Mathematics
Class – XII

Time allowed: **3** hours Maximum Marks: **100**

General Instructions:

a) All questions are compulsory.
b) The question paper consists of 26 questions divided into three sections A, B and C. Section A comprises of 6 questions of one mark each, Section B comprises of 13 questions of four marks each and Section C comprises of 7 questions of six marks each.
c) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
d) Use of calculators is not permitted.

Section A

(1 marks)

1. Write the principal value of $\text{Cos}^{-1}\left(\cos\left(\frac{-5\pi}{3}\right)\right)$

2. For what value of λ, $\vec{a}=\lambda\hat{i}+\hat{j}+4\hat{k}$ is perpendicular $\vec{b}=2\hat{i}+6\hat{j}+3\hat{k}$ **?**

3. If $\begin{vmatrix} 2x & 4 \\ -1 & x \end{vmatrix}=\begin{vmatrix} 6 & -3 \\ 2 & 1 \end{vmatrix}$ find X?

4. If f! A → B is bijective function such that n(A) = 10, then n(B) =?

Section B

(2 marks)

5. Show that $\text{Cos}\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right)$

6. Find $\int\tan^2 x\ \sec^2 x$ dx.

7. Three dice are thrown at the same time. Find the Probability of getting three dices, if it is known that the sum of the no. on the two's, was six.

8. If $x=e^{\text{Cos}\,2t}$ and $y=e^{\sin 2t}$, Prove that $\frac{dy}{dx}=\frac{y\log x}{x\log y}$.

9. Find the differential equation of system of concentric circle with centre (1, 2)?

10. A vector $\vec{r}$. inclined at equal angles to the three axes. If the magnitude of $\vec{r}$ is $2\sqrt{3}$ units find $\vec{r}$.

11. If $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$, then find (x – y).

12. find the approximate volume of metal in a follow spherical shell whose internal and external radii are 3cm and 3.0005cm respectively.

Section C

(4 marks)

13. If $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $\vec{a}\times\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$, find the value of $\vec{a}\cdot\vec{b}$.

14. Show that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

15. Solve $\int \frac{2x-1}{(x-1)(x+2)(x-3)}dx$

16. find the shortest distance between the lines given by
$\vec{r} = (8+3d\hat{i} - (9+16d)\hat{j} + (10+7d)\hat{k}$ and $\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$

17. A die is tossed twice. A 'success' is getting an even number on a toss. find the variance of the number of success.

18. There are two bags, one of which contains 3 black and 4 white balls while the other contains 4 black and 3 white balls. A die is thrown. If it shows up 1 or 3, a ball is taken from the Ist bag; but it shows up any other number, a ball is chosen from the 2nd bags. Find the probability of choosing a black ball.

19. Solve the differential equation

$$y - x\frac{dy}{dx} = 2\left(y^2 + \frac{dy}{dx}\right)$$

20. find the derivative of $\tan^{-1}\left(\frac{2x}{1-x^2}\right) w\cdot r\cdot t.\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

21. Prove the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

22. find the equation of the tangent to the curve $y=\sqrt{3x-2}$ which is parallel to the line 4x – y + 5 = 0.

Or

A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y co-ordinate is changing 8 times as fast as the x-coordinate.

Q23. If $A=\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$. Show that $A^2 -5A - 14I = 0$, Hence find A^{-1}

Or

Find a matrix A, if $A+\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}=\begin{bmatrix} 3 & -6 \\ -3 & 8 \end{bmatrix}$

Section D

(6 marks)

Q24. Complete the area bounded by line x + 2y = 2, y – x = 1 and 2x + y = 7

Q25. A manufacturer produces two models of bike-Model X and Model Y. Model X takes a 6 man-hours to make per unit, while Model y takes 10 man hours per unit. There is a total of 450 man-hour available per week. Handling and Marketing costs are Rs.2000 and Rs.1000 per unit of Model X and Y respectively. The total funds available for these purposes are Rs80,000 per week. Profit per unit for Model X and Y are Rs.1000 and Rs.500, respectively. How many bikes of each model should the manufacturer produce so as to yield a maximum profit? Find maximum profit.

26. find the equation of line passing through the point (3, 0, 1) and parallel to planes x + 2y = 0 and 3y – z = 0.

27. Show that the height of the cylinder of maximum volume that can be inscribed in a cone of height h is $\frac{1}{3}h$.

28. An urn contains 25 balls of which 10 bear mark 'A' and remaining 15 bears a mark 'B'.

A ball is drawn at random from the urn, its marks noted down and it is replaced. If 6 balls are drawn in this way, find the probability that

(i) All will bear mark 'A'

(ii) not more than two will bear ‘B’ mark.

(iii) at least one ball will bear ‘A’ mark and ‘B’ mark will be equal.

29. Evaluate : $\int_0^1 x(\tan^{-1} x)^2 \, dx$

OR

Evaluate $\int_1^2 (x^2 + x + 2) dx$ as a limit of sums.

(Solution)
Class – XII Mathematics

Ans.1. Cos is of period 2π, the angle associated with $\frac{-5\pi}{3}$ is equivalent to

$$\left(2\pi + \frac{-5\pi}{3}\right) = \frac{\pi}{3}$$

Hence $\text{Cos}^{-1}\left(\text{Cos}\left(\frac{-5\pi}{3}\right)\right)$ is same as $\text{Cos}^{-1}\left(\text{Cos}\frac{\pi}{3}\right) = \text{Cos}^{-1}\left(\frac{1}{2}\right)$

$$\frac{\pi}{3} = Ans.$$

Ans.2. if $\vec{a}$ and $\vec{b}$ are perpendicular then

$\vec{a} \cdot \vec{b} = 0$

$(\lambda\hat{i} + \hat{j} + 4\hat{k})(2\hat{i} + 6\hat{j} + 3\hat{k}) = 0$

$2\lambda + 6 + 12 = 0$

$2\lambda = - 18$

$\lambda = \frac{18}{2}$

$\lambda = -9$

Ans.3. $\begin{vmatrix} 2x & 4 \\ -1 & x \end{vmatrix} = \begin{vmatrix} 6 & -3 \\ 2 & 1 \end{vmatrix}$

$2x^2 + 4 = 6 + 6$

$2x^2 = 12 – 4$

$2x^2 = 8$

$x^2 = \frac{8}{2} = 4$

$x^2 = 4 \Rightarrow x = \pm\sqrt{4}$

$x = \pm 2$ Ans.

Ans.4. If A and B are bijective function,
then n(A) = n(B)
So, that n(B) = 10

Ans.5. Let $\alpha = \tan^{1}\frac{1}{7}$

Or $\tan\alpha = \frac{1}{7}$

Then $\sin\alpha = \frac{1}{\sqrt{1^2+7^2}} = \frac{1}{\sqrt{50}}$; $\text{Cos}\,\alpha = \frac{7}{\sqrt{1^2+7^2}} = \frac{7}{\sqrt{50}}$

Similarly $\beta = \tan^{-1}\frac{1}{3}$

$\Rightarrow \tan\beta = \frac{1}{3}$ $\sin\beta = \frac{1}{\sqrt{1^2+3^2}} = \frac{1}{\sqrt{10}}$

$\text{Cos}\,\beta = \frac{3}{\sqrt{1^2+3^2}} = \frac{3}{\sqrt{10}}$

New according to question $\text{Cos}\left(2\tan^{-1}\frac{1}{7}\right)$

$= \text{Cos}(2\alpha)$

$= \text{Cos}^2\alpha - \sin^2\alpha$

$= \left(\frac{7}{\sqrt{50}}\right)^2 - \left(\frac{1}{\sqrt{50}}\right)^2$

$= \frac{49}{50} - \frac{1}{50} = \frac{48}{50} = \frac{24}{25}$

$\text{Sin}\left(4\tan^{1}\left(\frac{1}{3}\right)\right)$

$= \text{Sin}(4\beta)$

$= 2\sin 2\beta\cos 2\beta$ using value of Cosβ and sinβ

=2(2sinβ cosβ) ($\cos^2\beta - \sin^2\beta$)

$= 2\left(2\times\frac{1}{\sqrt{10}}\times\frac{3}{\sqrt{10}}\right)\left(\left(\frac{3}{\sqrt{10}}\right)^2 - \left(\frac{1}{\sqrt{10}}\right)^2\right)$

$= 2\left(2\times\frac{3}{10}\right)\left(\frac{9}{10} - \frac{1}{10}\right)$

$= \frac{12}{10}\times\frac{8}{10} = \frac{24}{25}$.

Ans.6. $\int \tan^2 x \sec^2 x\, dx$

Put tan x = t

$\text{Sec}^2 x\, dx = dt$

$= \int (t)^2\, dt$

$= \left[\frac{t^3}{3}\right] + C$

$= \frac{1}{3}\tan^3 x + C$

Ans.7. There are 216 elements in the total sample space.

Let an element A as sum of the No. on the dice is 6.
A = (2, 2, 2) (2,1,3) (2,3,1) (1,3,2) (1,2,3) (1,1,4) (4,1,1) (1,4,1) (3,1,2) (3,2,1)
B = getting a No. 2 in all three dices (2,2,2)

$$P(B) = \frac{1}{10}$$

Ans.8. $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$

Applying log on both side in these two terms

$\log x = \cos 2t \log_e$; $\log y = \sin 2t \log_e$

logx = cos2t ....(i) ; logy = sin2t ......(2)

diff. on both side

$\frac{1}{x}\frac{dx}{dt} = -2\sin 2t$; $\frac{1}{y}\frac{dy}{dt}$ = 2cos2t

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y2\cos 2t}{x2\sin 2t}$$

$\frac{dy}{dx} = -\frac{y\cos 2t}{x\sin 2t}$ from Ist & 2nd

$$\frac{dy}{dx} = \frac{-y\log x}{x\log y}$$

Ans.9. Concentric circles with centre (1,2)

The equation of concentric circle with centre (1, 2) is

$(x - 1)^2 + (y - 2)^2 = a^2$

Now let us differentiate $w \cdot r \cdot t$ x

$$2(x-1)+2(y-2)\frac{dy}{dx} = 0$$

Dividing through by2,

$\Rightarrow (x-1)+(y-2)\frac{dy}{dx} = 0$ This is required equation.

Ans.10. Given $|\vec{r}| = 2\sqrt{3}\ units$

Also $\vec{r}$ is equally inclined with OX, OY and OZ

Hence it direction ratios are equal

Let the direction ratios be l, m and n

Since they are equal l = m = n

$\Rightarrow \quad \cos\alpha = \cos\beta = \cos\tau$

We know that $l^2 + m^2 + n^2 = 1$

$\Rightarrow \quad 3l^2 = 1$ therefore $l = \tau\frac{1}{3}$

Hence the direction cosines are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$

$\vec{r} = |\vec{r}|(l\hat{i} + m\hat{j} + n\hat{k})$

$= 2\sqrt{3}\left(\pm\frac{1}{\sqrt{3}}\hat{i} \pm \frac{1}{\sqrt{3}}\hat{j} \pm \frac{1}{\sqrt{3}}\hat{k}\right)$

$\vec{r} = 2(\pm\hat{i} \pm \hat{j} \pm \hat{k})$

Ans.11. $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$

$= \begin{bmatrix} 6 & 8 \\ 10 & 2x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0\cdot & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$

$= \begin{bmatrix} 7 & 8+y \\ 10 & 2x+1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$

8 + y = 0
Y = - 8

2x + 1 = 5
2x = 5 – 1
2x = 4
$x = \frac{4}{2}$
X = 2

X – y = 2 – (-8)
= 2 + 8 = 10

Ans.12. Volume of spherical shell $= \frac{4}{3}\pi(r_2^3 - r_1^3)$

Step 1:-
Let us find the volume of $(3\cdot005)^3$
Let x = 3 and x + Δx = 3·005
∴ Δx = 0·005
Step 2:-
$y = x^3$
$\frac{dy}{dx} = 3x^2$
$dy = 3x^2\,dx$
substituting for x and dx, we get
$dy = \Delta y = 3(3)^2 \times 0\cdot005$
= 27 × ·005
∴ dy = Δy = 0·135
Hence $(3.005)^3 = 27 + \cdot135 = 27.135$
Now volume is

$$U = \frac{4}{3}\pi\,(27\cdot135 - 27)$$

$$= \frac{4}{3}\pi\,(\cdot135) = 0\cdot018\pi cm^3$$

Ans.13. Given that $|\vec{a}| = 2$ and $|\vec{b}| = 5$

$$\vec{a}\times\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{(2)^2 + (1)^2 + (-2)^2}$$

$$= \sqrt{4+1+4} = \sqrt{9} = 3$$

We know that

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\ |\sin\theta|$$

$$3 = 2\times5\,|\sin\theta|$$

$$\pm\frac{3}{10} = \sin\theta$$

$$|\vec{a}\cdot\vec{b}| = |\vec{a}||\vec{b}|\cos\theta$$

$$= 2\times5\frac{\sqrt{91}}{10}$$

$$\vec{a}\cdot\vec{b} = \sqrt{91}$$

Ans.14. $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

$$\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

Taking out common factors a, b and c from
C_1, C_2 and C_3, we have

$$\Delta abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we have

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ b & b-c & -c \\ b-a & b & -a \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 + R_1$, we have

$$\Delta\ abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b-a & b & -a \end{vmatrix}$$

Apply $R_3 \rightarrow R_3 + R_2$, we have

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 2b & 2b & o \end{vmatrix}$$; now taking common from R3 and then C2=C2-C1

$$= 2ab^2c\begin{vmatrix} a & c-a & a+c \\ a+b & -a & a \\ 1 & o & o \end{vmatrix}$$

Expanding along R_3, we have!

$$\Delta = 2ab^2c\left[a(c-a)+a(a+c)\right]$$

$$= 2ab^2c\left[ac-a^2+a^2+ac\right]$$

$$= 2ab^2c\left[2ac\right] = 4a^2b^2c^2$$

Hence, the given result is proved.

Ans.15. $\int \frac{2x-1}{(x-1)(x+2)(x-3)}dx$

Let $\frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{x+2} + \frac{C}{x-3}$

2x – 1 = A(x + 2)(x – 3) + B(x – 1)(x – 3) + C(x – 1)(x + 2)

=A(x^2 + 2x – 3x – 6) + B(x^2 – x – 3x + 3) + C(x^2 – x + 2x – 2)

Equating the coefficient of x^2, x and constant

Term :-

0 = A + B + C ; 2 = - A – 4B + C ; -1 = -6A + 3B – 2C

On solving these 3 equation

$$\begin{array}{r} -A-4B+C=2 \\ A+B+C=0 \\ \hline -3B+2c=2 \end{array} \qquad \begin{array}{l} -1=-6A+3B-2C \\ 6(0=A+B+C) \end{array}$$

$$\begin{array}{r} -1=-6A+3B-2C \\ 0=6A+6B+6C \\ \hline -1=9B+4C \end{array}$$

Now Solve 4^{th} & 5^{th}

9B + 4C = - 1

3(-3B + 2C = 2) $\rightarrow$
$$\begin{array}{r} 9B+4C=-1 \\ -9B+6C=6 \\ \hline 10C=5 \end{array}$$

$-3B+2\times\frac{1}{2}=2 \qquad C=\frac{5}{10}=\frac{1}{2}$

B=-1/3   And A + B + C = 0

A=1/3-1/2 = -1/6

We get A=-1/6;B=-1/3;C=1/2

$\therefore \quad \frac{2x-1}{(x-1)(x+2)(x-3)}=\frac{1}{6(x-1)}-\frac{2}{(3x+2)}+1\frac{1}{2(x-3)}$

$\Rightarrow \int\frac{2x-1}{(x-1)(x+2)(x-3)}dx=\frac{1}{6}\int\frac{dx}{(x-1)}-\frac{2}{3}\int\frac{dx}{(x+2)}+\frac{1}{2}\int\frac{dx}{x-3}$

$=\frac{1}{6}\log|x-1|-\frac{2}{3}Log|x+2|+\frac{1}{2}L\text{og}|x-3|+C$

Ans.16. $\vec{r}=(8\hat{i}-9\hat{j}+10\hat{k})+\lambda(3\hat{i}-16\hat{j}+7\hat{k})$

$\vec{r}=(15\hat{i}-29\hat{j}+5\hat{k})+\mu(3\hat{i}-8\hat{j}+5\hat{k})$

It is known that the shortest distance between the lines, $\vec{r}=\overrightarrow{a_1}+\lambda\overrightarrow{b_1}$ and $\vec{r}=\overrightarrow{a_2}+\lambda\overrightarrow{b_2}$ is given by :-

$$d=\left|\frac{(\overrightarrow{b_1}\times\overrightarrow{b_2})\cdot(\overrightarrow{a_2}-\overrightarrow{a_1})}{|\overrightarrow{b_1}\times\overrightarrow{b_2}|}\right| \qquad \text{.......(i)}$$

$\overrightarrow{a_2}-\overrightarrow{a_1}=(15\hat{i}+29\hat{j}+5\hat{k})-(8\hat{i}-9\hat{j}+10\hat{k})$

$=7\hat{i}+38\hat{j}-5\hat{k}$

$$\overrightarrow{b_1}\times\overrightarrow{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -16 & 7\\ 3 & 8 & -5\end{vmatrix}$$

$=\hat{i}(+80-56)-\hat{j}(-15-21)+\hat{k}(24+48)$

$=24\hat{i}+36\hat{j}+72\hat{k}$

$\Rightarrow|\overrightarrow{b_1}\times\overrightarrow{b_2}|=\sqrt{(24)^2+(36)^2+(72)^2}=\sqrt{7056}=84$

Substituting all values in Ist.

$$d=\left|\frac{(7\hat{i}+38\hat{j}-5\hat{k})\cdot(24\hat{i}+36\hat{j}+72\hat{k})}{84}\right|$$

$$=\left|\frac{168+1368-360}{84}\right|=\left|\frac{1176}{84}\right|$$

$d=14\ unit.$

Ans.17. P(x = 0) **=** P(no success) = P(0, 0)

$\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$

P(x = 1) = P(1S, 1F) or P(1F, 1S)

= P (O, E) or P(E, O)

$$= 2\left(\frac{1}{2}\times\frac{1}{2}\right) = \frac{1}{2}$$

P (x = 2) -P(both failure) = P(F.F)

$$E(X) = \bar{2}PiXi$$

$$= 0\times\frac{1}{4} + 1\times\frac{1}{2} + 2\times\frac{1}{4}$$

= 1

$$\bar{2}PiXi^2$$

$$= 0\times\frac{1}{1} + 1^2\times\frac{1}{2} + 2^2\times\frac{1}{4}$$

$$= \frac{1}{2} + 1 = \frac{3}{2}$$

Variance $\bar{2}PiXi^2 - (\bar{2}PiXi)^2$

$$= \frac{3}{2} - 1^2 \quad = \frac{3}{2} - 1$$

$$= \frac{3-2}{2} = \frac{1}{2}$$

Ans.18. E_1 = 1 Bag is chosen

E_2 = 2 Bag is chosen

A chosing a black ball

P(A) = P(E_1) (P(A/E_1) + P(E_2) P(A/E_2)

Bag 1 is chosen if 1 or 3 appear in throw of die.

$P(E_1) = \frac{2}{6} = \frac{1}{3}$; bag 2 chosen otherwise $P(E_1) = 1 - \frac{1}{3} = \frac{2}{3}$;

P(A/E_1) = P(getting block ball / 1 bag is chosen) $= \frac{3}{7}$

P(A/E_2) = P(getting black ball/ 1 bag is chosen) $= \frac{4}{7}$

P(A) = P(getting black ball)

$$= \frac{1}{3}\times\frac{3}{7} + \frac{2}{3}\times\frac{4}{7} \qquad = \frac{11}{21}$$

Ans.19. $y - x\frac{dy}{dx} = 2\left(y^2 + \frac{dy}{dx}\right)$

$$\Rightarrow y - 2y^2 = (x+2)\frac{dy}{dx}$$

$$\Rightarrow \frac{dx}{x+2} = \frac{dy}{y-2y^2}$$

Let $\frac{1}{y(1-2y)} = \frac{r}{y} + \frac{S}{1-2y}$, where r + s are real No.

$$\Rightarrow \frac{1}{y(1-2y)} = \frac{r+(s-2r)y}{y(1-2y)}$$

Compare the coefficients of y and the constant terms of both sides r = 1 and S – 2r = 0

∴ s = 2r

We get $\frac{1}{y(1-2y)} = \frac{1}{y} + \frac{2}{1-2y}$

Now the equation obtain 1S

$$\frac{1}{x+2} = \frac{1}{y} + \frac{2}{1-2y}$$

Integrating on both sides

$$\log|x+2| = \log\ y - \frac{2\log}{2}|1-2y| + \log C$$

$$\log|x+2| = \log\left|\frac{Cy}{1-2y}\right|$$

Ans.20. Let $u = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Putting x = tanθ we get

$$u = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta) \qquad \left(\because \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}\right)$$

= 2θ $\qquad = 2\tan^{-1} x$

Differentiating both sides with respect to x, we get

$$\frac{du}{dx} = \frac{2}{1+x^2} \qquad .....(i)$$

$$V = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$= \text{Sin}^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \text{Sin}^{-1}(\sin 2\theta) \qquad \left(\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right)$$

= 2 θ $\qquad = 2\tan^{-1} x$

Differentiating both sides with respect to X, we get

$$\frac{du}{dx} = \frac{2}{1+x^2} \qquad .....(2)$$

From 1 and 2nd we get

$$\frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}} = 1$$

Ans.21. The given equations of curves are x = y^2 and xy = k.

Putting x = y^2 in xy = k, then we get

$$y^3 = k \Rightarrow y = k^{1/3}$$

$$\therefore \qquad x = k^{2/3}$$

The point of intersection of the Given curves is

$$\left(K^{2/3}, K^{1/3}\right)$$

Differentiating x = y^2 with respect to x

$$1 = 2y\frac{dy}{dx}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{1}{2y}$$

Therefore the slope of the tangent to the curve

X = y2 at $\left(K^{2/3}, K^{1/3}\right)$

Is $\left.\frac{dy}{dx}\right]_{\left(k^{2/3}, k^{1/3}\right)} = \frac{1}{2k^{1/3}}$

On diff. xy = k w·r·t· x, wehave

$$x\frac{dy}{dx} + y \cdot 1 = 0\,; \qquad \frac{dy}{dx} = \frac{-y}{x}$$

$\therefore$ Slope of tangent to the curve xy = k at $\left(x^{2/3}, k^{1/3}\right)$

Is given by $\frac{dy}{dx} = \frac{-y}{x} = \frac{-k^{1/3}}{k^{2/3}} = \frac{-1}{k^{1/3}}$

The two curves intersects at right angles if the tangents to the curves at the point of intersection tangent i.e; at $\left(k^{2/3}, k^{1/3}\right)$ are perpendicular to each other.

This implies that we should have the product of the tangents as -1

The product of the slopes of their respective tangent at $\left(k^{2/3}, k^{1/3}\right)$ is -1

$$\left(\frac{1}{2k^{2/3}}\right)\left(\frac{-1}{k^{2/3}}\right)=1$$

$\Rightarrow \quad 2k^{2/3}=1$

$\Rightarrow \quad \left(2k^{2/3}\right)^3=(1)^3$

$\Rightarrow \quad 8k^2=1$

Hence the given two curves cut at right angles if $8k^2 = 1$

Ans.22. Given that $y=\sqrt{3x-2}$

$\therefore \quad \frac{dy}{dx}=\frac{1}{2\sqrt{3x-2}}\times 3=\frac{3}{2\sqrt{3x-2}}$ ....(i)

Equation of line is

4x – y + 5 = 0

$\Rightarrow \quad 4-\frac{dy}{dx}=0$

$\boxed{4=\frac{dy}{dx}}$ ......(ii)

Thus the slope of the line is 4 and the tangent to the curve $y=\sqrt{3x-2}$ is parallel to the given line

4x – y + 5 = 0

∴ Slope of tangent = Slope of line

$\Rightarrow \quad \frac{3}{2\sqrt{3x-2}}=4$

$\Rightarrow \quad 3=8\sqrt{3x-2}$

Squaring on both sides

9 = 64 (3x – 2)

$\frac{9}{64}=3x-2$

$\frac{9}{64}+2=3x$

$\frac{9+128}{64}=3x$

$=\frac{137}{64}=3x$

$x=\frac{137}{64\times 3}=\frac{137}{192}$

$\therefore \quad y=\sqrt{3x-2}=\sqrt{3\times\frac{137}{192}-2}$

$$y=\sqrt{\frac{137-128}{64}}=\sqrt{\frac{9}{64}}=\frac{3}{8}$$

And $\frac{dy}{dx}=\frac{3}{2\sqrt{3x-2}}=\frac{3}{2\sqrt{3\times\frac{137}{192}-2}}$

$$=\frac{3\times8}{2\times3}=4$$

Pt. on the tangent is $\left(\frac{137}{192},\frac{3}{8}\right)$ and the slope of tangent is 4

The equation of tangent is

$$y-\frac{3}{8}=4\left(x-\frac{137}{192}\right)$$

$$\Rightarrow \quad \frac{8y-3}{8}=4\left(\frac{192x-137}{192}\right)$$

$$\Rightarrow \quad 8y-3=8\left(\frac{192x-137}{48}\right)$$

$$\Rightarrow \quad 6(8y-3)=(192x-137)$$

$$\Rightarrow \quad 48y-18=192x-137$$

$$\Rightarrow \quad 192x-48y-119$$

OR

Let the point be p(x, y). It is Given the rate of change of y-coordinate = 8 (Rate of change of x – coordinate)

$$\frac{dy}{dt}=8\left(\frac{dx}{dt}\right)$$

Is Given $6y = x^3 + 2$

On differentiating w·r·t· t on both sides,

$$\frac{6dy}{dt}=3x^2\frac{dx}{dt}+0$$

$$\Rightarrow \quad 6\left(8\frac{dx}{dt}\right)=3x^2\frac{dx}{dt}$$

$$\Rightarrow \quad 48=3x$$

$$\Rightarrow \quad x^2=16; \qquad \Rightarrow x=\pm4$$

When x = 4 then $6y = 4^3 + 2$

$$\Rightarrow \quad 6y=66$$

Therefore y = 11

When x = -4, then $6y = (-4)^3 + 2$

$$\Rightarrow \quad 6y=-62; \qquad \Rightarrow \qquad y=\frac{-31}{3}$$

Hence the required points are $\left(-4, \frac{-31}{3}\right)$ and (4, 11).

Ans.23. $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$

$\therefore \quad A^2 = A \cdot A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$

$= \begin{bmatrix} 9+20 & -15-10 \\ -12-8 & 20+4 \end{bmatrix}$

$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix}$

$\therefore \quad A^2 - 5A - 14I = 0$

$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} - 5\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} - 14\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} - \begin{bmatrix} 15 & -25 \\ -20 & 10 \end{bmatrix} - \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix}$

$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} - \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix}$

$= \begin{bmatrix} 29-29 & 25 \\ -20+20 & 24 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} R.H.S$

Multiply A^{-1} on both side of term

A-1(A·A) – 5AA^{-1} – 14I A^{-1} = 0

(A^{-1} A)·A – 5I – 14A^{-1} = 0

A – 5I = 14A^{-1}

$\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 14A^{-1}$

$\begin{bmatrix} 3-5 & -5-0 \\ -4-0 & 2-5 \end{bmatrix} = 14A^{-1}$

$\frac{1}{14}\begin{bmatrix} -2 & -5 \\ -4 & -3 \end{bmatrix} = A^{-1}$

OR

$A + \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ -3 & 8 \end{bmatrix}$

Let's Take $A = \begin{bmatrix} x & y \\ z & W \end{bmatrix}$

Then $\begin{bmatrix} x & y \\ z & w \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ -3 & 8 \end{bmatrix}$

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ -3 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 3-2 & -6-3 \\ -3+1 & 8-4 \end{bmatrix}$$

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & -9 \\ -2 & 4 \end{bmatrix}$$

Matrix $A = \begin{bmatrix} 1 & -9 \\ -2 & 4 \end{bmatrix}$

Ans.24. Given lines are

X + 2y = 2 ....(i)

Y – x = 1 ....(ii)

2x + y = 7 ....(iii)

Solving Ist and 2nd, we get

X = 0, y = 1

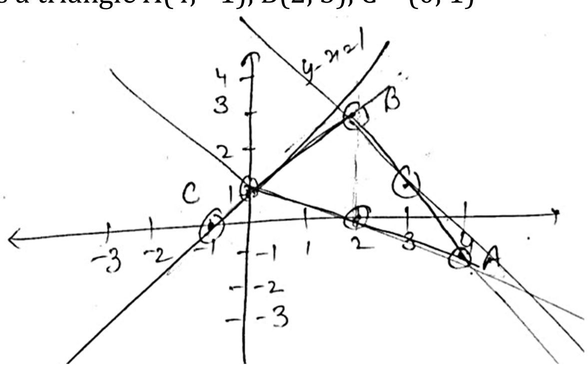
And, On solving (2) and (3) we get

X = 2, y = 3

And, On solving line (3) and (1), we get

X = 4, y = -1

Thus a triangle A(4, - 1), B(2, 3), C = (0, 1)



Bounded are = Area of ΔABC

$$= \int_0^2 \left[(x+1-\left(\frac{2-x}{2}\right) \right] dxt \int_2^4 \left[\left(\frac{2-x}{2}\right) + (7-2x) \right] dx$$

$$= \left[\frac{x^2}{2} + x - \frac{1}{2}\left(2x - \frac{x^2}{2}\right) \right]_0^2 + \left[\frac{1}{2}(2x - x^2) \right] + \left(7x - \frac{2x^2}{2} \right)_2^4$$

= [3] + [-2]

= 1

Ans.25.

Model of bike	X	Y
Hours/unit	6	10
Cost/unit	2000	1000
Profit/unit	1000	500

No. of hours available per week is 450

$6x + 10y \le 450$ ....(i)

The total funds available is Rs 80,000per week.

Therefore $2000x + 1000y \le 80,000$

OR

$2x + y \le 80$ .....(ii)

Profit are Rs.1000 and Rs.500

Hence $Z = 1000x + 500y$

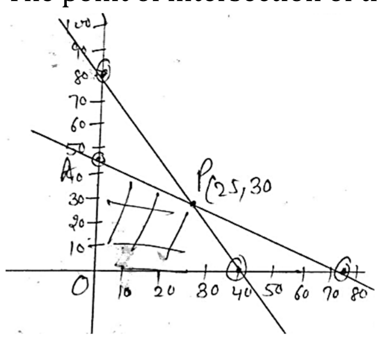
Now let us draw the graph for line AB = $6x + 10y = 450$

Put $x = y = 0$ in both

Then equation

$\le 450, \le 80$

The point of intersection of the line AB and CD is (25, 30)



The corner points of the feasible region are (0, 0) (0, 45) (25, 30) (40, 0)

At 0 (0, 0) value of Z = 0

At A(10, 45) $\Rightarrow$ $z = 1000 \times 0 + 500 \times = 22500$

At P(25, 30) the value of objective function

$Z = 1000 \times 25 + 500 \times 30$

$= 40000$

AD(40, 0) the value is

$Z = 1000 \times 40 + 0 = 40,000$

The maximum profit is 40,000

25 bikes of model X and 30 bike of model.

Y has to be manufactured.

Ans.26. Given planes $x + 2y = 0$ and $3y - z = 0$

Direction numbers of normal to these planes

Are <1, 2, 0> and <0, 3, -1> respectively. So, the normal to the given planes are along the vectors

$\overrightarrow{n_1} = \hat{i} + 2\hat{j}$ and $\overrightarrow{n_2} = 3\hat{j} - \hat{k}$

Let $\vec{b}$ be a vector along the required line As the line is parallel to the Given planes, $\vec{b}$ is perpendicular to both the vector $\overrightarrow{n_1} \times \overrightarrow{n_2}$.

$$\overrightarrow{n_1} \times \overrightarrow{n_2} = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 0 & 3 & -1 \end{vmatrix} = 2\hat{i} + \hat{j} + 3\hat{k}$$

The line passes through the point (3, 0, 1) i.e. the point with position vector

$$3\hat{i} + \hat{k}$$

∴ The vector equation of the required line is

$\vec{r} = (3\hat{i} + \hat{k}) + \lambda(-2\hat{i} + \hat{j} + 3\hat{k})$

Ans.27. Let height of cylinder be H and base radius be R Volume of cylinder

$V = \pi R^2 H$ .....(i)

$\Delta ABC \sim ADC \quad \Rightarrow \dfrac{AC}{AE} = \dfrac{BC}{DE}$

$$\Rightarrow \quad \frac{h}{h-4} = \frac{r}{R}$$

Therefore $R = r\left[\dfrac{h-H}{h}\right]$

$$= r\left[1 - \frac{H}{h}\right] \quad .....(ii)$$

Putting R in (i), we get

$$|| = \pi r^2 \left(1 - \frac{H}{h}\right) 2H \quad ......(iii)$$

Differentiating w·r·t h, we get

$$\frac{dU}{dH} = \pi r^2 \left[\left(1 - \frac{H}{h}\right) 2 \cdot 1 + H \cdot 2\left(1 - \frac{H}{h}\right)\left(\frac{-1}{h}\right)\right]$$

$$= \pi r^2 \left(1 - \frac{H}{h}\right)\left[1 - \frac{H}{h} - \frac{2H}{h}\right]$$

$$\frac{dU}{dH} = \pi r^2 \left(1 - \frac{H}{H}\right)\left(1 - \frac{3H}{3}\right)$$

For maximum or minimum $\frac{dU}{dH}=0$

As, $\pi r^2\left(1-\frac{H}{h}\right)\left(1-\frac{3H}{h}\right)=0$

$\Rightarrow$ H = h, H = h/3

But H = h is not possible. Hence $H=\frac{h}{3}$

Now $\frac{d^2U}{dh^2}=\pi r^2\left(1-\frac{H}{h}\right)\left(\frac{-3}{h}\right)+\left(1-\frac{3H}{h}\right)\left(\frac{-1}{h}\right)$

$=\pi r^2/h\left[0-3+\frac{3H}{h}-1+\frac{3h}{h}\right]$

$=\pi r^2/h\left[6H/h-4\right]$

Therefore $\left[\frac{d^2v}{dH^2}\right]$ at $H=\frac{h}{3}=\frac{\pi r^2}{h}\left[6/h\cdot\frac{h}{3}-4\right]$

$=\frac{\pi r^2}{h/-2}<0$

Therefore, volume is maximum at $H=\frac{h}{3}$

Ans.28. Given an urn contains 25 balls of which 10 balls bear a mark A and the remaining 15 bear a mark B.

P (drawing an A) $=P=\frac{10}{25}=\frac{2}{5}$

$\rightarrow q=1-p=\frac{3}{5}$

Given that 6 balls are to be drawn at random, it is a case of Bernoulli trails (n = 6)

P(x = r) $={}^6C_r\left(\frac{2}{5}\right)^r\left(\frac{3}{5}\right)^{6-r}$

i) $p(x=6)={}^6C_6\left(\frac{2}{5}\right)^6\left(\frac{3}{5}\right)^0=\left(\frac{2}{5}\right)^6$

ii) $=p(x\geq 4)=p(x=4)+p(x=5)+p(x=6)$

$={}^6C_4\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right)^2+{}^6C_4\left(\frac{2}{5}\right)^5\left(\frac{3}{5}\right)^1+{}^6C_6\left(\frac{2}{5}\right)^6\left(\frac{3}{5}\right)^0$

$=7\left(\frac{2}{5}\right)^4$

iii) 1 – p(x = 6)

$$=1-{}^6C_6\left(\frac{2}{5}\right)^6\left(\frac{3}{5}\right)^0$$

$$=1-\left(\frac{2}{5}\right)^6$$

Ans.29. $\int_0^1 x\left(\tan^{-1}x\right)^2 dx$

$$=\left(\tan^{-1}x\right)^2\int_0^1 xdx-\int_0^1\frac{d}{dx}\left(\tan^{-1}x\right)^2\left(\int_0^1 xdx\right)dx$$

$$=\left[(\tan^{-1}x)^2\left(\frac{x^2}{2}\right)\right]_0^1-\int_0^1 2(\tan^{-1}x)\frac{d}{dx}(\tan^{-1}x)\left(\left(\frac{x^2}{2}\right)_0^1\right)dx$$

$$=\left[(\tan^{-1}1)^2\left(\frac{1^2}{2}\right)-(\tan^{-1}0)^2\left(\frac{0^2}{2}\right)\right]-\int_0^1 2(\tan^{-1}x)\frac{1}{1+x^2}\left(\frac{1}{2}\right)dx$$

$$=\left[\left(\frac{\pi}{4}\right)^2\left(\frac{1}{2}\right)-0\right]-\int_0^1\frac{\tan^{-1}x}{1+x^2}dx$$

$$=\frac{1}{2}\frac{\pi^2}{16}-\int_0^1\frac{\tan^{-1}x}{1+x^2}dx \qquad ....(1)$$

Now $\int_0^1\frac{\tan^{-1}x}{1+x^2}dx=\int_0^{\pi} t\,dt$ let $\tan^{-1}x = t$ $\frac{1}{1+x^2}dx=dt$

$$=\left(\frac{t^2}{2}\right)_0^{\frac{\pi}{4}}$$

$$=\left[\left(\frac{\pi}{4}\right)^2\left(\frac{1}{2}\right)-0\right]$$

$$=\frac{1}{2}\frac{\pi^2}{16}$$

Substituting this value in equation ⊥, we have

$$=\int_0^1 x\left(\tan^{-1}x\right)^2 dx=\frac{1}{2}\left(\frac{\pi^{\,2}}{16}\right)-\frac{1}{2}\left(\frac{\pi^2}{16}\right)$$

= 0

Hence $\int_0^1 x(\tan^{-1}x)^2 dx=0$

OR

Evaluate $\int_1^2 (x^2+x+2)$

A = 1, b = 2

$\therefore$ nh = b – a = 2 – 1 = 1

We know that

$$\int_a^b f(x)dx = \lim h\int f(a) + f(a+h) + \ f(a+2h) + \ ....$$

$$\therefore \quad f(a) = f(1) = 1^2 + 1 + 2 = 4$$

$$f(a+h) = f(1+h) = (1+h)^2 + (1+h) + 2$$

$$f(a+(n-1)h) = f(1+(n-1)h) = (1+(n-1)h)^2 + (1+(n-1)h) + 2$$

$$\therefore \quad f(a) + f(a+h) + f(a+2h) + \ ......+ f(a+(n-1)h)$$

$$= 4 + \left[(h^2 + 1 + 2h) + (3+h)\right] + \left[(1 + 4h^2 + 4h) + (3+2h)\right]$$

$$+.........\left[1 + (n-1)^2 n^2 + 2(n-1)h\right] + \left[3 + (n-1)h\right]$$

$$= 4 + \left[h^2 + 3h + 4\right] + \left[4h^2 + 6h + 4\right] + ... + \left[4 + (n-1)^2 h^2 + 3(n-1)h\right]$$

$$= \left[h^2 + 4h^2 + ...(n-1)^2 h^2\right] + 3h\left[1 + 2 + ...(n-1)\right] + (4 + 4 + ...4) \text{ n times}$$

$$= h^2\left[1^2 + 2^2 + ...(n-1)^2\right] + 3h\left[1.....(n-1)\right] + 4n$$

$$= \frac{n(n-1)(2n-1)}{6}h^2 + 3h\frac{n(n-1)}{2} + 4n$$

$$\therefore \quad \int_1^2 (n^2 + n + 2)dx = \lim h\left[\frac{n(n-1)(2n-1)}{6}h^2 + \frac{3hn(n-1)}{2} + 4n\right]$$

= [2/6+3/2+4]

=5/2