two point (x_1, y_1) and (x_2, y_2) of parabola $y^2 = 4x$ meet again on the parabola where $x_1 + x_2 = 4$, then $|y_1 + y_2|$ equals to

a. $\sqrt{2}$ **b.** $2\sqrt{2}$

c. $4\sqrt{2}$ **d.** none of these

95. The end points of two normal chords of a parabola are concyclic, then the tangents at the feet of the normals will intersect at

a. tangent at vertex of the parabola

b. axis of the parabola

c. directrix of the parabola

d. none of these

96. The set of points on the axis of the parabola $(x-1)^2 = 8(y+2)$, from where three distinct normals can be drawn to the parabola is the set (h, k) of points satisfying

a. $h > 2$ **b.** $h > 1$

c. $k > 2$ **d.** none of these

97. The shortest distance between the parabola $2y^2 = 2x - 1$, $2x^2 = 2y - 1$ is

a. $2\sqrt{2}$ **b.** $\frac{1}{2\sqrt{2}}$ **c.** 4 **d.** $\sqrt{\frac{36}{5}}$

98. A tangent and normal is drawn at the point $P \equiv (16, 16)$ of the parabola $y^2 = 16x$ which cut the axis of the parabola at the points A and B, respectively. If the centre of the circle through P, A and B is C, then the angle between PC and the axis of x is

a. $\tan^{-1}\frac{1}{2}$ **b.** $\tan^{-1} 2$ **c.** $\tan^{-1}\frac{3}{4}$ **d.** $\tan^{-1}\frac{4}{3}$

99. Length of the shortest normal chord of the parabola $y^2 = 4ax$ is

a. $2a\sqrt{27}$ **b.** $9a$

c. $a\sqrt{54}$ **d.** None of these

100. From the point $(15, 12)$, three normals are drawn to the parabola $y^2 = 4x$, then centroid of triangle formed by three co-normals points is

a. $\left(\frac{16}{3}, 0\right)$ **b.** $(4, 0)$ **c.** $\left(\frac{26}{3}, 0\right)$ **d.** $(6, 0)$

101. 't_1' and 't_2' are two points on the parabola $y^2 = 4ax$. If the focal chord joining them coincides with the normal chord, then

a. $t_1(t_1 + t_2) + 2 = 0$ **b.** $t_1 + t_2 = 0$

c. $t_1 t_2 = -1$ **d.** none of these

102. The line $x - y = 1$ intersects the parabola $y^2 = 4x$ at A and B. Normals at A and B intersect at C. If D is the point at which line CD is normal to the parabola, then coordinates of D are

a. $(4, -4)$ **b.** $(4, 4)$

c. $(-4, -4)$ **d.** none of these

103. If normals are drawn from a point $P(h, k)$ to the parabola $y^2 = 4ax$, then the sum of the intercepts which the normals cutoff from the axis of the parabola is

a. $(h + a)$ **b.** $3(h + a)$

c. $2(h + a)$ **d.** none of these

104. The radius of circle touching parabola $y^2 = x$ at $(1, 1)$ and having directrix of $y^2 = x$ as its normal is

a. $\frac{5\sqrt{5}}{8}$ b. $\frac{10\sqrt{5}}{3}$

c. $\frac{5\sqrt{5}}{4}$ d. none of these

105. Normals drawn to $y^2 = 4ax$ at the points where it is intersected by the line $y = mx + c$ intersect at P. Foot of the another normal drawn to the parabola from the point 'P' is

a. $\left(\frac{a}{m^2}, -\frac{2a}{m}\right)$ b. $\left(\frac{9a}{m}, \frac{-6a}{m}\right)$

c. $(am^2, -2am)$ d. $\left(\frac{4a}{m^2}, -\frac{4a}{m}\right)$

106. If two different tangents of $y^2 = 4x$ are the normals to $x^2 = 4by$ then

a. $|b| > \frac{1}{2\sqrt{2}}$ b. $|b| < \frac{1}{2\sqrt{2}}$

c. $|b| > \frac{1}{\sqrt{2}}$ d. $|b| < \frac{1}{\sqrt{2}}$

107. Maximum number of common normals of $y^2 = 4ax$ and $x^2 = 4by$ is equal to

a. 3 b. 4 c. 6 d. 5

108. A ray of light travels along a line $y = 4$ and strikes the surface of a curves $y^2 = 4(x + y)$, then equations of the line along which reflected ray travel is

a. $x = 0$ b. $x = 2$

c. $x + y = 4$ d. $2x + y = 4$

109. The largest value of a for which the circle $x^2 + y^2 = a^2$ falls totally in the interior of the parabola $y^2 = 4(x + 4)$ is

a. $4\sqrt{3}$ b. 4 c. $4\frac{\sqrt{6}}{7}$ d. $2\sqrt{3}$

110. If two chords drawn from the point $A(4, 4)$ to the parabola $x^2 = 4y$ are bisected by line $y = mx$, the interval in which m lies is

a. $(-2\sqrt{2}, 2\sqrt{2})$

b. $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

c. $(-\infty, -2\sqrt{2}-2) \cup (2\sqrt{2}-2, \infty)$

d. none of these

111. The point of intersection of the tangents of the parabola $y^2 = 4x$, drawn at end points of the chord $x + y = 2$ lies on

a. $x - 2y = 0$ b. $x + 2y = 0$

c. $y - x = 0$ d. $x + y = 0$

112. The number of common chords of the parabolas $x = y^2 - 6y + 11$ and $y = x^2 - 6x + 1$ are

a. 1 b. 2 c. 4 d. 6

113. Two parabola have the focus $(3, -2)$. Their directrices are the x-axis, and the y-axis, respectively. Then the slope of their common chord is

a. -1 b. $-\frac{1}{2}$

c. $-\frac{\sqrt{3}}{2}$ d. none of these

114. If the tangent at the point $P(2, 4)$ to the parabola $y^2 = 8x$ meets the parabola $y^2 = 8x + 5$ at Q and R, then the mid-point of QR is

a. (4, 2) b. (2, 4) c. (7, 9) d. None of these

Multiple Correct Answers Type

Solutions on page 3.71

Each question has four choices a, b, c and d, out of which *one or more* answers are correct.

1. The equation of the directrix of the parabola with vertex at the origin and having the axis along the x-axis and a common tangent of slope 2 with the circle $x^2 + y^2 = 5$ is/are

a. $x = 10$ b. $x = 20$ c. $x = -10$ d. $x = -20$

2. Tangent is drawn at any point (x_1, y_1) other than vertex on the parabola $y^2 = 4ax$. If tangents are drawn from any point on this tangent to the circle $x^2 + y^2 = a^2$ such that all the chords of contact pass through a fixed point (x_2, y_2), then

a. x_1, a, x_2 are in G.P.

b. $\frac{y_1}{2}, a, y_2$ are in G.P.

c. $-4, \frac{y_1}{y_2}, \frac{x_1}{x_2}$ are in G.P.

d. $x_1x_2 + y_1y_2 = a^2$

3. If the focus of the parabola $x^2 - ky + 3 = 0$ is $(0, 2)$, then a value of k is/are

a. 4 b. 6 c. 3 d. 2

4. Let P be a point whose coordinates differ by unity and the point does not lie on any of the axes of reference. If the parabola $y^2 = 4x + 1$ passes through P, then the ordinate of P may be

a. 3 b. -1 c. 5 d. 1

5. If $y = 2$ be the directrix and $(0, 1)$ be the vertex of the parabola $x^2 + \lambda y + \mu = 0$ then

a. $\lambda = 4$ b. $\mu = 8$ c. $\lambda = -8$ d. $\mu = 4$

6. The extremities of latus rectum of a parabola are $(1, 1)$ and $(1, -1)$, then the equation of the parabola can be

a. $y^2 = 2x - 1$ b. $y^2 = 1 - 2x$

c. $y^2 = 2x - 3$ d. $y^2 = 2x - 4$

7. Parabola $y^2 = 4x$ and the circle having it's centre at (6, 5) intersect at right angle. Possible point of intersection of these curves can be

   a. (9, 6) b. $(2, \sqrt{8})$ c. (4, 4) d. $(3, 2\sqrt{3})$

8. A normal drawn to parabola $y^2 = 4ax$ meet the curve again at Q such that angle subtended by PQ at vertex is 90°, then coordinates of P can be

   a. $(8a, 4\sqrt{2}a)$ b. $(8a, 4a)$

   c. $(2a, -2\sqrt{2}a)$ d. $(2a, 2\sqrt{2}a)$

9. A quadrilateral is inscribed in parabola, then

   a. quadrilateral may be cyclic.

   b. diagonal of the quadrilateral may be equal.

   c. all possible pairs of adjacent sides may be perpendicular.

   d. none of these

10. The locus of the midpoint of the focal distance of a variable point moving on the parabola, $y^2 = 4ax$ is a parabola whose

    a. latus rectum is half the latus rectum of the original parabola.

    b. vertex is $(\frac{a}{2}, 0)$.

    c. directrix is y-axis.

    d. focus has the co-ordinates $(a, 0)$.

11. Which of the following line can be tangent to parabola $y^2 = 8x$?

    a. $x - y + 2 = 0$ b. $9x - 3y + 2 = 0$

    c. $x + 2y + 8 = 0$ d. $x + 3y + 12 = 0$

12. Which of the following line can be normal to parabola $y^2 = 12x$?

    a. $x + y - 9 = 0$ b. $2x - y - 32 = 0$

    c. $2x + y - 36 = 0$ d. $3x - y - 72 = 0$

13. A square has one vertex at the vertex of the parabola $y^2 = 4ax$ and the diagonal through the vertex lies along the axis of the parabola. If the ends of the other diagonal lie on the parabola, the coordinates of the vertices of the square are

    a. $(4a, 4a)$ b. $(4a, -4a)$

    c. $(0, 0)$ d. $(8a, 0)$

Reasoning Type

Solutions on page 3.72

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains Statement 1 and Statement 2.

a. Both the statements are true and Statement 1 is the correct explanation of Statement 2

b. Both the statements are true but Statement 1 is not the correct explanation of Statement 2

c. Statement 1 is true and Statement 2 is False

d. Statement 1 is false and Statement 2 is True

1. **Statement 1:** Slope of tangents drawn from (4, 10) to parabola $y^2 = 9x$ are $\frac{1}{4}, \frac{9}{4}$.

   Statement 2: Two tangents can be drawn to parabola from any point lying outside parabola.

2. **Statement 1:** Through $(\lambda, \lambda + 1)$ there can't be more than one normal to the parabola $y^2 = 4x$, if $\lambda < 2$.

   Statement 2: The point $(\lambda, \lambda + 1)$ lies outside the parabola for all $\lambda \neq 1$.

3. **Statement 1:** In parabola $y^2 = 4ax$, the circle drawn taking focal radii as diameter touches y-axis.

   Statement 2: The portion of the tangent intercepted between point of contact and directix subtends 90° angle at focus.

4. **Statement 1:** The line joining points (8, – 8) and $(\frac{1}{2}, 2)$, which are on parabola $y^2 = 8x$, passes through focus of parabola.

   Statement 2: Tangents drawn at (8, – 8) and $(\frac{1}{2}, 2)$ on the parabola $y^2 = 4ax$ are perpendicular.

5. **Statement 1:** The normals at the point (4, 4) and $\left(\frac{1}{4}, -1\right)$ of the parabola $y^2 = 4x$ are perpendicular.

   Statement 2: The tangents to the parabola at the end of a focal chord are perpendicular.

6. **Statement 1:** The line $y = x + 2a$ touches the parabola $y^2 = 4a(x + a)$.

   Statement 2: The line $y = mx + am + a/m$ touches $y^2 = 4a(x + a)$ for all real values of m.

7. **Statement 1:** Equation $(5x - 5)^2 + (5y + 10)^2 = (3x + 4y + 5)^2$ is parabola.

   Statement 2: If distance of the point from the given line and from the given point (not lying on the given line) is equal, then locus of variable point is parabola.

8. **Statement 1:** Length of focal chord of a parabola $y^2 = 8x$ making on angle of 60° with x-axis is 32.

   Statement 2: Length of focal chord of a parabola $y^2 = 4ax$ making an angle a with x-axis is $4a \operatorname{cosec}^2 \alpha$.

9. **Statement 1:** Circumcircle of a triangle formed by the lines $x = 0$, $x + y + 1 = 0$ and $x - y + 1 = 0$ also passes through the point (1, 0).

   Statement 2: Circumcircle of a triangle formed by three tangents of a parabola passes through its focus.

10. **Statement 1:** The point of intersection of the tangents at three distinct points A, B, C on the parabola $y^2 = 4x$ can be collinear.

Statement 2: If a line L does not intersect the parabola $y^2 = 4x$, then from every point of the line two tangents can be drawn to the parabola.

11. **Statement 1:** There are no common tangents between circle $x^2 + y^2 - 4x + 3 = 0$ and parabola $y^2 = 2x$.

    Statement 2: Given circle and parabola do not intersect.

12. **Statement 1:** The line $ax + by + c = 0$ is a normal to the parabola $y^2 = 4ax$, then the equation of tangent at the foot of this normal is $y = (b/a)x + (a^2/b)$.

    Statement 2: Equation of normal at any point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ is $y = -tx + 2at + at^3$.

13. **Statement 1:** The values of α for which the point (α, α^2) lies inside the triangle formed by the lines $x = 0, x + y = 2$ and $3y = x$ is $(0, 1)$.

    Statement 2: Parabola $y = x^2$ meets the line $x + y = 2$ at $(1, 1)$.

14. **Statement 1:** If there exists points on the circle $x^2 + y^2 = a^2$ from which two perpendicular tangents can be drawn to parabola $y^2 = 2x$, then $a \geq 1/2$.

    Statement 2: Perpendicular tangents to parabola meet on the directrix.

15. **Statement 1:** If straight line $x = 8$ meets the parabola $y^2 = 8x$ at P and Q, then PQ substends a right angle at the origin.

    Statement 2: Double ordinate equal to twice of latus rectum of a parabola subtends a right angle at the vertex.

16. **Statement 1:** Normal chord drawn at the point $(8, 8)$ of the parabola $y^2 = 8x$ subtends a right angle at the vertex of the parabola.

    Statement 2: Every chord of the parabola $y^2 = 4ax$ passing through the point $(4a, 0)$ subtends a right angle at the vertex of the parabola.

17. **Statement 1:** If end points of two normal chords AB and CD (normal at A and C) of a parabola $y^2 = 4ax$ are concyclic, then the tangents at A and C will intersect on the axis of the parabola.

    Statement 2: If four point on the parabola $y^2 = 4ax$ are concyclic, then sum of their ordinates is zero.

18. **Statement 1:** If parabola $y^2 = 4ax$ and circle $x^2 + y^2 + 2bx = 0$ touch each other externally, then roots of the equation, $f(x) = x^2 - (b + a + 1)x + a = 0$ has real roots.

    Statement 2: For parabola and circle externally touching a and b must have the same sign.

19. **Statement 1:** Line $x - y - 5 = 0$ cannot be normal to parabola $(5x - 15)^2 + (5y + 10)^2 = (3x - 4y + 2)^2$.

    Statement 2: Normal to parabola never passes through its focus.

20. **Statement 1:** AA' and BB' are double ordinates of the parabola. Then points A, A', B, B' are concyclic.

    Statement 2: Circle can cut parabola in maximum four points.

Linked Comprehension Type

Solutions on page 3.75

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

A tangent is drawn at any point $P(t)$ on the parabola $y^2 = 8x$ and on it is taken a point $Q(\alpha, \beta)$ from which pair of tangents QA and QB are drawn to the circle $x^2 + y^2 = 4$. Using this information answer the following questions

1. The locus of the point of concurrency of the chord of contact AB of the circle $x^2 + y^2 = 4$ is

   a. $y^2 - 2x = 0$ **b.** $y^2 - x^2 = 4$
   c. $y^2 + 4x = 0$ **d.** $y^2 - 2x^2 = 4$

2. The point from which perpendicular tangents can be drawn both to the given circle and the parabola is

   a. $(4, \pm\sqrt{3})$ **b.** $(-1, \sqrt{2})$
   c. $(-\sqrt{2}, -\sqrt{2})$ **d.** $(-2, \pm 2\sqrt{3})$

3. The locus of circumcentre of ΔAQB if $t = 2$ is

   a. $x - 2y + 2 = 0$ **b.** $x + 2y - 4 = 0$
   c. $x - 2y - 4 = 0$ **d.** $x + 2y + 4 = 0$

For Problems 4–6

Tangent to the parabola $y = x^2 + ax + 1$, at the point of intersection of y-axis also touches the circle $x^2 + y^2 = r^2$. Also no point of the parabola is below x-axis.

4. The radius of circle when a attains its maximum value

   a. $\frac{1}{\sqrt{10}}$ **b.** $\frac{1}{\sqrt{5}}$ **c.** 1 **d.** $\sqrt{5}$

5. The slope of the tangent when radius of the circle is maximum is

   a. -1 **b.** 1 **c.** 0 **d.** 2

6. The minimum area bounded by the tangent and the coordinate axes is

   a. 1 **b.** $\frac{1}{3}$ **c.** $\frac{1}{2}$ **d.** $\frac{1}{4}$

For Problems 7–9

If the locus of the circumcentre of a variable triangle having sides y-axis, $y = 2$ and $lx + my = 1$, where (l, m) lies on the parabola $y^2 = 4x$ is a curve C, then

7. Coordinates of the vertex of this curve C is

   a. $\left(-2, \frac{3}{2}\right)$ **b.** $\left(-2, -\frac{3}{2}\right)$
   c. $\left(2, \frac{3}{2}\right)$ **d.** $\left(-2, -\frac{3}{2}\right)$

8. The length of smallest focal chord of this curve C is

   a. $\frac{1}{4}$ **b.** $\frac{1}{12}$ **c.** $\frac{1}{8}$ **d.** $\frac{1}{16}$

9. The curve C is symmetric about the line

a. $x=\frac{3}{2}$ **b.** $y=-\frac{3}{2}$ **c.** $x=-\frac{3}{2}$ **d.** $y=\frac{3}{2}$

For Problems 10–12

$y=x$ is tangent to the parabola $y=ax^2+c$

10. If $a=2$, then the value of c is

a. 1 **b.** $-\frac{1}{2}$ **c.** $\frac{1}{2}$ **d.** $\frac{1}{8}$

11. If (1, 1) is point of contact, then a is

a. $\frac{1}{4}$ **b.** $\frac{1}{3}$ **c.** $\frac{1}{2}$ **d.** $\frac{1}{6}$

12. If $c=2$, then point of contact is

a. (3, 3) **b.** (2, 2) **c.** (6, 6) **d.** (4, 4)

For Problems 13–15

If l, m are variable real numbers such that $5l^2+6m^2-4lm+3l=0$, then variable line $lx+my=1$ always touches a fixed parabola, whose axes is parallel to x-axis.

13. Vertex of the parabola is

a. $\left(-\frac{5}{3},\frac{4}{3}\right)$ **b.** $\left(-\frac{7}{4},\frac{3}{4}\right)$

c. $\left(\frac{5}{6},-\frac{7}{6}\right)$ **d.** $\left(\frac{1}{2},-\frac{3}{4}\right)$

14. Focus of the parabola is

a. $\left(\frac{1}{6},-\frac{7}{6}\right)$ **b.** $\left(\frac{1}{3},\frac{4}{3}\right)$

c. $\left(\frac{3}{2},-\frac{3}{2}\right)$ **d.** $\left(-\frac{3}{4},\frac{3}{4}\right)$

15. Directrix of the parabola is

a. $6x+7=0$ **b.** $4x+11=0$

c. $3x+11=0$ **d.** none of these

For Problems 16–18

Consider the parabola whose focus is at (0, 0) and tangent at vertex is $x-y+1=0$.

16. The length of latus rectum is

a. $4\sqrt{2}$ **b.** $2\sqrt{2}$ **c.** $8\sqrt{2}$ **d.** $3\sqrt{2}$

17. The length of the chord of parabola on the x-axis is

a. $4\sqrt{2}$ **b.** $2\sqrt{2}$ **c.** $8\sqrt{2}$ **d.** $3\sqrt{2}$

18. Tangents drawn to the parabola at the extremities of the chord $3x+2y=0$ intersect at an angle

a. $\frac{\pi}{6}$ **b.** $\frac{\pi}{3}$

c. $\frac{\pi}{2}$ **d.** none of these

For Problems 19–21

Two tangents on a parabola are $x-y=0$ and $x+y=0$. If (2, 3) is focus of the parabola, then

19. The equation of tangent at vertex is

a. $4x-6y+5=0$ **b.** $4x-6y+3=0$

c. $4x-6y+1=0$ **d.** $4x-6y+3/2=0$

20. Length of latus rectum of the parabola is

a. $\frac{6}{\sqrt{3}}$ **b.** $\frac{10}{\sqrt{13}}$

c. $\frac{2}{\sqrt{13}}$ **d.** none of these

21. If P, Q are ends of focal chord of the parabola, then $\frac{1}{SP}+\frac{1}{SQ}=$

a. $\frac{2\sqrt{13}}{3}$ **b.** $2\sqrt{13}$

c. $\frac{2\sqrt{13}}{5}$ **d.** none of these

For Problems 22–24

$y^2=4x$ and $y^2=-8(x-a)$ intersect at points A and C. Points $O(0, 0)$, A, $B(a, 0)$, C are concyclic.

22. The length of common chord of parabolas is

a. $2\sqrt{6}$ **b.** $4\sqrt{3}$ **c.** $6\sqrt{5}$ **d.** $8\sqrt{2}$

23. The area of cyclic quadrilateral $OABC$ is

a. $24\sqrt{3}$ **b.** $48\sqrt{2}$ **c.** $12\sqrt{6}$ **d.** $18\sqrt{5}$

24. Tangents to parabola $y^2=4x$ at A and C intersect at point D and tangents to parabola $y^2=-8(x-a)$ intersect at point E, then the area of quadrilateral $DAEC$ is

a. $96\sqrt{2}$ **b.** $48\sqrt{3}$ **c.** $54\sqrt{5}$ **d.** $36\sqrt{6}$

For Problems 25–27

PQ is double ordinate of the parabola $y^2=4x$ which passes through the focus S. ΔPQA is isosceles right angle triangle, where A is on the axis of the parabola. Line PA meets the parabola at C and QA meets the parabola at B.

25. Area of the trapezium $PBCQ$ is

a. 96 sq. units **b.** 64 sq. units

c. 72 sq. units **d.** 48 sq. units

26. Circumradius of trapezium $PBCQ$ is

a. $6\sqrt{5}$ **b.** $3\sqrt{6}$ **c.** $2\sqrt{10}$ **d.** $5\sqrt{3}$

27. Ratio of inradius of ΔABC and that of ΔPAQ is

a. 2 : 1 **b.** 3 : 2

c. 4 : 3 **d.** 3 : 1

For Problems 28–30

Consider the inequality, $9^x-a.\,3^x-a+3\le 0$, where 'a' is a real parameter.

28. The given inequality has at least one negative solution for $a \in$

a. $(-\infty, 2)$ **b.** $(3, \infty)$ **c.** $(-2, \infty)$ **d.** $(2, 3)$

29. The given inequality has at least one positive solution for $a \in$

a. $(-\infty, -2)$ **b.** $[3, \infty)$ **c.** $(2, \infty)$ **d.** $[-2, \infty)$

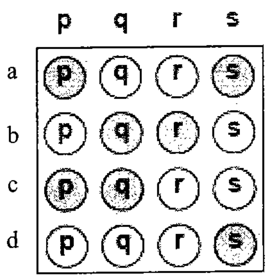
30. The given inequality has at least one real solution for $a \in$

a. $(-\infty, 3)$ **b.** $[2, \infty)$ **c.** $(3, \infty)$ **d.** $[-2, \infty)$

Matrix-Match Type

Solutions on page 3.79

Each question contains statements given in two columns which have to be matched. Statements (a, b, c, d) in column I have to be matched with statements (p, q, r, s) in column II. If the correct matches are a → p, a → s, , b → q, b → r, c → p, c → q and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:



1. Consider the parabola $(x-1)^2 + (y-2)^2 = \dfrac{(12x-5y+3)^2}{169}$

Column I	Column II
a. Locus of point of intersection of perpendicular tangent	**p.** $12x - 5y - 2 = 0$
b. Locus of foot of perpendicular from focus upon any tangent	**q.** $5x + 12y - 29 = 0$
c. Line along which minimum length of focal chord occurs	**r.** $12x - 5y + 3 = 0$
d. Line about which parabola is symmetrical	**s.** $24x - 10y + 1 = 0$

2. Consider the parabola $y^2 = 12x$.

Column I	Column II
a. Equation of tangent can be	**p.** $2x + y - 6 = 0$
b. Equation of normal can be	**q.** $3x - y + 1 = 0$
c. Equation of chord of contact w.r.t. any point on the directrix can be	**r.** $x - 2y - 12 = 0$
d. Equation of chord which subtends right angle at the vertex can be	**s.** $2x - y - 36 = 0$

3.

Column I	Column II
a. Tangents are drawn from point (2, 3) to the parabola $y^2 = 4x$, then points of contact are	**p.** $(9, -6)$
b. From a point P on the circle $x^2 + y^2 = 5$, the equation of chord of contact to the parabola $y^2 = 4x$ is $y = 2(x - 2)$, then the coordinate of point P will be	**q.** $(1, 2)$
c. $P(4, -4)$, Q are points on parabola $y^2 = 4x$ such that area of ΔPOQ is 6 sq. units where O is the vertex, then coordinates of Q may be	**r.** $(-2, 1)$
d. The common chord of circle $x^2 + y^2 = 5$ and parabola $6y = 5x^2 + 7x$ will pass through point(s)	**s.** $(4, 4)$

4.

Column I	Column II
a. Points from which perpendicular tangents can be drawn to parabola $y^2 = 4x$	**p.** $(-1, 2)$
b. Points from which only one normal can be drawn to parabola $y^2 = 4x$	**q.** $(3, 2)$
c. Point at which chord $x - y + 1 = 0$ of parabola $y^2 = 4x$ is bisected	**r.** $(-1, -5)$
d. Points from which tangents cannot be drawn to parabola $y^2 = 4x$	**s.** $(5, -2)$

Integer Type

Solutions on page 3.80

1. If the length of the latus rectum of the parabola $169\{(x-1)^2 + (y-3)^2\} = (5x - 12y + 17)^2$ is L then the value of $\dfrac{13L}{4}$ is

2. $y = x + 2$ is any tangent to the parabola $y^2 = 8x$. The ordinate of the point P on this tangent such that the other tangent from it which is perpendicular to it is

3. The focal chord of $y^2 = 16x$ is tangent to $(x-6)^2 + y^2 = 2$, then the possible value of the square of slope of this chord is

4. Two tangents are drawn from the point $(-2, -1)$ to the parabola $y^2 = 4x$. If θ is the angle between these tangents then $\tan\theta =$

5. The equation of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is $ax + by + c = 0$ then the value of $a + b + c$ is

6. If the point $P(4, -2)$ is the one end of the focal chord PQ of the $y^2 = x$, then the slope of the tangent at Q is

7. If the line $x + y = 6$ is a normal to the parabola $y^2 = 8x$ at point (a, b) then the value of $a + b$ is

8. The locus of the mid-points of the portion of the normal to the parabola $y^2 = 16x$ intercepted between the curve and the axis is another parabola whose latus rectum is

9. Consider locus of center of circle which touches circle $x^2 + y^2 = 4$ and line $x = 4$. The distance of the vertex of the locus from origin is

10. If on a given base BC ($B(0, 0)$ and $C(2, 0)$) a triangle be described such that the sum of the tangents of the base angles is 4, then equation of locus of opposite vertex A is parabola whose directrix is $y = k$, then the value of $8k - 9$ is

11. PQ is any focal chord of the parabola $y^2 = 8x$. Then the length of PQ can never be less than

12. If the length of focal chord to the parabola $y^2 = 12x$ drawn from the point $(3, 6)$ on it is L then the value of $L/3$ is

13. From the point $(-1, 2)$ tangent lines are drawn to the parabola $y^2 = 4x$. If the area of the triangle formed by the chord of contact & the tangents is A then the value of $\frac{A}{\sqrt{2}}$ is

14. Line $y = 2x - b$ cuts the parabola $y = x^2 - 4x$ at points A and B. Then the value of b for which the $\angle AOB$ is a right angle is

15. A line through the origin intersects the parabola $5y = 2x^2 - 9x + 10$ at two points whose x-coordinates add up to 17. Then the slope of the line is

16. If circle and $(x - 6)^2 + y^2 = r^2$ and parabola $y^2 = 4x$ have maximum number of common chord then least integral value of r is

Archives

Solutions on page 3.82

Subjective Type

1. Suppose that the normals drawn at three different points on the parabola $y^2 = 4x$ pass through the point (h, k). Show that $h > 2$. **(IIT-JEE, 1981)**

2. A is a point on the parabola $y^2 = 4ax$. The normal at A cuts the parabola again at point B. If AB subtends a right angle at the vertex of the parabola. find the slope of AB **(IIT-JEE, 1982)**

3. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point $(1, 2)$. **(IIT-JEE, 1984)**

4. Three normals are drawn from the point $(c, 0)$ to the curve $y^2 = x$. Show that c must be greater than $\frac{1}{2}$. One normal is always the x-axis. Find c for which the other two normals are perpendicular to each other. **(IIT-JEE, 1991)**

5. Through the vertex O of parabola $y^2 = 4x$, chords OP and OQ are drawn at right angles to one another. Show that for all positions of P, PQ cuts the axis of the parabola at a fixed point. Also find the locus of the middle point of PQ. **(IIT-JEE, 1994)**

6. Show that the locus of a point which divides a chord of slope 2 of the parabola $y^2 = 4x$ internally in the ratio 1:2 is a parabola. Find the vertex of this parabola. **(IIT-JEE, 1995)**

7. Points A, B and C lie on the parabola $y^2 = 4ax$. The tangents to the parabola at A, B and C, taken in pairs, intersect at points P, Q and R. Determine the ratio of the areas of the triangles ABC and PQR. **(IIT-JEE, 1996)**

8. From a point A common tangents are drawn to the circle $x^2 + y^2 = a^2/2$ and parabola $y^2 = 4ax$. Find the area of the quadrilateral formed by the common tangents, the chord of contact of the circle and the chord of contact of the parabola. **(IIT-JEE, 1996)**

9. The angle between a pair of tangents drawn from a point P to the parabola $y^2 = 4ax$ is 45°. Show that the locus of the point P is a hyperbola. **(IIT-JEE, 1998)**

10. Let C_1 and C_2 be, respectively, the parabola $x^2 = y - 1$ and $y^2 = x - 1$. Let P be any point on C_1 and Q be any point on C_2. Let P_1 and Q_1 be the reflections of P and Q, respectively, with respect to the line $y = x$. Prove that P_1 lies on C_2, Q_1 lies on C_1 and $PQ \geq \min\{PP_1, QQ_1\}$. Hence, or otherwise determine points P_0 and Q_0 on the parabolas C_1 and C_2, respectively, such that $P_0Q_0 \leq PQ$ for all pairs of points (P, Q) with P on C_1 and Q on C_2. **(IIT-JEE, 2000)**

11. Normals are drawn from the point P with slopes m_1, m_2, m_3 to that parabola $y^2 = 4x$. If locus of P with $m_1m_2 = \alpha$ is a part of the parabola itself, then find α. **(IIT-JEE, 2003)**

12. Tangent is drawn to parabola $y^2 - 2y - 4x + 5 = 0$ at a point P which cuts the directrix at the point Q. A point R is such that it divides QP externally in the ratio $\frac{1}{2}:1$. Find the locus of point R. **(IIT-JEE, 2004)**

Objective Type

Fill in the blanks

1. The point of intersection of the tangents at the ends of the latus rectum of the parabola $y^2 = 4x$ is____________.

(IIT-JEE, 1994)

Multiple choice questions with one correct answer

1. Consider a circle with its centre lying on the focus of the parabola $y^2 = 2px$ such that it touches the directrix of the parabola. Then a point of intersection of the circle and parabola is

a. $\left(\frac{p}{2}, p\right)$ or $\left(\frac{p}{2}, -p\right)$ **b.** $\left(\frac{p}{2}, -\frac{p}{2}\right)$

c. $\left(-\frac{p}{2}, p\right)$ **d.** $\left(-\frac{p}{2}, -\frac{p}{2}\right)$

(IIT-JEE, 1995)

2. The curve described parametrically by $x = t^2 + t + 1$, $y = t^2 - t + 1$ represents

a. a pair of straight lines

b. an ellipse

c. a parabola

d. a hyperbola

(IIT-JEE, 1999)

3. If $x + y = k$ is normal to $y^2 = 12x$, then k is

a. 3 **b.** 9 **c.** −9 **d.** −3

(IIT-JEE, 2000)

4. If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, then one of the values of k is

a. $\frac{1}{8}$ **b.** 8 **c.** 4 **d.** $\frac{1}{4}$

(IIT-JEE, 2000)

5. The equation of the common tangent touching the circle $(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis is

a. $\sqrt{3}y = 3x + 1$ **b.** $\sqrt{3}y = -(x + 3)$

c. $\sqrt{2}y = x + 3$ **d.** $\sqrt{3}\,y = -(3x + 1)$

(IIT-JEE, 2001)

6. The equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is

a. $x = -1$ **b.** $x = 1$ **c.** $x = -\frac{3}{2}$ **d.** $x = \frac{3}{2}$

(IIT-JEE, 2001)

7. The locus of the midpoint of the segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directix

a. $y = 0$ **b.** $x = -a$

c. $x = 0$ **d.** none of these

(IIT-JEE, 2002)

8. The focal chord to $y^2 = 16x$ is tangent to $(x - 6)^2 + y^2 = 2$, then the possible value of the slope of this chord, are

a. $\{-1. 1\}$ **b.** $\{-2, 2\}$

c. $\{-2, \frac{1}{2}\}$ **d.** $\{2, -\frac{1}{2}\}$

(IIT-JEE, 2003)

9. The angle between the tangents drawn from the point (1, 4) to the parabola $y^2 = 4x$ is

a. $\frac{\pi}{6}$ **b.** $\frac{\pi}{4}$ **c.** $\frac{\pi}{3}$ **d.** $\frac{\pi}{2}$

(IIT-JEE, 2004)

10. Tangent to the curve $y = x^2 + 6$ at a point (1, 7) touches the circle $x^2 + y^2 = 16x + 12y + c = 0$ at a point Q. Then the coordinate of Q are

a. (−6, −11) **b.** (− 9, −13)

c. (−10, −15) **d.** (−6, −7)

(IIT-JEE, 2005)

11. The axis of a parabola is along the line $y = x$ and the distance of its vertex and focus from origin are $\sqrt{2}$ and $2\sqrt{2}$, respectively. If vertex and focus both lie in the first quadrant, then the equation of the parabola is

a. $(x + y)^2 = (x - y - 2)$

b. $(x - y)^2 = (x + y - 2)$

c. $(x - y)^2 = 4\,(x + y - 2)$

d. $(x - y)^2 = 8\,(x + y - 2)$

12. Consider the two curves $C_1 : y^2 = 4x$, $C_2 : x^2 + y^2 - 6x + 1 = 0$. Then

a. C_1 and C_2 touch each other only at one point

b. C_1 and C_2 touch each other exactly at two points

c. C_1 and C_2 intersect (but do not touch) at exactly two points

d. C_1 and C_2 neither intersect nor touch each other

(IIT-JEE, 2008)

13. Let (x, y) by any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from (0, 0) to (x, y) in the ratio 1 : 3. Then the locus of P is

a. $x^2 = y$ b. $y^2 = 2x$

c. $y^2 = x$ d. $x^2 = 2y$ **(IIT-JEE, 2011)**

Multiple choice questions with one or more than one correct answer

1. The equations of the common tangents to the parabola $y = x^2$ and $y = -(x-2)^2$ is/are
   a. $y = 4(x-1)$ b. $y = 0$
   c. $y = -4(x-1)$ d. $y = -30x - 50$
   (IIT-JEE, 2006)

2. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the end points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latus rectum PQ are
   a. $x^2 + 2\sqrt{3}y = 3 + \sqrt{3}$
   b. $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$
   c. $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$
   d. $x^2 - 2\sqrt{3}y = 3 - \sqrt{3}$
   (IIT-JEE, 2008)

3. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at point T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose
   a. vertex is $\left(\frac{2a}{3}, 0\right)$
   b. directrix is $x = 0$
   c. latus rectum is $\frac{2a}{3}$
   d. focus is $(a, 0)$ **(IIT-JEE, 2009)**

4. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be
   a. $-\frac{1}{r}$ b. $\frac{1}{r}$
   c. $\frac{2}{r}$ d. $-\frac{2}{r}$ **(IIT-JEE, 2010)**

5. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point (9, 6), then L is given by
   a. $y - x + 3 = 0$ b. $y + 3x - 33 = 0$
   c. $y + x - 15 = 0$ d. $y - 2x + 12 = 0$
   (IIT-JEE,2011)

Match the following

1. (3, 0) is the point from which three normals are drawn to the parabola $y^2 = 4x$ which meet the parabola in the points P, Q and R. Then **(IIT-JEE, 2006)**

Column I	Column II
i. Area of ΔPQR	**a.** 2
ii. Radius of circumcircle of ΔPQR	**b.** $\frac{5}{2}$
iii. Centroid of ΔPQR	**c.** $\left(\frac{5}{2}, 0\right)$
iv. Circumcentre of ΔPQR	**d.** $\left(\frac{2}{3}, 0\right)$

Comprehension based questions

Consider the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$. They intersect at P and Q in the first and the fourth quadrants, respectively. Tangents to the circle at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S. **(IIT-JEE, 2007)**

1. The ratio of the areas of the triangles PQS and PQR is
   a. $1:\sqrt{2}$ b. 1:2 c. 1:4 d. 1:8
2. The radius of the circumcircle of the triangle PRS is
   a. 5 b. $3\sqrt{3}$ c. $3\sqrt{2}$ d. $2\sqrt{3}$
3. The radius of the incircle of the triangle PQR is
   a. 4 b. 3 c. $\frac{8}{3}$ d. 2

Assertion and Reasoning

1. **Statement 1:** The curve $y = \frac{-x^2}{2} + x + 1$ is symmetric with respect to the line $x = 1$.
   Statement 2 : A parabola is symmetric about its axis.
   a. Statement 2 is true, statement 2 is true; statement 2 is a correct explanation for statement 2
   b. Statement 2 is true, statement 2 is true; statement 2 is NOT a correct explanation for statement 2
   c. Statement 2 is true, statement 2 is false
   d. Statement 2 is false, statement 2 is true.
   (IIT-JEE, 2007)

Integer type

1. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P\left(\frac{1}{2}, 2\right)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is
   (IIT-JEE, 2011)

ANSWERS AND SOLUTIONS

Subjective Type

1.

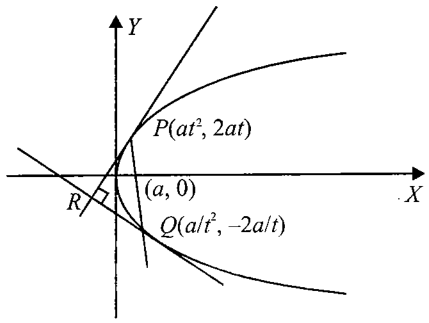


Fig. 3.63

Let the parabola be $y^2 = 4ax$.

ΔPQR is right angled at R as tangents at the extremities of the focal chord meet on the directrix at right angle.

Also coordinates of points P and Q are $P(at^2, 2at)$ and $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ respectively.

Hence, point of intersection of tangents at point $P(t)$ and $Q\left(-\frac{1}{t}\right)$ is $\left(-a, a\left(t-\frac{1}{t}\right)\right)$ and the coordinates of the centroid (G) is $\left(\frac{a}{3}\left(t^2-\frac{1}{t^2}-1\right), a\left(t-\frac{1}{t}\right)\right)$.

Hence, the slope of line $RG = 0$ (R is orthocentre).

2.

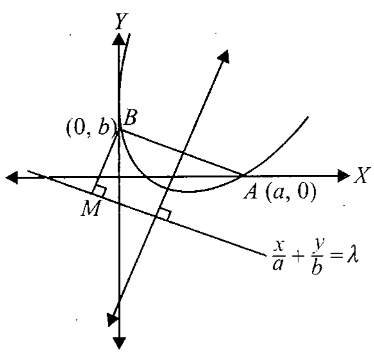


Fig. 3.64

AB is the shortest focal chord of the parabola, i.e., it is the latus rectum.

$\Rightarrow$ Focus of the parabola is the midpoint of AB

$$= \left(\frac{a}{2}, \frac{b}{2}\right)$$

Equation of the AB is $\frac{x}{a} + \frac{y}{b} = 1$

Equation of the directrix is $\frac{x}{a} + \frac{y}{b} = \lambda$

By definition of parabola $BD = BM = \frac{\sqrt{a^2+b^2}}{2}$

$$\Rightarrow \left|\frac{\frac{0}{a}+\frac{b}{b}-\lambda}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right| = \frac{\sqrt{a^2+b^2}}{2}$$

$$\Rightarrow \quad 2(1-\lambda)\, ab = \pm(a^2+b^2)$$

$$\Rightarrow \quad \lambda = \frac{-(a-b)^2}{2ab} \text{ or } \frac{(a+b)^2}{2ab}$$

$\Rightarrow$ Directrices are $\frac{x}{a} + \frac{y}{b} = \frac{-(a-b)^2}{2ab}$,

and $\quad \frac{x}{a} + \frac{y}{b} = \frac{(a+b)^2}{2ab}$

$\Rightarrow$ Two parabolas are possible whose equations are given by

$$\left(x-\frac{a}{2}\right)^2 + \left(y-\frac{b}{2}\right)^2 = \frac{\left(\frac{x}{a}+\frac{y}{b}+\frac{(a-b)^2}{2ab}\right)^2}{\frac{1}{a^2}+\frac{1}{b^2}}$$

$$\left(x-\frac{a}{2}\right)^2 + \left(y-\frac{b}{2}\right)^2 = \frac{\left(\frac{x}{a}+\frac{y}{b}-\frac{(a+b)^2}{2ab}\right)^2}{\frac{1}{a^2}+\frac{1}{b^2}}$$

3.

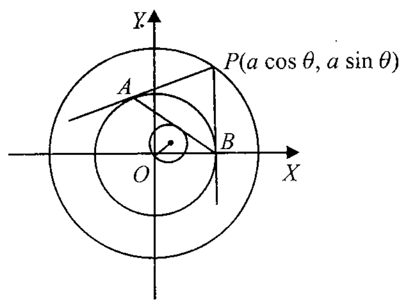


Fig. 3.65

Let centre of the variable circle be (h, k) and the point taken on the first circle $x^2 + y^2 = a^2$ be $(a \cos \theta, a \sin \theta)$.

Equation of the chord of contact AB will be

$$(a \cos \theta)x + (a \sin \theta)y = b^2$$

$$\Rightarrow \quad x \cos \theta + y \sin \theta = \frac{b^2}{a}$$

As this is tangent to the variable circle, centre of variable circle is at a constant distance r from AB and O where r is the radius (variable).

Hence, locus is a parabola.

4.

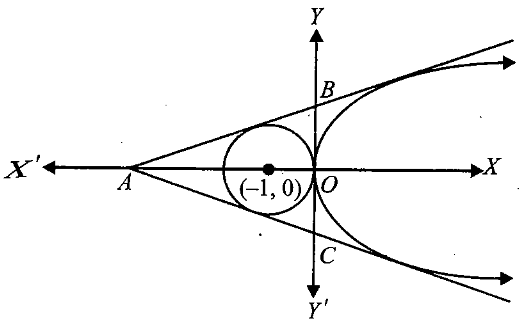


Fig. 3.66

As circle is $(x+1)^2 + y^2 = 1$, one of the common tangent is along y-axis.

Let the other common tangent has slope m,

Then, its equation is $\quad y = mx + \frac{1}{m}$

Solving it with the equation of circle,

$$x^2 + \left(mx + \frac{1}{m}\right)^2 + 2x = 0$$

$$\Rightarrow \quad (1+m^2)x^2 + 4x + \frac{1}{m^2} = 0$$

As the line touches the circle, $D = 0$

$$\Rightarrow \quad 16 - \frac{4}{m^2}(1+m^2) = 0 \Rightarrow 4m^2 = 1 + m^2$$

$$\Rightarrow \quad m = \pm\frac{1}{\sqrt{3}}$$

i.e. $\quad \angle BOA = \angle OAC = \frac{\pi}{6}$

Hence, the triangle is equilateral.

5.

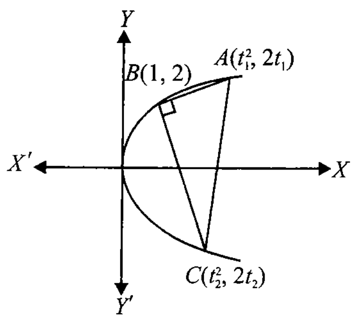


Fig. 3.67

Let points A and B be $(t_1^2, 2t_1)$ and $(t_2^2, 2t_2)$,

then $\quad m_{AB} m_{BC} = -1$

$$\Rightarrow \quad \frac{2}{(t_1+1)} \cdot \frac{2}{(t_2+1)} = -1$$

$$\Rightarrow \quad t_1 + t_2 + t_1 t_2 = -5 \quad \text{(i)}$$

Let the centroid of the ΔABC be (h, k), then

$$h = \frac{t_1^2 + t_2^2 + 1}{3}$$

and $\quad k = \frac{2t_1 + 2t_2 + 2}{3} \quad$ (ii)

From Eq. (ii), we get

$$t_1^2 + t_2^2 = 3h - 1 \text{ and } t_1 + t_2 = \frac{3k-2}{2} \quad \text{(iii)}$$

or $\quad (t_1 + t_2)^2 - 2t_1 t_2 = 3h - 1$

or $\quad \left(\frac{3k-2}{2}\right)^2 - 2t_1 t_2 = 3h - 1$

Hence, from Eq. (i), $\frac{3k-2}{2} + \frac{\left(\frac{3k-2}{2}\right)^2 - (3h-1)}{2} = -5$

Hence, locus is $3y - 2 + \left(\frac{3y-2}{2}\right)^2 - (3x-1) + 10 = 0$

6.

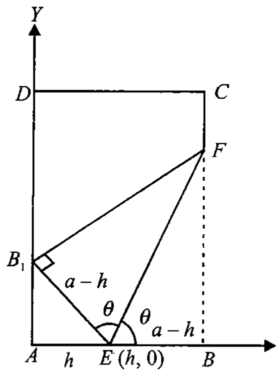


Fig. 3.68

Let $ABCD$ be a page with $AB = a$, $BC = b$.

The corner B of folded leaf is moving along AD.

Let A be origin, $AE = h$, then equation of crease EF is

$$y = (x - h)\tan\theta \quad \text{(i)}$$

Now $\quad \angle AEB_1 = \pi - 2\theta$

$$\Rightarrow \quad \cos(\pi - 2\theta) = \frac{h}{a-h}$$

$$\Rightarrow \quad \frac{\tan^2\theta - 1}{\tan^2\theta + 1} = \frac{h}{a-h}$$

$$\Rightarrow \quad \tan^2\theta = \frac{a}{a-2h}$$

$$\Rightarrow \quad \frac{y^2}{(x-h)^2} = \frac{a}{a-2h} \quad \text{[from Eq. (i)]}$$

$\Rightarrow \quad (a - 2h)y^2 = a(x - h)^2$

$\Rightarrow ah^2 + 2(y^2 - ax)h + a(x^2 - y^2) = 0$

For two coincident lines, discriminant $D = 0$

$4(y^2 - ax)^2 - 4a^2(x^2 - y^2) = 0$

$\Rightarrow \quad y^2(y^2 - 2ax + a^2) = 0$

Therefore, the crease is tangent to $y^2 - 2ax + a^2 = 0$, which is a parabola.

7. Let the fixed parabola is

$y^2 = 4ax$ (i)

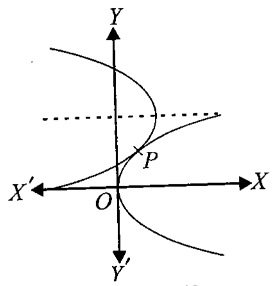


Fig. 3.69

and moving parabola is

$$(y - k)^2 = -4a(x - h) \quad \text{(ii)}$$

On solving Eqs. (i) and (ii), we get

$$(y - k)^2 = -4a\left(\frac{y^2}{4a} - h\right)$$

$\Rightarrow \quad y^2 - 2ky + k^2 = -y^2 + 4ah$

$\Rightarrow \quad 2y^2 - 2ky + k^2 - 4ah = 0$

Since the two parabolas touch each other, so $D = 0$

$\Rightarrow \quad 4k^2 - 8(k^2 - 4ah) = 0$

$\Rightarrow \quad -4k^2 + 32ah = 0$

$\Rightarrow \quad k^2 = 8ah$

$\Rightarrow$ Locus of the vertex of the moving parabola is

$$y^2 = 8ax.$$

8. Since the x-axis and the y-axis are two perpendicular tangents to the parabola and both meet at the origin, the directrix passes through the origin.

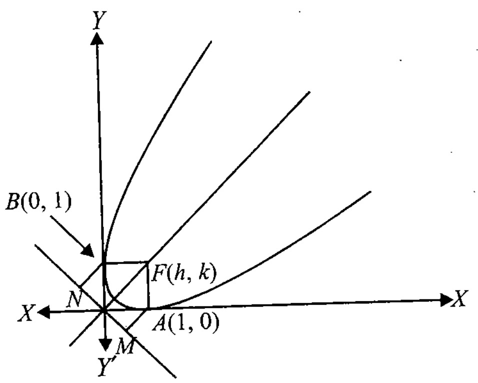


Fig. 3.70

Let $y = mx$ be the directrix and (h, k) be the focus

$\Rightarrow \quad FA = AM$

$\Rightarrow \quad \sqrt{(h-1)^2 + k^2} = \left|\frac{m}{\sqrt{1+m^2}}\right|$ (i)

and $\quad FB = BN$

$\Rightarrow \quad \sqrt{h^2 + (k-1)^2} = \left|\frac{1}{\sqrt{1+m^2}}\right|$ (ii)

Squaring and adding Eqs. (i) and (ii), we get

$$(h-1)^2 + h^2 + k^2 + (k-1)^2 = 1$$

$\Rightarrow \quad 2x^2 - 2x + 2y^2 - 2y + 1 = 0$, which is the required locus.

9. Tangent at the vertex is y-axis or $x = 0$ (i)

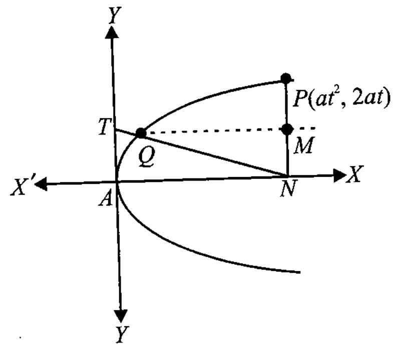


Fig. 3.71

Let P be any point $(at^2, 2at)$ on the parabola.

So, $PN = 2at$ and coordinates of N are $(at^2, 0)$.

If M is the midpoint of PN, then $NM = \frac{1}{2}PN = at$.

As MQ is parallel to x-axis, its equation is

$$y = at \quad \text{(ii)}$$

Solving (ii) with parabola $a^2t^2 = 4ax$

$\Rightarrow \quad x = \frac{1}{4}at^2$

So, the coordinates of Q are $\left(\frac{1}{4}at^2, at\right)$.

Equation of NQ will be

$$(y - 0) = \frac{at - 0}{\frac{1}{4}at^2 - at^2}(x - at^2)$$

$\Rightarrow \quad y = \frac{-4}{3t}(x - at^2)$ (iv)

If NQ meets the tangent at the vertex, i.e., $x = 0$ at T, then on putting $x = 0$ in Eq. (ii), we get $y = \frac{4at}{3}$.

So the coordinates of T are $\left(0, \frac{4at}{3}\right)$ or $AT = \frac{4at}{3}$.

$\Rightarrow \quad AT = \frac{2}{3} \times PN$

10. $y = mx + \frac{a}{m}$ (i)

is a tangent to $y^2 = 4ax$ and,

$x = m_1 y + \frac{b}{m_1}$ (ii)

is a tangent to $x^2 = 4by$.

Lines (i) and (ii) are perpendicular.

$\Rightarrow \quad m \times \frac{1}{m_1} = -1$

$\Rightarrow \quad m_1 = -m$

Let (h, k) be the point of intersection of Eqs. (i) and (ii), then

$$k = mh + \frac{a}{m} \text{ and } h = m_1 k + \frac{b}{m_1}$$

$\Rightarrow \quad k = mh + \frac{a}{m}$ and $h = -mk - \frac{b}{m}$

$\Rightarrow \quad m^2h - mk + a = 0$

and $\quad m^2k + mh + b = 0$

$\Rightarrow \quad \frac{m^2}{-kb - ah} = \frac{m}{ak - bh} = \frac{1}{h^2 + k^2}$

[By cross-multiplication]

Eliminating m, we have

$-(h^2 + k^2)(kb + ah) = (bh - ak)^2$

$\Rightarrow \quad (x^2 + y^2)(ax + by) + (bx - ay)^2 = 0$

11. $AS = 1 + t^2$ (focal distance of any point on the parabola $y^2 = 4ax$ is $a + at^2$)

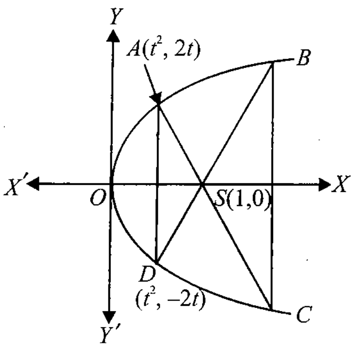


Fig. 3.72

$\therefore \quad CS = AC - AS = \frac{25}{4} - (1 + t^2)$

AC is a focal chord.

$\Rightarrow \quad \frac{1}{AS} + \frac{1}{CS} = \frac{1}{a}$

$\Rightarrow \quad \frac{1}{1 + t^2} + \frac{1}{\frac{25}{4} - (1 + t^2)} = 1$

$\Rightarrow \quad \frac{1}{1 + t^2} + \frac{4}{25 - 4(1 + t^2)} = 1$

$\Rightarrow \quad 25 - 4(1 + t^2) + 4(1 + t^2) = (1 + t^2)(25 - 4(1 + t^2))$

$\Rightarrow 4(1 + t^2)^2 - 25(1 + t^2) + 25 = 0$

$\Rightarrow \{(1 + t^2) - 5\}\{4(1 + t^2) - 5\} = 0$

$\Rightarrow \quad 1 + t^2 = 5, \frac{5}{4}$

$\Rightarrow \quad t^2 = 4, \frac{1}{4}$

$\Rightarrow \quad t = \pm 2, \pm \frac{1}{2}$

Thus, the coordinates of A, B, C, D are

$A \equiv (1/4, 1), B \equiv (4, 4),$

$C \equiv (4, -4)$

and $\quad D \equiv (1/4, -1)$

$\therefore \quad AD = 2, BC = 8$ and distance between AD and BC

$= \frac{15}{4}.$

Therefore, area of trapezium $ABCD$

$= \frac{1}{2}(2 + 8) \times \frac{15}{4}$

$= \frac{75}{4}$ sq. units

12. The given curves are $x^2 - y^2 = a^2$ (i)

and $\quad y = x^2$ (ii)

Solving Eqs. (i) and (ii) for y, we find that

$x^2 - y^2 = a^2$

$\Rightarrow \quad y - y^2 = a^2$

$\Rightarrow \quad y^2 - y + a^2 = 0$

Since y is real, $\quad 1 > 4a^2$

$\Rightarrow \quad -\frac{1}{2} < a < \frac{1}{2}$

The equation of any conic through the point of intersection of hyperbola and parabola is

$(x^2 - y^2 - a^2) + \lambda(y - x^2) = 0$

$\Rightarrow \quad x^2(1 - \lambda) - y^2 + \lambda y - a^2 = 0$ which is circle, then

$1 - \lambda = -1 \Rightarrow \lambda = 2$

Hence, the equation of the circle is

$x^2 + y^2 - 2y + a^2 = 0.$

13.

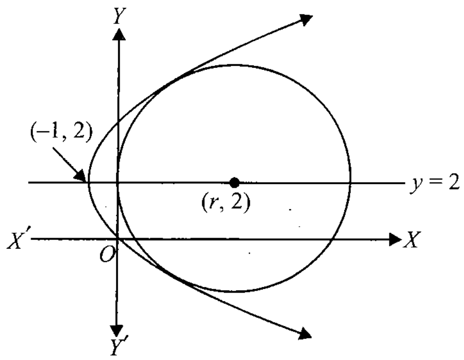


Fig. 3.73

$$y^2 = 4(x+y)$$

$$\Rightarrow \quad (y-2)^2 = 4(x+1) \qquad \text{(i)}$$

Its focus is (0, 2).

Let radius of the circle is r.

Then, equation of circle is

$$(x-r)^2 + (y-2)^2 = r^2 \qquad \text{(ii)}$$

Solving Eqs. (i) and (ii), we have

$$(x-r)^2 + 4(x+1) = r^2$$

$$x^2 + (4-2r)x + 4 = 0$$

$$(4-2r)^2 - 16 = 0 \qquad [\because D = 0]$$

$$\Rightarrow \quad 4 - 2r = \pm 4$$

$$\Rightarrow \quad r = 4$$

14. From the property of parabola, R is a focus.

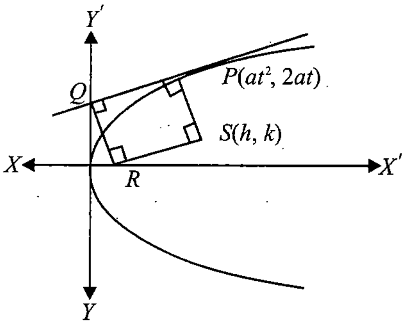


Fig. 3.74

$$\frac{2at-k}{at^2-h} \times \frac{k-0}{h-a} = -1$$

and $$\frac{k}{h-a} = \frac{1}{t} \Rightarrow t = \frac{h-a}{k}$$

So the required locus is

$$\left(2a\left(\frac{h-a}{k}\right) - k\right) = (a-h)\,\frac{(a-h)}{k}\left(a\left(\frac{h-a}{k}\right)^2 - h\right)$$

$$\Rightarrow \quad y^4 = (x-a)(a(x-a)^2 - y^2(x^2+2a))$$

15. Let the three points be $A\ (at_1^2,\ 2at_1)$, $B\ (at_2^2,\ 2at_2)$, $C\ (at_3^2, 2at_3)$

Tangents at these point are

$$t_1 y = x + at_1^2,\ t_2 y = x + at_2^2$$

$$t_3 y = x + at_3^2$$

Since t_1, t_2, t_3 are distinct, no two tangents are parallel or coincident.

Hence, these tangents will form a triangle.

The vertices of the triangle are $[at_1t_2,\ a(t_1+t_2)]$, $[at_2t_3,\ a(t_2+t_3)]$ and $[at_3t_1\ a(t_3+t_1)]$.

Equation of the two altitudes are

$$[y - a(t_2+t_3)] = -t_1(x - at_2t_3) \qquad \text{(i)}$$

and $$[y - a(t_3+t_1)] = -t_2(x - at_3\,t_1) \qquad \text{(ii)}$$

Subtracting Eq. (ii) from Eq. (i), we get $x = -a$

Hence, the locus of the orthocenter is $x + a = 0$ which is the directrix of the parabola.

16.

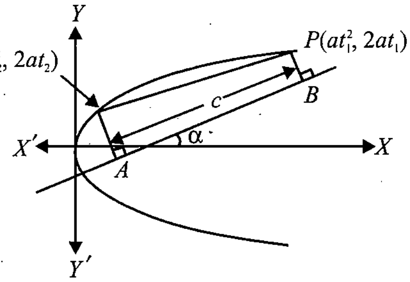


Fig. 3.75

$$\overrightarrow{PQ} = \vec{v} = (at_2^2 - at_1^2)\hat{i} + 2a(t_2 - t_1)\hat{j}$$

Also midpoint of PQ is (h, k).

$$\Rightarrow \quad 2h = a(t_1^2 + t_2^2)\ ;\ a(t_1 + t_2) = k$$

Also projection of $\vec{v}$ on $\overrightarrow{AB} = c$

$$\Rightarrow \quad \left|\frac{\vec{v}\,.\,\overrightarrow{AB}}{|\overrightarrow{AB}|}\right| = c$$

$$\Rightarrow \quad |\,a\,(t_2^2 - t_1^2)\cos\alpha + 2a(t_2 - t_2)\sin\alpha\,| = c$$

$$\Rightarrow \quad a^2\,(t_2 - t_1)^2\,[a(t_2 + t_1)\cos\alpha + 2a\sin\alpha]^2 = c^2$$

$$\Rightarrow \quad a^2\,[(t_2 + t_1)^2 - 4t_1t_2]\,[a(t_2 + t_1)\cos\alpha + 2a\sin\alpha]^2 = c^2$$

$$\Rightarrow \quad (y^2 - 4ax)\,(y\cos\alpha + 2a\sin\alpha)^2 + a^2c^2 = 0$$

17.

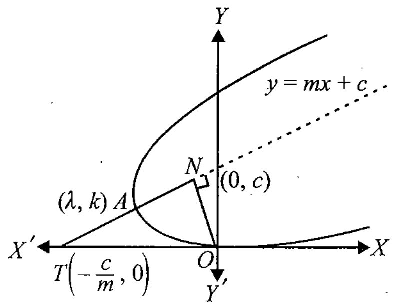


Fig. 3.76

Equation of ON is

$$y = -\frac{1}{m}x \text{ or } x + my = 0$$

Solving it with $\quad y = mx + c,$

coordinates of N are $\left(\dfrac{-cm}{1+m^2}, \dfrac{c}{1+m^2}\right)$

Also $\quad TA = AN$

$$\therefore \qquad \frac{-cm}{1+m^2} - \frac{c}{m} = 2\lambda$$

and $$\frac{c}{1+m^2} = 2k$$

$$\Rightarrow \qquad c = 2k(1+m^2)$$

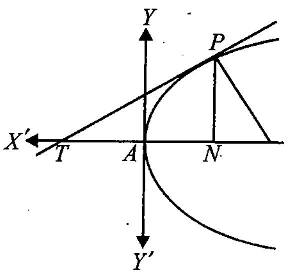


Fig. 3.77

Hence, equation of the axis of symmetry is

$$y = mx + 2k(1+m)^2$$

or $$2km^2 + mx + 2k - y = 0 \qquad \text{(i)}$$

This family of lines touch a curve which is obtained by equating to zero the discriminant of Eq. (i).

Hence, $$x^2 - 8k(2k - y) = 0$$

or $$x^2 = -8k(y - 2k)$$

18. Let the point P be $(p, 0)$ and equation of the chord through P be

$$\frac{x-p}{\cos\theta} = \frac{y-0}{\sin\theta} = r\ (r \in R) \qquad \text{(i)}$$

$\therefore$ $(r\cos\theta + p, r\sin\theta)$ lies on the parabola $y^2 = 4ax$

$$\therefore \quad r^2\sin^2\theta - 4ar\cos\theta - 4ap = 0 \qquad \text{(ii)}$$

If $AP = r_1$, $BP = -r_2$, then r_1 and r_2 are roots of Eq. (ii)

$$\therefore \qquad r_1 + r_2 = \frac{4a\cos\theta}{\sin^2\theta},\ r_1r_2 = \frac{-4ap}{\sin^2\theta}$$

Now $$\frac{1}{AP^2} + \frac{1}{BP^2} = \frac{1}{r_2^2} + \frac{1}{r_2^2}$$

$$= \frac{(r_1+r_2)^2 - 2r_1r_2}{r_1^2r_2^2}$$

$$= \frac{\cos^2\theta}{p^2} + \frac{\sin^2\theta}{2ap}$$

Since $\frac{1}{AP^2} + \frac{1}{BP^2}$ should be independent of θ, we take $p = 2a$.

Then $$\frac{1}{AP^2} + \frac{1}{BP^2} = \frac{1}{4a^2}(\cos^2\theta + \sin^2\theta)$$

$$= \frac{1}{4a^2}$$

Hence, $\frac{1}{AP^2} + \frac{1}{BP^2}$ is independent of θ for all positions of the chord if $P \equiv (2a, 0)$.

Objective Type

1. b. The coordinates of the focus and vertex of the required parabola are $S(a_1, 0)$ and $A(a, 0)$, respectively. Therefore, the distance between the vertex and focus is $AS = a_1 - a$ and so the length of the latus rectum $= 4(a_1 - a)$.

Thus, the equation of the parabola is

$$y^2 = 4(a_1 - a)(x - a).$$

2. d. Parabola having axis parallel to y-axis is

$$(x-a)^2 = 4A(y-b).$$

According to question, length of latus rectum $4A = l$.

Hence, equation of parabola is

$$(x-a)^2 = \frac{l}{4}(y-b)$$

or $$(x-a)^2 = \frac{l}{8}(2y-2b)$$

3. b. $x = 3\cos t$, $y = 4\sin t$

Eliminating t, we have

$$\frac{x^2}{9} + \frac{y^2}{16} = 1, \text{ which is ellipse}$$

$$\therefore \qquad x^2 - 2 = 2\cos t;\ y = 4\cos^2\frac{t}{2}$$

$$\Rightarrow \qquad y = 2(1 + 2\cos t)$$

$$y = 2\left(1 + \frac{x^2-2}{2}\right),$$

which is parabola

$$\sqrt{x} = \tan t; \sqrt{y} = \sec t$$

Eliminating t, we have

$$y - x = 1, \text{ which is straight line}$$

$$x = \sqrt{1 - \sin t};\ y$$

$$y = \sin\frac{t}{2} + \cos\frac{t}{2}$$

Eliminating t, we have

$x^2 + y^2 = 1 - \sin t + 1 + \sin t = 2$, which is circle.

4. c. Tangent at the vertex is

$$x - y + 1 = 0 \qquad \text{(i)}$$

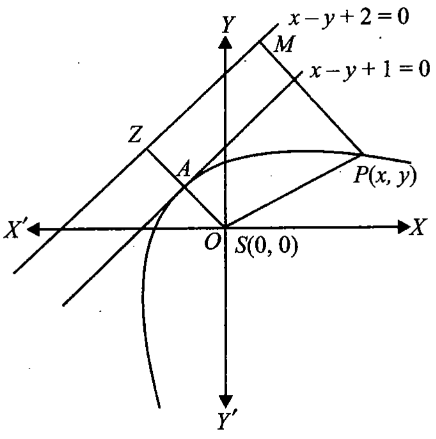


Fig. 3.78

Therefore, equation of axis of the parabola is

$$x + y = 0 \quad \text{(ii)}$$

Now solving Eqs. (i) and (ii), we get $A\left(-\frac{1}{2}, \frac{1}{2}\right)$

$\therefore$ Z is $(-1, 1)$

Now directrix is

$$x - y + k = 0$$

But this passes through $Z(-1, 1)$

$\Rightarrow$ $k = 2$

$\Rightarrow$ Directrix is $x - y + 2 = 0$

Therefore, by definition equation of parabola is given by

$$OP = PM$$

$\Rightarrow$ $OP^2 = PM^2$

$$\left(\frac{x-y+2}{\sqrt{2}}\right)^2 = x^2 + y^2$$

$\Rightarrow$ $(x - y + 2)^2 = 2x^2 + 2y^2$

$\Rightarrow x^2 + y^2 + 4 - 2xy + 4x - 4y = 2x^2 + 2y^2$

$\Rightarrow x^2 + y^2 + 2xy - 4x + 4y - 4 = 0$

5. **b.** We have, $\sqrt{px} + \sqrt{qy} = 1$

$\Rightarrow$ $(\sqrt{px} + \sqrt{qy})^2 = 1$

$\Rightarrow$ $px + qy + 2\sqrt{(pq)(xy)} = 1$

$\Rightarrow$ $(px + qy - 1)^2 = 4(pq)(xy)$

$\Rightarrow$ $p^2x^2 - 2(pq)(xy) + q^2y^2 - 2px - 2qy + 1 = 0$

On comparing this equation with the equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$$a = p^2, b = q^2, c = 1, g = -p,$$

$$f = -q \text{ and } h = -pq$$

$\therefore$
$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$
$$= p^2q^2 - 2p^2q^2 - p^2q^2 - p^2q^2 - p^2q^2$$
$$= -4p^2q^2 \neq 0$$

and, $h^2 - ab = p^2q^2 - p^2q^2 = 0$

Thus, we have $\Delta \neq 0$ and $h^2 = ab$

Hence, the given curve is parabola.

6. **d.** Two parabolas are equal if the length of their latus rectum are equal.

Length of the latus rectum of $y^2 = \lambda x$ is λ.

The equation of the second parabola is

$$25\{(x-3)^2 + (y+2)^2\} = (3x - 4y - 2)^2$$

$\Rightarrow$ $$\sqrt{(x-3)^2 + (y+2)^2} = \frac{|3x - 4y - 2|}{\sqrt{3^2 + 4^2}}$$

Here focus is $(3, -2)$ and equation of the directrix is

$$3x - 4y - 2 = 0.$$

Therefore, length of the latus rectum = 2 × distance between focus and directrix

$$= 2\left|\frac{3 \times 3 - 4 \times (-2) - 2}{\sqrt{3^2 + (-4)^2}}\right| = 6$$

Thus, the two parabolas are equal, if $\lambda = 6$.

7. **d.** Length of latus rectum

= 2 × distance of focus form directrix

$$= 2 \times \left|\frac{-\frac{u^2}{2g}\cos 2\alpha - \frac{u^2}{2g}}{\sqrt{1}}\right|$$

$$= \frac{2u^2}{g}\cos^2\alpha$$

8. **a.** Make homogeneous and put $A + B = 0$

$$y^2 = 4ax\left(\frac{lx + my}{-n}\right)$$

$\therefore$ $4al + n = 0$

9. **a.** $x^2 + y^2 - 2xy - 8x - 8y + 32 = 0$

$\Rightarrow$ $(x - y)^2 = 8(x + y - 4)$

is a parabola whose axis is $x - y = 0$ and the tangent at the vertex is $x + y - 4 = 0$.

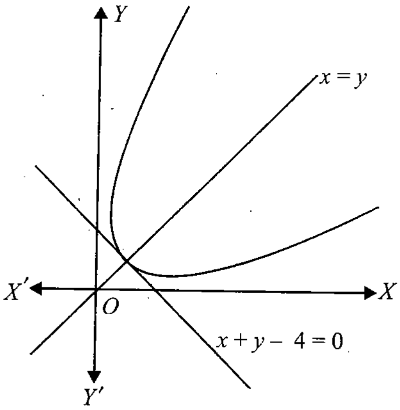


Fig. 3.79

Also, when $y = 0$, we have

$$x^2 - 8x + 32 = 0$$

which gives no real values of x.

when $x = 0$, we have $y^2 - 8y + 32 = 0$ which gives no real values of y.

So, the parabola does not intersect the axes. Hence, the graph falls in the first quadrant.

10. d. Eliminating θ from the given equations, we get

$$y^2 = -4a(x - a);$$

which is a parabola but $0 \le \cos^2\theta \le 1$

$\Rightarrow$ $0 \le x \le a$

and $-1 \le \sin\theta \le 1$

$\Rightarrow$ $-2a \le y \le 2a$

Hence, the locus of the point P is not exactly the parabola, rather it is a part of the parabola.

11. c. $(\sqrt{3h}, \sqrt{3k+2})$ lies on the line $x - y - 1 = 0$

$\Rightarrow$ $(\sqrt{3h})^2 = (\sqrt{3k+2} + 1)^2$

$\Rightarrow$ $3h = 3k + 2 + 1 + 2\sqrt{3k+2}$

$\Rightarrow$ $3^2(h - k - 1)^2 = 2^2(\sqrt{3k+2})^2$

$\Rightarrow$ $9(h^2 + k^2 + 1 - 2hk - 2h + 2k) = 4(3k + 2)$

$\Rightarrow$ $9(x^2 + y^2) - 18xy - 18x + 6y + 1 = 0$

Now $h^2 = ab$ and $\Delta \ne 0$

Therefore, locus is a parabola.

12. c.

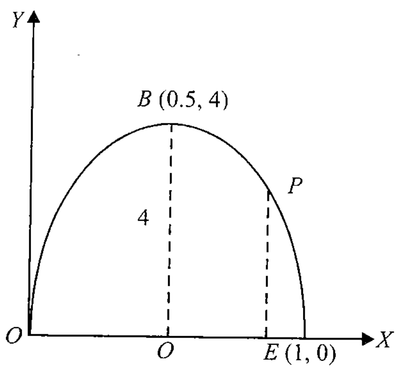


Fig. 3.80

The path of the water jet is a parabola.

Let its equation be

$$y = ax^2 + bx + c$$

It should pass through (0, 0), (0.5, 4), (1, 0)

$\Rightarrow$ $c = 0, a = -16, b = 16$

$\Rightarrow$ $y = -16x^2 + 16x$

If $x = 0.75$, we get $y = 3$.

13. a. $x = t^2 - t + 1, y = t^2 + t + 1$

$\Rightarrow$ $x + y = 2(t^2 + 1)$ and $y - x = 2t$

$\Rightarrow$ $\frac{x+y}{2} = 1 + \left(\frac{y-x}{2}\right)^2$

$\Rightarrow$ $(y - x)^2 = 2(x + y) - 4$

$\Rightarrow$ $(y - x)^2 = 2(x + y - 2)$

Vertex will be the point where lines $y - x = 0$ and $x + y - 2 = 0$ meet, i.e., the point (1, 1).

14. c. Let the point $P(h, k)$ on the parabola divides the line joining $A(4, -6)$ and $B(3, 1)$ in ratio λ.

Then, we have $(h, k) \equiv \left(\frac{3\lambda+4}{\lambda+1}, \frac{\lambda-6}{\lambda+1}\right)$

This point lies on the parabola,

$\therefore$ $\left(\frac{\lambda-6}{\lambda+1}\right)^2 = 4\left(\frac{3\lambda+4}{\lambda+1}\right)$

$\Rightarrow$ $(\lambda - 6)^2 = 4(3\lambda + 4)(\lambda + 1)$

$\Rightarrow$ $11\lambda^2 + 40\lambda - 20 = 0$

$\Rightarrow$ $\lambda = \frac{-20 \pm 2\sqrt{155}}{11} : 1$

15. d. Let $P(x, y)$ be the coordinates of the other end of the chord. OP.

Then $\frac{x+0}{2} = a, \frac{y+0}{2} = b$

But (x, y) lies on the parabola.

$\therefore$ $y^2 = 4x$

$\Rightarrow$ $(2b)^2 = 4(2a)$

$\Rightarrow$ $b^2 = 2a$

16. c. Let points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ lie on the parabola $y^2 = 4ax$.

Here points P and Q are variable but slope of the chord PQ,

$$m_{PQ} = \frac{2}{t_1 + t_2}.$$

Now let midpoint PQ be $R(h, k)$,

$$k = \frac{2at_1 + 2at_2}{2}$$

or $k = a(t_1 + t_2) = \frac{2}{m}$

$\Rightarrow$ $y = \frac{2}{m}$,

which is a line parallel to the axis of parabola.

17. d. Any line passing through focus other than axis always meets parabola in two distinct points.

Hence, $m \in R - \{0\}$.

18. c. Since the semi-latus rectum of a parabola is the harmonic mean between the segments of any focal chord of a parabola, therefore SP, 4, SQ are in HP.

$$\Rightarrow \quad 4 = \frac{2SP.SQ}{SP + SQ}$$

$$\Rightarrow \quad 4 = \frac{2(6)(SQ)}{6 + SQ}$$

$$\Rightarrow \quad 24 + 4(SQ) = 12(SQ)$$

$$\Rightarrow \quad SQ = 3$$

19. a. The graph shows $\lambda > 0$.

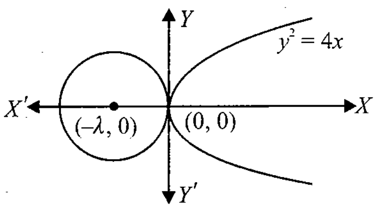


Fig. 3.81

20. c. Let x_1, x_2 and x_3 be the abscissae of the points on the parabola whose ordinates are y_1, y_2 and y_3, respectively. Then $y_1^2 = 4ax_1$, $y_2^2 = 4ax_2$ and $y_3^2 = 4ax_3$. Therefore, the area of the triangle whose vertices are (x_1, y_1) (x_2, y_2) and (x_3, y_3) is

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} \frac{y_1^2}{4a} & y_1 & 1 \\ \frac{y_2^2}{4a} & y_2 & 1 \\ \frac{y_3^2}{4a} & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{8a}\begin{vmatrix} y_1^2 & y_1 & 1 \\ y_2^2 & y_2 & 1 \\ y_3^2 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$$

21. d. The given parabolas are

$$y^2 = 4ax \quad \text{(i)}$$

and

$$x^2 = 4ay \quad \text{(ii)}$$

From Eq. (ii),

$$y = \frac{x^2}{4a}$$

Putting in Eq. (i),

$$\frac{x^4}{16a^2} = 4ax$$

$$\Rightarrow \quad x = 0 \text{ or } x = 4a$$

When $x = 0, y = 0$,

and when $x = 4a, y = \frac{16a^2}{4a} = 4a$

Thus, Eqs. (i) and (ii) meet at (0, 0) and $(4a, 4a)$.

Now $2bx + 3cy + 4d = 0$

passes through $(4a, 4a)$ and (0, 0).

$$\Rightarrow \quad d = 0$$

and

$$2b(4a) + 3c(4a) = 0$$

$$\Rightarrow \quad 2b + 3c = 0$$

$$\Rightarrow \quad d^2 + (2b + 3c)^2 = a^2$$

22. b. Let $R(h, k)$ be the midpoint of PQ. Therefore, Q is $(2h - 1, 2k)$

Since Q lies on $y^2 = 8x$

$$\therefore \quad (2k)^2 = 8(2h - 1)$$

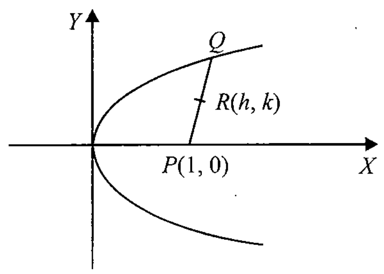


Fig. 3.82

Hence, locus of $Q(h, k)$ is

$$y^2 = 2(2x - 1)$$

or

$$y^2 = 4x - 2$$

$$\Rightarrow \quad y^2 - 4x + 2 = 0$$

23. a. The family of parabolas is $y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$ and the vertex is $A\left(\frac{-B}{2A}, \frac{-D}{4A}\right) = (h, k)$

$$\Rightarrow \quad h = -\frac{\frac{a^2}{2}}{2\frac{a^3}{3}} = -\frac{3}{4a}$$

and

$$k = \frac{\left(\frac{a^2}{2}\right)^2 - \frac{4a^3(-2a)}{3}}{4\frac{a^3}{3}}$$

$$\Rightarrow \quad h = -\frac{3}{4a} \text{ and } k = -\frac{35a}{16}$$

Eliminating a, we have $hk = 105/64$.

Hence, the required locus is $xy = 105/64$.

24. c. Let $C_1(h, k)$ be the centre of the circle.

Circle touches the x-axis then its radius is $r_1 = k$.

Also circle touches the circle with centre $C_2(0, 3)$ and radius $r_2 = 2$.

$\therefore \quad |C_1 C_2| = r_1 + r_2$

$\Rightarrow \quad \sqrt{(h-0)^2 + (k-3)^2} = |k+2|$

Squaring

$$h^2 - 10k + 5 = 0$$

$\Rightarrow$ Locus is $x^2 - 10y + 5 = 0$, which is parabola.

25. b. Let $P(x, y)$ be the point of contact. At 'P' both of them must have same slope. We have,

$$2y\frac{dy}{dx} = 4a, 2x = 4a\frac{dy}{dx}$$

Eliminating $\frac{dy}{dx}$, we get $xy = 4a^2$.

26. a. Let the point be $P(at^2, 2at)$.

Then according to question, $SP = at^2 + a = k$ (i)

Let (α, β) is the moving point, then $\alpha = at^2, \beta = 2at$

$\Rightarrow \quad \frac{\alpha}{\beta} = \frac{t}{2}$

and $\quad a = \frac{\beta^2}{4\alpha}$

($\because$ point (α, β) lies on $y^2 = 4ax$)

On substituting these values in Eq. (i),

$$\frac{\beta^2}{4\alpha}\left(1 + \frac{4\alpha^2}{\beta^2}\right) = k$$

$\Rightarrow \quad \beta^2 + 4\alpha^2 = 4k\alpha$

$\Rightarrow \quad 4x^2 + y^2 - 4kx = 0$ is the required locus.

27. d. $y - \sqrt{3}x + 3 = 0$ can be rewritten as

$$\frac{y-0}{\sqrt{3}/2} = \frac{x-\sqrt{3}}{1/2} = r \quad \text{(i)}$$

On solving Eq. (i) with the parabola $y^2 = x + 2$

$$\frac{3r^2}{4} = \frac{r}{2} + \sqrt{3} + 2$$

$\Rightarrow \quad 3r^2 - 2r - (4\sqrt{3} + 8) = 0$

$\Rightarrow \quad AP \cdot AQ = |r_1 r_2|$

$$= \frac{4(\sqrt{3}+2)}{3}$$

(product of roots)

28. a. Let $P(-2 + r\cos\theta, r\sin\theta)$ and Q lies on parabola

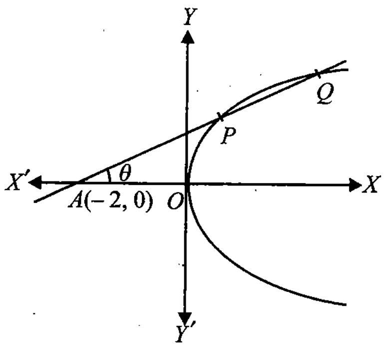


Fig. 3.83

$\Rightarrow \quad r^2\sin^2\theta - 4(-2 + r\cos\theta) = 0$

$\Rightarrow \quad r_1 + r_2 = \frac{4\cos\theta}{\sin^2\theta}, r_1 r_2 = \frac{8}{\sin^2\theta}$

Now $\quad \frac{1}{AP} + \frac{1}{AQ} = \frac{r_1 + r_2}{r_1 r_2} = \frac{\cos\theta}{2}$

Given that $\quad \frac{1}{AP} + \frac{1}{AQ} < \frac{1}{4}$

$\Rightarrow \quad \cos\theta < \frac{1}{2}$

$\Rightarrow \quad \tan\theta > \sqrt{3}$

($\because$ $\cos\theta$ is decreasing and $\tan\theta$ is increasing in $(0, \pi/2)$)

$\Rightarrow \quad m > \sqrt{3}$

29. a. The general equation of a parabola having its axis parallel to y-axis is

$$y = ax^2 + bx + c \quad \text{(i)}$$

This is touched by the line $y = x$ at $x = 1$.

Therefore, slope of the tangent at $(1, 1)$ is 1 and, $x = ax^2 + bx + c$ must have equal roots.

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,1)} = 1$ and $(b-1)^2 = 4ac$

$\Rightarrow \quad 2a + b = 1$ and $(b-1)^2 = 4ac$

Also, $(1, 1)$ lies on Eq. (i)

$\Rightarrow \quad a + b + c = 1$

From $\quad 2a + b = 1$ and $a + b + c = 1$, $a - c = 0$

$\Rightarrow \quad a = c$

Then from $\quad a + b + c = 1, 2c + b = 1$

$\Rightarrow \quad 2f(0) + f'(0) = 1$

[$\because f(0) = c$ and $f'(0) = b$]

30. b.

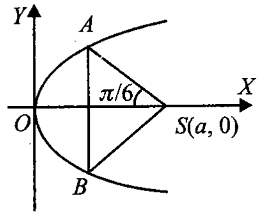


Fig. 3.84

Let $A = (at_1^2, 2at_1)$, $B \equiv (at_2^2, -2at_1)$.

We have

$$m_{AS} = \tan\left(\frac{5\pi}{6}\right)$$

$$\Rightarrow \quad \frac{2at_1}{at_1^2 - a} = -\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad t_1^2 + 2\sqrt{3}t_1 - 1 = 0$$

$$\Rightarrow \quad t_1 = -\sqrt{3} \pm 2$$

Clearly $t_1 = -\sqrt{3} - 2$ is rejected.

Thus, $t_1 = (2 - \sqrt{3})$

Hence, $AB = 4at_1 = 4a(2 - \sqrt{3})$

31. c. $\vec{V} = (T^2 - 1)\hat{i} + 2T\hat{j}$

$$\vec{n} = \hat{j} - \hat{i}$$

Projection of $\vec{V}$ on $\vec{n}$

$$y = \frac{\vec{V}.\vec{n}}{|\vec{n}|} = \frac{(1 - T^2) + 2T}{\sqrt{2}}$$

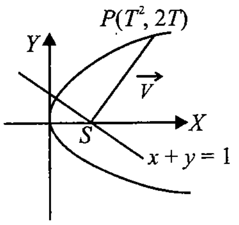


Fig. 3.85

Given $\frac{dx}{dt} = 4$; but $x = T^2$

$$\Rightarrow \quad \frac{dx}{dt} = 2T\frac{dT}{dt}$$

When $P(4, 4)$ then $T = 2$

$$\Rightarrow \quad 4 = 2(2)\frac{dT}{dt}$$

$$\Rightarrow \quad \frac{dT}{dt} = 1$$

Now $\frac{dy}{dt} = \left(\frac{-2T + 2}{\sqrt{2}}\right)\frac{dT}{dt}$

Therefore, at $T = 2$,

$$\sqrt{2}\frac{dy}{dt} = -4 + 2 = -2$$

$$\Rightarrow \quad \frac{dy}{dt} = -\sqrt{2}$$

32. a. Let focus be (a, b).

Equations are

$$S_1 : (x - a)^2 + (y - b)^2 = x^2$$

and $S_2 : (x - a)^2 + (y - b)^2 = y^2$

Common chord $S_1 - S_2 = 0$ gives $x^2 - y^2 = 0$

$$\Rightarrow \quad y = \pm x.$$

33. d. Put $x^2 = \frac{y}{a}$ in circle, $x^2 + (y - 1)^2 = 1$, we get (Note that for $a < 0$ they cannot intersect other than origin) $\frac{y}{a} + y^2 - 2y = 0$. Hence, we get $y = 0$ or $y = 2 - \frac{1}{a}$

Substituting $y = 2 - \frac{1}{a}$ in $y = ax^2$, we get

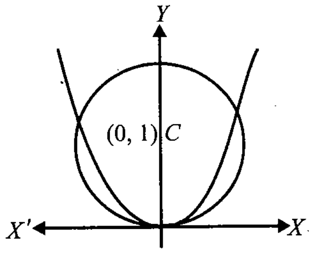


Fig. 3.86

$$ax^2 = 2 - \frac{1}{a}$$

$$\Rightarrow \quad x^2 = \frac{2a - 1}{a^2} > 0$$

$$\Rightarrow \quad a > \frac{1}{2}$$

34. d. Chord through (2, 1) is $\frac{x - 2}{\cos\theta} = \frac{y - 1}{\sin\theta} = r$ (i)

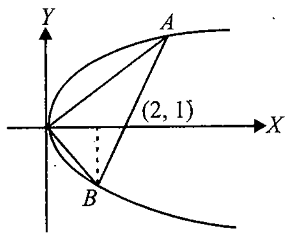


Fig. 3.87

Solving Eq. (i) with parabola $y^2 = x$, we have

$$(1 + r\sin\theta)^2 = 2 + r\cos\theta$$

$\Rightarrow \sin^2\theta\, r^2 + (2\sin\theta - \cos\theta)r - 1 = 0$

This equation has two roots $r_1 = AC$ and $r_2 = -BC$

Then, sum of roots $r_1 + r_2 = 0$

$\Rightarrow \quad 2\sin\theta - \cos\theta = 0 \Rightarrow \tan\theta = \frac{1}{2}$

$$AB = |r_1 - r_2|$$

$$= \sqrt{(r_1+r_2)^2 - 4r_1r_2}$$

$$= \sqrt{4\frac{1}{\sin^2\theta}} = 2\sqrt{5}$$

35. c. Solving circle $x^2 + y^2 = 5$ and parabola $y^2 = 4x$, we have

$$x^2 + 4x - 5 = 0$$

$\Rightarrow \quad x = 1$

or $\quad x = -5$ (not possible)

$\Rightarrow \quad y^2 = 4$ or $y = \pm 2$

$\Rightarrow$ Points of intersection are $P(1, 2)$; $Q(1, -2)$

Hence, $\quad PQ = 4$

36. c. Difference of the ordinate

$$d = \left|2at + \frac{2a}{t}\right| = 2a\left|t + \frac{1}{t}\right|$$

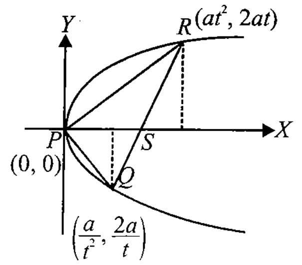


Fig. 3.88

Now area $\quad A = \frac{1}{2}\begin{vmatrix} at^2 & 2at & 1 \\ \frac{a}{t^2} & -\frac{2a}{t} & 1 \\ 0 & 0 & 1 \end{vmatrix} = a^2\left(t + \frac{1}{t}\right)$

$\Rightarrow \quad 2a\left(t + \frac{1}{t}\right) = \frac{2A}{a}$

37. a. A_1B_1 is a focal chord, then $A_1(at_1^2, 2at_1)$ and $B_1\left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$.

A_2B_2 is a focal chord, then $A_2(at_2^2, 2at_2)$ and $B_2\left(\frac{a}{t_2^2}, \frac{-2a}{t_2}\right)$

Now equation of chord A_1A_2 is

$$y(t_1 + t_2) - 2x - 2at_1t_2 = 0 \quad \text{(i)}$$

Chord B_1B_2 is

$$y\left(-\frac{1}{t_1} - \frac{1}{t_2}\right) - 2x - 2a\left(-\frac{1}{t_1}\right)\left(-\frac{1}{t_2}\right) = 0$$

or $\quad y(t_1 + t_2) + 2x\,t_1t_2 + 2a = 0 \quad$ (ii)

For their intersection, we subtract them and get

$$2x(t_1t_2 + 1) + 2a(t_1t_2 + 1) = 0$$

or $\quad (x + a)(1 + t_1t_2) = 0$

$\Rightarrow \quad x + a = 0$

Hence, they interset on directrix.

38. b. Joint equation of OA and OB is

$x^2 - 4x(y - 3x) - 4y(y - 3x) + 20(y - 3x)^2 = 0$

Making equation of parabola homogeneous using straight line.

$\Rightarrow \quad x^2(1 + 12 + 180) - y^2(4 - 20) - xy(4 - 12 + 120) = 0$

$\Rightarrow \quad 193x^2 + 16y^2 - 112xy = 0$

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$= \frac{2\sqrt{56^2 - 193 \times 16}}{193 + 16} = \frac{8\sqrt{3}}{209}$$

39. a. Let the midpoint of PQ be (α, β)

$\Rightarrow \quad \alpha = x + \frac{c}{2}$ and $\beta = y + \frac{c}{2}$

$\Rightarrow \quad \left(\beta - \frac{c}{2}\right)^2 = 4a\left(\alpha - \frac{c}{2}\right)$

$\Rightarrow \quad \left(y - \frac{c}{2}\right)^2 = 4a\left(x - \frac{c}{2}\right)$

which is required locus.

40. c. Latus rectum of $y^2 = 2bx$ is $2b$.

Semi-latus rectum is b.

We know that semi-latus rectum is H.M. of segments of focal chord.

Then $\quad \frac{1}{a} + \frac{1}{c} = \frac{2}{b}$

$\Rightarrow \quad b = \frac{2ac}{a + c}$

Now for $\quad ax^2 + bx + c = 0,$

$$D = b^2 - 4ac$$

$$= \left(\frac{2ac}{a + c}\right)^2 - 4ac$$

$$= -4ac\left(\frac{a^2 + c^2 - ac}{(a + c)^2}\right) < 0$$

Hence, roots are imaginary.

41. c. $\tan\theta = \frac{y}{x}$

Projection of BC on the x-axis

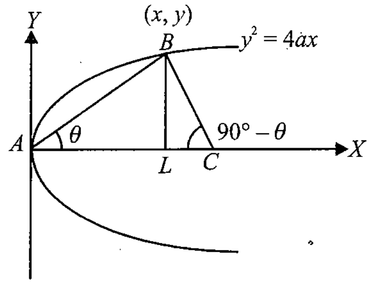


Fig. 3.89

$$LC = \frac{y}{\tan(90° - \theta)} = y\tan\theta$$

$$\frac{y^2}{x} = 4a$$

42. c. $\alpha^2 + 1 - 4 < 0$

$\Rightarrow \quad \alpha^2 < 3, |\alpha| < \sqrt{3}$

$\Rightarrow \quad 1 - 4\alpha < 0$

$\Rightarrow \quad \alpha > \frac{1}{4}$

43. c. $y^2 = 6\left(x - \frac{3}{2}\right)$

Equation of directrix is

$$x - \frac{3}{2} = -\frac{3}{2}, \text{ i.e, } x = 0$$

Let coordinates of P be $\left(\frac{3}{2} + \frac{3}{2}t^2, 3t\right)$

Therefore, coordinate of M are $(0, 3t)$

$\Rightarrow \quad MS = \sqrt{9 + 9t^2}$

$$MP = \frac{3}{2} + \frac{3}{2}t^2$$

$\therefore \quad 9 + 9t^2 = \left(\frac{3}{2} + \frac{3}{2}t^2\right)^2 = \frac{9}{4}(1 + t^2)^2$

$\therefore \quad 4 = 1 + t^2$

$\therefore \quad$ Length of side $= 6$

44. d. Equation of tangent to given parabola having slope m is

$$y = m(x + a) + \frac{a}{m}$$

or $\quad y = mx + am + \frac{a}{m}$

Comparing Eq. (i) with $y = mx + c$, we have

$$c = am + \frac{a}{m}$$

45. d. The coordinates of the focus of the parabola $y^2 = 4ax$ are $(a, 0)$. The line $y - x - a = 0$ pass through this point. Therefore, it is a focal chord of the parabola. Hence, the tangent intersect at right angle.

46. c.

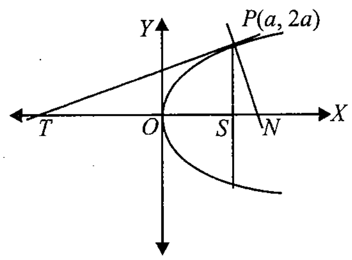


Fig. 3.90

One end of the latus rectum, $P(a, 2a)$.

The equation of the tangent PT at $P(a, 2a)$ is

$$2ya = 2a(x + a), \text{ i.e., } y = x + a$$

The equation of normal PN at $(a, 2a)$ is

$$y + x = 2a + a, \text{ i.e., } y + x = 3a$$

Solving $\quad y = 0$ and $y = x + a$, we get

$$x = -a, y = 0.$$

Solving $\quad y = 0, y + x = 3a$, we get

$$x = 3a, y = 0.$$

The area of the triangle with vertices $P(a, 2a)$, $T(-a, 0)$, $N(3a, 0)$ is $4a^2$.

47. a. Let $A \equiv (at^2, 2at)$, $B \equiv (at^2, -2at)$.

$$m_{OA} = \frac{2}{t}, m_{OB} = \frac{-2}{t}$$

Thus, $\quad \left(\frac{2}{t}\right)\left(\frac{-2}{t}\right) = -1$

$\Rightarrow \quad t^2 = 4.$

Thus, tangents will intersect at $(-4a, 0)$.

48. b. Clearly P is the point of intersection of two perpendicular tangents to the parabola $y^2 = 8x$.

Hence, P must lie on the directrix $x + a = 0$ or $x + 2 = 0$

$\therefore \quad x = -2.$

Hence, the point is $(-2, 0)$.

49. c.

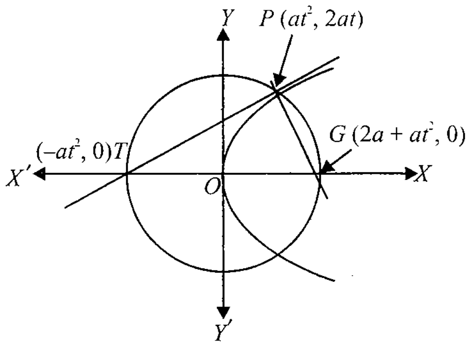


Fig. 3.91

Tangent and normal at $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ is

$$ty = x + at^2 \quad \text{(i)}$$

and $$y = -tx + 2at + at^3 \quad \text{(ii)}$$

Equations (i) and (ii) meet the x-axis where $y = 0$

From Eq. (i), $x = -at^2$

$\Rightarrow T$ is $(-at^2, 0)$

From Eq. (ii), $tx = 2at + at^3$

$\Rightarrow G$ is $(2a + at^2, 0)$

Midpoint of $TG = \left(\frac{2a + at^2 - at^2}{2}, 0\right)$

$= O\,(a, 0)$

Since $\angle TPG = 90°$, therefore centre of the circle through PTG is $(a, 0)$.

If θ is the angle between tangents at P to the parabola and circle through P, T, G, then $(90° - \theta)$ is the angle between PT and OP.

Slope of $PT = \frac{2at}{2at^2} = \frac{1}{t}$

Slope of $OP = \frac{2at}{a(t^2-1)} = \frac{2t}{t^2-1}$

$$\therefore \quad \tan(90° - \theta) = \left|\frac{\frac{1}{t} - \frac{2t}{t^2-1}}{1 + \frac{1}{t}\left(\frac{2t}{t^2-1}\right)}\right| = \frac{1}{t}$$

$$\therefore \quad \cot\theta = \frac{1}{t} \Rightarrow \tan\theta = t$$

$$\Rightarrow \quad \theta = \tan^{-1}(t)$$

50. a. $\frac{dy}{dx} = 2x - 5$

$$\therefore \quad m_1 = \left(\frac{dy}{dx}\right)_{(2,0)} = 4 - 5 = -1 \text{ and}$$

$$m_2 = \left(\frac{dy}{dx}\right)_{(3,0)} = 6 - 5 = 1$$

$\Rightarrow m_1 m_2 = -1 \Rightarrow$ angle between tangents $= \frac{\pi}{2}$

51. a.

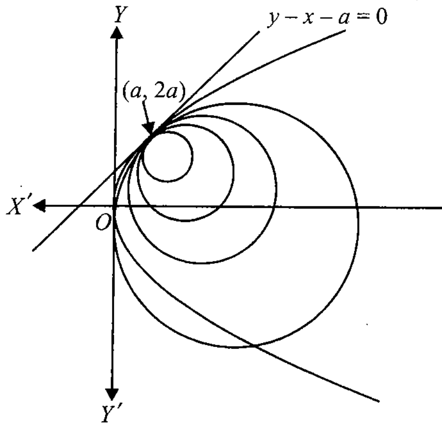


Fig. 3.92

Equation of tangent of parabola at $(a, 2a)$ is $2ya = 2a(x + a)$, i.e., $y - x - a = 0$.

Equation of circle touching the parabola at $(a, 2a)$ is

$(x - a)^2 + (y - 2a)^2 + \lambda(y - x - a) = 0$

It passes through $(0, 0)$

$$\Rightarrow \quad a^2 + 4a^2 + \lambda(-a) = 0 \Rightarrow \lambda = 5a$$

Thus, required circle is $x^2 + y^2 - 7ax - ay = 0$

it's radius is $\sqrt{\frac{49}{4}a^2 + \frac{a^2}{4}} = \frac{5}{\sqrt{2}}a$

52. c. The required point is obtained by solving $x + y = 1$ and $y^2 - y + x = 0$.

53. c. Any tangent to $y^2 = 4a(x + a)$ is

$$y = m(x + a) + \frac{a}{m} \quad \text{(i)}$$

Any tangent to $y^2 = 4b(x + b)$ which is perpendicular to Eq. (i) is

$$y = -\frac{1}{m}x + (x + b) - bm \quad \text{(ii)}$$

Subtracting, we get

$$\left(m + \frac{1}{m}\right)x + (a + b)\left(m + \frac{1}{m}\right) = 0$$

or $x + a + b = 0$ which is a locus of their point of intersection.

54. c. Any point on the given parabola is $(t^2, 2t)$.

The equation of the tangent at $(1, 2)$ is $x - y + 1 = 0$

The image (h, k) of the point $(t^2, 2t)$ in $x - y + 1 = 0$ is given by

$$\frac{h - t^2}{1} = \frac{k - 2t}{-1} = -\frac{2(t^2 - 2t + 1)}{1 + 1}$$

$$\therefore \quad h = t^2 - t^2 + 2t - 1 = 2t - 1$$

and $$k = 2t + t^2 - 2t + 1 = t^2 + 1$$

Eliminating t from $h = 2t - 1$ and $k = t^2 + 1$, we get

$$(h + 1)^2 = 4(k - 1)$$

The required equation of reflection is $(x + 1)^2 = 4(y - 1)$.

55. a.

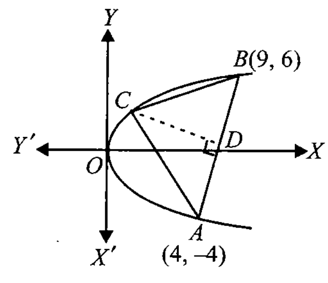


Fig. 3.93

Area of triangle ABC is maximum if CD is maximum, because AB is fixed.

That means tangent drawn to parabola at 'C' should be parallel to AB.

Slope of $\qquad AB = \frac{6+4}{9-4} = 2$

For $\qquad y^2 = 4x, \frac{dy}{dx} = \frac{2}{y} = 2$

$\Rightarrow \qquad y = 1$

$\Rightarrow \qquad x = \frac{1}{4}$

56. b. Let $y_1 = 5 + \sqrt{1 - x_1^2}$ and $y_2 = \sqrt{4x_2}$ or

$x_1^2 + (y_1 - 5)^2 = 1$ and $y_2^2 = 4x_2$

Thus (x_1, y_1) lies on the circle $x^2 + (y - 5)^2 = 1$ and (x_2, y_2) lies on the parabola $y^2 = 4x$.

Thus, given expression is the shortest distance between the curves $x^2 + (y - 5)^2 = 1$ and $y^2 = 4x$.

Now the shortest distance always occur along common normal to the curves and normal to circle passes through the centre of the circle.

Normal to the parabola $y^2 = 4x$ is $y = mx - 2m - m^3$ passes through (0, 5) gives

$m^3 + 2m + 5 = 0$, which has only one root $m = -2$.

Hence, corresponding point on the parabola is (4, 4).

Thus, required minimum distance $= \sqrt{4^2 + 8^2} - 1 = 4\sqrt{5} - 1$.

57. c. $SP_1 = a(1 + t_1^2)$; $SP_2 = a(1 + t_2^2)$

$\Rightarrow \qquad t_1 t_2 = -1$

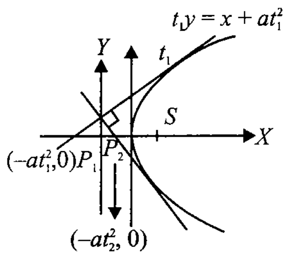


Fig. 3.94

$$\frac{1}{SP_1} = \frac{1}{a(1 + t_1^2)}$$

$$\frac{1}{SP_2} = \frac{1}{a(1 + t_2^2)} = \frac{t_1^2}{a(t_1^2 + 1)}$$

$$\therefore \quad \frac{1}{SP_1} + \frac{1}{SP_2} = \frac{1}{a}$$

58. b. Tangent at point P is $ty = x + t^2$, where slope of tangent is $\tan\theta = \frac{1}{t}$.

Now required area is $A = \frac{1}{2}(AN)(PN) = \frac{1}{2}(2t^2)(2t)$

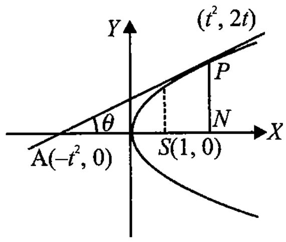


Fig. 3.95

$$A = 2t^3 = 2(t^2)^{3/2}$$

Now $t^2 \in [1, 4]$, then A_{max} occurs when $t^2 = 4$

$\Rightarrow \qquad A_{max} = 16$

59. d. $OT^2 = OA \cdot OB = \alpha\beta = \frac{c}{a} \Rightarrow OT = \sqrt{\frac{c}{a}}$

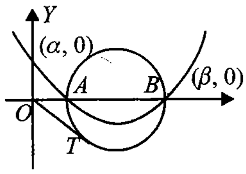


Fig. 3.96

60. b. Tangent at point P is $ty = x + at^2$. (i)

Line perpendicular to Eq. (i) passes through $(a, 0)$

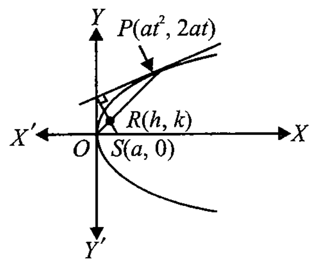


Fig. 3.97

$\therefore y - 0 = -t(x - a)$ or $tx + y = ta$ or $y = t(a - x)$ (ii)

Equation of OP

$$y - \frac{2}{t}x = 0 \text{ or } y = \frac{2}{t}x \quad \text{(iii)}$$

From Eqs. (ii) and (iii), eliminating t, we get

$$y^2 = 2x(a - x)$$

or $\qquad 2x^2 + y^2 - 2ax = 0$

61. a. Slope of tangent at P is $\frac{1}{t_1}$ and at Q is $\frac{1}{t_2}$.

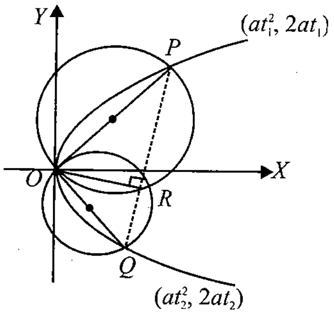


Fig. 3.98

$\Rightarrow \quad \cot\theta_1 = t_1$ and $\cot\theta_2 = t_2$

Slope of $\quad PQ = \dfrac{2}{t_1 + t_2}$

$\Rightarrow$ Slope of OR is $-\dfrac{t_1 + t_2}{2} = \tan\phi$

(Note angle in a semicircle is 90°)

$\Rightarrow \quad \tan\phi = -\frac{1}{2}(\cot\theta_1 + \cot\theta_2)$

$\Rightarrow \quad \cot\theta_1 + \cot\theta_2 = -2\tan\phi$

62. b.

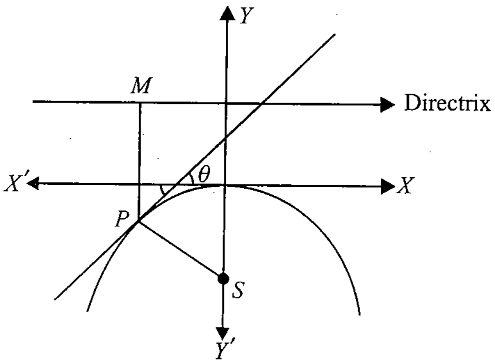


Fig. 3.99

Slope of line $\quad \lambda = \tan\theta$

$\Rightarrow \tan(\angle MPS) = \tan 2\left(\frac{\pi}{2} - \theta\right) = \tan(\pi - 2\theta) = -\tan 2\theta$

$= \dfrac{2\lambda}{\lambda^2 - 1}$

63. c. Solving $y = 2x - 3$ and $y^2 = 4a\left(x - \frac{1}{3}\right)$, we have

$$(2x-3)^2 = 4a\left(x - \frac{1}{3}\right)$$

$\Rightarrow \quad 4x^2 + 9 - 12x = 4ax - \dfrac{4a}{3}$

$\Rightarrow 4x^2 - 4(3 + a)x + 9 + \dfrac{4a}{3} = 0$

This equation must have equal roots $\Rightarrow D = 0$

$\Rightarrow 16(3 + a)^2 - 16\left(9 + \frac{4a}{3}\right) = 0$

$\Rightarrow \quad 9 + a^2 + 6a = 9 + \dfrac{4a}{3}$

$\Rightarrow \quad a^2 + \dfrac{14a}{3} = 0$

$\Rightarrow \quad a = 0$ or $a = \dfrac{14}{3}$

64. c.

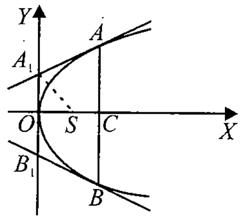


Fig. 3.100

Let $\quad A \equiv (at_1^2, 2at_1)$,

$B \equiv (at_1^2, -2at_1)$

Equation of tangents at A and B are,

$t_1 y = x + at_1^2$

and $\quad -t_1 y = x + at_1^2$, respectively.

These tangents meet y-axis at

$A_1 \equiv (0, at_1)$

and $\quad B_1 \equiv (0, -at_1)$.

Area of trapezium $\quad AA_1B_1B = \frac{1}{2}(AB + A_1B_1) \times OC$

$\Rightarrow \quad 24a^2 = \frac{1}{2}(4at_1 + 2at_1)(at_1^2)$

$\Rightarrow \quad t_1^3 = 8 \Rightarrow t_1 = 2$

$\Rightarrow \quad A_1 \equiv (0, 2a)$

If $\quad \angle OSA_1 = \theta \Rightarrow \tan\theta = \dfrac{2a}{a} = 2$

$\Rightarrow \quad \theta = \tan^{-1}(2)$

Thus, required angle is $2\tan^{-1}(2)$.

65. b. Here $a = 2$ for parabola and the two tangents pass through the points $(-2, -3)$, which lie on the directrix, then tangents are perpendicular or $m_1 m_2 = -1$.

66. b.

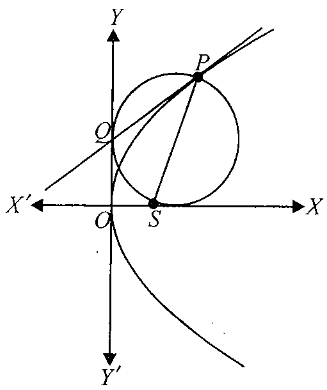


Fig. 3.101

Tangent at P intersects y-axis at $Q \equiv (0, at)$.

Also circle with PS as diameter touches the y-axis at $(0, at)$.

$\Rightarrow$ y-axis is the tangent to circumcircle of ΔPQS at Q.

67. d. Tangents $y = m_1x + c$ and $y = m_2x + c$ intersect at $(0, c)$ which lies on the directrix of the given parabola.

Hence, tangents are perpendicular for which, $m_1m_2 = -1$.

68. b. $y = ax^2 - 6x + b$ passes through $(0, 2)$

Here, $2 = a(0^2) - 6(0) + b$

$\therefore$ $b = 2$

Also, $\frac{dy}{dx} = 2ax - 6$

$\therefore$ $\left(\frac{dy}{dx}\right)_{x=\frac{3}{2}} = 2a\left(\frac{3}{2}\right) - 6$

$= 3a - 6 = 0$

$\therefore$ $a = 2$

69. b. Since tangents are perpendicular, they intersect on the directrix.

$\Rightarrow$ $(\lambda, 1)$ lies on the line $x = -4$

$\Rightarrow$ $\lambda = -4$

70. b. Parabolas $y = x^2 + 1$ and $x = y^2 + 1$ are symmetrical about $y = x$.

Therefore, tangent at point A is parallel to $y = x$.

$\Rightarrow$ $\frac{dy}{dx} = 2x \Rightarrow 2x = 1$

$\Rightarrow$ $x = \frac{1}{2}$ and $y = \frac{5}{4}$

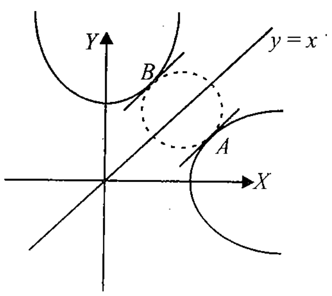


Fig. 3.102

$A\left(\frac{1}{2}, \frac{5}{4}\right)$ and $B\left(\frac{5}{4}, \frac{1}{2}\right)$

Therefore, radius $= \frac{1}{2}\sqrt{\left(\frac{1}{2} - \frac{5}{4}\right)^2 + \left(\frac{5}{4} - \frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{\frac{9}{16} + \frac{9}{16}}$

$= \frac{3}{8}\sqrt{2}$

Therefore, area $= \frac{9\pi}{32}$

71. b.

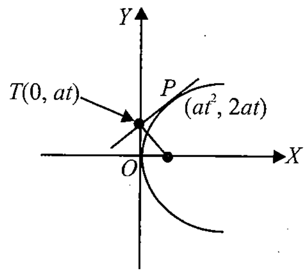


Fig. 3.103

Let middle point of P and T be (h, k)

$\therefore$ $2h = at^2$

and $2k = 3at$

$\therefore$ $2h = a \cdot \frac{4k^2}{9a^2}$

Locus of (h, k) is $2y^2 = 9ax$

As $a = 2 \therefore y^2 = 9x$

72. c. Tangent to parabola $y^2 = 4x$ having slope m is

$$y = mx + \frac{1}{m}$$

Tangent to circle $(x-1)^2 + (y+2)^2 = 16$ having slope m is

$$(y + 2) = m(x - 1) + 4\sqrt{1 + m^2}$$

Distance between tangents

$$= \left|\frac{4\sqrt{1+m^2} - m - 2 - 1/m}{\sqrt{1+m^2}}\right|$$

$$= \left|4 - \frac{2}{\sqrt{1+m^2}} - \frac{\sqrt{m^2+1}}{m}\right|$$

As $m > 0 \Rightarrow d < 4$

73. d.

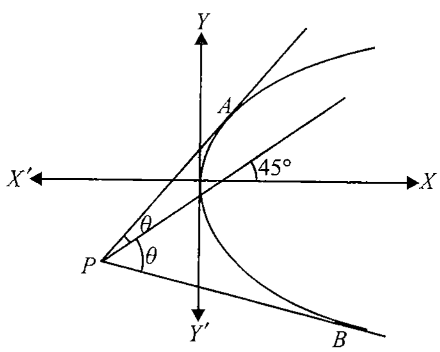


Fig. 3.104

Here $\frac{1}{t_1} = \tan\left(\frac{\pi}{4} + \theta\right)$

and $\quad \frac{1}{t_2} = \tan\left(\frac{\pi}{4} - \theta\right)$

So, $\quad t_1 t_2 = 1$

$\Rightarrow$ the x-coordinate of $\quad P = at_1t_2 = a.$

74. a.

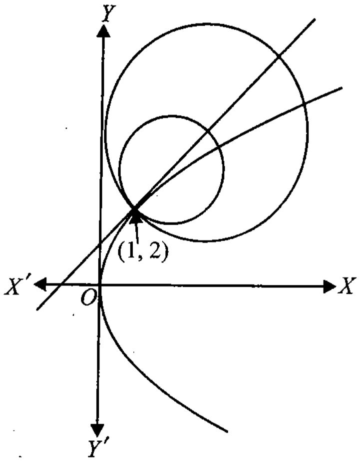


Fig. 3.105

Tangent to parabola $y^2 = 4x$ at (1, 2) will be the locus

i.e., $\quad 2y = 2(x + 1)$

$\Rightarrow \quad y = x + 1$

75. c.

Circle S_2, taking focal chord AB as diameter will touch directrix at point P and circle S_1, taking AP as diameter will pass through focus S (since AP subtends angle 90° at focus of parabola).

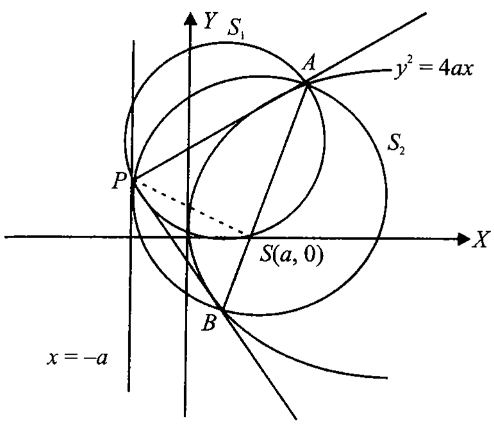


Fig. 3.106

Hence, common chord of given circles is line AP (which is intercept of tangent at point 'A' between point A and directrix).

76. a. Since the normal at $(ap^2, 2ap)$ to $y^2 = 4ax$ meets the parabola at $(aq^2, 2aq)$,

$$\therefore \quad q = -p - \frac{2}{p} \qquad \text{(i)}$$

Since $\quad OP \perp OQ,$

$$\therefore \quad \frac{2ap - 0}{ap^2 - 0} \times \frac{2aq - 0}{aq^2 - 0} = -1 \Rightarrow pq = -4.$$

$$\Rightarrow \quad p\left(-p - \frac{2}{p}\right) = -4 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad p^2 = 2$$

77. d. Given parabola is $y^2 = 4x + 8$ or $y^2 = 4(x + 2)$.

Equation of normal to parabola at any point $P(t)$ is $y = -t(x + 2) + 2t + t^3$.

It passes through $(k, 0)$ if $tk = t^3 \Rightarrow t(t^2 - k) = 0$

Hence, it has three real values of t if $k > 0$.

78. c. Equations of tangent and normal at $P(at^2, 2at)$ are $ty = x + at^2$ and $y = -tx + 2at + at^3$, respectively.

Thus, $\quad T \equiv (-at^2, 0),$

$\quad N \equiv (2a + at^2, 0)$

Also, $\quad S \equiv (a, 0)$

Hence, $\quad SP = a + at^2, ST = a + at^2$

and $\quad SN = a + at^2$

Thus, $\quad SP = ST = SN$

79. b. Let AB be a normal chord where $A \equiv (at_1^2, 2at_1)$, $B \equiv (at_2^2, 2at_2)$. If it's midpoint is $P(h, k)$, then

$$2h = a\left(t_1^2 + t_2^2\right)$$

$$= a\left[(t_1 + t_2)^2 - 2t_1t_2\right]$$

and $\quad 2k = 2a(t_1 + t_2)$

We also have

$$t_2 = -t_1 - \frac{2}{t_1}$$

$$\Rightarrow \quad t_1 + t_2 = \frac{-2}{t_1} \text{ and } t_1t_2 = -t_1^2 - 2$$

$$\Rightarrow \quad t_1 = -\frac{2a}{k} \text{ and } h = a\left(t_1^2 + 2 + \frac{2}{t_1^2}\right)$$

Thus, required locus is $\quad x = a\left(\frac{4a^2}{y^2} + 2 + \frac{y^2}{2a^2}\right).$

80. c.

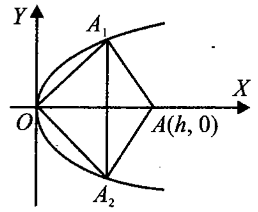


Fig. 3.107

Let $A_1 \equiv (2at_1^2, 4t_1), A_2 \equiv (2t_1^2, -4t_1)$

Clearly, $\angle A_1OA = \frac{\pi}{6}$

$\Rightarrow \frac{2}{t_1} = \frac{1}{\sqrt{3}}$

$\Rightarrow t_1 = 2\sqrt{3}$

Equation of normal at A_1 is $y = -t_1x + 4t_1 + 2t_1^3$

$\Rightarrow h = 4 + 2t_1^2 = 4 + 2\,(12) = 28$

81. c. $y = mx + c$ is a normal to $y^2 = 4ax$ if $c = -2am - am^3$, $y = -2x - \lambda$

$\Rightarrow m = -2, a = -2$

$\Rightarrow -\lambda = -2am - am^3 = -2(-2)(-2) - (-2)(-2)^3 = -24$

$\Rightarrow \lambda = 24$

82. d. We have $a = 1$

Normal at $(m^2, -2m)$ is $y = mx - 2m - m^3$

Given that normal makes equal angle with axes, then its slope $m = \pm 1$

Therefore, point P is $(m^2, -2m) = (1, \pm 2)$.

83. c. Let $A = (\alpha, \beta)$

The normal at $(at^2, 2at)$ is $y = -tx + 2at + at^3$

$\therefore \quad at^3 + (2a - \alpha)t - \beta = 0 \quad$ (i)

Let t_1, t_2, t_3 be roots of Eq. (i), then

$at^3 + (2a - \alpha)t - \beta = a(t - t_1)(t - t_2)(t - t_3) \quad$ (ii)

Let $P = (at_1^2, 2at_1)$, $Q = (at_2^2, 2at_2)$, and $R = (at_3^2, 2at_3)$.

Since the focus S is $(a, 0)$

$\therefore \quad SP = a(t_1^2 + 1)$

Similarly, $SQ = a(t_2^2 + 1)$,

and $SR = a(t_3^2 + 1)$

Put $t = i = \sqrt{-1}$ in Eq. (ii), we have

$-ai + (2a - \alpha)\,i - \beta = a(i - t_1)(i - t_2)(i - t_3)$

$\Rightarrow |(a - \alpha)\,i - \beta| = a|(i - t_1)(i - t_2)(i - t_3)|$

$\Rightarrow \sqrt{(a - \alpha)^2 + \beta^2} = a\sqrt{1 + t_1^2}\,\sqrt{1 + t_2^2}\,\sqrt{1 + t_3^2}$

$\Rightarrow a\sqrt{(a - \alpha)^2 + \beta^2} = \sqrt{a + at_1^2}\,\sqrt{a + at_2^2}\,\sqrt{a + at_3^2}$

$\Rightarrow aSA^2 = SP \cdot SQ \cdot SR$

84. d. Ends of latus rectum are $P(a, 2a)$ and $P'(a, -2a)$.

Point P has parameter $t_1 = 1$ and point P' has parameter $t_2 = -1$.

Normal at point P meets the curve again at point Q whose parameter $t_1' = -t_1 - \frac{2}{t_1} = -3$.

Normal at point P' meets the curve again at point Q' whose parameter $t_2' = -t_2 - \frac{2}{t_2} = 3$.

Hence, point Q and Q' have coordinates $(9a, -6a)$ and $(9a, 6a)$, respectively.

Hence, $QQ' = 12a$

85. d. Point $(\sin\theta, \cos\theta)$ lies on the circle $x^2 + y^2 = 1$ for $\forall\, \theta \in R$.

Now three normals can be drawn to the parabola $y^2 = 4ax$ if $x = |2a|$ meets this circle.

Hence, we must have $\cos\theta > |2a|$.

$\Rightarrow 0 < |2a| < 1$

$\Rightarrow 0 < |a| < \frac{1}{2}$

$\Rightarrow a \in \left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$

86. c. Normal at point $P(t_1)$ meets the parabola again at point $R(t_3)$, then

$$t_3 = -t_1 - \frac{2}{t_1}$$

Also normal at point $Q(t_2)$ meets the parabola at the same point $R(t_3)$, then

$$t_3 = -t_2 - \frac{2}{t_2}$$

Comparing these values of t_3, we have

$$-t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2} \text{ or } t_1t_2 = 2$$

87. b. Normal at point $(t^2, 2t)$ is $y = -tx + 2t + t^3$

Slope of the tangent is 1.

Hence, $-t = 1 \Rightarrow t = -1$

$\Rightarrow$ Coordinates of P are $(1, -2)$.

Hence, parameter at Q is $t_2 = -t_1 - 2/t_1 = 1 + 2$
$= 3$

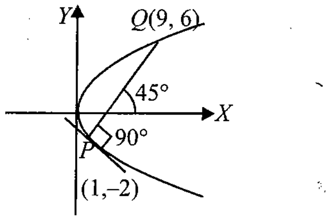


Fig. 3.108

Therefore, coordinates at Q are (9, 6).

$\therefore \quad l(PQ) = \sqrt{64+64} = 8\sqrt{2}$

88. b. $\tan\alpha = -t_1$ and $\tan\beta = -t_2$

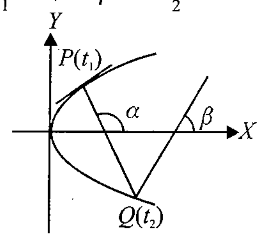


Fig. 3.109

also $\quad t_2 = -t_1 - \dfrac{2}{t_1}$

$$t_1t_2 + t_1^2 = -2$$

$$\tan\alpha\tan\beta + \tan^2\alpha = -2$$

89. b. Slope of normal at point $P(t_1)$ and $Q(t_2)$ is $-t_1$ and $-t_2$, respectively.

Equation of chord joining $P(t_1)$ and $Q(t_2)$ is

$$y - 2at_1 = \frac{2}{t_1+t_2}\left(x - at_1^2\right)$$

or $\quad 2x - (t_1+t_2)y + 2at_1t_2 = 0$

But $\quad t_1t_2 = -1$

Chord PQ is $2x - (t_1+t_2)y - 2a = 0$

or $\quad (2x - 2a) - (t_1+t_2)y = 0$

which passes through the fixed point $(a, 0)$.

90. c. $t_2 = -t_1 - \dfrac{2}{t_1} \Rightarrow t_1t_2 = -t_1^2 - 2$

Equation of the line through P parallel to AQ

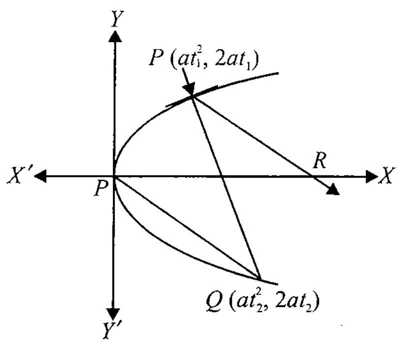


Fig. 3.110

$$y - 2at_1 = \frac{2}{t_2}\left(x - at_1^2\right)$$

Put $\quad y = 0 \Rightarrow x = at_1^2 - at_1t_2$

$$= at_1^2 - a\left(-2 - t_1^2\right)$$

$$= 2a + 2at_1^2$$

$$= 2(a + at_1^2)$$

= twice the focal distance of P.

91. b. Equations of tangent and normal at A are $yt = x + at^2$ and $y = -tx + 2at + at^3$

$\Rightarrow B \equiv (-at^2, 0), D \equiv (2a + at^2, 0)$. If $ABCD$ is a rectangle, then midpoints of BD and AC will be coincident.

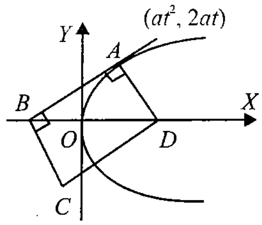


Fig. 3.111

$\Rightarrow \quad h + at^2 = 2a + at^2 - at^2, k + 2at = 0$

$\Rightarrow \quad h = 2a, t = -\dfrac{k}{2a}$

92. d. $4y = x^2 - 8$

$$4\frac{dy}{dx} = 2x$$

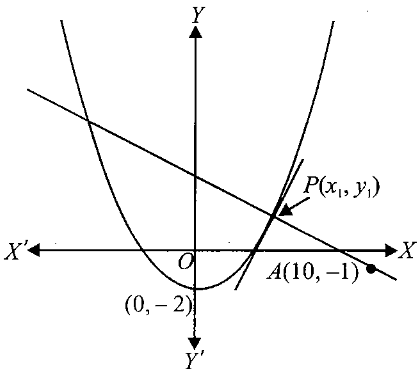


Fig. 3. 112

Therefore, slope of normal $= -\dfrac{2}{x_1}$; but slope of normal

$$= \frac{y_1 + 1}{x_1 - 10}$$

$$\therefore \quad \frac{y_1+1}{x_1-10} = -\frac{2}{x_1}$$

$\Rightarrow \quad x_1y_1 + x_1 = -2x_1 + 20$

$\Rightarrow \quad x_1y_1 + 3x_1 = 20$

Substituting $y_1 = \frac{x_1^2 - 8}{4}$

(from the given equation)

$$x_1\left(\frac{x_1^2-8}{4}+3\right) = 20$$

$\Rightarrow$ $x_1(x_1^2+4) = 80$

$\Rightarrow$ $x_1^3 + 4x_1 - 80 = 0,$

which has one root $x_1 = 4$

Hence, $x_1 = 4; y_1 = 2$

$\therefore$ $P = (4, 2)$

Therefore, equation of PA is

$$y + 1 = -\frac{1}{2}(x - 10)$$

$\Rightarrow$ $2y + 2 = -x + 10$

$\Rightarrow$ $x + 2y - 8 = 0$

93. a. A circle through three co-normal points of a parabola always passes through the vertex of the parabola. Hence, the circle through P, Q, R, S out of which P, Q, R are co-normals points will always pass through vertex (2, 3) of parabola.

94. c. Normal at point $P(x_1, y_1) \equiv (at_1^2, 2at_1)$ meets the parabola at $R(at^2, 2at)$

$\Rightarrow$ $t = -t_1 - \frac{2}{t_1}$ (i)

Normal at point $Q(x_2, y_2) \equiv (at_2^2, 2at_2)$ meets the parabola at $R(at^2, 2at)$

$\Rightarrow$ $t = -t_2 - \frac{2}{t_2}$ (ii)

From Eqs. (i) and (ii)

$$-t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2}$$

$\Rightarrow$ $t_1t_2 = 2$

Now given that $x_1 + x_2 = 4$

$\Rightarrow$ $t_1^2 + t_2^2 = 4$

$\Rightarrow$ $(t_1 + t_2)^2 = 4 + 4 = 8$

$\Rightarrow$ $|t_1 + t_2| = 2\sqrt{2}$

$\Rightarrow$ $|y_1 + y_2| = 4\sqrt{2}$

95. b. Let the concyclic points be t_1, t_2, t_3 and t_4

$\Rightarrow$ $t_1 + t_2 + t_3 + t_4 = 0$

Here, t_1 and t_3 are feet of the normals.

$\Rightarrow$ $t_2 = -t_1 - \frac{2}{t_1}$ and $t_4 = -t_3 - \frac{2}{t_3}$

$\Rightarrow$ $t_1 + t_2 = -\frac{2}{t_1}$ and $t_4 + t_3 = -\frac{2}{t_3}$

Adding,

$$-2\left(\frac{1}{t_1} + \frac{1}{t_3}\right) = 0$$

$\Rightarrow$ $t_1 + t_3 = 0$

$\Rightarrow$ Point of intersection of tangents at t_1 and t_3 $(at_1t_3, a(t_1 + t_3)) \equiv (at_1t_3, 0)$.

$\Rightarrow$ This point lies on the axis of the parabola.

96. c. Axis of the parabola is $x = 1$. Any point on it is $(1, k)$. Now distance of $(1, k)$ from $(1, -2)$ should be more than the semi-latus rectum and $(1, k)$ should be inside the parabola, hence $k > 2$.

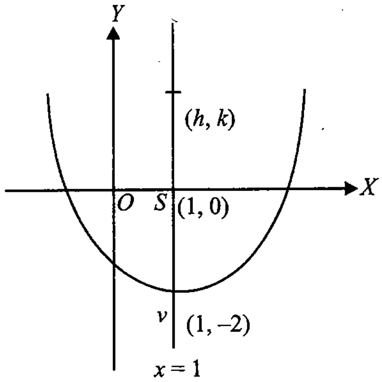


Fig. 3.113

97. b.

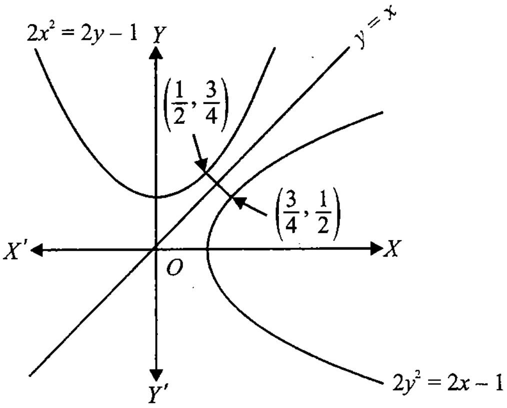


Fig. 3.114

Given parabolas $2y^2 = 2x - 1$, $2x^2 = 2y - 1$ are symmetrical about the line $y = x$.

Also shortest distance occurs along the common normal which perpendicular to the line $y = x$.

Differentiating $2y^2 = 2x - 1$ w.r.t. x, we have

$$2y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y} = 1 \Rightarrow y = \frac{1}{2}$$

Hence, points are as shown in the figure.

Then, the shortest distance, $d = \sqrt{\frac{1}{16} + \frac{1}{16}} = \frac{1}{2\sqrt{2}}$

98. d.

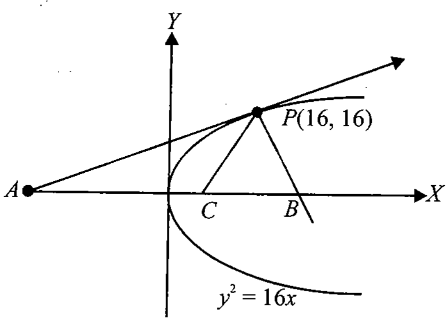


Fig. 3.115

By property centre of circle coincides with focus of parabola

$\Rightarrow$ $C \equiv (4, 0)$

$$\tan\alpha = \text{slope of } PC = \frac{16}{12}$$

$\Rightarrow$ $\alpha = \tan^{-1}\left(\frac{4}{3}\right)$

99. a. Let AB be a normal chord where $A \equiv (at^2, 2at)$, $B \equiv (at_1^2, 2at_1)$.

We have $t_1 = -t - \frac{2}{t}$.

Now,

$$AB^2 = [a^2(t^2 - t_1^2)]^2 + 4a^2(t - t_1)^2$$

$$= a^2(t - t_1)^2 [(t + t_1)^2 + 4]$$

$$= a^2\left(t + t + \frac{2}{t}\right)^2\left(\frac{4}{t^2} + 4\right)$$

$$= \frac{16a^2(1 + t^2)^3}{t^4}$$

$$\Rightarrow \frac{d(AB)^2}{dt} = 16a^2\left(\frac{t^4[3(1 + t^2)^2 . 2t] - (1 + t^2)^3 . 4t^3}{t^8}\right)$$

$$= 32a^2(1 + t^2)^2\left(\frac{3t^2 - 2 - 2t^2}{t^5}\right)$$

$$= \frac{a^2 \times 32(1 + t^2)^2}{t^5}(t^2 - 2)$$

For $\frac{d(AB^2)}{dt} = 0 \Rightarrow t = \sqrt{2}$ for which AB^2 is minimum.

Thus, $AB_{min} = \frac{4a}{2}(1 + 2)^{3/2} = 2a\sqrt{27}$ units

100. c. The equation of any normal be $y = -tx + 2t + t^3$

Since it passes through the points (15, 12)

$\therefore$ $12 = -15t + 2t + t^3$

$\Rightarrow$ $t^3 - 13t - 12 = 0$

One root is -1, then

$$(t + 1)(t^2 + t - 12) = 0$$

$\Rightarrow$ $t = 1, 3, 4$

Therefore, the co-normal points are (1, – 2), (9, – 6), (16, 8).

Therefore, centroid is $\left(\frac{26}{3}, 0\right)$.

101. d. For a focal chord $t_1 t_2 = -1$ and for the normal $t_1(t_1 + t_2) + 2 = 0$.

$\therefore$ $t_1^2 + t_1 t_2 + 2 = 0 \Rightarrow t_1^2 = -1$

Therefore, t_1 is imaginary.

102. b. Solving the line $y = x - 1$ and parabola $y^2 = 4x$, we have

$$(x - 1)^2 = x$$

$\Rightarrow$ $x^2 - 6x + 1 = 0$

$\Rightarrow$ $x = 3 \pm \sqrt{8}$

$\therefore$ $y = 2 \pm \sqrt{8}$

Suppose point D is (x_3, y_3), then

$$y_1 + y_2 + y_3 = 0$$

$\Rightarrow$ $2 + \sqrt{8} + 2 - \sqrt{8} + y_3 = 0$

$\Rightarrow$ $y_3 - 4$, then $x_3 = 4$

Therefore, the point is (4, 4).

103. c. Equation of normal $y = mx - 2am - am^3$

Put $y = 0$, we get

$$x_1 = 2a + am_1^2$$
$$x_2 = 2a + am_2^2$$
$$x_3 = 2a + am_3^2$$

where x_1, x_2, x_3 are the intercepts on the axis of the parabola.

The normal passes through (h, k).

$\Rightarrow$ $am^3 + (2a - h)m + k = 0$,

which has roots m_1, m_2, m_3 which are slopes of the normals.

$\Rightarrow$ $m_1 + m_2 + m_3 = 0$

and $m_1 m_2 + m_2 m_3 + m_3 m_1 = \frac{2a - h}{a}$

$\Rightarrow$ $m_1^2 + m_2^2 + m_3^2 = (m_1 + m_2 + m_3)^2 - 2(m_1 m_2 + m_2 m_3 + m_3 m_1)$

$$= -\frac{2(2a - h)}{a}$$

$\Rightarrow$ $x_1 + x_2 + x_3 = 6a - 2(2a - h) = 2(h + a)$

104.c.

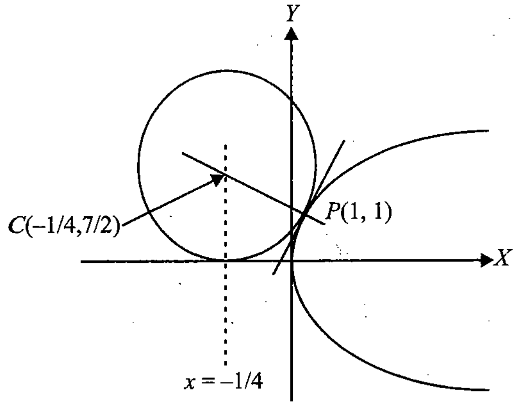


Fig. 3.116

Equation of normal at $P(1, 1)$ is

$$y - 1 = -2(x - 1)$$

or $\quad y + 2x = 3 \quad$ (i)

Directrix of the parabola $y^2 = x$ is

$$x = -\frac{1}{4}. \quad \text{(ii)}$$

Centre of the circle is intersection of two normals to the circle, i.e., Eqs. (i) and (ii) which is $\left(-\frac{1}{4}, \frac{7}{2}\right)$.

Hence, radius of the circle is

$$\sqrt{\left(1+\frac{1}{4}\right)^2 + \left(1-\frac{7}{2}\right)^2} = \sqrt{\frac{25}{16} + \frac{25}{4}} = \frac{5\sqrt{5}}{4}$$

105.d. Let $y = mx + c$, intersect $y^2 = 4ax$ at $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$.

Then, $\quad \frac{2}{t_1 + t_2} = m$

$\Rightarrow \quad t_1 + t_2 = \frac{2}{m}$

Let the foot of another normal be $C(at_3^2, 2at_3)$.

Then,

$$t_1 + t_2 + t_3 = 0$$

$\Rightarrow \quad t_3 = -(t_1 + t_2) = -\frac{2}{m}$

Thus, other foot is $\left(\frac{4a}{m^2}, \frac{-4a}{m}\right)$.

106.b. Tangent to $y^2 = 4x$ in terms of 'm' is

$$y = mx + \frac{1}{m}$$

Normal to $\quad x^2 = 4by$ in terms of 'm' is

$$y = mx + 2b + \frac{b}{m^2}$$

If these are same lines, then

$$\frac{1}{m} = 2b + \frac{b}{m^2}$$

$\Rightarrow \quad 2bm^2 - m + b = 0$

For two different tangents

$$1 - 8b^2 > 0$$

$\Rightarrow \quad |b| < \frac{1}{\sqrt{8}}$

107.d. Normals to $y^2 = 4ax$ and $x^2 = 4by$ in terms of 'm' are

$$y = mx - 2am - am^3$$

and $\quad y = mx + 2b + \frac{b}{m^2}$.

For a common normal,

$$2b + b/m^2 + 2am + am^3 = 0$$

$\Rightarrow \quad am^5 + 2am^3 + 2bm^2 + b = 0$

This means there can be most '5' common normals.

108. a.

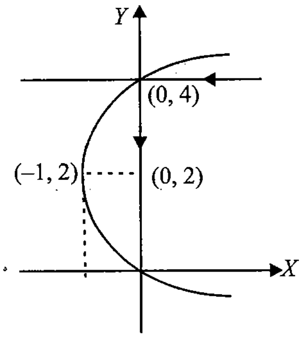


Fig. 3.117

Given curve is $\quad (y - 2)^2 = 4(x + 1)$

Focus is (0, 2).

Point of intersection of the curve and $y = 4$ is (0, 4).

From the reflection property of parabola, reflected ray passes through the focus.

$\therefore \quad x = 0$ is required line.

109.d. Solving the equations,

$$x^2 + 4(x + 4) = a^2$$

If circle and parabola touch each other, then

$$D = 0$$

$\Rightarrow \quad 16 - 4(16 - a^2) = 0$

$\Rightarrow \quad a = 2\sqrt{3}$

110.c.

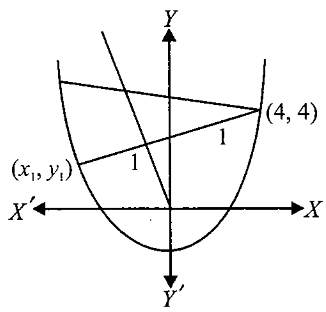


Fig. 3.118

Point (4, 4) lies on the parabola.

Let the point of intersection of the line $y = mx$ with the chords be $(\alpha, m\alpha)$, then

$$\alpha = \frac{4+x_1}{2}$$

$$\Rightarrow \quad x_1 = 2\alpha - 4$$

and

$$m\alpha = \frac{4+y_1}{2}$$

$$\Rightarrow \quad y_1 = 2m\alpha - 4$$

as (x_1, y_1) lies on the curve

$$\therefore \quad (2\alpha-4)^2 = 4(2m\alpha-4)$$

$$\Rightarrow \quad 4\alpha^2 + 16 - 16\alpha = 8(m\alpha - 2)$$

$$\Rightarrow \quad 4\alpha^2 - 8\alpha(2+m) + 32 = 0$$

For two distinct chords

$$\therefore \quad D > 0$$

$$(8(2+m))^2 - 4(4)(32) > 0$$

$$\Rightarrow \quad (2+m)^2 - 8 > 0$$

$$2 + m > 2\sqrt{2}$$

or $\quad 2 + m < -2\sqrt{2}$

$$\Rightarrow \quad m > 2\sqrt{2} - 2$$

or $\quad m < -2\sqrt{2} - 2$

111. c. Let point of intersection be (α, β).

Therefore, chord of contact w.r.t. this point is

$$\beta y = 2x + 2\alpha$$

which is same as $\quad x + y = 2.$

$$\Rightarrow \quad \alpha = \beta = -2$$

which satisfy $\quad y - x = 0.$

112. d. The given parabolas are symmetrical about the line $y = x$ as shown in the figure.

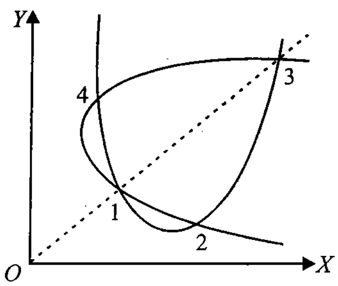


Fig. 3.119

They intersect to each other at four distinct points.

Hence, the number of common chords $= {}^4C_2 = \frac{4\times3}{2} = 6$

113. a. Let the focus be F. The parabolas are open down and open right, respectively. Let the parabolas intersect at points P and Q. From P perpendiculars are drawn on the x-axis and y-axis at A and B, respectively, then

$$PA = PF = PB$$

$\Rightarrow \quad P$ lies on the line $y = -x$.

Similarly, Q lies on the line $y = -x$

$\Rightarrow$ slope of $PQ = -1$

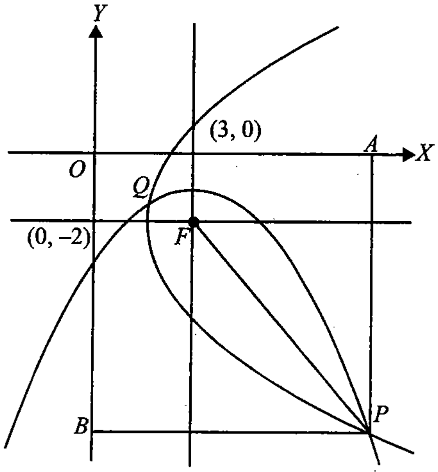


Fig. 3.120

114. b.

Equation of tangent to the parabola $y^2 = 8x$ at $P(2, 4)$ is

$$4y = 4(x+2)$$

or $\quad x - y + 2 = 0 \quad$ (i)

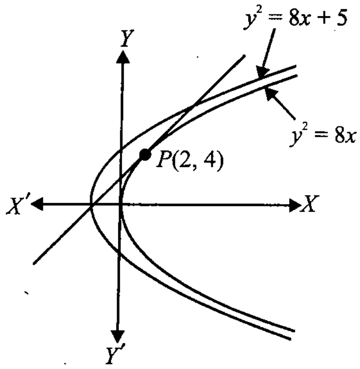


Fig. 3.121

Equation of chord of parabola $y^2 = 8x + 5$ whose middle point is (h, k) is $T = S_1$.

i.e., $\quad ky - 4(x+h) - 5 = k^2 - 8h - 5$

or $\quad 4x - ky + k^2 - 4h = 0 \quad$ (ii)

Equations (i) and (ii) must be identical

$$\therefore \quad \frac{4}{1} = \frac{k}{1} = \frac{k^2-4h}{2}$$

By comparing Eqs. (i), (ii)

$$k = 4$$

and $\quad 8 = k^2 - 4h$

Hence, the required point is (2, 4).

Multiple Correct Answers Type

1. a., c. The line $y = 2x + c$ is a tangent to $x^2 + y^2 = 5$.

If $c^2 = 25$

$\Rightarrow$ $c = \pm 5$

Let the equation of parabola be $y^2 = 4ax$. Then

$$\frac{a}{2} = \pm 5$$

$\Rightarrow$ $a = \pm 10$

$\Rightarrow$ Equation of the parabola is $y^2 = \pm 40x$.

$\Rightarrow$ Equations of the directrix are $x = \pm 10$.

2. b., c., d. Let $(x_1, y_1) \equiv (at^2, 2at)$

Tangent at this point is $ty = x + at^2$.

Any point on this tangent is $\left(h, \left(\frac{h + at^2}{t}\right)\right)$.

Chord of contact of this point with respect to the circle $x^2 + y^2 = a^2$ is

$$hx + \left(\frac{h + at^2}{t}\right)y = a^2$$

or $(aty - a^2) + h\left(x + \frac{y}{t}\right) = 0$

which is a family of straight lines passing through point of intersection of

$$ty - a = 0 \text{ and } x + \frac{y}{t} = 0$$

So, the fixed point is $\left(-\frac{a}{t^2}, \frac{a}{t}\right)$.

$\therefore$ $x_2 = -\frac{a}{t^2}, y_2 = \frac{a}{t}$

Clearly, $x_1x_2 = -a^2, y_1y_2 = 2a^2$

Also, $\frac{x_1}{x_2} = -t^4$

$$\frac{y_1}{y_2} = 2t^2$$

$\Rightarrow$ $4\frac{x_1}{x_2} + \left(\frac{y_1}{y_2}\right)^2 = 0$

3. b., d. Given parabola is

$$x^2 - ky + 3 = 0$$

or $x^2 = k\left(y - \frac{3}{k}\right)$

Le $x = Y, y - \frac{3}{k} = X$

then the parabola is

$$Y^2 = kX$$

whose focus is $\left(0, \frac{k}{4}\right)$.

Therefore, the focus of $x^2 = k\left(y - \frac{3}{k}\right)$ is

$$\left(0, \frac{3}{k} + \frac{k}{4}\right) \equiv (0, 2)$$

$\therefore$ $\frac{3}{k} + \frac{k}{4} = 2$

$\Rightarrow$ $12 + k^2 = 8k$

$\Rightarrow$ $k^2 - 8k + 12 = 0$

$\Rightarrow$ $(k - 6)(k - 2) = 0$

$\Rightarrow$ $k = 2, 6$

4. a., c. $P = (\alpha, \alpha + 1)$ where $\alpha \neq 0, -1$

or $P = (\alpha, \alpha - 1)$ where $\alpha \neq 0, 1$

$(\alpha, \alpha + 1)$ is on $y^2 = 4x + 1$

$\Rightarrow$ $(\alpha + 1)^2 = 4\alpha + 1$

$\Rightarrow$ $\alpha^2 - 2\alpha = 0$

$\Rightarrow$ $\alpha = 2$ $(\because \alpha \neq 0)$

Therefore, ordinate of P is 3

$(\alpha, \alpha - 1)$ is on $y^2 = 4x + 1$

$\Rightarrow$ $(\alpha - 1)^2 = 4\alpha + 1$

$\Rightarrow$ $\alpha^2 - 6\alpha = 0$

$\Rightarrow$ $\alpha = 6$ $(\because \alpha \neq 0)$

Therefore, ordinate of P is 5

5. a., d. Here $x^2 = -\lambda\left(y + \frac{\mu}{\lambda}\right)$

Therefore, vertex $= \left(0, -\frac{\mu}{\lambda}\right)$

And the directrix is

$$\left(y + \frac{\mu}{\lambda}\right) + \frac{-\lambda}{4} = 0.$$

Comparing with the given data, $-\frac{\mu}{\lambda} = 1$ and $\frac{\mu}{\lambda} - \frac{\lambda}{4} = -2$

$\therefore$ $-1 - \frac{\lambda}{4} = -2$

or $\lambda = 4 \Rightarrow \mu = 4$.

6. a., c. Given that the extremities of the latus rectum are (1, 1) and (1, −1)

$\Rightarrow$ $4a = 2 \Rightarrow a = \frac{1}{2}$

$\Rightarrow$ The focus of the parabola is (1, 0).

$\Rightarrow$ The vertex can be $\left(\frac{1}{2}, 0\right)$ and $\left(\frac{3}{2}, 0\right)$.

$\Rightarrow$ The equations of the parabola can be

$$y^2 = 2\left(x - \frac{1}{2}\right)$$

or $y^2 = 2\left(x - \frac{3}{2}\right)$

$\Rightarrow$ $y^2 = 2x - 1$

or $y^2 = 2x - 3$

7. a., c. Let the possible point be $(t^2, 2t)$. Equation of tangent at this point is

$$yt = x + t^2.$$

It must pass through (6, 5). (Since normal to circle always passes through its centre)

$$\Rightarrow \quad t^2 - 5t + 6 = 0$$

$$\Rightarrow \quad t = 2, 3$$

$\Rightarrow$ Possible points are (4, 4), (9, 6).

8. c., d.

$$t_2 = -t_1 - \frac{2}{t_1}$$

Also, $$\frac{2at_1}{at_1^2} \times \frac{2at_2}{at_2^2} = -1$$

$$\Rightarrow \quad t_1 t_2 = -4$$

$$\therefore \quad \frac{-4}{t_1} = -t_1 - \frac{2}{t_1}$$

$$\Rightarrow \quad t_1^2 + 2 = 4 \text{ and } t_1 = \pm\sqrt{2}$$

So point can be $(2a, \pm 2\sqrt{2}a)$.

9. a., b. As a circle can intersect a parabola in four points, so quadrilateral may be cyclic.

The diagonals of the quadrilateral may be equal as the quadrilateral may be an isosceles trapezium.

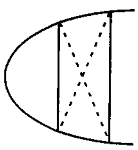


Fig. 3.122

A rectangle cannot be inscribed in a parabola. So (**C**) is wrong.

10. a., b., c., d. Any point on the parabola is $P(at^2, 2at)$.

Therefore, midpoint of $S(a, 0)$ and $P(at^2, 2at)$ is

$$R\left(\frac{a + at^2}{2}, at\right) \equiv (h, k).$$

$$\therefore \quad h = \frac{a + at^2}{2}, \; k = at$$

Eliminate 't'

i.e., $$2x = a\left(1 + \frac{y^2}{a^2}\right) = a + \frac{y^2}{a}$$

i.e., $$2ax = a^2 + y^2$$

i.e., $$y^2 = 2a\left(x - \frac{a}{2}\right)$$

It's a parabola with vertex at $\left(\frac{a}{2}, 0\right)$, latus rectum = $2a$

Directrix is

$$x - \frac{a}{2} = -\frac{a}{2},$$

i.e., $$x = 0$$

Focus is

$$x - \frac{a}{2} = \frac{a}{2},$$

i.e., $$x = a$$

i.e., $(a, 0)$

11. a., b., c. Equation of tangent to parabola $y^2 = 8x$ having slope m is $y = mx + \frac{2}{m}$

Options (a), (b), (c) are tangents for $m = 1, 3, -\frac{1}{2}$ respectively.

12. a., c., d. Equation of normal to parabola $y^2 = 12x$ having slope m is $y = mx - 6m - 3m^3$. Options (a), (c), (d) are normal for $m - 1, -2$ and 3 respectively.

13. a., b., c., d.

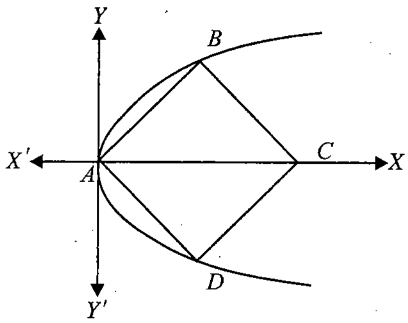


Fig. 3.123

AC is one diagonal along x-axis, then the other diagonal is BD where both B and D lie on parabola. Also slope of AB is $\tan\frac{\pi}{4} = 1$. If B is $(at^2, 2at)$ then the slope of AB

$$= \frac{2at}{at^2} = \frac{2}{t} = 1$$

$$\therefore \quad t = 2$$

Therefore, B is $(4a, 4a)$ and hence D is $(4a, -4a)$.

Clearly, C is $(8a, 0)$.

Reasoning Type

1. b. Any tangent having slope m is

$$y = mx + \frac{a}{m}$$

or $$y = mx + \frac{9/4}{m}$$

It passes through the point (4, 10), then

$$10 = 4m + \frac{9/4}{m}$$

$$\Rightarrow \quad 16m^2 - 40m + 9 = 0$$

$\Rightarrow \quad m_1 = \frac{1}{4}, m_2 = \frac{9}{4}$

$\Rightarrow$ Statement 1 is correct.

Also statement 2 is correct but it does not say anything about slope of the tangents, hence it is not correct explanation of statement 1.

2. **b.** Any normal to $y^2 = 4x$ is

$$y = -tx + 2t + t^3$$

If only one normal can be drawn to parabola from $(\lambda, \lambda + 1)$, then $\lambda < 2$.

Hence, statement 1 is true.

Statement 2 is also true as $(\lambda + 1)^2 > 4\lambda$ is true $\forall \lambda \in R - \{1\}$, but does not explain statement 1, as it is not necessary that from every outside points only one normal can be drawn.

3. **b.** Both the statements are true (see properties of focal chord), but statement 2 is not correct explanation of statement 1.

4. **a.** Differentiating $y^2 = 8x$ w.r.t. x, we have

$$2y\frac{dy}{dx} = 8$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{4}{y}$$

Now slopes of tangents at $(8, -8)$ and $(\frac{1}{2}, 2)$ are $-\frac{1}{2}$ and 2. Hence, tangents are perpendicular.

Also tangents at the extremities of the focal chord are perpendicular and meet on the directrix. Hence, both the statements are true and statement 2 is correct explanation of statement 1.

5. **a.** For parabola $y^2 = 4x$, $(4, 4)$ and $\left(\frac{1}{4}, -1\right)$ are extremities of the focal chord. Hence, tangents are perpendicular.

Then obviously normals at these points are also perpendicular.

6. **c.** Any tangent having slope m is

$$y = m(x + a) + \frac{a}{m}$$

or

$$y = mx + am + \frac{a}{m}$$

is tangent to the given parabola for all $m \in R - \{0\}$.

Hence, statement 2 is false.

However, statement 1 is true as when $m = 1$, tangent is

$$y = x + 2a.$$

7. **d.** Statement 2 is true as it is the definition of parabola.

From statement 1, we have

$$\sqrt{(x-1)^2 + (y+2)^2} = \frac{|3x + 4y + 5|}{5},$$

which is not parabola as point $(1, -2)$ lies on the line $3x + 4y + 5 = 0$. Hence, statement 1 is false.

8. **d.** Statement 2 is correct (see properties of the focal chord).

Then length of the focal chord according to the statement 1 is

$$4(2)\left(\frac{4}{3}\right) = \frac{32}{3}.$$

9. **a** Let parabola be $y^2 = 4x$

Clearly $x = 0$ is tangent to the parabola at $(0, 0)$.

And lines $y = -x - 1$ and $y = x + 1$ are tangents to the parabola at $(1, 2)$ and $(1, -2)$ which are extremities of the latus rectum. These tangents meet on the directrix at right angle at $(-a, 0)$. Hence, circle passing through the point A, B, C also passes through its focus, as shown in the figure.

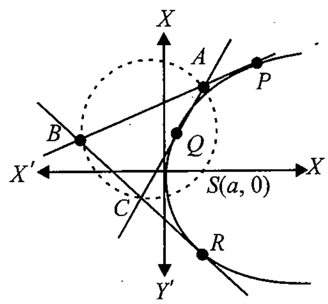


Fig. 3.124

Now consider a parabola $y^2 = 4ax$

Let $P(t_1)$, $Q(t_2)$ and $R(t_3)$ be three points on it.

Tangents are drawn at these points which intersect at

$$A \equiv (at_1t_2, a(t_1 + t_2))$$

$$B \equiv (at_1t_3, a(t_1 + t_3))$$

$$C \equiv (at_2t_3, a(t_2 + t_3))$$

Let $\quad \angle SAC = \alpha$ and $\angle SBC = \beta$

$$\Rightarrow \quad \tan\alpha = \left|\frac{\frac{1}{t_2} - \frac{t_1 + t_2}{t_1t_2 - 1}}{1 + \frac{1}{t_2}\left(\frac{t_1 + t_2}{t_1t_2 - 1}\right)}\right| = \left|\frac{1}{t_1}\right|$$

Similarly $\quad \tan\beta = \left|\frac{1}{t_1}\right|$

$$\Rightarrow \quad \alpha = \beta \text{ or } \alpha + \beta = \pi$$

$\Rightarrow A, B, C$ and S are concyclic.

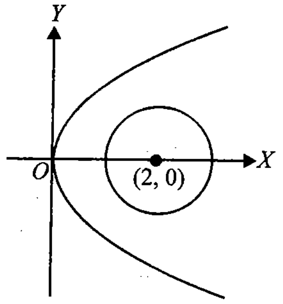
10. d. Area of the triangle formed by the intersection points of tangents at point $A(t_1)$, $B(t_2)$ and $C(t_3)$ is

$$\frac{1}{2}|t_1 - t_2|\,|t_2 - t_3|\,|t_3 - t_1| \neq 0.$$

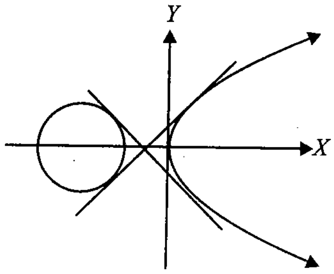
Hence, statement 1 is wrong. However, statement 2 is correct.

11. b. Statement 2 is true as circle is lying inside parabola without intersecting it.

But this cannot be considered the explanation of the statement 1, as even if they are not intersecting we can have common tangents as shown in the figure.



a.



b.

Fig. 3.125

12. a. Let the foot of normal be $P(at^2, 2at)$, then

$$ax + by + c = 0$$

and $\quad y = -tx + 2at + at^3$

are identical line,

$\Rightarrow \quad \frac{1}{b} = \frac{t}{a} = \frac{2at + at^3}{-c}$

$\Rightarrow \quad t = \frac{a}{b}.$

Thus 'P' is $\left(\frac{a^3}{b^2}, \frac{2a^2}{b}\right)$.

Hence, equation of required tangent is

$$ty = x + at^2$$

or $\quad \frac{a}{b}y = x + a\left(\frac{a}{b}\right)^2$

or $\quad y = \frac{b}{a}x + \frac{a^2}{b}$

13. a.

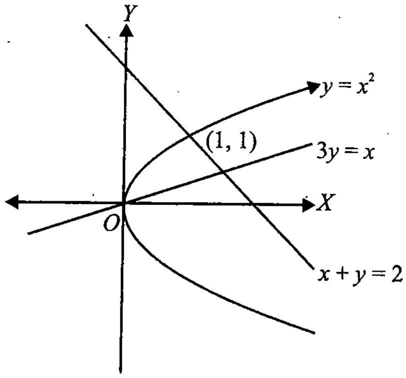


Fig. 3.126

Point (α, α^2) lies on the parabola $y^2 = x$.

As shown in the figure, we have to find the value of α for which the part of the parabola lies inside the triangle formed by three lines.

Now line $x + y = 2$ meets the parabola at point (0, 0) and (1, 1).

Hence, $\alpha \in (0, 1)$

14. a.

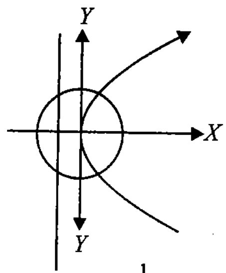


Fig. 3.127

Statement 2 is true as it is the property of the parabola.

Now such points exists on the circle $x^2 + y^2 = a^2$ if it meets the directrix at least one point, for which radius of the circle $a \geq 1/2$.

15. a. Let $y^2 = 4ax$ be a parabola. Consider a line $x = 4a$ (this is a double ordinate which is twice of latus rectum), which cuts the parabola at $A(4a, 4a)$ and $B(4a, -4a)$.

Slope of $\quad OA = 1,$

Slope of $\quad OB = -1$, where O is given.

Therefore, AB subtends 90° at the origin.

$\Rightarrow$ Statement 2 is correct and it clearly explains statement 1.

16. d. Statement 1 is false

Since here $t^2 = 4$

Therefore, the normal chord subtends a right angle at the focus (not at the vertex).

However, statement 2 true (a standard result).

17. a. Let normals at points $A(at_1^2, 2at_1)$ and $C(at_3^2, 2at_3)$ meets the parabola again at points $B(at_2^2, 2at_2)$ and $D(at_4^2, 2at_4)$, then $t_2 = -t_1 - \frac{2}{t_1}$ and $t_4 = -t_3 - \frac{2}{t_3}$

Adding $t_2 + t_4 = -t_1 - t_3 - \frac{2}{t_1} - \frac{2}{t_3}$

$\Rightarrow$ $t_1 + t_2 + t_3 + t_4 = -\frac{2}{t_1} - \frac{2}{t_3}$

$\Rightarrow$ $\frac{1}{t_1} + \frac{1}{t_3} = 0$

$\Rightarrow$ $t_1 + t_3 = 0$

Now, point of intersection of tangent at A and C will be $(at_1 t_3, a(t_1 + t_3))$

Since $t_1 + t_3 = 0$, so this point will lie on x-axis, which is axis of parabola.

18. a.

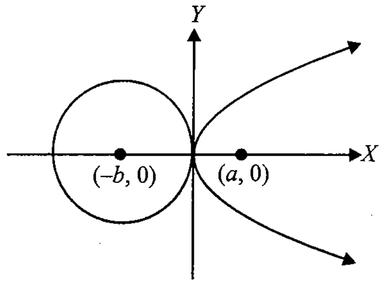


Fig. 3.128

As shown in the figure, circle and parabola touch when a and b have same sign.

Now for

$f(x) = x^2 - (b + a + 1)x + a,$

$\Rightarrow$ $f(0) = a$

and $f(1) = 1 - (b + a + 1) + a = -b$

$\Rightarrow$ $f(0) \cdot f(1) = -ab < 0$

Hence, one root lies in (0, 1).

$\Rightarrow$ Both the statements are true and statement 2 is correct explanation of statement 1.

19. c. Statement 2 is false, as axis of parabola is normal to parabola which passes through the focus. However, normal other than axis never passes through focus. Statement 1 is correct as $x - y - 5 = 0$ passes through focus $(3, -2)$, hence it cannot be normal.

20. b. Obviously, statement 2 is true, but it is not the correct explanation of statement 1 as A, A', B, B' form an isosceles trapezium, hence points are concyclic.

Linked Comprehension Type

For Problems 1–3

1. c., 2. d., 3. a.

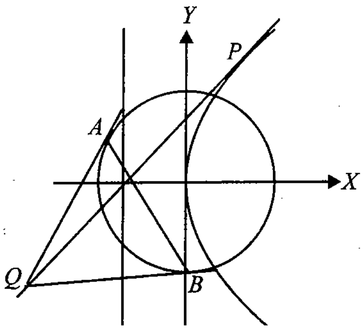


Fig. 3.129

Sol. 1. c. Equation of the tangent at point P of the parabola $y^2 = 8x$ is

$$yt = x + 2t^2 \quad \text{(i)}$$

Equation of the chord of contact of the circle $x^2 + y^2 = 8$ w.r.t. $Q(\alpha, \beta)$ is

$$x\alpha + y\beta = 8 \quad \text{(ii)}$$

$Q(\alpha, \beta)$ lies in Eq. (i)

Hence, $\beta t = \alpha + 2t^2$ (iii)

$$x\alpha + y\left(\frac{\alpha}{t} + 2t\right) - 8 = 0 \text{ [from Eqs. (ii) and (iii)]}$$

$$2(ty - 4) + \alpha\left(x + \frac{y}{t}\right) = 0$$

For point of concurrency

$$x = -\frac{y}{t} \text{ and } y = \frac{4}{t}$$

Therefore, locus is $y^2 + 4x = 0$

2. d. Required point will lie on the director circle of the given circle as well as on the directrix of parabola.

$\Rightarrow$ $x_1^2 + y_1^2 = 16$ and $x_1 + 2 = 0$

$\Rightarrow$ $4 + y_1^2 = 16$

$\Rightarrow$ $y = \pm 2\sqrt{3}$

Therefore, point is $(-2, \pm 2\sqrt{3})$.

3. a. Equation of circumcentre of ΔAQB is

$$x^2 + y^2 - 4 + \lambda(x\alpha + y\beta - 8) = 0$$

Because it passes through (0, 0), i.e., centre of circle

$$\Rightarrow \quad \lambda = -\frac{1}{2}$$

Let circumcentre be (h, k).

$$\therefore \quad h = \frac{\alpha}{4}, k = \frac{\beta}{4}$$

$$\Rightarrow \quad \alpha = 4h, \beta = 4k$$

Also $\quad \beta t = \alpha + 2t^2$

or $\quad \alpha - 2\beta + 8 = 0 \quad (\because t = 2)$

Substituting $\alpha = 4h$ and $\beta = 4k$, we get

$$h - 2k + 2 = 0$$

Therefore, locus is $x - 2y + 2 = 0$.

For Problems 4–6

4. b., 5. c., 6. d.

Sol. 4. b. Since no point of the parabola is below x-axis

$$\therefore \quad D = a^2 - 4 \le 0$$

Therefore, maximum value of a is 2.

Equation of the parabola, when $a = 2$, is

$$y = x^2 + 2x + 1$$

It intersects y-axis at (0, 1).

Equation of the tangent at (0, 1) is

$$y = 2x + 1$$

Since $y = 2x + 1$ touches the circle $x^2 + y^2 = r^2$

$$\therefore \quad r = \frac{1}{\sqrt{5}}$$

5. c. Equation of the tangent at (0, 1) to the parabola $y = x^2 + ax + 1$ is

$$y - 1 = a(x - 0)$$

or $\quad ax - y + 1 = 0$

As it touches the circle,

$$\therefore \quad r = \frac{1}{\sqrt{a^2 + 1}}$$

Radius is maximum when $a = 0$

Therefore, equation of the tangent is $y = 1$.

Therefore, slope of the tangent is 0.

6. d. Equation of tangent is $y = ax + 1$.

Intercepts are $-\frac{1}{a}$ and 1.

Therefore, area of the triangle bounded by tangent and the axes $= \frac{1}{2}\left|-\frac{1}{a}\cdot 1\right| = \frac{1}{2|a|}$

It is minimum when $a = 2$

Therefore, minimum area $= \frac{1}{4}$

For Problems 7–9

7. a., 8. c., 9. d.

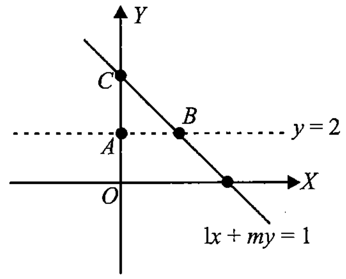


Fig. 3.130

Sol. $C \equiv \left(0, \frac{1}{m}\right)$, $B \equiv \left(\frac{1-2m}{l}, 2\right)$, $A \equiv (0, 2)$

Let (h, k) be the circumcentre of $\Delta\, ABC$ which is mid-point of BC

$$\Rightarrow \quad h = \frac{1-2m}{2l}; k = \frac{1+2m}{2m},$$

$$\Rightarrow \quad m = \frac{1}{2k-2}; l = \frac{k-2}{2h(k-1)}$$

Given that (l, m) lies on $\quad y^2 = 4x$

$$\therefore \quad m^2 = 4l$$

$$\Rightarrow \quad \left(\frac{1}{2k-2}\right)^2 = 4\left\{\frac{k-2}{2h(k-1)}\right\}$$

$$\Rightarrow \quad h = 8(k^2 - 3k + 2)$$

Therefore, locus of (h, k) is

$$x = 8(y^2 - 3y + 2)$$

or $\quad \left(y - \frac{3}{2}\right)^2 = \frac{1}{8}(x + 2)$

Therefore, vertex is $\left(-2, \frac{3}{2}\right)$.

Length of smallest focal chord = length of latus rectum $= \frac{1}{8}$.

From the equation of curve C, it is clear that it is symmetric about line $y = \frac{3}{2}$.

For Problems 10–12

10. d., 11. c., 12. d.

Sol. 10. d. $y = ax^2 + c$

$$\therefore \quad \frac{dy}{dx} = 2ax = 1$$

Therefore, point of contact of the tangent is $\left(\frac{1}{2a}, \frac{1}{4a} + c\right)$

Since it lies on $y = x$

$$\therefore \quad c = \frac{1}{4a}, \text{ thus } c = \frac{1}{8} \text{ for } a = 2.$$

11. c. If (1, 1) is point of contact, then $a = \frac{1}{2}$

12. d. If $c = 2$, then point of contact is $\left(\frac{1}{2a}, \frac{1}{4a} + 2\right)$.
Since it lies on line $y = x$,

$$\therefore \quad \frac{1}{2a} = \frac{1}{4a} + 2,$$

$$\text{i.e.,} \quad a = \frac{1}{8}$$

Therefore, point of contact is (4, 4).

For Problems 13–15

13. a., 14. b., 15. c.

Sol. Any parabola whose axes is parallel to x-axis will be of the form

$$(y - a)^2 = 4b(x - c) \quad \text{(i)}$$

Now, $lx + my = 1$, can be rewritten as

$$y - a = -\frac{l}{m}(x - c) + \frac{1 - am - lc}{m} \quad \text{(ii)}$$

Equation (ii) will touch Eq. (i) if

$$\frac{1 - am - lc}{m} = \frac{b}{-l/m}$$

$$\Rightarrow \quad -\frac{l}{m} = \frac{bm}{1 - am - lc}$$

$$\Rightarrow \quad cl^2 - bm^2 + alm - l = 0 \quad \text{(iii)}$$

But given that $5l^2 + 6m^2 - 4lm + 3l = 0$ (iv)

Comparing Eqs. (iii) and (iv), we get

$$\frac{c}{5} = \frac{-b}{6} = \frac{a}{-4} = \frac{-1}{3}$$

$$\Rightarrow \quad c = \frac{-5}{3}, b = 2 \text{ and } a = \frac{4}{3}$$

So parabola is $\left(y - \frac{4}{3}\right)^2 = 8\left(x + \frac{5}{3}\right)$ whose focus is $\left(\frac{1}{3}, \frac{4}{3}\right)$ and directrix is $3x + 11 = 0$.

For Problems 16–18

16. b., 17. a., 18. c.

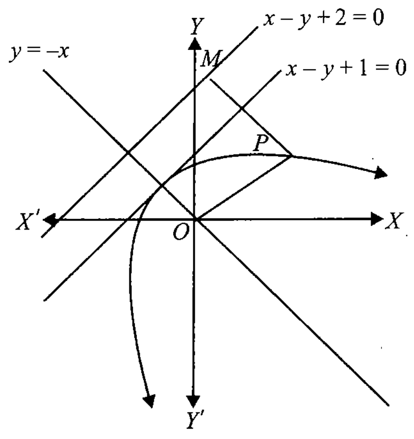


Fig 3.131

Sol. The distance between the focus and the tangent at the vertex $= \frac{|0 - 0 + 1|}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}$.

The directrix is the line parallel to the tangent at vertex and at a distance $2 \times \frac{1}{\sqrt{2}}$ from the focus.

Let equation of directrix is

$$x - y + \lambda = 0,$$

$$\text{where} \quad \frac{\lambda}{\sqrt{1^2 + 1^2}} = \frac{2}{\sqrt{2}}$$

$$\Rightarrow \quad \lambda = 2$$

Let $P(x, y)$ be any moving point on the parabola, then

$$OP = PM$$

$$\Rightarrow \quad x^2 + y^2 = \left(\frac{x - y + 2}{\sqrt{1^2 + 1^2}}\right)^2$$

$$\Rightarrow \quad 2x^2 + 2y^2 = (x - y + 2)^2$$

$$\Rightarrow x^2 + y^2 + 2xy - 4x + 4y - 4 = 0.$$

Latus rectum length = 2 × (distance of focus from directrix)

$$= 2\left|\frac{0 - 0 + 2}{\sqrt{1^2 + 1^2}}\right|$$

$$= 2\sqrt{2}$$

Solving parabola with x-axis,

$$x^2 - 4x - 4 = 0$$

$$\Rightarrow \quad x = \frac{4 \pm \sqrt{32}}{2} = 2 \pm 2\sqrt{2}$$

$\Rightarrow$ Length of chord on x-axis is $4\sqrt{2}$.

Since the chord $3x + 2y = 0$ passes through the focus, it is focal chord.

Hence, tangents at the extremities of chord are perpendicular.

For Problems 19–21

19. a., 20. b., 21. c.

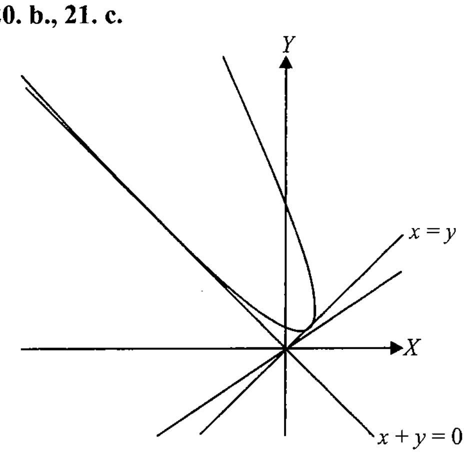


Fig. 3.132

Sol. We know that foot of perpendicular from focus upon tangent lies on the tangent at vertex of the parabola.

Now, if foot of perpendicular of (2, 3) on the line $x - y = 0$ is (x_1, y_1), then

$$\frac{x_1 - 2}{1} = \frac{y_1 - 3}{-1} = \frac{2-3}{2}$$

$$\Rightarrow \quad x_1 = \frac{5}{2} \text{ and } y_1 = \frac{5}{2}$$

If foot of perpendicular of (2, 3) on the line $x + y = 0$ is (x_2, y_2), then

$$\frac{x_2 - 2}{1} = \frac{y_2 - 3}{1} = -\frac{2+3}{2}$$

$$\Rightarrow \quad x_2 = -\frac{1}{2} \text{ and } y_2 = \frac{1}{2}$$

Now tangent at vertex passes through the points $\left(\frac{5}{2}, \frac{5}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Then, its equation is

$$y - \frac{1}{2} = \frac{2}{3}\left(x + \frac{1}{2}\right)$$

or $\quad 4x - 6y + 5 = 0$

Latus rectum of the parabola

$$= 4\times \text{(distance of focus from tangent at vertex)}$$

$$= 4\times \left|\frac{8 - 18 + 5}{\sqrt{52}}\right| = \frac{10}{\sqrt{13}}$$

Also, distance between the focus and tangent at vertex $= \frac{5}{\sqrt{13}}$

Since tangents $x + y = 0$ and $x - y = 0$ are perpendicular, they meet at (0, 0) which lies on the directrix.

Also, it is parallel to the tangent at vertex, hence its equation is $4x - 6y = 0$.

We know that $\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$, where a is $\left(\frac{1}{4}\right)$th of latus rectum.

$$\Rightarrow \quad \frac{1}{SP} + \frac{1}{SQ} = \frac{2\sqrt{13}}{5}$$

For Problems 22–24

22. d., 23. b., 24. a.

Sol. Solving given parabolas, we have

$$-8(x - a) = 4x$$

$$\Rightarrow \quad x = \frac{2a}{3}$$

$\Rightarrow$ Points of intersection are $\left(\frac{2a}{3}, \pm\sqrt{\frac{8a}{3}}\right)$

Now $OABC$ is concyclic.

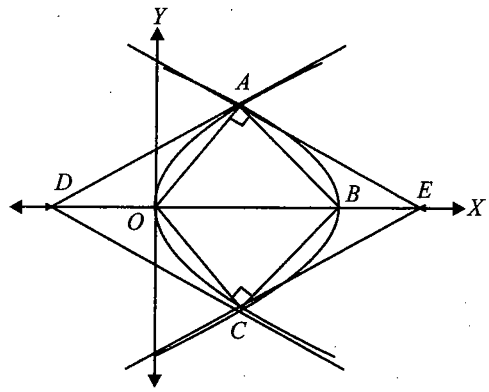


Fig. 3.133

Hence, $\angle OAB$ must be right angle.

$\Rightarrow$ Slope of OA $\times$ Slope of $AB = -1$

$$\Rightarrow \quad \frac{\sqrt{\frac{8a}{3}}}{\frac{2a}{3}} \times \frac{\sqrt{\frac{8a}{3}}}{a - \frac{2a}{3}} = -1$$

$$\Rightarrow \quad a = 12$$

$\Rightarrow$ Coordinates of A and B are $(8, 4\sqrt{2})$ and $(8, -4\sqrt{2})$ respectively

$\Rightarrow$ Length of common chord $= 8\sqrt{2}$.

$$\text{Area of quadrilateral} = \frac{1}{2} OB \times AC$$

$$= \frac{1}{2} \times 12 \times 8\sqrt{2}$$

$$= 48\sqrt{2}$$

Tangent to parabola $y^2 = 4x$ at point $(8, 4\sqrt{2})$ is $4\sqrt{2}y = 2(x + 8)$ or $x - 2\sqrt{2}y + 8 = 0$ which meets the x-axis at $D(-8, 0)$.

Tangent to parabola $y^2 = -8(x - 12)$ at point $(8, 4\sqrt{2})$ is $4\sqrt{2}y = -4(x + 8) + 96$ or $x + \sqrt{2}y - 16 = 0$, which meets the x-axis at $E(16, 0)$.

Hence, area of quadrilateral $DAEC = \frac{1}{2} DE \times AC$

$$= \frac{1}{2} \times 24 \times 8\sqrt{2}$$

$$= 96\sqrt{2}$$

For Problems 25–27

25.b., 26. c., 27. d.

Sol. For $y^2 = 4x$, coordinates of end of latus rectum are $P(1, 2)$ and $Q(1, -2)$.

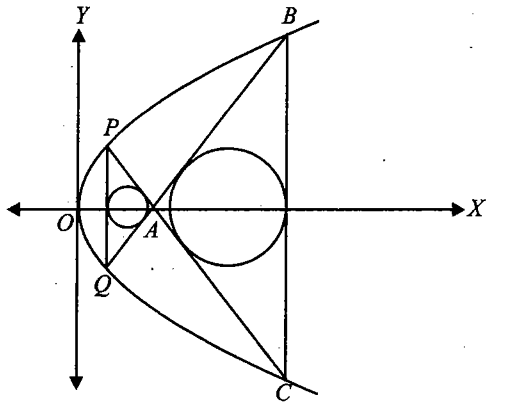


Fig. 3.134

ΔPAQ is isosceles right angled. Therefore, slope of PA is -1 and its equation is $y - 2 = -(x - 1)$ or $x + y - 3 = 0$.

Similarly, equation of line QB is $x - y - 3 = 0$.

Solving $x + y - 3 = 0$ with the parabola $y^2 = 4x$, we have

$$(3 - x)^2 = 4x \text{ or } x^2 - 10x + 9 = 0$$

$$\therefore \quad x = 1, 9$$

Therefore, coordinates of B and C are $(9, -6)$ and $(9, 6)$ respectively.

Area of trapezium $PBCQ = \frac{1}{2} \times (12 + 4) \times 8$

$= 64$ sq. units

Let the circumcentre of trapezium $PBCQ$ is $T(h, 0)$

Then $\quad PT = PB$

$$\Rightarrow \quad \sqrt{(h-1)^2 + 4} = \sqrt{(h-9)^2 + 36}$$

$$\Rightarrow \quad -2h + 5 = -18h + 81 + 36$$

$$\Rightarrow \quad 16h = 112$$

$$\Rightarrow \quad h = 7$$

Hence, radius is $\sqrt{40} = 2\sqrt{10}$

Let inradius of ΔAPQ be r_1, then $r_1 = \frac{\Delta_1}{s_1}$

$$= \frac{\frac{1}{2} \times 4 \times 2}{4 + 2\sqrt{4 + 4}}$$

$$= \frac{1}{1 + \sqrt{2}} = \sqrt{2} - 1$$

Let inradius of ΔABC be r_2, then

$$r_2 = \frac{\Delta_2}{s_2}$$

$$= \frac{\frac{1}{2} \times 12 \times 6}{12 + 2\sqrt{36 + 36}}$$

$$= \frac{3}{1 + \sqrt{2}} = 3(\sqrt{2} - 1)$$

$$\Rightarrow \quad \frac{r_2}{r_1} = 3$$

For Problems 28–30

28.d., 29. c., 30. b.

Sol. $9x - a.\ 3^x - a + 3 \le 0$

Let $t = 3^x$

$\Rightarrow t^2 - at - a + 3 \le 0$

or $t^2 + 3 \le a(t + 1)$

where $t \in R^+$ for $\forall x \in R$

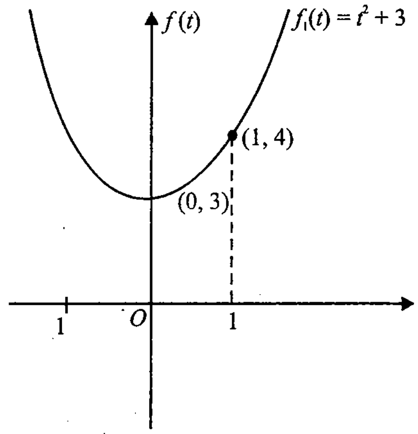


Fig. 3.135

Let $f_1(t)$ be $t^2 + 3$ and $f_2(t)$ be $a(t + 1)$.

28. d. From $x < 0$, $t \in (0, 1)$. That means (1) should have at least one solution in $t \in (0, 1)$.

From (1), it is obvious that $a \in R^+$.

Now $f_2(t) = a(t + 1)$ represents a straight line. It should meet the curve.

$f_1(t) = t^2 + 3$, at least once in $t \in (0, 1)$.

$f_1(0) = 3, f_1(1) = 4, f_2(0) = a, f_2(1) = 2a$.

If $f_1(0) = f_2(0) \Rightarrow a = 3$; if $f_1(1) = f_2(1) = a = 2$

Hence, required a $\in (2, 3)$.

29. c. For at least one positive solution, $t \in (1, \infty)$. That means graphs of $f_1(t) = t^2 + 3$ and $f_2(t) = a(t + 1)$ should meet at least once in $t \in (1, \infty)$.

If $a = 2$, both curve touch each other at $(1, 4)$.

Hence, the required $a \in (2, \infty)$.

30 b. In this case, both graphs should meet at least once in $t \in (0, \infty)$.

For $a = 2$, both curves touch; hence, the required $a \in [2, \infty)$.

Matrix-Match Type

1. a. → r., b. → s., c. → p., d. → q.

Sol. Locus of point of intersection of perpendicular tangent is directrix which is $12x - 5y + 3 = 0$.

Parabola is symmetrical about its axis, which is a line passing through the focus (1, 2) and perpendicular to the directrix, which has equation $5x + 12y - 29 = 0$.

Minimum length of focal chord occurs along the latus rectum line, which is a line passing through the focus and parallel to directrix, i.e,. $12x - 5y - 2 = 0$.

Locus of foot of perpendicular from foucs upon any tangent is tangent at the vertex, which is parallel to directrix and equidistant from directrix and latus rectum line, i.e., $12x - 5y + \lambda = 0$

where $\frac{|\lambda - 3|}{\sqrt{12^2+5^2}} = \frac{|\lambda + 2|}{\sqrt{12^2+5^2}} \Rightarrow \lambda = \frac{1}{2}$.

Hence, equation of tangent at vertex is $24x - 10y + 1 = 0$.

2. **a. → q., b. → s., c. → p., d. → r.**

Equation of tangent having slope m is $y = mx + \frac{3}{m}$.

Line $3x - y + 1 = 0$ is tangent for $m = 3$.

Equation of normal having slope m is $y = mx - 6m - 3m^3$.

Line $2x - y - 36 = 0$ is normal for $m = 2$.

Chord of contact w.r.t. any point on the directrix is the focal chord which passes through the focus (3, 0).

Line $2x - y - 36 = 0$ passes through the focus.

Chords which subtends right angle at the vertex are concurrent at point $(4 \times 3, 0)$ or (12, 0).

Line $x - 2y - 12 = 0$ passes through the point (12, 0).

3. **a. → q., s; b. → r.; c. → p., q.; d. → q., r**

a. Tangent to parabola having slope m is $ty = x + t^2$, it passes through point (2, 3) then $3t = 2 + t^2 \Rightarrow t = 1$ or $2 \Rightarrow$ point of contact $(t^2 + 2t) \equiv (1, 2)$ or (4, 4)

b. Let point on the circle be $P(x_1, y_1)$, then chord of contact of parabola w.r.t. P is $yy_1 = 2(x + x_1)$. Comparing with $y = 2(x - 2)$, we have $y_1 = 1$ and $x_1 = -2$, which also satisfy the circle.

c. Point Q on the parabola $Q(t^2, 2t)$

Now area of triangle OPQ is $\left\| \frac{1}{2}\begin{vmatrix} 0 & 0 \\ 4 & -4 \\ t^2 & 2t \\ 0 & 0 \end{vmatrix} \right\| = 6 \Rightarrow 8t + 4t^2 = \pm 12$

For $t^2 + 2t - 3 = 0$, $(t - 1)(t + 3) = 0$, then $t = 1$ or $t = -3$.

Then point Q are (1, 2) or (9, – 6).

d. Points (1, 2) and (– 2, 1) satisfy both the curves.

4. **a. → p, r.; b. → p., r.; c. → q; d. → q., s.**

Points through which perpendicular tangent can be drawn to the parabola $y^2 = 4x$ lie on the directrix. Points (–1, 2) and (–1, –5) lie on the directirx. Also from these points only one normal can be drawn.

Integer type

1. **(7)** Here, $(x - 1)^2 + (y - 3)^2 = \left\{\frac{5x - 12y + 17}{\sqrt{5^2 + (-12)^2}}\right\}^2$

$\therefore$ the focus = (1, 3) and the directrix is $5x - 12y + 17 = 0$

The distance of the focus from the directrix

$$= \left|\frac{5 \times 1 - 12 \times 3 + 17}{\sqrt{5^2 + (-12)^2}}\right| = \frac{14}{13}$$

$\therefore$ latus rectum $= 2 \times \frac{14}{13} = \frac{28}{13}$

2. **(0)** Clearly P is the point of intersection of two perpendicular tangents to the parabola $y^2 = 8x$, $4a = 8$ or $a = 2$. Hence P must lie on the directrix $x + a = 0$ or $x + 2 = 0$ $\therefore x = -2$. hence the point is (–2, 0)

3. **(1)** Focus of $y^2 = 16x$ is (4, 0).

Any focal chord is $y - 0 = m(x - 4)$

or $mx - y - 4m = 0$

This focal chord touches the circle $(x - 6)^2 + y^2 = 2$

Then distance from the center of circle to this chord is equal to radius of the circle

or $$\frac{|6m - 4m|}{\sqrt{m^2+1}} = \sqrt{2} \text{ or } 2m = \sqrt{2}.\sqrt{m^2+1}$$

or $$2m^2 = m^2 + 1 \Rightarrow m^2 = 1$$

$\therefore$ $$m = \pm 1$$

4. **(3)** Any tangent to parabola $y^2 = 4x$, $(a = 1)$ is

$y = mx + \frac{1}{m}$. It passes through (–2, – 1)

$\therefore -1 = -2m + \frac{1}{m}$ or $2m^2 - m - 1 = 0$

Or $(2m + 1)(m - 1) = 0$

Or $m = 1/2$ and $m = 1$

Then angle between lines is

$$\tan\theta = \left|\frac{m_1 + m_2}{1 - m_1 m_2}\right| = 3$$

5. **(3)** Equation of tangent in terms of slope of parabola $y^2 = 4x$ is

$$y = mx + \frac{1}{m}$$

$\because$ Eq. (i) is also tangent of $x^2 = -32y$

then $x^2 = -32\left(mx + \frac{1}{m}\right)$

$\Rightarrow x^2 + 32mx + \frac{32}{m} = 0$

Above equation must have equal roots,

Hence its discriminant must be zero

$\Longrightarrow (32\,m)^2 = 4.1\,.\,\dfrac{32}{m}$

$\Longrightarrow m^3 = \dfrac{1}{8}$ or $m = \dfrac{1}{2}$

From Eq. (i), $y = \dfrac{x}{2} + 2$

$\Longrightarrow x - 2y + 4 = 0$

6. **(4)** $y^2 = x \quad \therefore 4a = 1,$

$P\ (at_1^2, 2at_1) = (4, -2)$

$\therefore t_1 = -4$

Also $t_1 t_2 = -1$ as PQ is a focal chord.

Slope of tangent at t_2 is $\dfrac{1}{t_2} = -t_1 = 4$

7. **(6)** Slope of the line -1

From the curve, $\dfrac{dy}{dx} = \dfrac{4}{y}$

Hence slope of normal $= -\dfrac{y}{4} = -1$ or $y = 4$.

Putting $y = 4$ in equation of curve we have $x = 2$

Hence point is (4, 2)

8. **(4)**

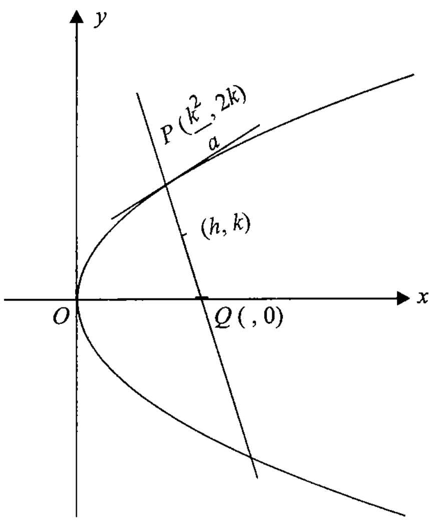


Fig. 3.136

Consider the parabola $y^2 = 4ax$

We have to find the locus of $R(h, k)$, since Q has ordinate 'O', ordinate of P is $2k$.

Also P is on the curve, then abscissa of P is k^2/a.

Now PQ is normal to curve

Slope of tangent to curve at any point $\dfrac{dy}{dx} = \dfrac{2a}{y}$

Hence slope of normal at point P is $-\dfrac{k}{a}$

Also slope of normal joining P and R(h, k) is $\dfrac{2k-k}{\dfrac{k^2}{a} - h}$

Hence comparing slopes $\dfrac{2k-k}{\dfrac{k^2}{a} - h} = -\dfrac{k}{a}$

Or $y^2 = a\,(x - a)$

For $y^2 = 16x$, $a = 4$, hence locus us $y^2 = 4(x - 4)$

9. **(3)**

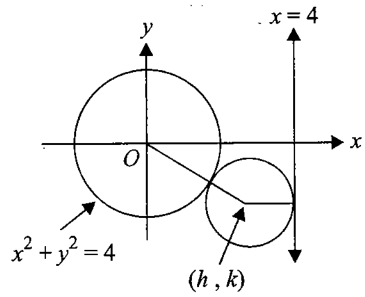


Fig. 3.137

Radius of variable circle is $4 - h$

It touches $x^2 + y^2 = 4$

$\therefore 2 + 4 - h = \sqrt{h^2 + k^2}$

or $x^2 + y^2 = x^2 - 12x + 36$

$\Rightarrow \quad y^2 = -12(x - 3)$

The vertex (3, 0)

10. **(8)**

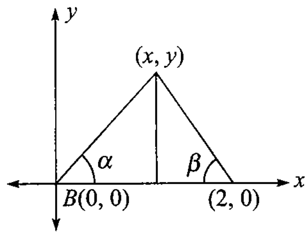


Fig. 3.138

Given $\tan\alpha \cdot \tan\beta = 4$

$\Rightarrow y\,.\,\dfrac{y}{x} + \dfrac{y}{2-x} = 4 \Rightarrow y = 2x\,(2 - x)$

$\Rightarrow -\dfrac{y}{2} = x^2 - 2x = (x - 1)^2 - 1$

$\Rightarrow (x - 1)^2 = -\dfrac{1}{2}(y - 2)$

$\Rightarrow$ Directrix $y - 2 = \dfrac{1}{8} \Rightarrow y = \dfrac{17}{8}$

11. **(8)** Length of focal chord having one extremity $(at^2, 2at)$ is

$a\left(t + \dfrac{1}{t}\right)^2$

$\left|r + \dfrac{1}{t}\right| \geq 2 \Rightarrow a\left(1 + \dfrac{1}{t}\right)^2 \geq 4a = 8 \Rightarrow$ length of coal chord $\not< 8$.

12. (4) $a = 3$, comparing point (3, 6) with $(3t^2, 6t)$, we have $t = 1$,

then length of chord $= a\left(t+\frac{1}{t}\right)^2 = 3(1+1)^2 = 12$

13. (8) Chord of the contact w.r.t. point $O(-1, 2)$ is

$y = (x - 1)$ (using $yy_1 = 2a(x + x_1)$)

solving $y = x - 1$, with parabola, we get the points of intersection as

$$P(3+2\sqrt{2}, 2+2\sqrt{2}) \text{ and } Q(3-2\sqrt{2}, 2-2\sqrt{2})$$

$\therefore PQ^2 = 32 + 32 = 64$

$\therefore PQ = 8$

Also length of perpendicular from $O(-1, 2)$ on PQ

$$= \frac{4}{\sqrt{2}}$$

Then required area of triangle is

$$A = \frac{1}{2}.8.\left(\frac{4}{\sqrt{2}}\right) = 8\sqrt{2} \text{ sq. units}$$

14. (7) Line $y = 2x - b$

$\Rightarrow \quad 1 = \frac{2x-y}{b}$

Homogenising parabola with line

$$x^2 - 4x\left(\frac{2x-y}{b}\right) - y\left(\frac{2x-y}{b}\right) = 0$$

Since $\angle AOB = 90°$

$\therefore$ coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow 1 - \frac{8}{b} + \frac{1}{b} = 0$

$\Rightarrow \quad b = 7$

15.(5) Let the line be $y = mx$ (1)

Solving it with

$5y = 2x^2 - 9x + 10$, we get

$5mx = 2x^2 - 9x + 10$

$2x^2 - (9 + 5m)x + 10 = 0$

sum of the roots $= \frac{9+5m}{2} = 17$

$\Rightarrow 9 + 5m = 34$

$\Rightarrow \quad 5m = 25$

$\Rightarrow \quad m = 5$

16. (5) For maximum number of common chord, circle and parabola must intersect in 4 distinct points.

Let us first find the value of r when circle and parabola touch each other.

For that solving the given curves we have $(x - 6)^2 + 4x = r^2$ or $x^2 - 8x + 36 - r^2 = 0$

Curves touch if discriminant $D = 64 - 4(36 - r^2) = 0$ or $r^2 = 20$

Hence least integral value of r for which the curves intersect is 5.

Archives

Subjective Type

1. The equation of a normal to the parabola $y^2 = 4x$ in its slope from is given by

$$y = mx - 2am - am^3$$

Therefore, equation of normal to $y^2 = 4x$ is,

$$y = mx - 2m - m^3 \quad \text{(i)}$$

Since the normal drawn at three different points on the parabola passes through (h, k), it must satisfy Eq. (i).

$\therefore \quad k = mh - 2m - m^3$

$\Rightarrow \quad m^3 - (h - 2)m + k = 0$

This cubic equation in m has three different roots say m_1, m_2, m_3.

$\therefore \quad m_1 + m_2 + m_3 = 0$ (ii)

$m_1m_2 + m_2m_3 + m_3m_1 = -(h - 2)$ (iii)

Now, $(m_1 + m_2 + m_3)^2 = 0$ [Squaring Eq. (ii)]

$\Rightarrow \quad m_1^2 + m_2^2 + m_3^2 = -2(m_1m_2 + m_2m_3 + m_3m_1)$

$\Rightarrow \quad m_1^2 + m_2^2 + m_3^2 = 2(h - 2)$ [Using Eq. (iii)]

Since LHS of this equation is the sum of perfect squares, therefore, it is positive.

$\therefore \quad h - 2 > 0$

$\Rightarrow \quad h > 2$

2. Normal at point $A(at_1^2, 2at_1)$ meets the parabola at point $B(at_2^2, 2at_2)$.

$\Rightarrow \quad t_2 = -t_1 - \frac{2}{t_1}$ (i)

Also AB subtends right angle at vertex,

$\Rightarrow \quad t_1t_2 = -4$ (ii)

Eliminating t_2 from Eqs. (i) and (ii),

$\Rightarrow \quad -\frac{4}{t_1} = -t_1 - \frac{2}{t_1}$

$\Rightarrow \quad \frac{2}{t_1} = t_1$

Then slope of AB = slope of normal at $A = \pm\sqrt{2}$

3. Equation of normal to the parabola $x^2 = 4y$ having slope m is

$$x = my - 2m - m^3$$

Since it passes through the point (1, 2), we have

$$1 = 2m - 2m - m^3$$

$\Rightarrow$ $m = -1$

Hence, equation of normal is $x = -y + 2 + 1$ or $x + y - 3 = 0$.

4. Using the result of problem 1,

We have $h \geq 2a$, if three normals can be drawn to $y^2 = 4ax$ from (h, k)

Hence for the given question $c \geq 2(1/4)$ or $c \geq \frac{1}{2}$

Alternative Method:

We know that from any point normal to $y^2 = 4ax$ is given by $y = mx - 2am - am^3$, here $a = \frac{1}{4}$,

$\therefore$ Normal is $y = mx - \frac{m}{2} - \frac{m^3}{4}$

This normal passes through $(c, 0)$

$\Rightarrow$ $mc - \frac{m}{2} - \frac{m^3}{4} = 0$ (i)

$\Rightarrow$ $m\left[c - \frac{1}{2} - \frac{m^2}{4}\right] = 0$

$\Rightarrow$ $m = 0$

or $m^2 = 4\left(c - \frac{1}{2}\right)$

$m = 0$ shows normal is $y = 0$, i.e., x-axis is always a normal

Also $m^2 \geq 0$

$\Rightarrow$ $4\left(c - \frac{1}{2}\right) \geq 0$

$\Rightarrow$ $c \geq \frac{1}{2}$

At $c = \frac{1}{2}$, from Eq. (i), $m = 0$

Therefore, for other real value of m, $c > \frac{1}{2}$.

Now for other two normals to be perpendicular to each other, we must have $m_1 \cdot m_2 = -1$.

$\Rightarrow$ Product of roots of the equation

$$\frac{m^2}{4} + \frac{1}{2} - c = 0 \text{ is } -1.$$

$\Rightarrow$ $\dfrac{\left(\frac{1}{2} - c\right)}{\frac{1}{4}} = -1$

$\Rightarrow$ $\frac{1}{2} - c = -\frac{1}{4}$

$\Rightarrow$ $c = \frac{3}{4}$

5.

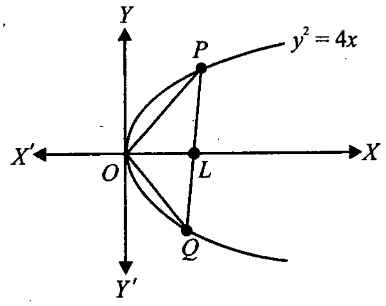


Fig. 3.139

Chord joining $P(t_1^2, 2t_1)$ and $Q(t_2^2, 2t_2)$ subtends right angle at origin.

$\Rightarrow$ $t_1t_2 = -4$

Also slope of the chord $= \frac{2}{t_1 + t_2}$

$\Rightarrow$ Equation of chord PQ is

$$y - 2t_1 = \frac{2}{t_1 + t_2}(x - t_1^2)$$

or $(t_1 + t_2)y - 2t_1^2 - 2t_1t_2 = 2(x - t_1^2)$

or $(t_1 + t_2)y - 2t_1^2 + 8 = 2(y - t_1^2)$

or $(t_1 + t_2)y + 8 = 2x$

or $2(x - 4) = (t_1 + t_2)y$

which always passes through point (4, 0).

If (h, k) is midpoint of PQ, then

$$h = \frac{t_1^2 + t_2^2}{2} \text{ and } k = t_1 + t_2$$

$\Rightarrow$ $h = \dfrac{(t_1 + t_2)^2 - 2(t_1t_2)}{2}$

$= \dfrac{k^2 + 8}{2}$

or $y^2 = 2(x - 4)$

6. Let $P\left(t_1^2, 2t_1\right)$ and $Q\left(t_2^2, 2t_2\right)$ be the ends of the chord PQ of the parabola

$$y^2 = 4x. \quad \text{(i)}$$

Therefore, slope of chord $PQ = \frac{2}{t_2 + t_1} = 2$

$\Rightarrow$ $t_2 + t_1 = 1$ (ii)

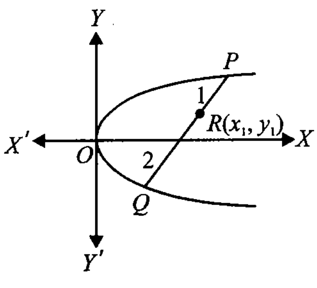


Fig. 3.140

If $R(x_1, y_1)$ is a point dividing PQ internally in the ratio 1 :2, then

$$x_1 = \frac{t_2^2 + 2t_1^2}{1+2},$$

$$y_1 = \frac{1(2t_2) + 2(2t_1)}{1+2}$$

$\Rightarrow$ $$t_2^2 + 2t_1^2 = 3x_1 \quad \text{(iii)}$$

and $$t_2 + 2t_1 = \frac{(3y_1)}{2} \quad \text{(iv)}$$

From Eqs. (ii) and (iv), we get

$$t_1 = \frac{3}{2}y_1 - 1, t_2 = 2 - \frac{3}{2}y_1$$

Substituting in Eq. (iii), we get

$$\left(2 - \frac{3}{2}y_1\right)^2 + 2\left(\frac{3}{2}y_1 - 1\right)^2 = 3x_1$$

$\Rightarrow$ $$\left(\frac{9}{4}\right)y_1^2 - 4y_1 = x_1 - 2$$

$$\left(y_1 - \frac{8}{9}\right)^2 = \left(\frac{4}{9}\right)\left(x_1 - \frac{2}{9}\right)$$

Therefore, locus of the point $R(x_1, y_1)$ is

$$\left(y - \frac{8}{9}\right)^2 = \left(\frac{4}{9}\right)\left(x - \frac{2}{9}\right)$$

which is a parabola having vertex at the point $\left(\frac{2}{9}, \frac{8}{9}\right)$.

7. Let the three points on the parabola $y^2 = 4ax$ be $A(at_1^2, 2at_1)$, $B(at_2^2, 2at_2)$ and $C(at_3^2, 2at_3)$.
Then equations of tangents at A, B and C are

$$y = \frac{x}{t_1} + at_1 \quad \text{(i)}$$

$$y = \frac{x}{t_2} + at_2 \quad \text{(ii)}$$

$$y = \frac{x}{t_3} + at_3 \quad \text{(iii)}$$

Solving the above equations pairwise, we get the points $P(at_1t_2, a(t_1 + t_2))$, $Q(at_2t_3, a(t_2 + t_3))$, $R(at_3t_1, a(t_3 + t_1))$

Now area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} 1 & at_1^2 & 2at_1 \\ 1 & at_2^2 & 2at_2 \\ 1 & at_3^2 & 2at_3 \end{vmatrix}$

$$= a^2\begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{vmatrix}$$

$$= \left|a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)\right| \quad \text{(iv)}$$

Also area of $\Delta PQR = \frac{1}{2}\begin{vmatrix} 1 & a_1t_2 & a(t_1 + t_2) \\ 1 & a_2t_3 & a(t_2 + t_3) \\ 1 & a_3t_1 & a(t_3 + t_1) \end{vmatrix}$

$$= \frac{a^2}{2}\begin{vmatrix} 1 & t_1t_2 & t_1 + t_2 \\ 1 & t_2t_3 & t_2 + t_3 \\ 1 & t_3t_1 & t_3 + t_1 \end{vmatrix}$$

$$= \frac{a^2}{2}\begin{vmatrix} 0 & (t_1 - t_3)t_2 & t_1 - t_3 \\ 0 & (t_2 - t_1)t_3 & t_2 - t_1 \\ 1 & t_3t_1 & t_3 + t_1 \end{vmatrix}$$

Expanding along C_1,

$$= \left|\frac{a^2}{2}(t_1 - t_3)(t_2 - t_1)(t_2 - t_3)\right| \quad \text{(v)}$$

From Eqs. (iv) and (v), we get

$$\frac{\text{Ar}(\Delta ABC)}{\text{Ar}(\Delta ABC)} = \frac{a^2 \,|\, (t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|}{\frac{a^2}{2}\,|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|}$$

$$= \frac{2}{1}$$

Therefore, the required ratio is 2:1.

8. Equation of any tangent to the parabola $y^2 = 4ax$ is

$$y = mx + \frac{a}{m}$$

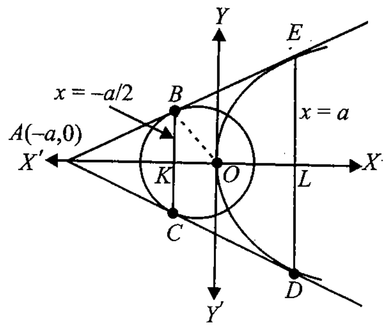


Fig. 3.141

This line will touch the circle $x^2 + y^2 = \frac{a^2}{2}$

if $$\frac{a}{m} = \pm\frac{a}{\sqrt{2}}\sqrt{m^2 + 1}$$
$$[c = \pm r\sqrt{1 + m^2}]$$

$\Rightarrow$ $$\frac{a^2}{m^2} = \frac{a^2}{2}(m^2 + 1)$$

$\Rightarrow$ $$2 = m^4 + m^2$$

$$\Rightarrow \quad m^4 + m^2 - 2 = 0$$

$$\Rightarrow \quad (m^2 + 2)(m^2 - 1) = 0$$

$$\Rightarrow \quad m = 1, -1$$

Thus, the two tangents (common one) are $y = x + a$ and $y = -x - a$

These two intersect each other at $(-a, 0)$.

The chord of contact of circle w.r.t. $A(-a, 0)$ is

$$(-a)x + (0)y = \frac{a^2}{2}$$

or $$x = -\frac{a}{2}$$

and the chord of contact of parabola w.r.t. $A(-a, 0)$ is

$$(0)y = 2a(x - a)$$

or $$x = a$$

Note that DE is latus return of parabola $y^2 = 4ax$, therefore, its length is $4a$.

Chords of contact are clearly parallel to each other, so required quadrilateral is a trapezium.

$$\text{Ar(trap } BCDE) = \frac{1}{2}(BC + DE) \times KL$$

$$= \frac{1}{2}(a + 4a)\left(\frac{3a}{2}\right)$$

$$= \frac{15a^2}{4}$$

9. Point of intersection of tangents at points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ is

$$R(h, k) \equiv (at_1t_2, a(t_1 + t_2))$$

$$\Rightarrow \quad t_1 + t_2 = \frac{k}{a} \text{ and } t_1t_2 = \frac{h}{a}$$

Now $$\tan 45° = \left|\frac{\frac{1}{t_1} - \frac{1}{t_2}}{1 + \frac{1}{t_1t_2}}\right|$$

$$\Rightarrow \quad 1 = \left|\frac{t_2 - t_1}{t_1t_2 + 1}\right|$$

$$= \left|\frac{\sqrt{(t_1 + t_2)^2 - 4t_1t_2}}{t_1t_2 + 1}\right|$$

$$\Rightarrow \quad 1 = \left|\frac{\sqrt{\frac{k^2}{a^2} - 4\frac{h}{a}}}{\frac{h}{a} + 1}\right|$$

$$\Rightarrow \quad k^2 - 4ah = (h + a)^2$$

$$\Rightarrow \quad x^2 - y^2 + 6ay + a^2 = 0$$

which is parabola.

10. Given that $C_1: x^2 = y - 1$, $C_2: y^2 = x - 1$

Here C_1 and C_2 are symmetrical about the line $y = x$.

Let $P(x_1, x_1^2 + 1)$ on C_1 and $Q(y_2^2 + 1, y^2)$ on C_2

Then image of P in $y = x$ is $P_1(x_1^2 + 1, x_1)$ on C_2 and image of Q in $y = x$ is $Q_1(y_2, y_2^2 + 1)$ on C_1.

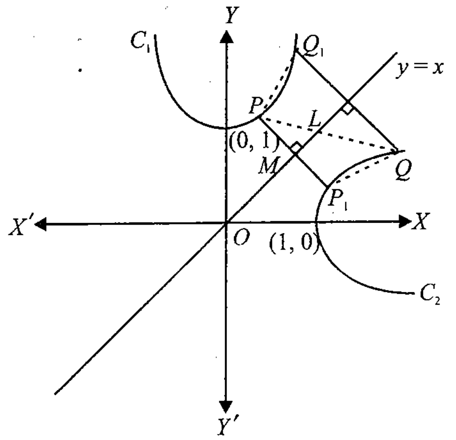


Fig. 3.142

Now PP_1 and QQ_1 both are perpendicular to mirror line $y = x$.

Also M is midpoint of PP_1

($\because P_1$ is mirror image of P in $y = x$)

$$\therefore \quad PM = \frac{1}{2}PP_1$$

In ΔPML, $$PL > PM$$

$$\Rightarrow \quad PL > \frac{1}{2}PP_1 \quad \text{(i)}$$

Similarly, $$LQ > \frac{1}{2}QQ_1 \quad \text{(ii)}$$

Adding Eqs. (i) and (ii), we get

$$PL + LQ > \frac{1}{2}(PP_1 + QQ_1)$$

$$\Rightarrow \quad PQ > \frac{1}{2}(PP_1 + QQ_1)$$

$\Rightarrow PQ$ is more then mean of PP_1 and QQ_1.

$$\Rightarrow \quad PQ \ge \min(PP_1, QQ_1)$$

Let $\min(PP_1, QQ_1) = PP_1$

Then

$$PQ^2 \ge PP_1^2 = \left(x_1^2 + 1 - x_1\right)^2 + \left(x_1^2 + 1 - x_1\right)^2$$

$$= 2\left(x_1^2 + 1 - x_1\right)^2$$

$$= f(x_1)$$

$\Rightarrow \quad f'(x_1) = 4(x_1^2 + 1 - x_1)(2x_1 - 1)$

$$= 4\left(\left(x_1 - \tfrac{1}{2}\right)^2 + \tfrac{3}{4}\right)(2x_1 - 1)$$

$\therefore \quad f'(x_1) = 0$ when $x_1 = \frac{1}{2}$

Also $\quad f'(x_1) < 0$ if $x_1 < \frac{1}{2}$

and $\quad f'(x_1) > 0$ if $x_1 > \frac{1}{2}$

$\Rightarrow f(x_1)$ is minimum when $x_1 = \frac{1}{2}$

Thus, at $x_1 = \frac{1}{2}$, point P is P_0 on C_1

$$P_0\left(\tfrac{1}{2}, \left(\tfrac{1}{2}\right)^2 + 1\right) \equiv \left(\tfrac{1}{2}, \tfrac{5}{4}\right)$$

Similarly, Q_0 on C_2 will be image of P_0 with respect to $y = x$.

$\therefore \quad Q_0 \equiv \left(\frac{5}{4}, \frac{1}{2}\right)$

11. Equation of normal to parabola $y^2 = 4x$ having slope m is

$$y = mx - 2m - m^3$$

It passes through the point $P(h, k)$

$\Rightarrow \quad mh - k - 2m - m^3 = 0$

$\Rightarrow \quad m^3 + (2 - h)m + k = 0 \qquad$ (i)

which is cubic in m and has three roots such that product of roots

$$m_1 m_2 m_3 = -k \qquad \text{(from Eq. (i))}$$

But given that $m_1 m_2 = \alpha$

$\therefore \quad m_3 = -\frac{k}{\alpha}$

But m_3 must satisfy Eq. (i)

$\Rightarrow \frac{-k^3}{\alpha^3} + (2 - h)\left(\frac{-k}{\alpha}\right) + k = 0$

$\Rightarrow \quad k^2 - 2\alpha^2 - h\alpha^2 - \alpha^3 = 0$

$\therefore$ Locus of $P(h, k)$ is $\quad y^2 = \alpha^2 x + (\alpha^3 - 2\alpha^2)$

But given that, locus of P is a part of parabola $y^2 = 4x$, therefore comparing the two, we get

$$\alpha^2 = 4 \text{ and } \alpha^3 - 2\alpha^2 = 0$$

$\Rightarrow \quad \alpha = 2$

12. Parabola is $(y - 1)^2 = 4(x - 1)$ whose directrix is y-axis or $x = 0$.

Any point on this parabola is $P(t^2 + 1, 2t + 1)$, $t \in R$

$\therefore$ Equation of tangent at $P(t^2 + 1, 2t + 1)$ is

$$t(y - 1) = (x - 1) + t^2$$

or $\quad x - ty + (t^2 + t - 1) = 0 \qquad$ (i)

$\Rightarrow$ Tangent meets directrix at $Q\left(0, \frac{t^2 + t - 1}{t}\right)$

Now Q is mid point of PR

$\therefore \quad \frac{h + t^2 + 1}{2} = 0$

and $\quad \frac{k + 2t + 1}{2} = \frac{t^2 + t - 1}{t}$

$\Rightarrow \quad t^2 = -1 - h$

and $\quad kt + 2t^2 + t = 2t^2 + 2t - 2$

$\Rightarrow \quad t^2 = -1 - h$

and $\quad 2 = t(1 - k)$

Eliminating t, we have

$$4 = (-1 - h)(1 - k)^2$$

or $\quad (x + 1)(y - 1)^2 + 4 = 0$

Objective Type

Fill in the blanks

1. Tangents at the extremities of the focal chord intersect on directrix and tangents at the end of latus rectum intersect at foot of directrix $(-1, 0)$.

Multiple choice questions with one correct answer

1. a. The focus of parabola $y^2 = 2px$ is $\left(\frac{p}{2}, 0\right)$ and directrix $x = -\frac{p}{2}$.

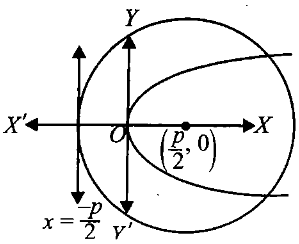


Fig. 3.143

$\therefore$ Centre of circle is $\left(\frac{p}{2}, 0\right)$ and radius $= \frac{p}{2} + \frac{p}{2} = p$

$\therefore$ Equation of circle is $\left(x - \frac{p}{2}\right)^2 + y^2 = p^2$

or $\quad 4x^2 + 4y^2 - 4px - 3p^2 = 0$

Solving this circle with the given parabola, we have (eliminating y)

$$4x^2 + 8px - 4px - 3p^2 = 0$$

$\Rightarrow \quad 4x^2 + 4px - 3p^2 = 0$

$\Rightarrow \quad (2x + 3p)(2x - p) = 0$

$\Rightarrow \quad x = \frac{-3p}{2}, \frac{p}{2}$

$\Rightarrow \quad y^2 = -3p^2$ (not possible),

$\Rightarrow \quad y^2 = 2p.\frac{p}{2} \Rightarrow = \pm p$

Therefore, required points are $\left(\frac{p}{2}, p\right), \left(\frac{p}{2}, -p\right)$.

2. c. $\frac{x+y}{2} = t^2 + 1, \frac{x-y}{2} = t$

Eliminating t, $2(x + y) = (x - y)^2 + 4$

Since 2nd degree terms form a perfect square, it represents a parabola (also $\Delta \neq 0$).

3. b. $y = mx + c$ is normal to the parabola $y^2 = 4ax$ if $c = -2am - am^3$

Here $m = -1$ and $c = k$ and $a = 3$

$\therefore \quad c = k = -2(3)(-1) - 3(-1)^3$

$= 9$

4. c. $y^2 = kx - 8$

$\Rightarrow \quad y^2 = k\left(x - \frac{8}{k}\right)$

Directrix of parabola is $\quad x = \frac{8}{k} - \frac{k}{4}$

Now $x = 1$ coincides with $x = \frac{8}{k} - \frac{k}{4}$

$\Rightarrow \quad \frac{8}{k} - \frac{k}{4} = 1$, we get $k = 4$

5. c. Equation of tangent to the given parabola having slope m is

$$y = mx + \frac{1}{m} \quad \text{(i)}$$

Equation of tangent to the given circle having slope m is

$$y = m(x - 3) \pm 3\sqrt{1 + m^2} \quad \text{(ii)}$$

Equations (i) and (ii) are identical,

$\Rightarrow \quad \frac{1}{m} = -3m \pm 3\sqrt{1 + m^2}$

$\Rightarrow \quad 1 + 3m^2 = \pm 3m\sqrt{1 + m^2}$

$\Rightarrow \quad 1 + 6m^2 + 9m^4 = 9(m^2 + m^4)$

$\Rightarrow \quad 3m^2 = 1$

$\Rightarrow \quad m = \pm \frac{1}{\sqrt{3}}$

Hence, equation of common tangent is $\sqrt{3}y = x + 3$ (as tangent is lying above x-axis).

6. d. $\quad y^2 + 4y + 4x + 2 = 0$

$y^2 + 4y + 4 = -4x + 2$

$(y + 2)^2 = -4(x - 1/2)$

It is of the form $Y^2 = 4AX$ whose directrix is given by $X = A$

$\therefore \quad$ Required equation is $x - \frac{1}{2} = 1$

$\Rightarrow \quad x = \frac{3}{2}$

7. c. If (h, k) is the midpoint of line joining focus $(a, 0)$ and $Q(at^2, 2at)$ on parabola then

$$h = \frac{a + at^2}{2} \quad k = at$$

Eliminating t, we get,

$$2h = a + a\left(\frac{k^2}{a^2}\right)$$

$\Rightarrow \quad k^2 = a(2h - a)$

$\Rightarrow \quad k^2 = 2a(h - \frac{a}{2})$

Therefore, locus of (h, k) is

$$y^2 = 2a(x - \frac{a}{2})$$

whose directrix is $\quad (x - \frac{a}{2}) = -\frac{a}{2}$

$\Rightarrow \quad x = 0$

8. a. For parabola $y^2 = 16x$, focus $= (4, 0)$. Let m be the slope of focal chord, then equation is

$$y = m(x - 4) \quad \text{(i)}$$

Given that above line is a tangent to the circle $(x - 6)^2 + y^2 = 2$ for which centre $C(6, 0)$ and radius $r = \sqrt{2}$

Therefore, length of perpendicular from $(6, 0)$ to Eq. (i) $= r$

$\Rightarrow \quad \frac{|6m - 4m|}{\sqrt{m^2 + 1}} = \sqrt{2}$

$\Rightarrow \quad 2m^2 = m^2 + 1$

$\Rightarrow \quad m^2 = 1$

$\Rightarrow \quad m = \pm 1$

9. c. $\quad y = mx + \frac{1}{m}$

Above tangent passes through $(1, 4)$

$\Rightarrow \quad 4 = m + \frac{1}{m}$

$\Rightarrow \quad m^2 - 4m + 1 = 0$

Now angle between the lines is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2}$$

$$= \frac{\sqrt{16-4}}{1+1} = \sqrt{3}$$

$\Rightarrow \quad \theta = \frac{\pi}{3}$

Alternative Solution:

The combined equation of tangents drawn from (1, 4) to the parabola $y^2 = 4x$ is

$$(y^2 - 4x)(4^2 - 4 \times 1) = [y \times 4 - 2(x + 1)]^2$$

[Using $SS_1 = T^2$]

$\Rightarrow \quad 12(y^2 - 4x) = 4(2y - x - 1)^2$

$\Rightarrow \quad 3(y^2 - 4x) = 4y^2 + x^2 + 1 - 4xy + 2x - 4y$

$\Rightarrow x^2 + y^2 - 4xy + 14x - 4y + 1 = 0$

Now we know angle between two lines, given by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$\theta = \tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{a+b}\right)$$

$\therefore$ The required angle is

$$\theta = \tan^{-1}\left(\frac{2\sqrt{2^2 - 1 \times 1}}{1+1}\right)$$

$$= \tan^{-1}(\sqrt{3}) = \pi/3$$

10. d. The given curve is

$$y = x^2 + 6$$

Equation of tangent at (1, 7) is

$$\frac{1}{2}(y + 7) = x(1) + 6$$

$\Rightarrow \quad 2x - y + 5 = 0 \quad$ (i)

According to question, this tangent Eq. (i) touches the circle $x^2 + y^2 + 16x + 12y + C = 0$ at Q (centre of circle (−8, − 6))

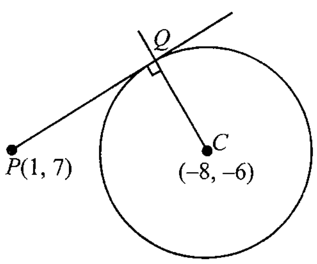


Fig. 3.144

Then equation of CQ which is perpendicular to Eq. (i) and passes through (−8, − 6) is

$$y + 6 = -\frac{1}{2}(x + 8)$$

$\Rightarrow \quad x + 2y + 20 = 0 \quad$ (ii)

Now Q is point of intersection of Eqs. (i) and (ii),

i.e., $\quad x = -6, y = -7$

Therefore, required point is (−6, − 7).

11. d. Vertex is (1, 1), focus (2, 2), directrix $x + y = 0$

$\therefore$ Equation of parabola is

$$(x-2)^2 + (y-2)^2 = \left(\frac{x+y}{\sqrt{2}}\right)^2$$

$\Rightarrow \quad x^2 + y^2 - 2xy = 8(x + y - 2)$

$\Rightarrow \quad (x - y)^2 = 8(x + y - 2)$

12. b. Solving the curves, we have

$$x^2 + 4x - 6x + 1 = 0$$

or $\quad (x - 1)^2 = 0$

or $\quad x = 1.$

Hence, two curves touch each other. For $x = 1, y = \pm 2$.

The circle and the parabola touch each other at the point (1, 2) and (1, − 2) as shown in the figure.

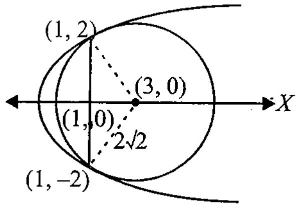


Fig. 3.145

13. c. $\quad y^2 = x$

and Q will lie on it

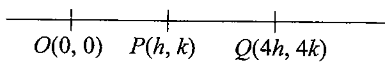


Fig. 3.146

$\Rightarrow (4k)^2 = 4 \times 4h$

$\Rightarrow y^2 = x$ (replacing h by x and k by y)

Multiple choice questions with one or more than one correct answer

1. a., b. If $y = mx + c$ is tangent to $y = x^2$, then $x^2 - mx - c = 0$ has equal roots.

$\Rightarrow \quad m^2 + 4c = 0$

$\Rightarrow \quad c = -\frac{m^2}{4}$

$\therefore \quad y = mx - \frac{m^2}{4}$

is tangent to $y = x^2$

$\because$ This is also tangent to $y = -(x-2)^2$

$\Rightarrow mx - \frac{m^2}{4} = -x^2 + 4x - 4$ has equal roots.

$\Rightarrow x^2 + (m-4)x + \left(4 - \frac{m^2}{4}\right) = 0$ has equal roots.

$\Rightarrow \quad (m-4)^2 - 4\left(4 - \frac{m^2}{4}\right) = 0$

$\Rightarrow \quad m^2 - 8m + 16 + m^2 - 16 = 0$

$\Rightarrow \quad m = 0, 4$

$\Rightarrow \quad y = 0$

or $\quad y = 4x - 4$ are the tangents.

2. b., c.

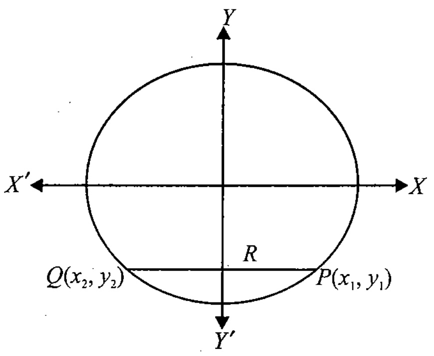


Fig 3.147

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

$$b^2 = a^2(1 - e^2)$$

$\Rightarrow \quad e = \frac{\sqrt{3}}{2}$

$\Rightarrow P\left(\sqrt{3}, -\frac{1}{2}\right)$ and $Q\left(-\sqrt{3}, -\frac{1}{2}\right)$ (given y_1 and y_2 less than 0).

Coordinates of midpoint of PQ are $R \equiv \left(0, -\frac{1}{2}\right)$

$$PQ = 2\sqrt{3}$$
$$= \text{length of latus rectum.}$$

$\Rightarrow$ Two parabola are possible whose vertices are $\left(0, -\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$ and $\left(0, \frac{\sqrt{3}}{2} - \frac{1}{2}\right)$

Hence, the equations of the parabolas are

$$x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$$

and $\quad x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$

3. a., d.

Tangent at point $P(at^2, 2at)$ is

$$ty = x + at^2.$$

It meets the x-axis at $(-at^2, 0)$

Normal at point P is $y = -tx + 2at + at^3$

It meets the x-axis at $(2a + at^2, 0)$

Let centroid of triangle PNT is $G \equiv (h, k)$

$\Rightarrow \quad h = \frac{2a + at^2}{3}$ and $k = \frac{2at}{3}$

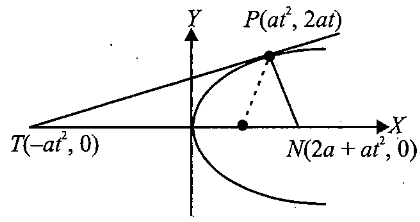


Fig. 3.148

$\Rightarrow \quad \left(\frac{3h - 2a}{a}\right) = \frac{9k^2}{4a^2}$

$\Rightarrow$ Required parabola is

$$\frac{9y^2}{4a^2} = \frac{(3x - 2a)}{a} = \frac{3}{a}\left(x - \frac{2a}{3}\right)$$

$\Rightarrow \quad y^2 = \frac{4a}{3}\left(x - \frac{2a}{3}\right)$

Vertex $\equiv \left(\frac{2a}{3}, 0\right)$; focus $\equiv (a, 0)$

4. c., d.

$$A = (t_1^2, 2t_1) B = (t_2^2, 2t_2)$$

Centre $= \left[\frac{t_1^2 + t_2^2}{2}, (t_1 + t_2)\right]$

$t_1 + t_2 = \pm r$

$$m = \frac{2(t_1 - t_2)}{t_1^2 - t_2^2} = \frac{2}{t_1 + t_2} = \pm\frac{2}{r}.$$

5. a., b., d.

$y^2 = 4x$

Equation of normal is $y = mx - 2m - m^3$.

It passes through (9, 6)

$\Rightarrow m^3 - 7m + 6 = 0$

$\Rightarrow m = 1, 2, -3$

$\Rightarrow y - x + 3 = 0, y + 3x - 33 = 0, y - 2x + 12 = 0$

Match the following

1. i. → a.; ii → b.; iii → d.; iv → c.

Equation of normal to $y^2 = 4x$ is

$$y = mx - 2m - m^3$$

As it passes through (3, 0), we get $m = 0, 1, -1$

Then three points on parabola are given by $(m^2, -2m)$ for $m = 0, 1, -1$

$\therefore$ $P(0, 0)$, $Q(1, -2)$, $R(1, 2)$

$$\therefore \quad \text{Area of } \Delta PQR = \frac{1}{2} \times \begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 2 \text{ sq. units}$$

$$R = \frac{abc}{4\Delta} = \frac{\sqrt{5} \times \sqrt{5} \times 4}{4 \times 2} = \frac{5}{2}$$

Centroid of $\Delta PQR = \left(\frac{2}{3}, 0\right)$, (where a, b, c are the sides of ΔPQR)

Circumcentre $= \left(\frac{5}{2}, 0\right)$

Comprehension based questions

Sol. Solving circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$

$$x^2 + 8x - 9 = 0$$

$$\Rightarrow \quad x = 1, -9$$

$$\Rightarrow \quad x = 1 \ (x = -9 \text{ not possible})$$

$$\Rightarrow \quad y^2 = 8$$

$$\Rightarrow \quad y = \pm 2\sqrt{2}$$

Hence, points of intersection are $P(1, 2\sqrt{2})$ and $Q(1, -2\sqrt{2})$.

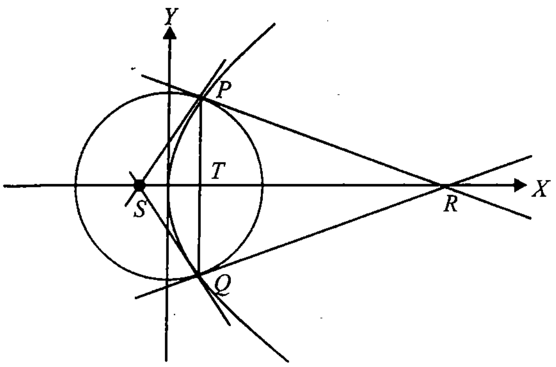


Fig. 3.149

Tangent to the parabola at point P is

$$2\sqrt{2}y = 4(x + 1)$$

It meets the x-axis at $S(-1, 0)$.

Tangent to the circle at point P is $(1)x + 2\sqrt{2}\,y = 9$

It meets the x-axis at $R(9, 0)$.

1. c.
$$\frac{\text{Ar}(\Delta PQS)}{\text{Ar}(\Delta PQR)} = \frac{\frac{1}{2}PQ \times ST}{\frac{1}{2}PQ \times TR}$$

$$= \frac{ST}{TR} = \frac{2}{8} = \frac{1}{4}$$

2. b. For ΔPRS,

$$\text{Ar}(\Delta PRS) = \Delta = \frac{1}{2} \times SR \times PT$$

$$= \frac{1}{2} \times 10 \times 2\sqrt{2}$$

$$\therefore \quad \Delta = 10\sqrt{2},\ a = PS = 2\sqrt{3}$$

$$b = PR = 6\sqrt{2},\ c = SR = 10$$

$$\therefore \quad \text{Radius of circumference} = R = \frac{abc}{2\Delta}$$

$$= \frac{2\sqrt{3} \times 6\sqrt{2} \times 10}{4 \times 10\sqrt{2}} = 3\sqrt{3}$$

3. d. Radius of incircle of triangle PQR is

$$= \frac{\text{Area of } \Delta PQR}{\text{Semi-perimeter of } \Delta PQR}$$

$$= \frac{\Delta}{s}$$

We have $a = PR = 6\sqrt{2}$, $b = QP = PR = 6\sqrt{2}$ and $c = PQ = 4\sqrt{2}$

Also
$$\Delta = \frac{1}{2} \times PQ \times TR = 16\sqrt{2}$$

$$\therefore \quad s = \frac{6\sqrt{2} + 6\sqrt{2} + 4\sqrt{2}}{2} = 8\sqrt{2}$$

$$\therefore \quad r = \frac{16\sqrt{2}}{8\sqrt{2}} = 2$$

Assertion and reasoning

1. a. The given curve is

$$y = -\frac{x^2}{2} + x + 1$$

or
$$(x - 1)^2 = -2\left(y - \frac{3}{2}\right)$$

which is a parabola, so should be symmetric with respect to its axis $x - 1 = 0$.

$\therefore$ Both the statement are true and statement 2 is a correct explanation for statement 1.

Integer type

1. (2)

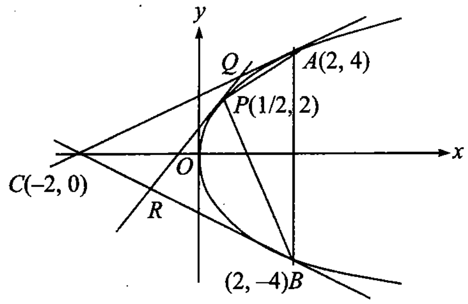


Fig. 3.150

$y^2 = 8x = 4{\cdot}2{\cdot}x$

Tangents at end points of latus- rectum meet on directrix on x-axis $(-2, 0)$

Δ_1 = Area of ΔABC

$$= \frac{1}{2}(8)\left(2-\frac{1}{2}\right) = 6$$

Δ_2 = Area of ΔCQR

Now tangent at point $A(2, 4)$ is $4y = 4(x + 2)$ or $y = x + 2$

Tangent at point $(2, -4)$ is $-4y = 4(x + 2)$ or $y = -x - 2$

Also tangent at point $P(1/2, 2)$ is $2y = 4(x + 1/2)$ or $y = 2x + 1$

Solving for Q and R we get $Q(1, 3)$ and $R(-1, -1)$

Hence area of ΔCQR is

$$\frac{1}{2}\left\|\begin{matrix} -1 & -1 & 1 \\ 1 & 3 & 1 \\ -2 & 0 & 1 \end{matrix}\right\| = \frac{1}{2}(-3+2+6+1) = 3$$

Hence the ratio of area is $6/3 = 2$