09

Integration

9.01 Introduction

We have already studied how to find the derivative of a given function. As a consequence, a natural question arises : given a function say $f(x)$, can we find a function $g(x)$ such that $g'(x) = f(x)$. If such a functions $g(x)$ exist, we shall call it anti-derivative of $f(x)$ or indefinite integral of $f(x)$. Therefore, integration is an inverse process of differentiation. It is also called antiderivative or primitive.

9.02 Integration of a function

If the given function is $f(x)$ and its integral is $F(x)$, then

$$\frac{d}{dx}[F(x)] = f(x) \tag{1}$$

Here, $F(x)$ is called integration of function $f(x)$ with respect to x. In symbols, it is expressed as

$$\int f(x)dx = F(x) \tag{2}$$

where symbol $\int$ is used for integration and dx means to integrate with respect to variable x. Also, the function $f(x)$, whose integration is to be done, is called Integrand and $F(x)$ is called integral.

Since integration and differentiation are inverse process of each other. Therefore, then differentiating eq. (2) with respect to x, we get

$$\frac{d}{dx}\left[\int f(x)dx\right] = \frac{d}{dx}[F(x)]$$

or $$\frac{d}{dx}\left[\int f(x)dx\right] = f(x) \qquad \text{[From (1)]}$$

For example: $\frac{d}{dx}(\sin x) = \cos x$ so $\int \cos x\, dx = \sin x$

$\frac{d}{dx}(x^2) = 2x$ so $\int 2x dx = x^2$

Remark : If $\int f(x)dx = F(x)$, then $f(x)$ is called integrand, $F(x)$ is called integral and the process of finding the integral is known as integration.

9.03 Indefinite integral and constant of integration

We know that differential coefficient of any constant is zero.

That means, $\frac{d}{dx}(c) = 0$, where c is any constant

Let $$\frac{d}{dx}[F(x)] = f(x)$$

then, $$\frac{d}{dx}[F(x)+c] = \frac{d}{dx}[F(x)] + \frac{d}{dx}(c)$$
$$= f(x) + 0$$

so $$\frac{d}{dx}[F(x)+c] = f(x)$$

On integrating both sides with respect to x,

$$\int\left[\frac{d}{dx}\{F(x)+c\}\right]dx = \int f(x)dx$$

or $$\int f(x)dx = F(x) + c\,,$$ (by definition)

where c is an arbitrary constant, which is called coefficient of integration. This is independent of x. Antiderivative of any continuous function is not unique. Actually, there exist infinitely many anti-derivatives of each of these functions which can be obtained by choosing c arbitrarily from the set of real numbers. In fact, c is the parameter by varying which one gets different antiderivatives (or integrals) of the given function.

For example, $$\frac{d}{dx}(x^2+1) = 2x \Rightarrow \int 2x dx = x^2 + 1$$

$$\frac{d}{dx}(x^2+4) = 2x \Rightarrow \int 2x dx = x^2 + 4$$

but (x^2+1) and (x^2+4) are not same, they are differ by a constant.

Remark : In indefinite integration, the constant of integration should be added at the end of the process of integration.

9.04 Theorems on Integration

Theorem 1: For any constant k, $\int \kappa f(x)\,dx = \kappa \int f(x)dx$

$\therefore$ The integration of product of a constant function and variable function is equal to the product of constant function and integral of variable function.

Proof : We know by theorem of differentiation

$$\frac{d}{dx}\left[k\int f(x)dx\right] = k\frac{d}{dx}\left[\int f(x)dx\right] = k\,f(x)$$ [by definition]

Integrating both sides,

$$\int \frac{d}{dx}\left[k\int f(x)dx\right]dx = \int k\,f(x)dx$$

$$k\int f(x)dx = \int k\,f(x)dx$$

or $$\int k\,f(x)dx = k\int f(x)\,dx$$

Theorem 2 : $\int [f_1(x) \pm f_2(x)]\,dx = \int f_1(x)\,dx \pm \int f_2(x)dx$

$\therefore$ The integral of sum or difference of any two variable functions is equal to the sum or difference of their integrals.

Proof : Let $\int f_1(x)dx = F_1(x)$ and $\int f_2(x)dx = F_2(x)$

$\therefore$ $\frac{d}{dx}[F_1(x)] = f_1(x)$ and $\frac{d}{dx}[F_2(x)] = f_2(x)$

Also, $$\frac{d}{dx}[F_1(x) \pm F_2(x)] = \frac{d}{dx}[F_1(x)] \pm \frac{d}{dx}[F_2(x)]$$

$$= f_1(x) \pm f_2(x)$$

Integrating both the sides,

$$\int \frac{d}{dx}[F_1(x) \pm F_2(x)]dx = \int [f_1(x) \pm f_2(x)]\,dx$$

or, $$F_1(x) \pm F_2(x) = \int [f_1(x) \pm f_2(x)]dx$$

or $$\int [f_1(x) \pm f_2(x)]dx = F_1(x) \pm F_2(x)$$

$$= \int f_1(x)\,dx \pm \int f_2(x)\,dx$$

This rule can be applied for two or more terms but not necessarily applicable on sum of infinite terms.

Generalization

$$\int [k_1 f_1(x) \pm k_2 f_2(x)]\,dx = \int k_1 f_1(x)dx \pm \int k_2 f_2(x)dx$$

$$= k_1 \int f_1(x)dx \pm k_2 \int f_2(x)dx$$

9.05 Standard formulae of Integration

We already know the formulae for the derivatives of many important functions. From these formulae, we can write down the corresponding formulae for the integrals of these functiosn, as listed below which will be used to find integrals of other functions.

For example $$\frac{d}{dx}(x^n) = nx^{n-1} (n \neq o)$$

$\Rightarrow$ $$\int nx^{n-1}dx = x^n + c$$

Putting n as $(n+1)$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c (n \neq -1)$$

Similarly following formulae can be proved

	Derivatives		**Integrals**
1.	$\frac{d}{dx}(c)=0$	$\Rightarrow$	$\int 0\cdot dx=c$
2.	$\frac{d}{dx}(x^n)=nx^{n-1},\quad n\neq 0$	$\Rightarrow$	$\int x^n dx=\frac{x^{n+1}}{n+1}+c,\quad n\neq -1$
3.	$\frac{d}{dx}(\log\|x\|)=\frac{1}{x},\quad x\neq 0$	$\Rightarrow$	$\int \frac{1}{x}dx=\log\|x\|+c,\quad x\neq 0$
4.	$\frac{d}{dx}(e^x)=e^x$	$\Rightarrow$	$\int e^x dx=e^x+c$
5.	$\frac{d}{dx}(a^x)=a^x\log_e a$	$\Rightarrow$	$\int a^x dx=\frac{a^x}{\log_e a}+c$
6.	$\frac{d}{dx}(\sin x)=\cos x$	$\Rightarrow$	$\int \cos x dx=\sin x+c$
7.	$\frac{d}{dx}(-\cos x)=\sin x$	$\Rightarrow$	$\int \sin x dx=-\cos x+c$
8.	$\frac{d}{dx}(\tan x)=\sec^2 x$	$\Rightarrow$	$\int \sec^2 x dx=\tan x+c$
9.	$\frac{d}{dx}(-\cot x)=\operatorname{cosec}^2 x$	$\Rightarrow$	$\int \operatorname{cosec}^2 x\, dx=-\cot x+c$
10.	$\frac{d}{dx}(\sec x)=\sec x\tan x$	$\Rightarrow$	$\int \sec x\tan x dx=\sec x+c$
11.	$\frac{d}{dx}(-\operatorname{cosec} x)=\operatorname{cosec} x\cot x$	$\Rightarrow$	$\int \operatorname{cosec} x\cot x\, dx=-\operatorname{cosec} x+c$
12.	$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}},(\|x\|<1)$	$\Rightarrow$	$\int \frac{1}{\sqrt{1-x^2}}dx=\sin^{-1}x+c$
13.	$\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}},(\|x\|<1)$	$\Rightarrow$	$\int \frac{-1}{\sqrt{1-x^2}}dx=\cos^{-1}x+c$
14.	$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$	$\Rightarrow$	$\int \frac{1}{1+x^2}dx=\tan^{-1}x+c$
15.	$\frac{d}{dx}(-\cot^{-1}x)=\frac{1}{1+x^2}$	$\Rightarrow$	$\int \frac{1}{1+x^2}dx=-\cot^{-1}x+c$
16.	$\frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{x^2-1}}$	$\Rightarrow$	$\int \frac{1}{x\sqrt{x^2-1}}dx=\sec^{-1}x+c$

17. $\frac{d}{dx}(-\text{cosec}^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$ $\Rightarrow$ $\int \frac{1}{x\sqrt{x^2-1}} = -\text{cosec}^{-1} x + c$

18. $\frac{d}{dx}|x| = \frac{|x|}{x}, (x \neq 0)$ $\Rightarrow$ $\int \frac{|x|}{x} dx = |x| + c, \quad x \neq 0$

Particularly $\frac{d}{dx}(x) = 1$ $\Rightarrow$ $\int 1.dx = x + c$

Note (a) $\frac{d}{dx}\int f(x)dx = f(x)$ (b) $\int \frac{d}{dx} f(x)dx = f(x) + c$

hence there is a difference of integral constant between differentiation of integral and integral of derivative.

Remarks :

(1) We should not conclude for formula 12 and 13 that $\sin^{-1} x = -\cos^{-1} x$ because they are differ by constant term only, because we know that $\sin^{-1} x + \cos^{-1} x = \pi / 2$.

(2) In practice, we normally do not mention the interval over whcih the various functions are defined. However, in any specific problem one has to keep it in mind.

9.06 About Differentiation and Integration

(1) Both are operations on functions, the result of each is also a function.

(2) Both satisfy the property of linearity.

(3) All functions are not differentiable and integralble.

(4) The derivative fo a function, when it exist, is a unique function. The integral of a function is not so due to integral constant.

(5) We can speak of the derivastive at a point. We never speak of the integral at a point, We speak of the integral of a function over an interval on whcih the integral is defined.

(6) The derivative of a function has a geometrical meaning, namely, the slope of the tangent to the corresponding curve at a point similarly, the indefinite integral of a function represents geometrically, are of some region, or area under curve.

(7) The derivative is used for finding some physical quantities like the velocity of a moving particle, acceleration whereas integration is used for finding, centre of mass, momentum etc.

(8) The process of differentiation and integration are inverse operation of each other.

9.07 Methods of Integration

Some prominent methods to find out the integration are :

(I) Usign standard formulae

(II) Integration by substitution

(III) Integration using Partial fractions

(IV) Integration by parts

I Integration by the use of standard formulae

Here by using the standard formulae or other trigonometric formulae, We can find integral of given function. We can illustrate with the following examples.

Illustrative Examples

Example 1. Integrate the following functions with respect to x

(i) x^6 (ii) $\sqrt{x}$ (iii) $\dfrac{x^2+1}{x^4}$ (iv) $\dfrac{1}{\sqrt{x}}$

Solution : We know that $\int x^n dx = \dfrac{x^n+1}{n+1} + c, n \neq -1$

(i) Let $I = \int x^6 dx = \dfrac{x^{6+1}}{6+1} + c = \dfrac{x^7}{7} + c$

(ii) Let $I = \int \sqrt{x} dx = \int x^{1/2} dx = \dfrac{x^{1/2+1}}{(1/2)+1} + c = \dfrac{x^{3/2}}{3/2} + c = \dfrac{2}{3} x^{3/2} + c$

(iii) Let $I = \int \dfrac{x^2+1}{x^4} dx = \int \left(\dfrac{x^2}{x^4} + \dfrac{1}{x^4} \right) dx = \int \dfrac{1}{x^2} dx + \int \dfrac{1}{x^4} dx$

$$= \int x^{-2} dx + \int x^{-4} dx = \frac{x^{-2+1}}{-2+1} + \frac{x^{-4+1}}{-4+1} + c$$

$$= \frac{x^{-1}}{-1} + \frac{x^{-3}}{-3} + c = -\frac{1}{x} - \frac{1}{3x^3} + c$$

(iv) Let $I = \int \dfrac{1}{\sqrt{x}} dx = \int x^{-1/2} dx = \left[\dfrac{x^{-1/2+1}}{-1/2+1} \right] + c$

$$= \frac{x^{1/2}}{(1/2)} + c = 2\sqrt{x} + c$$

Example 2. Evaluate $\int \dfrac{ax^2+bx+c}{x} dx$

Solution : $\int \dfrac{ax^2+bx+c}{x} dx = \int \left[\dfrac{ax^2}{x} + \dfrac{bx}{x} + \dfrac{c}{x} \right] dx$

$$= \int \left(ax + b + \frac{c}{x} \right) dx$$

$$= \int ax\, dx + \int b\, dx + \int \frac{c}{x} dx$$

$$= a \int x\, dx + b \int dx + c \int \frac{1}{x} dx$$

$$= \frac{ax^2}{2} + bx + c \log|x| + k$$

Example 3. Evaluate $\int \frac{\sin^2 x}{1+\cos x} dx$

Solution :

$$\int \frac{\sin^2 x}{1+\cos x} dx = \int \frac{1-\cos^2 x}{1+\cos x} dx$$

$$= \int \frac{(1-\cos x)(1+\cos x)}{(1+\cos x)} dx$$

$$= \int (1-\cos x)\, dx = \int 1.dx - \int \cos x\, dx$$

$$= x - \sin x + c$$

Example 4. Evaluate $\int \frac{x^2}{x+1} dx$

Solution :

$$\int \frac{x^2}{x+1} dx = \int \frac{(x^2-1)+1}{(x+1)} dx$$

$$= \int \left[\frac{x^2-1}{x+1} + \frac{1}{x+1}\right] dx$$

$$= \int \left[(x-1) + \frac{1}{(x+1)}\right] dx = \int \left(x-1+\frac{1}{1+x}\right) dx$$

$$= \frac{x^2}{2} - x + \log|x+1| + c, (x \neq -1)$$

Example 5. Evaluate $\int \sqrt{1+\sin 2x}\ dx$

Solution :

$$\int \sqrt{1+\sin 2x}\ dx = \int \sqrt{[(\sin^2 x + \cos^2 x) + 2\sin x \cos x]}\ dx$$

$$= \int \sqrt{(\sin x + \cos x)^2}\ dx$$

$$= \int (\sin x + \cos x)\ dx$$

$$= -\cos x + \sin x + c$$

Example 6. Evaluate $\int \frac{1-\cos 2x}{1+\cos 2x} dx$

Solution :

$$\int \frac{1-\cos 2x}{1+\cos 2x} dx = \int \frac{2\sin^2 x}{2\cos^2 x} dx \qquad [\because \cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1]$$

$$= \int \tan^2 x\ dx = \int (\sec^2 x - 1) dx$$

$$= \tan x - x + c$$

Example 7. Evaluate $\int \frac{1}{1+\sin x} dx$

Solution : $$\int \frac{1}{1+\sin x} dx = \int \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx$$

$$= \int \frac{1-\sin x}{1-\sin^2 x} dx = \int \frac{1-\sin x}{\cos^2 x} dx$$

$$= \int \left[\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right] dx$$

$$= \int (\sec^2 x - \sec x \tan x) dx$$

$$= \tan x - \sec x + c$$

Example 8. The slope of a curve is given by $\frac{dy}{dx} = 2x - \frac{3}{x^2}$. It passes through (1, 1). Find the equation of curve.

Solution : $\because \frac{dy}{dx} = 2x - \frac{3}{x^2}$

Integrating both the sides with respect tot x

$$\int \frac{dy}{dx} dx = \int \left(2x - 3x^{-2}\right) dx$$

$$\Rightarrow \quad \int dy = 2\int x dx - 3\int x^{-2} dx$$

$$\Rightarrow \quad y = \frac{2x^2}{2} - 3\frac{x^{-1}}{-1} + c$$

$$\Rightarrow \quad y = x^2 + \frac{3}{x} + c$$

$\because$ It passes through (1, 1)

$$1 = (1)^2 + \frac{3}{(1)} + c \Rightarrow c = -3$$

$\therefore$ required equation of curve

$$y = x^2 + \frac{3}{x} - 3$$

Exercise 9.1

1. Integrate the following functions with respect to x

(i) $3\sqrt{x^2}$ (ii) e^{3x} (iii) $(1/2)^x$ (iv) $a^{2\log_a x}$

Evaluate the following :

2. $\int\left(5\cos x-3\sin x+\frac{2}{\cos^2 x}\right)dx$ 3. $\int\frac{x^3-1}{x^2}dx$ 4. $\int\sec^2 x\cos ec^2 x\,dx$

5. $\int(1+x)\sqrt{x}\,dx$ 6. $\int a^x da$ 7. $\int\frac{x^2}{1+x^2}dx$

8. $\int\frac{\cos^2 x}{1+\sin x}dx$ 9. $\int\sec x(\sec x+\tan x)\,dx$

10. $\int(\sin^{-1}x+\cos^{-1}x)\,dx$ 11. $\int\frac{x^2-1}{x^2+1}dx$

12. $\int\tan^2 x\,dx$ 13. $\int\cot^2 x\,dx$ 14. $\int\frac{dx}{\sqrt{1+x}-\sqrt{x}}$

15. $\int(\tan^2 x-\cot^2 x)\,dx$ 16. $\int\frac{\sin x}{1+\sin x}dx$ 17. $\int\frac{1}{1-\cos x}dx$

18. $\int\left[1+\frac{1}{1+x^2}+\frac{3}{x\sqrt{x^2-1}}+2^x\right]dx$ 19. $\int\cot x(\tan x-\operatorname{cosec}x)dx$

20. $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2dx$ 21. $\int\log_x x\,dx$ 22. $\int\sqrt{1+\cos 2x}\,dx$

23. $\int\frac{\cos 2x}{\sin^2 x\cos^2 x}dx$ 24. $\int\frac{3\cos x+4}{\sin^2 x}dx$

II Integration by substitution

(a) Substitution of Variables : The given variable cna be transformed into antoher form or independent variable, then doing integrationis called integration by substitutiion.

Theorem : If x is substituted by new variable in $\int f(x)\,dx$ then $x=\phi(t)$

$$\int f(x)\,dx=\int f\phi\{t\}\phi'(t)dt,\ \text{where}\ \phi'(t)=\frac{d\phi}{dt}$$

Proof : Let $\int f(x)\,dx=F(x)$ then $\frac{d}{dx}\int f(x)dx=\frac{d}{dx}F(x)$ (From differentiation) (1)

Now if $x=\phi(t)$ then $\frac{dx}{dt}=\phi'(t)$ (2)

again $$\frac{d}{dt}F(x)=\frac{d}{dx}F(x).\frac{dx}{dt} \quad \text{(Chain rule)}$$

$$= f(x).\phi'(t) \qquad \text{[From (1) and (2)]}$$

$$= f\{\phi(t)\}\phi'(t)$$

Now by definition of integration

$$\int \frac{d}{dx}F(x)\,dt = \int f\{\phi(t)\}.\phi'(t)\,dt$$

Or $$F(x) = \int f\{\phi(t)\}.\phi'(t)\,dt$$

Or $$\int f(x)\,dx = \int f\{\phi(t)\}.\phi'(t)\,dt$$

Some integrands for substitution

(a) $$\int \frac{f'(x)}{f(x)}dx = \log|f(x)|+c \qquad \text{(Let } f(x)=t \text{ etc.)}$$

(b) $$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}+c \qquad \text{(Let } f(x)=t \text{ etc.)}$$

(c) For linear function $f(ax+b)$

$$\int f(ax+b)\,dx = \frac{f(ax+b)}{a}+c \qquad \text{(where a, b are constants)}$$

whereas $$\int f(x)\,dx = F(x)+c$$

Formulae for linear functions

If $a \neq o$ then

(i) $$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)}+c, \quad n \neq -1$$

(ii) $$\int \frac{1}{ax+b}dx = \frac{1}{a}\log|ax+b|+c, \ a>0$$

(iii) $$\int e^{ax+b}dx = \frac{e^{ax+b}}{a}+c$$

(iv) $$\int \sin(ax+b)\,dx = -\frac{\cos(ax+b)}{a}+c$$

(v) $$\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a}+c$$

Remark : There is no general ruel for substitution, it depends on the nature of integral. The success of substitution method depends that we make a substitution such that a function whose derivative also occurs in the integrand in product form.

Illustrative Examples

Example 9. Integrate the following functions with respect to x

(i) $\dfrac{\cos[\log(x)]}{x}$ (ii) $\dfrac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$ (iii) $\dfrac{\sin\sqrt{x}}{\sqrt{x}}$ (iv) $\dfrac{1}{\cos^2(5x+2)}$

Solution : (i) Let $\log x = t$ then $\dfrac{1}{x}dx = dt$

$\therefore$ $$I = \int \frac{\cos(\log x)}{x}dx = \int \cos t\, dt = \sin t + c = \sin(\log x) + c$$

(ii) Let $$I = \int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx$$

Let $$\sin^{-1}x = t \Rightarrow \frac{1}{\sqrt{1-x^2}}dx = dt$$

$\therefore$ $$I = \int e^t dt = e^t + c = e^{\sin^{-1}x} + c$$

(iii) $$I = \int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$$

Let $$\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}}dx = dt \Rightarrow \frac{1}{\sqrt{x}}dx = 2dt$$

$\therefore$ $$I = \int \sin t \times 2dt = 2\int \sin t\, dt$$

$$= 2\times(-\cos t) + c = -2\cos\sqrt{x} + c$$

(iv) $$I = \int \frac{1}{\cos^2(5x+2)}dx$$

$$= \int \sec^2(5x+2)\,dx$$

Let $$5x+2 = t \Rightarrow 5dx = dt \Rightarrow dx = \frac{1}{5}dt$$

$\therefore$ $$I = \int \sec^2 t \times \frac{1}{5}dt$$

$$= \frac{1}{5}\int \sec^2 t\, dt = \frac{1}{5}\tan t + c = \frac{1}{5}\tan(5x+2) + c$$

Example 10. Integrate the following functions with respect to x

(i) $\dfrac{\log[x+\sqrt{1+x^2}]}{\sqrt{1+x^2}}$ (ii) $\sec x \log(\sec x + \tan x)$ (iii) $\dfrac{1}{1+\tan x}$

Solution : (i) $$I=\int\frac{\log[x+\sqrt{1+x^2}]}{\sqrt{1+x^2}}dx$$

Let $$\log[x+\sqrt{1+x^2}]=t$$

$$\therefore \quad \frac{1}{x+\sqrt{1+x^2}}\times\left[1+\frac{2x}{2\sqrt{1+x^2}}\right]dx=dt$$

$$\Rightarrow \quad \frac{1}{[x+\sqrt{1+x^2}]}\times\frac{[\sqrt{1+x^2}+x]}{\sqrt{1+x^2}}dx=dt$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+x^2}}dx=dt$$

$$\therefore \quad I=\int t\,dt$$

$$=\frac{t^2}{2}+c$$

$$=\frac{1}{2}[\log\{x+\sqrt{1+x^2}\}]^2+c$$

(ii) $$I=\int \sec x\,.\log(\sec x+\tan x)dx$$

Let $$\log(\sec x+\tan x)=t$$

$$\therefore \quad \frac{1}{(\sec x+\tan x)}\times(\sec x\tan x+\sec^2 x)dx=dt$$

$$\sec x\,dx=dt$$

$$\therefore \quad I=\int t\,dt=\frac{t^2}{2}+c=\frac{1}{2}\left[\log(\sec x+\tan x)\right]^2+c$$

(iii) $$I=\int\frac{1}{1+\tan x}dx=\int\frac{1}{1+\frac{\sin x}{\cos x}}dx=\int\frac{\cos x}{\cos x+\sin x}dx$$

$$=\frac{1}{2}\int\frac{2\cos x}{\cos x+\sin x}dx=\frac{1}{2}\int\frac{(\cos x+\sin x)+(\cos x-\sin x)}{\cos x+\sin x}$$

$$=\frac{1}{2}\int\frac{\cos x+\sin x}{\cos x+\sin x}dx+\frac{1}{2}\int\frac{\cos x-\sin x}{\cos x+\sin x}dx$$

$$=\frac{1}{2}\int dx+\frac{1}{2}\int\frac{\cos x-\sin x}{\cos x+\sin x}dx$$

In second integral, Let $\cos x + \sin x = t$

$\therefore$ $(-\sin x + \cos x)dx = dt$

$$\therefore \quad I = \frac{1}{2}\int dx + \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}x + \frac{1}{2}\log|t| + c$$

$$= \frac{x}{2} + \frac{1}{2}\log|\cos x + \sin x| + c$$

(b) Integration of trigonometric functions $\tan x, \cot x, \sec x$ and $\cos ecx$

(i) Let $$I = \int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$$

Let $\cos x = t \Rightarrow -\sin x\, dx = dt \Rightarrow \sin x dx = -dt$

$$\therefore \quad I = \int \frac{-dt}{t} = -\log|t| + c = -\log|\cos x| + c$$

$$= \log|\sec x| + c$$

$$\therefore \quad \int \tan x\, dx = \log|\sec x| + c = -\log|\cos x| + c$$

(ii) Let $$I = \int \cot x\, dx = \int \frac{\cos x}{\sin x} dx$$

Let $\sin x = t \Rightarrow \cos x\, dx = dt$

$$\therefore \quad I = \int \frac{dt}{t} = \log|t| = \log|\sin x| + c$$

$$\therefore \quad \int \cot t\, dx = \log|\sin x| + c$$

(iii) Let $$I = \int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx$$

Let $\sec x + \tan x = t$

$$\therefore \quad (\sec x \tan x + \sec^2 x)dx = dt \Rightarrow \sec x(\sec x + \tan x)dx = dt$$

$$\therefore \quad I = \int \frac{dt}{t} = \log|t| + c = \log|\sec x + \tan x| + c \qquad \ldots (1)$$

$$= \log\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| + c$$

$$= \log\left|\frac{1 + \sin x}{\cos x}\right| + c$$

$$= \log\left|\frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}\right| + c$$

$$= \log\left|\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right|+c$$

$$= \log\left|\frac{1+\tan x/2}{1-\tan x/2}\right|+c$$

$$= \log\tan\left|\frac{\pi}{4}+\frac{x}{2}\right|+c$$

$\therefore$ $$\int \sec x\,dx = \log\left|\sec x+\tan x\right|+c = \log\tan\left|\frac{\pi}{4}+\frac{x}{2}\right|+c$$

(iv) Let $$I = \int \operatorname{cosec} x\,dx = \int \frac{\operatorname{cosec} x\left(\operatorname{cosec} x-\cot x\right)}{\left(\operatorname{cosec} x-\cot x\right)}dx$$

Let $$\operatorname{cosec} x-\cot x = t \Rightarrow \left(-\operatorname{cosec} x\cot x+\operatorname{cosec}^2 x\right)dx = dt$$

$\therefore$ $$\operatorname{cosec} x\left(\operatorname{cosec} x-\cot x\right)dx = dt$$

$\therefore$ $$I = \int\frac{dt}{t} = \log|t|+c = \log\left|\operatorname{cosec} x-\cot x\right|+\text{c}$$

$$= \log\left|\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right|+c = \log\left|\frac{1-\cos x}{\sin x}\right|+c$$

$$= \log\left|\frac{1-1+2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)}\right|+c = \log\left|\tan\frac{x}{2}\right|+c$$

$\therefore$ $$\int \operatorname{cosec} x\,dx = \log\left|\operatorname{cosec} x-\cot x\right|+c = \log\left|\tan x/2\right|+c$$

$$\left(\because \operatorname{cosec} x-\cot x = \tan x/2\right)$$

Example 11. Integrate $\dfrac{1}{\sqrt{1+\cos 2x}}$ w.r.t. x

Solution : Let $$I = \int\frac{1}{\sqrt{1+\cos 2x}}dx = \int\frac{1}{\sqrt{2\cos^2 x}}dx$$

$$= \frac{1}{\sqrt{2}}\int\frac{1}{\cos x}dx = \frac{1}{\sqrt{2}}\int\sec x\,dx$$

$$= \frac{1}{\sqrt{2}}\log|\sec x+\tan x|+c$$

Example 12. Integrate $\sqrt{\sec x+1}$ with respect to x

Solution : Let $I=\int\sqrt{\sec x+1}dx=\int\sqrt{\left(\frac{1}{\cos x}+1\right)}dx$

$$=\int\sqrt{\frac{1+\cos x}{\cos x}}dx=\int\sqrt{\frac{2\cos^2 x/2}{1-2\sin^2 x/2}}dx=\int\frac{\sqrt{2}\cos x/2}{\sqrt{1-\{\sqrt{2}\sin(x/2)\}^2}}dx$$

Let $\sqrt{2}\sin(x/2)=t\Rightarrow\sqrt{2}\cos(x/2)\times 1/2dx=dt$

$\Rightarrow$ $\sqrt{2}\cos(x/2)dx=2dt$

$\therefore$ $I=\int\frac{2dt}{\sqrt{1-t^2}}=2\sin^{-1}t+=2\sin^{-1}(\sqrt{2}\sin x/2)+c$

(c) Using substitution method by trigonometric identities.

Many times when the integrand involves some trigonometric functiosn, we use some known identities to make it integrable and then find integral by suitable substitution.

Illustrative Examples

Example 13. Evaluate the following:

(i) $\int\cos 3x\cos 4x\,dx$ (ii) $\int\sin^2 x\,dx$ (iii) $\int\cos^3 x\,dx$ (iv) $\int\sin^4 x\,dx$

Solution :(i) Let $I=\int\cos 3x\cos 4x\,dx=\frac{1}{2}\int 2\cos 4x\cos 3x\,dx$

$$=\frac{1}{2}\int(\cos 7x+\cos x)dx=\frac{1}{2}\left[\frac{\sin 7x}{7}+\sin x\right]+c$$

(ii) Let $I=\int\sin^2 x\,dx=\int\frac{1-\cos 2x}{2}dx=\frac{1}{2}\int(1-\cos 2x)dx$

$$=\frac{1}{2}\left[x-\frac{\sin 2x}{2}+c\right]$$

(iii) Let $I=\int\cos^3 x\,dx=\frac{1}{4}\int(\cos 3x+3\cos x)\,dx$

$$\left(\because\cos 3x=4\cos^3 x-3\cos x\Rightarrow\cos^3 x=1/4(\cos 3x+3\cos x)\right)$$

$$=\frac{1}{4}\left[\frac{\sin 3x}{3}+3\sin x\right]+c$$

(iv) Let $I=\int\sin^4 x\,dx=\int(\sin^2 x)^2dx=\int\left(\frac{1-\cos 2x}{2}\right)^2dx$

$$=\frac{1}{4}\int(1+\cos^2 2x-2\cos 2x)\,dx$$

$$= \frac{1}{4}\int\left[1 + \frac{1+\cos 4x}{2} - 2\cos 2x\right]dx = \frac{1}{8}\int(3+\cos 4x - 4\cos 2x)\,dx$$

$$= \frac{1}{8}\left[3x + \frac{\sin 4x}{4} - 2\sin 2x\right] + c$$

Exercise 9.2

Integrate the following functions with respect to x

1. (i) $x\sin x^2$ (ii) $x\sqrt{x^2+1}$

2. (i) $\dfrac{e^x - \sin x}{e^x + \cos x}$ (ii) $\dfrac{e^x}{\sqrt{1+e^x}}$

3. (i) $\sqrt{e^x+1}$ (ii) $\dfrac{e^{\sqrt{x}}\cos e^{\sqrt{x}}}{\sqrt{x}}$

4. (i) $\dfrac{1}{x(1+\log x)}$ (ii) $\dfrac{(1+\log x)^3}{x}$

5. (i) $\dfrac{e^{m\tan^{-1}x}}{1+x^2}$ (ii) $\dfrac{\sin^p x}{\cos^{p+2} x}$

6. (i) $\dfrac{1}{\sqrt{1+\cos 2x}}$ (ii) $\dfrac{1+\cos x}{\sin x\cos x}$

7. (i) $\sin 3x \sin 2x$ (ii) $\sqrt{1-\sin x}$

8. (i) $\cos^4 x$ (ii) $\sin^3 x$

9. (i) $\dfrac{1}{\sin x\cos^3 x}$ (ii) $\dfrac{(1+x)e^x}{\cos^2(xe^x)}$

10. (i) $\dfrac{1}{1-\tan x}$ (ii) $\dfrac{1}{1+\cot x}$

11. (i) $\dfrac{\sec^4 x}{\sqrt{\tan x}}$ (ii) $\dfrac{1-\tan x}{1+\tan x}$

12. (i) $\dfrac{\sin(x+a)}{\sin(x-a)}$ (ii) $\dfrac{\sin x}{\sin(x-a)}$

13. (i) $\dfrac{\sin 2x}{\sin 5x \sin 3x}$ (ii) $\dfrac{\sin 2x}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}$

$\left[\text{Hint} = \sin 2x = \sin(5x-3x)\right]$ $\left[\text{Hint} = 2x = \left(x-\frac{\pi}{6}\right)+\left(x+\frac{\pi}{6}\right)\right]$

14. (i) $\frac{1}{3\sin x+4\cos x}$ $[\text{Hint: } 3=r\cos\theta, 4=r\sin\theta]$ (ii) $\frac{1}{\sin(x-a)\sin(x-b)}$

15. (i) $\frac{\sin x\cos x}{a\cos^2 x+b\sin^2 x}$ (ii) $\frac{\sec x}{\sqrt{\sin(2x+\alpha)+\sin\alpha}}$

16. (i) $\frac{1}{\sqrt{\cos^3 x\sin(x+a)}}$ (ii) $\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}$

(d) Integration by substitution of variables by trigonometric functions.

(i) $\frac{1}{a^2+x^2}$ (ii) $\frac{1}{\sqrt{a^2-x^2}}$ (iii) $\frac{1}{\sqrt{x^2+a^2}}$ (iv) $\frac{1}{\sqrt{x^2-a^2}}$

(i) Let, $$I=\int\frac{1}{a^2+x^2}dx$$

If, $x=a\tan\theta$ then $dx=a\sec^2\theta d\theta$

Now $$I=\int\frac{a\sec^2\theta d\theta}{a^2+a^2\tan^2\theta}=\frac{1}{a}\int\frac{\sec^2\theta}{1+\tan^2\theta}d\theta$$

$$=\frac{1}{a}\int\frac{\sec^2\theta}{\sec^2\theta}d\theta=\frac{1}{a}\int d\theta=\frac{1}{a}(\theta)+c=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$$

$$\int\frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$$

(ii) Let $$I=\int\frac{1}{\sqrt{a^2-x^2}}dx$$

If $x=a\sin\theta$ then $dx=a\cos\theta d\theta$

$\therefore$ $$I=\int\frac{a\cos\theta d\theta}{\sqrt{a^2-a^2\sin^2\theta}}=\int\frac{a\cos\theta d\theta}{a\cos\theta}=\int d\theta=\theta+c=\sin^{-1}\frac{x}{a}+c$$

$$\int\frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\frac{x}{a}+c$$

(iii) Let $$I=\int\frac{1}{\sqrt{x^2+a^2}}dx$$

Let $x=a\tan\theta\Rightarrow dx=a\sec^2\theta d\theta$

$\therefore$ $$I=\int\frac{a\sec^2\theta d\theta}{\sqrt{a^2\tan^2\theta+a^2}}=\int\frac{a\sec^2\theta}{a\sec\theta}d\theta$$

$$=\int\sec\theta d\theta=\log|\sec\theta+\tan\theta|+c_1$$

$$= \log\left|\sqrt{1+\frac{x^2}{a^2}}+\frac{x}{a}\right|+c_1$$

$$= \log\left|\frac{\sqrt{x^2+a^2}+x}{a}\right|+c_1 = \log\left|x+\sqrt{x^2+a^2}\right|-\log a+c_1$$

$$= \log\left|x+\sqrt{x^2+a^2}\right|+c, \text{ where } c=c_1-\log a$$

$$\therefore \quad \int\frac{1}{\sqrt{x^2+a^2}}\,dx = \log|x+\sqrt{x^2+a^2}|+c$$

(iv) Let $I = \int\frac{1}{\sqrt{x^2-a^2}}\,dx$

Let $x = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta\,d\theta$

$$\therefore \quad I = \int\frac{1}{\sqrt{a^2\sec^2\theta-a^2}}\times a\sec\theta\tan\theta\,d\theta = \int\frac{a\sec\theta\tan\theta}{a\tan\theta}$$

$$= \int\sec\theta\,d\theta = \log|\sec\theta+\tan\theta|+c_1$$

$$= \log\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right|+c_1 = \log\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|+c_1$$

$$= \log|x+\sqrt{x^2-a^2}|-\log a+c_1 = \log|x+\sqrt{x^2-a^2}|+c \qquad (\text{where } c=c_1-\log a)$$

$$\therefore \quad \int\frac{1}{\sqrt{x^2-a^2}}\,dx = \log|x+\sqrt{x^2-a^2}|+c$$

Some Suitable trigonometric substitutions

	Integrands	**Substitution**
(i)	$\sqrt{x^2+a^2}$ or $\frac{1}{\sqrt{x^2+a^2}}$	$x=a\tan\theta$
(ii)	$\sqrt{a^2-x^2}$ or $\frac{1}{\sqrt{a^2-x^2}}$	$x=a\sin\theta$ or $x=a\cos\theta$
(iii)	$\sqrt{x^2-a^2}$ or $\frac{1}{\sqrt{x^2-a^2}}$	$x=a\sec\theta$
(iv)	$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x=a\cos 2\theta$ or $x=a\cos\theta$
(v)	$\sqrt{x+a}$	$x=a\cos 2\theta$ or $x=a\cos\theta$

(vi) $\sqrt{2ax-x^2}$ $\qquad x=2a\sin^2\theta$ or $x=a(1-\cos 2\theta)$

(vii) $\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$ $\qquad x^2=a^2\cos 2\theta$

(viii) $\sqrt{\dfrac{x+a}{x}}$ or $\sqrt{\dfrac{x}{x+a}}$ $\qquad x=a\tan^2\theta$

Illustrative Examples

Example 14. Integrate the following with respect to x

(i) $\dfrac{x}{1+x^4}$ $\qquad$ (ii) $\dfrac{1}{\sqrt{9-25x^2}}$

Solution : (i) Let $\qquad I=\int \dfrac{x}{1+x^4}dx$

Let $\qquad x^2=t \Rightarrow xdx=\dfrac{dt}{2}$

$\therefore \qquad I=\dfrac{1}{2}\int\dfrac{dt}{1+t^2}=\dfrac{1}{2}\tan^{-1}(t)+c=\dfrac{1}{2}\tan^{-1}(x^2)+c$

(ii) Let $\qquad I=\int\dfrac{1}{\sqrt{9-25x^2}}dx=\dfrac{1}{5}\int\dfrac{1}{\sqrt{(3/5)^2-x^2}}dx$

$$=\frac{1}{5}\sin^{-1}\left(\frac{x}{3/5}\right)+c=\frac{1}{5}\sin^{-1}\frac{5x}{3}+c$$

Example 15. Integrate $\dfrac{1}{\sqrt{x^2-4x+5}}$ with respect to x

Solution : $\qquad I=\int\dfrac{1}{\sqrt{x^2-4x+5}}dx=\int\dfrac{1}{\sqrt{(x-2)^2+1}}dx$

$$=\log|(x-2)+\sqrt{(x-2)^2+1}|+c$$

$$=\log|(x-2)+\sqrt{x^2-4x+5}|+c$$

Example 16. Evaluate: $\int\dfrac{1}{x^2+2x+5}dx$

Solution : Let $\qquad I=\int\dfrac{1}{x^2+2x+5}dx=\int\dfrac{1}{(x+1)^2+(2)^2}dx$

$$=\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c$$

Example 17. Integrate $\frac{1}{\sqrt{5x-6-x^2}}$ with respect to x

Solution : Let $I=\int\frac{1}{\sqrt{5x-6-x^2}}dx=\int\frac{1}{\sqrt{-6-(x^2-5x)}}dx$

$$=\int\frac{1}{\sqrt{(25/4-6)-(x^2-5x+25/4)}}dx=\int\frac{1}{\sqrt{(1/2)^2-(x-5/2)^2}}dx$$

$$=\sin^{-1}\left[\frac{x-5/2}{1/2}\right]+c=\sin^{-1}\left(\frac{2x-5}{1}\right)+c$$

Example 18. Integrate $\frac{(1+x)^2}{x+x^3}$ with respect to x

Solution : Let $I=\int\frac{(1+x)^2}{x+x^3}dx=\int\frac{1+x^2+2x}{x(1+x^2)}dx$

$$=\int\left[\frac{(1+x^2)}{x(1+x^2)}+\frac{2x}{x(1+x^2)}\right]dx=\int\frac{1}{x}dx+\int\frac{2}{1+x^2}dx$$

$$=\log|x|+2\tan^{-1}x+c$$

Example 19. Integrate $\frac{\sin 2x\cos 2x}{\sqrt{9-\cos^4 2x}}$ with respect to x

Solution : Let $I=\int\frac{\sin 2x\cos 2x}{\sqrt{9-\cos^4 2x}}dx$

Let $\cos^2 2x=t\Rightarrow 2\cos 2x.(-\sin 2x)2.dx=dt$

$\Rightarrow$ $\sin 2x\cos 2x\,dx=-\frac{dt}{4}$

$\therefore$ $I=-\frac{1}{4}\int\frac{dt}{\sqrt{9-t^2}}=-\frac{1}{4}\sin^{-1}\left(\frac{t}{3}\right)+c$

$$=-\frac{1}{4}\sin^{-1}\left(\frac{\cos^2 2x}{3}\right)+c$$

Example 20. If $\int\frac{2^x}{\sqrt{1-4^x}}dx=k\sin^{-1}2^x+c$, then find the vlaue of k

Solution : Let $$I = \int \frac{2^x}{\sqrt{1-4^x}}dx = \int \frac{2^x}{\sqrt{1-(2^x)^2}}dx$$

Let $$2^x = t \Rightarrow 2^x \log_e 2dx = dt \Rightarrow 2^x dx = \frac{dt}{\log_e 2}$$

$\therefore$ $$I = \frac{1}{\log_e 2}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{\log_e 2}\sin^{-1}(t) + c = \log_2 e.(\sin^{-1} 2^x) + c$$

$\therefore$ $$\int \frac{2^x}{\sqrt{1-4^x}}dx = \log_2 e.(\sin^{-1} 2^x) + c$$

but it is given that $$\int \frac{2^x}{\sqrt{1-4^x}}dx = k(\sin^{-1} 2^x) + c$$

$\therefore$ On comparison, $k = \log_2 e$

Exercise 9.3

Integrate the following function with respect to *x*

1. (i) $\dfrac{1}{50+2x^2}$ (ii) $\dfrac{1}{\sqrt{32-2x^2}}$

2. (i) $\dfrac{1}{\sqrt{1-e^{2x}}}$ (ii) $\dfrac{1}{\sqrt{1+4x^2}}$

3. (i) $\dfrac{1}{\sqrt{a^2-b^2x^2}}$ (ii) $\dfrac{1}{\sqrt{(2-x)^2+1}}$

4. (i) $\dfrac{x^2}{\sqrt{x^6+4}}$ (ii) $\dfrac{x^4}{\sqrt{1-x^{10}}}$

5. (i) $\dfrac{1}{x^2+6x+8}$ (ii) $\dfrac{1}{\sqrt{2x^2-x+2}}$

6. (i) $\dfrac{e^x}{e^{2x}+2e^x\cos\propto+1}$ (ii) $\dfrac{1+\tan^2 x}{\sqrt{\tan^2 x+3}}$

7. (i) $\dfrac{1}{\sqrt{3x-2-x^2}}$ (ii) $\dfrac{1}{\sqrt{4+8x-5x^2}}$

8. (i) $\dfrac{\sin x+\cos x}{\sqrt{\sin 2x}}$ (ii) $\dfrac{1}{\sqrt{x^2+2ax+b^2}}$

9. (i) $\sqrt{\dfrac{a-x}{x}}$ (ii) $\sqrt{\dfrac{a+x}{a-x}}$

10. (i) $\dfrac{\sqrt{x}}{\sqrt{a^3 - x^3}}$ (ii) $\dfrac{1}{(a^2 + x^2)^{3/2}}$

11. (i) $\dfrac{1}{(1 - x^2)^{3/2}}$ (ii) $\dfrac{x+1}{\sqrt{x^2+1}}$

12. (i) $\dfrac{1}{\sqrt{(x-\alpha)(\beta - x)}}$ (ii) $\dfrac{1}{\sqrt{2x - x^2}}$

13. (i) $\dfrac{1}{\sqrt{(x-1)(x-2)}}$ (ii) $\dfrac{\cos x}{\sqrt{4 - \sin^2 x}}$

III. Integration by resolving into partial fractions

(a) Rational algebraic function

Definition : If $f(x)$ and $g(x)$ are polynomials of x then fraction $\dfrac{f(x)}{g(x)}$ is called rational algebraic function.

For example: $\dfrac{x^2 - x - 6}{x^3 + x^2 - 3x + 4}, \dfrac{2x+1}{2x^2 + x + 1}, \dfrac{x^2}{x^2+1}, \dfrac{2x^3}{(x-1)(x^2+1)}, \dfrac{x^4}{x^3 + 2x - 4}$

Proper Rational Fraction : If in a rational algebraic fraction the power of numerator is less than the power of denominator then it is called a proper rational fraction.

Improper Rational Fraction : If in a rational algebraic fraction the power of numerator is more than or equal to the power of denominator then it is called an improper rational fraction.

For example : $\dfrac{2x+3}{3x^2 + x + 4}$, is a proper fraction.

For example : $\dfrac{3x^3 + x^2 + 5x - 4}{x^2 + x + 2}$ and $\dfrac{3x^2 + x + 2}{(x+1)(x+3)}$ are improper fractions.

Remark : An improper rational fraction can be expressed into a proper rational fraction by division process.

For example $\dfrac{3x^3 + 2x + 7}{x^2 + 5x + 9} = 3(x-5) + \dfrac{50x + 142}{x^2 + 5x + 9}$

The above rational algebraci function may be expressed or convert into partial fraction and then integrate each fraction.

Partial Fraction : It is always possible to write the integrand as a sum of simpler rational functiosn by a method called partial fraction decomposition.

For example $\dfrac{2x-5}{x^2 - 5x + 6} = \dfrac{1}{x-2} + \dfrac{1}{x-3}$

Rules of resolving a rational fraction into partial fraction

[A] First of all if the fraction is not proper then convert it into a proper fraction by using division method. So that an improper fraction will be decompose into a polynomial and proper fraction. Keep the polynomial same and decompose the real fraction into partial fraction.

[B] If denominator of proper fraction is not in the form of factors then factorize it.

[C] Now assume the constant term as equal to the power of denominator. The following indicates the types of simpler partial fraction that is associated with various kind of rational functions.

(a) If denominator contains linear factors without repetition then the form of partial fraction will be according tot he following example:

$$\frac{x}{(x-1)(x+2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x+2)}+\frac{C}{(x-3)}$$

(b) If denominaot rcontains linear factors with repetition then the form of partial fraction will be according tot he following example:

$$\frac{x}{(x-1)^2(x+3)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x+3)}$$

(c) If denominator contains quadratic factors then the form of partial fraction will be according to the following example:

$$\frac{x}{(x-1)(x^2+2)}=\frac{A}{(x-1)}+\frac{Bx+C}{(x^2+2)}$$

Remark : If in a partial fraction both numerator and denominator contain x^2 i.e. quadratic then x^2 must be considedred as linear and the partial fraction may be written as

$$\frac{x^2+2}{(x^2+1)(x^2+3)}=\frac{A}{x^2+1}+\frac{B}{x^2+3}$$

[D] Finding the values of constant A, B and C

(a) As discussed in [C] take LCM of denominators of partial fractions in RHS and find their sum.

(b) Fractions of both the sides are equal and denominators are also equal. hence by comparing their numerators and factors of all powers of x and constant terms find equations. The number of such equations should be same as number of unknown constants. Find the vlaues of unknown constants from equations and get the required partial fraction.

Let $$\frac{2x+3}{(x+2)(x+1)}=\frac{A}{(x+2)}+\frac{B}{(x+1)}$$

or $$\frac{2x+3}{(x+2)(x+1)}=\frac{A(x+1)+B(x+2)}{(x+2)(x+1)}$$

or $$2x+3=A(x+1)+B(x+2) \quad (1)$$

or $$2x+3=(A+B)x+(A+2B)$$

On comparision of coefficients of equal terms

$$\left.\begin{array}{l} A+B \quad =2 \\ A+2B=3 \end{array}\right] \begin{array}{l}\text{on solving} \\ A=1,\ B=1\end{array}$$

so $$\frac{2x+3}{(x+2)(x+1)}=\frac{1}{(x+2)}+\frac{1}{(x+1)}$$

Alternative Methods :

(i) **Short Method :** In the above example the corresponding vlaues of x of factors $(x + 1)$ and $(x + 2)$ as $x = -1$ and $x = -2$ can be substituted in equation (1) to find the values of A and B.

(ii) **Division Method :** Division method is more sutiable for repeatign factors of denominator in fractions, in this repeathin factor may be considered as y and the division process id done so that we can get integrable terms.

For example $\frac{x^2}{(x+1)^3(x+2)}$ Let $(x+1)=y$ then

$$\frac{x^2}{(x+1)^3(x+2)}=\frac{(y-1)^2}{y^3(y+1)}=\frac{(1-2y+y^2)}{y^3(1+y)}$$

$$=\frac{1}{y^3}\left[1-3y+4y^2-\frac{4y^3}{1+y}\right]$$

$$=\frac{1}{y^3}-\frac{3}{y^2}+\frac{4}{y}-\frac{4}{1+y}$$

$$=\frac{1}{(x+1)^3}-\frac{3}{(x+1)^2}+\frac{4}{(x+1)}-\frac{4}{(x+2)}$$

which can easily be integrated

(iii) **By inspection :** If there is 1 as numerator in a real fraction and the difference of parts is a constant quantity then this methdo can be used. For this divide by difference of parts and subtract the reciprocal of bigger part from the reciprocal of smaller part.

For example $\frac{1}{(x+2)(x-3)}=\frac{1}{5}\left[\frac{1}{x-3}-\frac{1}{x+2}\right]$ here difference of parts $=(x+2)-(x-3)=5$

Some Standard Integrals

(i) $$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c \qquad (x>a)$$

(ii) $$\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c \qquad (x<a)$$

Proof :

(i) $$\frac{1}{x^2-a^2}=\frac{1}{(x-a)(x+a)}=\frac{1}{2a}\left[\frac{1}{x-a}-\frac{1}{x+a}\right] \qquad \text{(By insepection)}$$

$\therefore$ $$\int\frac{1}{x^2-a^2}dx=\frac{1}{2a}\int\left[\frac{1}{x-a}-\frac{1}{x+a}\right]dx\,\frac{1}{2a}\int\left[\frac{1}{x-a}-\frac{1}{x+a}\right]dx$$

$$=\frac{1}{2a}\int\frac{1}{x-a}dx-\frac{1}{2a}\int\frac{1}{x+a}dx$$

$$=\frac{1}{2a}\log|x-a|-\frac{1}{2a}\log|x-a|+c$$

$$=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$$

Similarly

(ii) $$\frac{1}{a^2-x^2}=\frac{1}{(a+x)(a-x)}=\frac{1}{2a}\left[\frac{1}{a+x}+\frac{1}{a-x}\right]$$

$\therefore$ $$\int\frac{1}{a^2-x^2}dx=\frac{1}{2a}\int\left[\frac{1}{a+x}+\frac{1}{a-x}\right]dx$$

$$=\frac{1}{2a}\left[\log|a+x|+\frac{\log|a-x|}{-1}\right]+c$$

$$=\frac{1}{2a}[\log|a+x|-\log|a-x|]+c$$

$$=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$$

Remark : In some cases substittion makes the task easy. Specially when there is any power of x, Let x^{n-1} is a part of numerator and remaining fraction is a rational function of x^n then substitute $x^n=t$ and then decompose in partial fraction.

Illustrative Examples

Example 21. Integrate the following functions with respect to x

(i) $\dfrac{1}{16x^2-9}$ (ii) $\dfrac{1}{9-4x^2}$

Solution : (i) Let, $$I=\int\frac{1}{16x^2-9}dx=\int\frac{1}{(4x)^2-(3)^2}dx$$

Let $4x=t\Rightarrow 4dx=dt$ or $dx=\frac{1}{4}dt$

$\therefore$ $$I=\frac{1}{4}\int\frac{dt}{t^2-3^2}=\frac{1}{4}\times\frac{1}{2\times3}\log\left|\frac{t-3}{t+3}\right|+c$$

$$=\frac{1}{24}\log\left|\frac{4x-3}{4x+3}\right|+c$$

Solution : (ii) Let $$I=\int\frac{1}{9-4x^2}dx=\int\frac{1}{(3)^2-(2x)^2}dx$$

Let $$2x=t\Rightarrow dx=\frac{dt}{2}$$

$\therefore$ $$I=\frac{1}{2}\int\frac{dt}{3^2-t^2}=\frac{1}{2}\times\frac{1}{2\times3}\log\left|\frac{3+t}{3-t}\right|+c$$

$$=\frac{1}{12}\log\left|\frac{3+2x}{3-2x}\right|+c$$

Example 22. Integrate $\frac{1}{x^2-x-2}$ with respect to x.

Solution : $$\frac{1}{x^2-x-2}=\frac{1}{(x-2)(x+1)}=\frac{1}{3}\left[\frac{1}{x-2}-\frac{1}{x+1}\right]$$ (by mathod of inspection)

$\therefore$ $$\int\frac{1}{x^2-x-2}dx=\frac{1}{3}\int\left[\frac{1}{(x-2)}-\frac{1}{(x+1)}\right]dx$$

$$=\frac{1}{3}[\log|(x-2)|-\log|x+1|]+c$$

$$=\frac{1}{3}\log\left|\frac{x-2}{x+1}\right|+c$$

Example 23. Evaluate: $\int\frac{x^2+x+2}{(x-1)(x-2)}dx$

Solution : $$\frac{x^2+x+2}{(x-1)(x-2)}=1+\frac{4x}{(x-1)(x-2)}$$ (on dividing)

Let $$\frac{4x}{(x-1)(x-2)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}$$

or $$4x=A(x-2)+B(x-1) \qquad (1)$$

Now in (1)

Put $x=2$ $8 = B(2-1)$ or $B = 8$

Put $x=1$ $4 = -A$ or $A = -4$

$$\therefore \quad \frac{4x}{(x-1)(x-2)} = \frac{-4}{x-1} + \frac{8}{x-2}$$

$$\therefore \quad \frac{x^2+x+2}{(x-1)(x-2)} = 1 + \left[\frac{-4}{x-1} + \frac{8}{x-2}\right]$$

or

$$\int \frac{x^2+x+2}{(x-1)(x-2)}dx = \int\left[1 - \frac{4}{x-1} + \frac{8}{x-2}\right]dx$$

$$= x - 4\log|x-1| + 8\log|x-2| + c$$

$$= x + 4\left[2\log|x-2| - \log|x-1|\right] + c$$

$$= x + 4\log\frac{(x-2)^2}{|x-1|} + c.$$

Example 24. Integrate $\dfrac{1}{(x+1)^2(x^2+1)}$ with respect to *x*.

Solution : Let $\dfrac{1}{(x+1)^2(x^2+1)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+1)^2} + \dfrac{Cx+D}{(x^2+1)}$

$$\Rightarrow \quad 1 = A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2$$

$$\Rightarrow \quad 1 = A(x^3+x^2+x+1) + B(x^2+1) + (Cx^3+2Cx^2+Dx^2+2Dx+Cx+D)$$

$$\Rightarrow \quad 1 = x^3(A+C) + x^2(A+B+2C+D) + x(A+C+2D) + (A+B+D)$$

On comparison

$A + C = 0$ (1) $\quad$ $A+B+2C+D = 0$ (2)

$A + C + 2D = 0$ (3) $\quad$ $A+B+D = 0$ (4)

From (1) and (3), $2D = O \Rightarrow D = 0$

From (1) and (2), $B + C + D = 0$ $\quad$ on solving, $2C = -1 \Rightarrow C = -1/2 \therefore A = 1/2$

From (1) and (4), $B - C + D = 1$

From (4), $1/2 + B + 0 = 1 \Rightarrow B = 1/2$

$$\therefore \quad \frac{1}{(x+1)^2(x^2+1)} = \frac{1}{2}.\frac{1}{(x+1)} + \frac{1}{2}.\frac{1}{(x+1)^2} - \frac{1}{2}.\frac{x}{(x^2+1)}$$

$$\therefore \qquad \int \frac{1}{(x+1)^2(x^2+1)}dx = \frac{1}{2}\int \frac{1}{(x+1)}dx + \frac{1}{2}\int \frac{1}{(x+1)^2}dx - \frac{1}{4}\int \frac{2x}{(x^2+1)}dx$$

$$= \frac{1}{2}\log|x+1| - \frac{1}{2}\frac{1}{(x+1)} - \frac{1}{4}\log(x^2+1) + c$$

[here $x^2+1=t \Rightarrow 2xdx = dt$]

$$= \frac{1}{2}\log|x+1| - \frac{1}{4}\log(x^2+1) - \frac{1}{2(x+1)} + c$$

Example 25. Integrate $\frac{x^2+x+1}{(x-1)^3}$ with respect to *x*.

Solution : Let $(x-1) = y \therefore \frac{x^2+x+1}{(x-1)^3} = \frac{(y+1)^2+(y+1)+1}{y^3}$

$$= \frac{y^2+3y+3}{y^3} = \frac{1}{y} + \frac{3}{y^2} + \frac{3}{y^3}$$

$$= \frac{1}{(x-1)} + \frac{3}{(x-1)^2} + \frac{3}{(x-1)^3}$$

$$\therefore \qquad \int \frac{x^2+x+1}{(x-1)^3}dx = \int \frac{1}{(x-1)}dx + \int \frac{3}{(x-1)^2}dx + \int \frac{3}{(x-1)^3}dx$$

$$= \log|x-1| - \frac{3}{(x-1)} - \frac{3}{2(x-1)^2} + c$$

Example 26. Integrate $\frac{1}{\sin x + \sin 2x}$ with respect to *x*.

Solution : Let $I = \int \frac{1}{\sin x + \sin 2x}dx$

$$= \int \frac{1}{\sin x(1+2\cos x)}dx = \int \frac{\sin x}{\sin^2 x(1+2\cos x)}dx$$

$$= \int \frac{\sin x}{(1-\cos^2 x)(1+2\cos x)}dx$$

$$= \int \frac{-dt}{(1-t^2)(1+2t)} \qquad \text{[where } \cos x = t \Rightarrow -\sin x\,dx = dt\text{]}$$

$$= -\int \frac{dt}{(1-t)(1+t)(1+2t)}$$

Again, let $\dfrac{1}{(1-t)(1+t)(1+2t)} = \dfrac{A}{(1-t)} + \dfrac{B}{(1+t)} + \dfrac{C}{(1+2t)}$

or $1 = A(1+t)(1+2t) + B(1-t)(1+2t) + C(1-t)(1+t)$

On putting on both sides,

$$\left.\begin{array}{lll} \text{put } t=1, & 1 = A(2)(3) & \Rightarrow A = 1/6 \\ \text{put } t=-1, & 1 = B(1+1)(1-2) & \Rightarrow B = -1/2 \\ \text{put } t=-1/2, & 1 = C(1+1/2)(1-1/2) & \Rightarrow C = 4/3 \end{array}\right]$$

$\therefore$ $$\frac{1}{(1-t)(1+t)(1+2t)} = \frac{1}{6}\cdot\frac{1}{(1-t)} - \frac{1}{2}\cdot\frac{1}{(1+t)} + \frac{4}{3}\cdot\frac{1}{(1+2t)}$$

$\therefore$ $$I = -\int\left[\frac{1}{6}\cdot\frac{1}{(1-t)} - \frac{1}{2}\cdot\frac{1}{(1+t)} + \frac{4}{3}\cdot\frac{1}{(1+2t)}\right]dt$$

$$= -\frac{1}{6}\frac{\log|1-t|}{(-1)} + \frac{1}{2}\log|1+t| - \frac{4}{3}\frac{\log|1+2t|}{2} + c$$

$$= \frac{1}{6}\log|1-\cos x| + \frac{1}{2}\log|1+\cos x| - \frac{2}{3}\log|1+2\cos x| + c$$

Example 27. Integrate $\dfrac{2x}{(x^2+1)(x^2+3)}$ with respect to x

Solution : Let $I = \displaystyle\int \frac{2x}{(x^2+1)(x^2+3)}dx$

$$= \int \frac{dt}{(t+1)(t+3)} \qquad [\text{where } x^2 = t \Rightarrow 2xdx = dt]$$

$$= \frac{1}{2}\int\left[\frac{1}{t+1} - \frac{1}{t+3}\right]dt$$

$$= \frac{1}{2}\left[\log|t+1| - \log|t+3|\right] + c$$

$$= \frac{1}{2}\log\left|\frac{t+1}{t+3}\right| + c = \frac{1}{2}\log\left(\frac{x^2+1}{x^2+3}\right) + c$$

Exampler 28. Integrate $\frac{1}{x(x^n-1)}$ with respect to x.

Solution : Let $I = \int \frac{1}{x\left(x^n-1\right)} dx$

$$= \int \frac{x^{n-1}}{x^n\left(x^n-1\right)} \quad \text{(multiplying numerator and denominator by } x^{n-1}\text{)}$$

Again, let $x^n = t \Rightarrow nx^{n-1}dx = dt \Rightarrow x^{n-1}dx = \frac{dt}{n}$

$$I = \frac{1}{n}\int \frac{dt}{t(t-1)} = \frac{1}{n}\int\left[\frac{1}{t-1}-\frac{1}{t}\right]dt = \frac{1}{n}[\log|t-1|-\log|t|]+c$$

$$= \frac{1}{n}\log\left|\frac{t-1}{t}\right|+c = \frac{1}{n}\log\left|\frac{x^n-1}{x^n}\right|+c$$

Exercise 9.4

Integrate the following functions with respect to x.

(1) $\frac{1}{16-9x^2}$ (2) $\frac{1}{x^2-36}$ (3) $\frac{3x}{(x+1)(x-2)}$ (4) $\frac{3x-2}{(x+1)^2(x+3)}$

(5) $\frac{x^2}{(x+1)(x-2)(x-3)}$ (6) $\frac{x^2}{x^4-x^2-12}$ (7) $\frac{1}{x^3-x^2-x+1}$ (8) $\frac{x^2}{(x+1)(x-2)}$

(9) $\frac{x^2}{(x^2+a^2)(x^2+b^2)}$ (10) $\frac{x+1}{x^3+x^2-6x}$ (11) $\frac{x^2+8x+4}{x^3-4x}$ (12) $\frac{1}{(x-1)^2(x+2)}$

(13) $\frac{1-3x}{1+x+x^2+x^3}$ (14) $\frac{1+x^2}{x^5-x}$ (15) $\frac{x^2+5x+3}{x^2+3x+2}$ (16) $\frac{x-1}{(x+1)(x^2+1)}$

(17) $\frac{1}{(1+e^x)(1-e^{-x})}$ (18) $\frac{1}{(e^x-1)^2}$ (19) $\frac{e^x}{e^{2x}+5e^x+6}$ (20) $\frac{\sec^2 x}{(2+\tan x)(3+\tan x)}$

(21) $\frac{1}{x(x^5+1)}$ (22) $\frac{1}{x(a+bx^n)}$ (23) $\frac{8}{(x+2)(x^2+4)}$ (24) $\frac{(1-\cos x)}{\cos x(1+\cos x)}$

(b) Integration of special forms of rational functions

(i) $\int \frac{1}{ax^2+bx+c}dx$ (ii) $\int \frac{px+q}{ax^2+bx+c}dx$

where a, b, c, p and q are constants.

Proof : (i) $ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]$

$$= a\left[\left(x+\frac{b}{2a}\right)^2 - \left(\frac{b^2-4ac}{4a^2}\right)\right]$$

Case : (1) When $b^2 - 4ac > o$

then, $$\int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2-4ac}{4a^2}}\right)^2}$$

$$= \frac{1}{a}\int \frac{dt}{t^2-\lambda^2} \qquad \text{(where } x+\frac{b}{2a} = t \text{ and } \sqrt{\frac{b^2-4ac}{4a^2}} = \lambda \text{ etc.)}$$

$$= \frac{1}{a}\cdot\frac{1}{2\lambda}\log\left|\frac{t-\lambda}{t+\lambda}\right| + c$$

Case : (2) : When $b^2 - 4ac < o$

then $$\int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dt}{t^2+\lambda^2}$$

$$= \frac{1}{a\lambda}\tan^{-1}\left(\frac{t}{\lambda}\right) + c$$

on again substituting the values of t and λ the required integration can be done

(ii) Let numerator $px + q = \lambda$ (differential coefficient of denominator + μ)

or $$px + q = \lambda(2ax+b) + \mu$$

On comparing the coefficients of equal terms

$$2a\lambda = p \Rightarrow \lambda = \frac{p}{2a}$$

$$b\lambda + \mu = q \Rightarrow \mu = q - \frac{bp}{2a}$$

Hence the given integral $\int \frac{px+q}{ax^2+bx+c}dx = \frac{p}{2a}\int\frac{2ax+b}{ax^2+bx+c}dx + \left(q-\frac{bp}{2a}\right)\int\frac{dx}{ax^2+bx+c}$

$$= \frac{p}{2a}\log|ax^2+bx+c| + \left(q-\frac{bp}{2a}\right)\int\frac{dx}{ax^2+bx+c}$$

Where secodn integral can be solved by method (i)

(C) Integration of irrational algebraic function

Irrational function : A function in which power of variable is fraction :

For example ; $f(x) = x^{3/2} + x + 1,\ g(x) = 2\sqrt{x} + 3,\ h(x) = \frac{x^2+\sqrt{x}}{1-x^{1/3}}$ etc.

Integration of standard irrational functions

(i) $\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$ (ii) $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

First Method : (i) Term $I = \int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$ m there are two methods of integration.

(a) where a > o then $I = \frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{x^2+\frac{bx}{a}+\frac{c}{a}}} = \frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ac}{4a^2}\right)}}$

It has three steps :

(i) where $b^2-4ac>o$ then

$$I = \frac{1}{\sqrt{a}}\int \frac{dt}{\sqrt{t^2-\lambda^2}}, \text{ where } t = x+\frac{b}{2a}, \lambda = \sqrt{\frac{b^2-4ac}{4a^2}}$$

$$= \frac{1}{\sqrt{a}}\log\left|t+\sqrt{t^2-\lambda^2}\right|+c$$

(ii) when $b^2-4ac<o$ then

$$I = \frac{1}{\sqrt{a}}\int \frac{dt}{\sqrt{\left(x+\frac{b}{2a}\right)^2+\left(\frac{\sqrt{4ac-b^2}}{2a}\right)^2}}$$

$$= \frac{1}{\sqrt{a}}\int \frac{dt}{\sqrt{t^2+\lambda^2}}, \text{ where } t = x+\frac{b}{2a}, \ \lambda = \frac{\sqrt{4ac-b^2}}{2a}$$

$$= \frac{1}{\sqrt{a}}.\log\left|t+\sqrt{t^2+\lambda^2}\right|+c$$

(iii) when $b^2-4ac=o$

then, $I = \frac{1}{\sqrt{a}}\int \frac{dx}{x+\frac{b}{2a}} = \frac{1}{\sqrt{a}}\log\left|x+\frac{b}{2a}\right|+c$

(b) when $a<o$ let $a=-\propto$

then, $I = \int \frac{dx}{\sqrt{-\propto x^2+bx+c}} = \frac{1}{\sqrt{\propto}}\int \frac{dx}{\sqrt{\left(\frac{b^2+4c\propto}{4\alpha^2}\right)-\left(x-\frac{b}{2\propto}\right)^2}}$

$$= \frac{1}{\sqrt{\alpha}} \int \frac{dt}{\sqrt{\lambda^2 - t^2}}, \text{ where } t = x - \frac{b}{2\alpha}, \lambda^2 = \frac{b^2 + 4c\alpha}{4\alpha^2}$$

$$= \frac{1}{\sqrt{\propto}} \sin^{-1}\left(\frac{t}{\lambda}\right) + c$$

Second method : $I = \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Let $px + q = A\frac{d}{dx}(ax^2 + bx + c) + B$

or $px + q = A(2ax + b) + B$

on comparing and solving $A = \frac{p}{2a}, B = q - \frac{bp}{2a}$

then, $I = \frac{p}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}} dx + \left(q - \frac{bp}{2a}\right)\int \frac{1}{\sqrt{ax^2+bx+c}} dx,$

where in I integral put $ax^2 + bx + c = t$ and II integral can be solved by case I discussed earlier.

Illustrative Examples

Example 29. Integrate $\frac{1}{x^2+4x+1}$ with respect to x.

Solution : Let $I = \int \frac{1}{x^2+4x+1} dx = \int \frac{1}{(x+2)^2 - 3} dx$

$$= \int \frac{1}{(x+2)^2 - (\sqrt{3})^2} dx = \frac{1}{2\sqrt{3}} \log\left|\frac{x+2-\sqrt{3}}{x+2+\sqrt{3}}\right| + c.$$

Example 30. Integrate $\frac{1}{1-6x-9x^2}$ with respect to x.

Solution : Here $1 - 6x - 9x^2 = 9\left[\frac{1}{9} - \frac{6x}{9} - x^2\right]$

$$= 9\left[\frac{2}{9} - \left(x^2 + \frac{2x}{3} + \frac{1}{9}\right)\right]$$

$$= 9\left[2/9 - (x + 1/3)^2\right]$$

$\therefore$ $I = \int \frac{1}{1-6x-9x^2} dx$

$$= \frac{1}{9}\int \frac{1}{2/9 - (x+1/3)^2} dx = \frac{1}{9}\int \frac{1}{(\sqrt{2}/3)^2 - (x+1/3)^2} dx$$

$$= \frac{1}{9 \times 2 \times \frac{\sqrt{2}}{3}} \log \left| \frac{\sqrt{2}/3 + x + 1/3}{\sqrt{2}/3 - x - 1/3} \right| + c$$

$$= \frac{1}{6\sqrt{2}} \log \left| \frac{\sqrt{2} + 1 + 3x}{\sqrt{2} - 1 - 3x} \right| + c.$$

Example 31. Integrate $\frac{5x-2}{3x^2+2x+1}$ with respect to *x*.

Solution : Let $5x - 2 = A\frac{d}{dx}(3x^2 + 2x + 1) + B$

or $5x - 2 = A(6x + 2) + B$

on comparing $6A = 5 \;\therefore A = \frac{5}{6}$ and $B = -2 - 2A = -2 - 5/3 = -11/3$

$\therefore$ $$5x - 2 = \frac{5}{6}(6x+2) - \frac{11}{3}$$

$\therefore$ $$I = \int \frac{5x-2}{3x^2+2x+1} dx$$

$$= \int \frac{5/6(6x+2) - 11/3}{3x^2+2x+1} dx = \frac{5}{6}\int \frac{6x+2}{3x^2+2x+1} dx - \frac{11}{3}\int \frac{1}{3x^2+2x+1} dx$$

$$= \frac{5}{6}\log|3x^2+2x+1| - \frac{11}{3\times 3}\int \frac{1}{x^2+2x/3+1/3} dx$$

$$= \frac{5}{6}\log|3x^2+2x+1| - \frac{11}{9}\int \frac{1}{(x+1/3)^2+(\sqrt{2}/3)^2} dx$$

$$= \frac{5}{6}\log|3x^2+2x+1| - \frac{11}{9} \times \frac{1}{\sqrt{2}/3}\tan^{-1}\left(\frac{x+1/3}{\sqrt{2}/3}\right) + c$$

$$= \frac{5}{6}\log|3x^2+2x+1| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + c.$$

Example 32. Integrate $\frac{1}{\sqrt{x^2-8x+15}}$ with respect to *x*.

Solution : Here $$I = \int \frac{1}{\sqrt{x^2-8x+15}} dx = \int \frac{1}{\sqrt{(x-4)^2-1}} dx$$

$$= \log|(x-4) + \sqrt{x^2-8x+15}| + c$$

Example 33. Integrate $\frac{1}{\sqrt{1+3x-4x^2}}$ with respect to x

Solution : Let

$$I=\int\frac{1}{\sqrt{1+3x-4x^2}}dx$$

$$=\frac{1}{2}\int\frac{dx}{\sqrt{1/4+3x/4-x^2}}$$

$$=\frac{1}{2}\int\frac{dx}{\sqrt{25/64-(x^2-3x/4+9/64)}}$$

$$=\frac{1}{2}\int\frac{dx}{\sqrt{\left(\frac{5}{8}\right)^2-\left(x-\frac{3}{8}\right)^2}}$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{x-3/8}{5/8}\right)+c=\frac{1}{2}\sin^{-1}\left(\frac{8x-3}{5}\right)+c$$

Example 34. Integrate $\frac{2x+5}{\sqrt{x^2+3x+1}}$ with respect to x

Solution : Let $2x+5=(2x+3)+2$

(On changing numerator into differential coefficient of (x^2+3x+1) by inspection)

$$\therefore\quad =\int\frac{2x+5}{\sqrt{x^2+3x+1}}dx=\int\frac{2x+3}{\sqrt{x^2+3x+1}}dx+\int\frac{2}{\sqrt{x^2+3x+1}}dx$$

$$=\int\frac{dt}{\sqrt{t}}+\int\frac{2}{\sqrt{(x+3/2)^2+(\sqrt{5}/2)^2}},\quad \text{where}\quad x^2+3x+1=t$$

$$=2\sqrt{t}+2\log\left|(x+3/2)+\sqrt{x^2+3x+1}\right|+c$$

$$=2\sqrt{x^2+3x+1}+2\log\left|(x+3/2)+\sqrt{x^2+3x+1}\right|+c$$

Exercise 9.5

Integrate the following functions with respect to x

(1) $\frac{1}{x^2+2x+10}$ (2) $\frac{1}{2x^2+x-1}$ (3) $\frac{1}{9x^2-12x+8}$ (4) $\frac{1}{3+2x-x^2}$

(5) $\frac{x}{x^4+x^2+1}$ (6) $\frac{\cos x}{\sin^2 x+4\sin x+5}$ (7) $\frac{x-3}{x^2+2x-4}$ (8) $\frac{3x+1}{2x^2-2x+3}$

(9) $\frac{x+1}{x^2+4x+5}$ (10) $\frac{(3\sin x-2)\cos x}{5-\cos^2 x-4\sin x}$ (11) $\frac{1}{2e^{2x}+3e^x+1}$ (12) $\frac{1}{\sqrt{4x^2-5x+1}}$

(13) $\frac{1}{\sqrt{5x-6-x^2}}$ (14) $\frac{1}{\sqrt{1-x-x^2}}$ (15) $\frac{1}{\sqrt{4+3x-2x^2}}$ (16) $\frac{x+2}{\sqrt{x^2-2x+4}}$

(17) $\frac{x+1}{\sqrt{x^2-x+1}}$ (18) $\frac{x+3}{\sqrt{x^2+2x+2}}$ (19) $\sqrt{\sec x-1}$ (20) $\sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}$

(21) $\frac{x^3}{x^2+x+1}$ (22) $\frac{e^x}{e^{2x}+6e^x+5}$

IV Integration of Parts:

We have studied the methods of integration by substitution, trigonometric identities and algebraic methods. But integral of some functions is either difficult or impossible with above methods. Such functions can be expressed in parts and then their integration is can be found.

Here the main functions are non algebraic functions like exponential, logarithmic and inverse trigonometric functions.

Rule of integration by parts or integration of product of functions:

Theorem : If u and v are two functions of x then

$$\int u.v\,dx = u\left(\int v\,dx\right) = \int\left[\frac{du}{dx}.\int v\,dx\right]dx$$

Proof : For any two functions $f(x)$ and $g(x)$

$$\frac{d}{dx}\{f(x).g(x)\} = f(x)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x)$$

integrating both sides with respect to x

$$f(x).g(x)=\int\left[f(x)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x)\right]dx$$

or
$$\int\left[f(x)\frac{d}{dx}g(x)\right]dx = f(x)g(x)-\int\left[g(x)\frac{d}{dx}f(x)\right]dx \qquad (1)$$

Now let
$$f(x)=u,\ \frac{d}{dx}[g(x)]=v \Rightarrow g(x)=\int v\,dx$$

Put this value in (1)

$\therefore$
$$\int u.v\,dx = u\int v\,dx-\int\left[\frac{du}{dx}\int v\,dx\right]dx$$

If we take u as first function and v as the second function, then this formula may be stated as follows:

"The integral of the product of two functions = (First function) × ∫ (second function) dx– ∫ (Differential coefficient of first function) × integral of second function dx.

Remark : The success of integration by parts method depends on selection of first and second function. Function should be selected in a manner so that the integral of second function can be done easily. Although there is no specific rule for selection of functions but following points may be kept in mind.

(i) if integrand is a product of algebraic function of x and exponential or trigonometric function them exponential or trigonometric function should be selected as second function.

(ii) In integration of single inverse trigonometric functions or logarithmic functions, unit (1) should be taken as second function.

(iii) If integral obtained in original form in right hand side then integration should be done by transposing.

(iv) Integration by parts may be used more than once in an integral as per necessity.

Note : We can select the function as they appear in word 'ILATE'

Where : I = Inverse trigonometric functions sicj as $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x$

L = Logarithmic functions such as $\log x, \log(x^2 + a^2)$

A = Algebraic functions such as $x, x+1, 2x, \sqrt{x}$

T = Trigonometric functions such as $\sin x, \cos x, \tan x$

E = Exponential function such as $a^x, e^x, 2^x, 3^{-x}$

Application of ingegration by parts

In Integral of the type $\int e^x [f(x) + f'(x)]\,dx$ and $\int [x f'(x) + f(x)]\,dx$

(i) Let $I = \int e^x [f(x) + f'(x)]\,dx,$ where $f'(x) = \dfrac{d}{dx} f(x)$

$$= \int \underset{\text{II}}{e^x} \underset{\text{I}}{f(x)}\,dx + \int e^x f'(x)\,dx \quad \text{(on taking } e^x \text{ as II function)}$$

$$= f(x).e^x - \int f'(x) e^x dx + \int e^x f'(x) dx + c$$

(Integration by parts of first integral)

$$= e^x f(x) + c$$

similarly $\int e^x [f(x) + f'(x)]\,dx = e^x f(x) + c$

(ii) Let $I = \int [x f'(x) + f(x)]\,dx$

$$= \int \underset{\text{I}}{x} \underset{\text{II}}{f'(x)}\,dx + \int f(x)\,dx$$

put $f'(x)$ as second function in first integral and then integrating by parts

$$= x f(x) - \int 1 \times f(x)\,dx + \int f(x) dx$$

$$= x f(x) + c$$

$\therefore \quad \int [x f'(x) + f(x)]\,dx = x f(x) + c$

Illustrative Examples

Example 35. Integrate $x^2 e^x$ with respect to x

Solution : Let $I = \int \underset{\text{I}}{x^2} \underset{\text{II}}{e^x}\, dx$

On taking e^x as II function, Integration by parts gives

$$= x^2 e^x - \int \underset{\text{I}}{2x} \underset{\text{II}}{e^x}\, dx$$

$$= x^2 e^x - 2[xe^x - \int 1 \times e^x dx]$$

$$= x^2 e^x - 2xe^x + 2e^x$$

$$= e^x (x^2 - 2x + 2) + c$$

Example 36. Integrate $x \log x$ with respect to x

Solution : Let $I = \int \underset{\text{II}}{x} \underset{\text{I}}{\log x}\, dx$

On taking $\log x$ as I function and x as second function, Integration by parts gives

$$I = (\log x)\frac{x^2}{2} - \int \frac{1}{x} \times \frac{x^2}{2} dx$$

$$= \frac{x^2}{2}(\log x) - \frac{1}{2}\int x dx + c$$

$$= \frac{x^2}{2}\log x - \frac{x^2}{4} + c$$

Example 37. Integrate $x^2 \sin 2x$ with respect to x

Solution : Let $I = \int \underset{\text{I}}{x^2} \underset{\text{II}}{\sin 2x}\, dx$

Taking x^2 as I and $\sin 2x$ as II function respectively, Integration by parts, gives

$$I = x^2\left(\frac{-\cos 2x}{2}\right) - \int 2x \times \frac{-\cos 2x}{2} dx$$

$$= \frac{-x^2}{2}\cos 2x + \int \underset{\text{I}}{x} \cdot \underset{\text{II}}{\cos 2x}\, dx$$

Taking x as I and $\cos 2x$ as II functions respectively, again Integration by parts gives

$$= \frac{-x^2}{2}\cos 2x + x\left(\frac{\sin 2x}{2}\right) - \int 1 \times \frac{\sin 2x}{2} dx$$

$$= \frac{-x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{\cos 2x}{4} + c$$

Example 38. Integrate $\log x$ with respect to x

Solution : Let $I = \int \underset{\text{II}}{1} \cdot \underset{\text{I}}{\log x}\, dx$

Taking one as second function, Integration by parts gives

$$= (\log x)(x) - \int \frac{1}{x} \times x\, dx$$

$$= x \log x - x + c$$

$$= x(\log x - 1) + c$$

$$= x[\log x - \log e] + c = x \log(x/e) + c$$

Example 39. Integrate $\tan^{-1} x$ with respect to x

Solution : Let $$I = \int \tan^{-1} x\, dx$$

$$I = \int \underset{\text{II}}{1} \cdot \underset{\text{I}}{\tan^{-1}} x\, dx$$

Taking $\tan^{-1} x$ as I and one as II function respectively, Integration by parts gives

$$= \left(\tan^{-1} x\right)\left(x\right) - \int \frac{1}{1+x^2} \times x\, dx$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$$

$$= x \tan^{-1} x - \frac{1}{2} \log\left(1+x^2\right) + c \qquad \text{(where, let } 1+x^2 = t\text{)}$$

Example 40. Integrate $\cos^{-1} \sqrt{\frac{x}{a+x}} dx$ with respect to x

Solution : Let $$I = \int \cos^{-1} \sqrt{\frac{x}{a+x}} dx$$

Let $$x = a \tan^2 \theta \Rightarrow dx = 2a \tan\theta \sec^2\theta\, d\theta$$

$\therefore$ $$I = \int \cos^{-1} \sqrt{\left(\frac{a \tan^2\theta}{a + a\tan^2\theta}\right)} \times 2a \tan\theta \sec^2\theta\, d\theta$$

$$= \int \cos^{-1} \left(\frac{\tan\theta}{\sec\theta}\right) \times 2a \tan\theta \sec^2\theta\, d\theta$$

$$= 2a \int \cos^{-1}(\sin\theta).\tan\theta \sec^2\theta\, d\theta$$

$$= 2a \int \cos^{-1}[\cos(\frac{\pi}{2} - \theta)].\tan\theta \sec^2\theta\, d\theta$$

$$= 2a \int (\frac{\pi}{2} - \theta).\tan\theta \sec^2\theta\, d\theta$$

Taking $(\frac{\pi}{2} - \theta)$ as I and $\tan\theta \sec^2\theta$ as II function, integration by parts gives

$$I = 2a\left[\left(\frac{\pi}{2}-\theta\right)\frac{\tan^2\theta}{2} - \int -1\times\frac{\tan^2\theta}{2}\,d\theta\right]$$

$$\left[\because \int \tan\theta\sec^2\theta\,d\theta = \frac{\tan^2\theta}{2}\right]$$

$$= a(\frac{\pi}{2}-\theta)\tan^2\theta + \text{a}\int(\sec^2\theta-1)d\theta$$

$$= a(\frac{\pi}{2}-\theta)\tan^2\theta + a[\tan\theta-\theta]+c$$

$$= a\left[\pi/2-\tan^{-1}\sqrt{x/a}\right](x/a)+a\left[\sqrt{x/a}-\tan^{-1}\sqrt{x/a}\right]+c$$

$$= x\cdot\frac{\pi}{2} - x\tan^{-1}\sqrt{x/a}+\sqrt{ax}-a\tan^{-1}\sqrt{x/a}+c$$

या $$I = x\cdot\frac{\pi}{2}-(a+x)\tan^{-1}\sqrt{x/a}+\sqrt{ax}+c$$

Example 41. Evaluate $\int \log[x+\sqrt{x^2+a^2}]\,dx$

Solution : Here $$I = \int \underset{\text{II}}{1}\cdot\underset{\text{I}}{\log(x+\sqrt{x^2+a^2})}\,dx$$

Taking one as second function, integration by parts, gives

$$I = \log[x+\sqrt{x^2+a^2}].x - \int\frac{1}{[x+\sqrt{x^2+a^2}]}\times\left[1+\frac{2x}{2\sqrt{x^2+a^2}}\right]x\,dx$$

$$= x\log[x+\sqrt{x^2+a^2}] - \int\frac{1}{(x+\sqrt{x^2+a^2})}\times\frac{(\sqrt{x^2+a^2}+x)}{\sqrt{x^2+a^2}}\times x\,dx$$

$$= x\log[x+\sqrt{x^2+a^2}] - \int\frac{x}{\sqrt{x^2+a^2}}dx$$

(On putting $x^2+a^2=t$ and solving)

$$= x\log[x+\sqrt{x^2+a^2}] - \frac{1}{2}\times 2\sqrt{x^2+a^2}+c$$

$$= x\log[x+\sqrt{x^2+a^2}] - \sqrt{x^2+a^2}+c$$

Example 42. Integrate $\dfrac{x^2}{(x\sin x+\cos x)^2}$ with respect to x

Solution : Let $$I = \int\frac{x^2}{(x\sin x+\cos x)^2}dx$$

$$= \int \frac{x}{\cos x} \cdot \frac{x \cos x}{(x \sin x + \cos x)^2} dx \qquad \left(\text{Put } x^2 = \frac{x}{\cos x} \times x \cos x \text{ in numerator}\right)$$

Taking $\frac{x}{\cos x}$ as I and remaining as II function, integration by parts gives

$$I = \frac{x}{\cos x} \times \int \frac{x \cos x}{(x \sin x + \cos x)^2} dx - \int \left[\frac{d}{dx} \left(\frac{x}{\cos x} \right) \times \int \frac{x \cos x}{(x \sin x + \cos x)^2} dx \right] dx$$

Let $x \sin x + \cos x = t \Rightarrow x \cos x \, dx = dt$

$$= \frac{x}{\cos x} \times \left[\frac{-1}{x \sin x + \cos x} \right] + \int \frac{[\cos x + (\sin x) x]}{\cos^2 x} \times \frac{1}{(x \sin x + \cos x)} dx$$

$$= \frac{-x}{\cos x (x \sin x + \cos x)} + \int \sec^2 x \, dx$$

$$= \frac{-x}{\cos x (x \sin x + \cos x)} + \tan x + c$$

$$= \frac{-x}{\cos x (x \sin x + \cos x)} + \frac{\sin x}{\cos x} + c$$

$$= \frac{-x + \sin x (x \sin x + \cos x)}{\cos x (x \sin x + \cos x)} + c$$

$$= \frac{-x + x \sin^2 x + \sin x \cos x}{\cos x (x \sin x + \cos x)} + c$$

$$= \frac{-x(1 - \sin^2 x) + \sin x \cos x}{\cos x (x \sin x + \cos x)} + c$$

$$= \frac{-x \cos^2 x + \sin x \cos x}{\cos x (x \sin x + \cos x)} + c$$

$$= \frac{\sin x - x \cos x}{x \sin x + \cos x} + c$$

Example 43. Integrate $\frac{x + \sin x}{1 + \cos x}$ with respect to x.

Solution : Let $I = \int \frac{x + \sin x}{1 + \cos x} dx = \int \frac{x + 2 \sin(x/2) \cos(x/2)}{2 \cos^2(x/2)} dx$

$$= \frac{1}{2} \int \underset{\text{I}}{x} \underset{\text{II}}{\sec^2} (x/2) \, dx + \int \tan(x/2) \, dx$$

Taking x as I function in first integral, integration by parts gives

$$= \frac{1}{2}\left[2x\tan(x/2) - \int 1\times 2\tan(x/2)\,dx\right] + \int \tan(x/2)\,dx$$

$$= x\tan(x/2) - \int \tan(x/2)\,dx + \int \tan(x/2)\,dx$$

$$= x\tan(x/2) + c$$

Example 44. Evaluate $\int \frac{xe^x}{(x+1)^2}dx$

Solution : Let $I = \int \frac{xe^x}{(x+1)^2}dx = \int \frac{(\overline{x+1}-1)e^x}{(x+1)^2}dx$

$$= \int \left[\frac{1}{(x+1)} - \frac{1}{(x+1)^2}\right]e^x dx$$

$$= \int \frac{e^x}{(x+1)}dx - \int \frac{e^x}{(x+1)^2}dx$$

(Taking $\frac{1}{x+1}$ as I function in first integral, Integration by parts gives)

$$= \left[\frac{1}{(x+1)}\times e^x - \int -\frac{1}{(x+1)^2}e^x dx\right] - \int \frac{e^x}{(x+1)^2}dx$$

$$= \frac{e^x}{x+1} + \int \frac{e^x}{(x+1)^2}dx - \int \frac{e^x}{(x+1)^2}dx = \frac{e^x}{x+1} + c$$

Exercise 9.6

Integrate the following functions with respect to *x*

1. (i) $x\cos x$ (ii) $x\sec^2 x$
2. (i) $x^3 e^{-x}$ (ii) $x^3 \sin x$
3. (i) $x^3(\log x)^2$ (ii) $x^3 e^{x^2}$
4. (i) $e^{2x}e^{e^x}$ (ii) $(\log x)^2$
5. (i) $\cos^{-1} x$ (ii) $\operatorname{cos}ec^{-1}\sqrt{\frac{x+a}{x}}$
6. (i) $\sin^{-1}(3x-4x^3)$ (ii) $\frac{x}{1+\cos x}$
7. (i) $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$ $(\text{Hint}: x = \cos\theta)$ (ii) $\cos\sqrt{x}$
8. (i) $\frac{x}{1+\sin x}$ (ii) $x^2\tan^{-1} x$
9. $\frac{x\sin^{-1} x}{\sqrt{1-x^2}}$
10. $\frac{x\tan^{-1} x}{(1+x^2)^{3/2}}$
11. $e^x(\cot x + \log\sin x)$
12. $\frac{2x+\sin 2x}{1+\cos 2x}$

13. $e^x\left(\frac{1-\sin x}{1-\cos x}\right)$

14. $e^x\left[\log x+\frac{1}{x^2}\right]$

15. $e^x[\log(\sec x+\tan x)+\sec x]$

16. $e^x(\sin x+\cos x)\sec^2 x$

17. $e^x\left(\frac{1}{x^2}-\frac{2}{x^3}\right)$

18. $e^x\left(\frac{1-x}{1+x^2}\right)^2 \left(\text{Hint}=\left(\frac{1-x}{1+x^2}\right)^2=\frac{1}{(1+x^2)}-\frac{2x}{(1+x^2)^2}\right)$

19. $\cos 2\theta.\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$

20. $\frac{x^2}{(x\cos x-\sin x)^2}$

21. $\cos^{-1}(1/x)$

22. $(\sin^{-1} x)^2$

9.08 Some special type of Integral

Many times while integrating the product of two functions, integration does not come to an end, whaterver the first or second function is. This happens in the case of exponential and trigonometric functions. In such cases using transpose we can calculate the integral.

For Example :

Integration of $e^{ax}\sin bx$ and $e^{ax}\cos bx$

Let, $I=\int \underset{\text{II}}{e^{ax}}\,\underset{\text{I}}{\sin bx}\,dx$

taking $\sin bx$ as I and e^{ax} as II function, Integration by parts, gives

$$I=\sin bx\left(\frac{e^{ax}}{a}\right)-\int b\cos bx\times\frac{e^{ax}}{a}dx$$

or
$$I=\frac{1}{a}e^{ax}\sin bx-\frac{b}{a}\int \underset{\text{II}}{e^{ax}}\,\underset{\text{I}}{\cos bx}\,dx$$

Taking $\cos bx$ as I and e^{ax} as II function, Integration by parts gives.

$$I=\frac{1}{a}e^{ax}\sin bx-\frac{b}{a}\left[\cos bx.\frac{e^{ax}}{a}-\int -b\sin bx\times\frac{e^{ax}}{a}dx\right]$$

or
$$I=\frac{1}{a}e^{ax}\sin bx-\frac{b}{a^2}e^{ax}\cos bx-\frac{b^2}{a^2}\int e^{ax}\sin bx\,dx$$

or
$$I=\frac{1}{a}e^{ax}\sin bx-\frac{b}{a^2}e^{ax}\cos bx-\frac{b^2}{a^2}I$$

or
$$I\left(1+\frac{b^2}{a^2}\right)=\frac{e^{ax}}{a^2}(a\sin bx-b\cos bx)$$ [transposing the last term]

or
$$I=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+c$$

or $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]+c$$

similarly $$\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]+c$$

9.09 Three Important Integrals

(i) $\int \sqrt{x^2+a^2}\,dx$ (ii) $\int \sqrt{x^2-a^2}\,dx$ (iii) $\int \sqrt{a^2-x^2}\,dx$

(i) Let $$I = \int \sqrt{x^2+a^2}\,dx = \int \underset{\text{I}}{\sqrt{x^2+a^2}}\cdot \underset{\text{II}}{1}\,dx$$

Here, we will take $\sqrt{a^2+x^2}$ as I and 1 as II function, Integration by parts gives

$$I = \sqrt{x^2+a^2}\times x - \int \frac{2x}{2\sqrt{x^2+a^2}}\times x\,dx$$

or $$I = x\sqrt{x^2+a^2} - \int \frac{x^2}{\sqrt{x^2+a^2}}\,dx$$

$$= x\sqrt{x^2+a^2} - \int \frac{(x^2+a^2)-a^2}{\sqrt{x^2+a^2}}\,dx$$

$$= x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}\,dx + a^2\int \frac{1}{\sqrt{x^2+a^2}}dx$$

or $$I = x\sqrt{x^2+a^2} - I + a^2\log|x+\sqrt{x^2+a^2}|+c_1$$

or $$2I = x\sqrt{x^2+a^2} + a^2\log|x+\sqrt{x^2+a^2}|+c_1$$

or $$I = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|+\frac{c_1}{2}$$

or $$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|+c$$ (where $c_1/2 = c$)

similarly

(ii) $$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|+c$$

(iii) $$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$$

Illustrative Examples

Example 45. Integrate $e^{3x}\sin 4x$ with respec tot x

Solution : Let $I = \int \underset{\text{II}}{e^{3x}} \underset{\text{I}}{\sin 4x}\, dx$

Taking $\sin 4x$ as I and e^{3x} as II function, Integration by parts gives,

$$I = \sin 4x . \frac{e^{3x}}{3} - \int 4\cos 4x \times \frac{e^{3x}}{3} dx$$

$$= \frac{1}{3} e^{3x} \sin 4x - \frac{4}{3} \int \underset{\text{II}}{e^{3x}} \underset{\text{I}}{\cos 4x}\, dx$$

Taking $\cos 4x$ as I function, Integration by parts gives

$$I = \frac{1}{3} e^{3x} \sin 4x - \frac{4}{3}\left[\cos 4x . \frac{e^{3x}}{3} - \int -4\sin 4x \times \frac{e^{3x}}{3} dx\right]$$

or $$I = \frac{1}{3} e^{3x} \sin 4x - \frac{4}{9} e^{3x} \cos 4x - \frac{16}{9} \int e^{3x} \sin 4x\, dx$$

or $$I = \frac{e^{3x}}{9}\left[3\sin 4x - 4\cos 4x\right] - \frac{16}{9} I + c_1$$

or $$\frac{25}{9} I = \frac{1}{9} e^{3x}\left(3\sin 4x - 4\cos 4x\right) + c_1$$

or $$I = \frac{e^{3x}}{25}\left[3\sin 4x - 4\cos 4x\right] + c$$

Example 46. Evaluate $\int \frac{\sin(\log x)}{x^3} dx$

Solution : Let $I = \int \frac{\sin(\log x)}{x^3} dx$

Let $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$

$$= \int \frac{(\sin t) e^t dt}{(e^t)^3} = \int e^{-2t} \sin t\, dt$$

$$= \frac{e^{-2t}}{(-2)^2 + (1)^2}[-2\sin t - \cos t] + c$$

$$\left[\because \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}[a \sin bx - b\cos bx]\right]$$

$$= \frac{x^{-2}}{5}\left[-2\sin(\log x) - \cos(\log x)\right] + c$$

$$I = -\frac{1}{5x^2}\left[2\sin(\log x) + \cos(\log x)\right] + c$$

Example 47. Integrate $\frac{xe^{\sin^{-1}x}}{\sqrt{1-x^2}}$ with respect to x.

Solution : Let $I=\int\frac{xe^{\sin^{-1}x}}{\sqrt{1-x^2}}dx$

Let $\sin^{-1}x=t\Rightarrow x=\sin t\Rightarrow dx=\cos t\,dt$

$$=\int\frac{\sin t.e^t}{\cos t}\times\cos t\,dt=\int e^t\sin t\,dt$$

$$=\frac{e^t}{2}[\sin t-\cos t]+c=\frac{e^{\sin^{-1}x}}{2}\left[x-\sqrt{1-x^2}\right]+c$$

Example 48. Integrate $e^{3x}\cos(4x+5)dx$ with respect to x

Solution : Let $I=\int \underset{\text{II}}{e^{3x}}\underset{\text{I}}{\cos(4x+5)}dx$

Integration by parts gives,

$$I=\cos(4x+5).\frac{e^{3x}}{3}-\int -4\sin(4x+5)\times\frac{e^{3x}}{3}dx$$

$$=\frac{1}{3}e^{3x}\cos(4x+5)+\frac{4}{3}\int \underset{\text{II}}{e^{3x}}\underset{\text{I}}{\sin(4x+5)}dx$$

Again, Integration by parts gives,

$$I=\frac{1}{3}e^{3x}\cos(4x+5)+\frac{4}{3}\left[\sin(4x+5)\times\frac{e^{3x}}{3}-\int 4\cos(4x+5)\times\frac{e^{3x}}{3}dx\right]$$

or $$I=\frac{1}{3}e^{3x}\cos(4x+5)+\frac{4}{9}e^{3x}\sin(4x+5)-\frac{16}{9}\int e^{3x}\cos(4x+5)dx$$

or $$I=\frac{1}{9}e^{3x}\left[3\cos(4x+5)+4\sin(4x+5)\right]-\frac{16}{9}I+c_1$$

or $$\frac{25}{9}I=\frac{1}{9}e^{3x}\left[3\cos(4x+5)+4\sin(4x+5)\right]+c_1$$

or $$I=\frac{e^{3x}}{25}\left[3\cos(4x+5)+4\sin(4x+5)\right]+c$$

Example 49. Integrate the following functions with respect to x

(i) $\sqrt{x^2+2x+5}$ (ii) $\sqrt{3-2x-x^2}$ (iii) $\sqrt{x^2+8x-6}$

Solution : (i) $I=\int\sqrt{x^2+2x+5}\,dx=\int\sqrt{(x+1)^2+(2)^2}dx$

$$=\frac{(x+1)}{2}\sqrt{(x+1)^2+(2)^2}+\frac{(2)^2}{2}\log\left|(x+1)+\sqrt{(x+1)^2+2^2}\right|+c$$

$$= \frac{x+1}{2}\sqrt{x^2+2x+5} + 2\log\left|(x+1)+\sqrt{x^2+2x+5}\right| + c$$

(ii) $$I = \int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-\left(x^2+2x+1\right)}dx$$

$$= \int \sqrt{(2)^2-(x+1)^2}\,dx$$

$$= \frac{(x+1)}{2}\sqrt{(2)^2-(x+1)^2} + \frac{(2)^2}{2}\sin^{-1}\frac{(x+1)}{2} + c$$

$$= \frac{x+1}{2}\sqrt{3-2x-x^2} + 2\sin^{-1}\left(\frac{x+1}{2}\right) + c$$

(iii) Let $$I = \int \sqrt{x^2+8x-6}\;dx$$

$$= \int \sqrt{(x+4)^2-22}\,dx$$

$$= \frac{x+4}{2}\sqrt{(x+4)^2-22} - \frac{22}{2}\log\left|(x+4)+\sqrt{(x+4)^2-22}\right| + c$$

$$= \frac{(x+4)}{2}\sqrt{x^2+8x-6} - 11\log\left|(x+4)+\sqrt{x^2+8x-6}\right| + c$$

Example 50. Integrate $\sec^3 x$ with respect to x

Solution : Let $I = \int \sec x.\sec^2 x\,dx$

$$= \int \sqrt{1+\tan^2 x}.\sec^2 x dx$$

Let $\tan x = t$ $\quad\therefore \sec^2 x\,dx = dt$

$$I = \int \sqrt{1+t^2}.dt$$

$$= \frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\log\left|t+\sqrt{1+t^2}\right| + c$$

$$= \frac{\tan x}{2}\sqrt{1+\tan^2 x} + \frac{1}{2}\log\left|\tan x+\sqrt{1+\tan^2 x}\right| + c$$

$$= \frac{1}{2}\tan x\sec x + \frac{1}{2}\log\left|\tan x+\sec x\right| + c$$

Example 51. Integrate $e^{\sin x}\cos x\sqrt{4-e^{2\sin x}}\,dx$ with respect to x

Solution : Let $I = \int e^{\sin x}\cos x\sqrt{4-e^{2\sin x}}\,dx$

Let $e^{\sin x} = t \Rightarrow \cos x.e^{\sin x}dx = dt$

$$\therefore \qquad I = \int \sqrt{4-t^2}\,dt$$

$$= \frac{t}{2}\sqrt{4-t^2} + \frac{4}{2}\sin^{-1}\frac{t}{2} + c$$

$$= \frac{1}{2}e^{\sin x}\sqrt{4-e^{2\sin x}} + 2\sin^{-1}\left(\frac{e^{\sin x}}{2}\right) + c$$

Exercise 9.7

Integrate the following functions with respect to *x*

1. $e^{2x}\cos x$
2. $\sin(\log x)$
3. $\frac{e^{a\tan^{-1}x}}{(1+x^2)^{3/2}}$
4. $e^{x/\sqrt{2}}\cos(x+\alpha)$
5. $e^x\sin^2 x$
6. $e^{a\sin^{-1}x}$
7. $\cos\left(b\log x/a\right)$
8. $e^{4x}\cos 4x\cos 2x$
9. $\sqrt{2x-x^2}$
10. $\sqrt{x^2+4x+6}$
11. $\sqrt{x^2+6x-4}$
12. $\sqrt{2x^2+3x+4}$
13. $x^2\sqrt{a^6-x^6}$
14. $(x+1)\sqrt{x^2+1}$
15. $\sqrt{1-4x-x^2}$
16. $\sqrt{4-3x-2x^2}$

Miscellaneous Examples

Example 52. Integrate $\frac{1}{a^2\cos^2 x + b^2\sin^2 x}$ with respect to x

Solution : Let
$$I = \int \frac{1}{a^2\cos^2 x + b^2\sin^2 x}dx$$

Dividing numberator and denominator by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x dx}{a^2 + b^2\tan^2 x}$$

Let $\tan x = t$ then $\sec^2 x\,dx = dt$

$$\therefore \qquad I = \int \frac{dt}{a^2+b^2t^2} = \frac{1}{b^2}\int \frac{dt}{t^2+(a/b)^2}$$

$$= \frac{1}{b^2}\times\frac{1}{(a/b)}\tan^{-1}\left(\frac{t}{a/b}\right) + c$$

$$= \frac{1}{ab}\tan^{-1}\left(\frac{bt}{a}\right) + c$$

$$= \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right) + c$$

Example 53. Integrate $\frac{1}{x^{1/2}+x^{1/3}}$ with respect to x

Solution : Here $I = \int \frac{1}{x^{1/2} + x^{1/3}} dx$

Let $x = t^6 \Rightarrow dx = 6t^5 \, dt$

$$\therefore \quad I = \int \frac{6t^5}{t^3 + t^2} dt$$

$$= \int \frac{6t^3}{t+1} dt = 6\int \left[t^2 - t + 1 - \frac{1}{t+1} \right] dt$$

$$= 6\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \log|t+1| \right] + c$$

$$= 6\left[\frac{\sqrt{x}}{3} - \frac{x^{1/3}}{2} + x^{1/6} - \log(x^{1/6} + 1) \right] + c$$

Example 54. Integrate $\cos\sqrt{x}$ with respect to x

Solution : Let $I = \int \cos\sqrt{x} \, dx$

Let $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} dx = dt \Rightarrow dx = 2t \, dt$

$$\therefore \quad I = \int \cos t \times 2t \, dt$$

$$= 2\int \underset{\text{I}}{t} \underset{\text{II}}{\cos t} \, dt$$

$$= 2\left[t \sin t - \int 1 \times \sin t \, dt \right]$$

$$= 2[t \sin t + \cos t] + c$$

$$= 2\left[\sqrt{x} \sin\sqrt{x} + \cos\sqrt{x} \right] + c$$

Example 55. Integrate $\frac{\sqrt{\tan x}}{\sin x \cos x} dx$ with respect to x

Solution : Let $I = \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx = \int \frac{\sqrt{\tan x}}{\tan x \cos^2 x} dx$

On multiplying and dividing by $\cos x$ in denominator

$$= \int \frac{\sec^2 x}{\sqrt{\tan x}} dx \qquad \text{Let } \tan x = t \qquad \therefore \sec^2 x dx = dt$$

$$= \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + c = 2\sqrt{\tan x} + c$$

Example 56. Integrate $\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$ with respect to x

Solution : Let
$$I=\int\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx=\int\left[\frac{\sqrt{\sin x}}{\sqrt{\cos x}}+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right]dx$$
$$=\int\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}dx=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{2\sin x\cos x}}dx$$
$$=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{1-(1-2\sin x\cos x)}}dx=\sqrt{2}\int\frac{(\sin x+\cos x)}{\sqrt{1-(\sin x-\cos x)^2}}dx$$

Let $\sin x-\cos x=t\Rightarrow(\cos x+\sin x)dx=dt$

$\therefore$
$$I=\sqrt{2}\int\frac{dt}{\sqrt{1-t^2}}=\sqrt{2}\sin^{-1}t+c$$
$$=\sqrt{2}\sin^{-1}(\sin x-\cos x)+c$$

Example 57. Integrate $\dfrac{(x^5-x)^{1/5}}{x^6}$ with respect to x

Solution :
$$I=\int\frac{(x^5-x)^{1/5}}{x^6}dx=\int\frac{x(1-1/x^4)^{1/5}}{x^6}dx$$
$$=\int\frac{(1-1/x^4)^{1/5}}{x^5}dx$$

Let $\left(1-\dfrac{1}{x^4}\right)=t\Rightarrow\dfrac{4}{x^5}dx=dt\Rightarrow\dfrac{1}{x^5}dx=\dfrac{dt}{4}$

$\therefore$
$$I=\frac{1}{4}\int t^{1/5}dt=\frac{1}{4}\frac{t^{1/5+1}}{(1/5+1)}+c$$
$$=\frac{1}{4}\times\frac{5}{6}t^{6/5}+c=\frac{5}{24}\left(1-\frac{1}{x^4}\right)^{6/5}+c$$

Miscellaneous Exercise 9

Integrate the following functions with respect to x

1. $1+2\tan x(\tan x+\sec x)$
2. $e^x\sin^3 x$
3. $x^2\log(1-x^2)$
4. $\dfrac{\sqrt{x}-\sqrt{a}}{\sqrt{(x+a)}}$ $\left[\text{Hint}: x=a\tan^2\theta\right]$
5. $\dfrac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}$
6. $\dfrac{x}{1+\sin x}$
7. $\dfrac{1}{x+\sqrt{a^2-x^2}}$
8. $\dfrac{2x-1}{(1+x)^2}$
9. $\dfrac{1}{\cos 2x+\cos 2\alpha}$
10. $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

11. $\dfrac{\sin x-\cos x}{\sqrt{\sin 2x}}$

12. $\dfrac{\sin 2x}{\sin^4 x+\cos^4 x}$

13. $\dfrac{1+x}{(2+x)^2}$

14. $\dfrac{\sin^6 x+\cos^6 x}{\sin^2 x\cos^2 x}$

15. $\dfrac{\tan^{-1} x}{x^2}$

16. $\dfrac{1}{\sin^2 x+\sin 2x}$

17. $\dfrac{1}{4x^2-4x+3}$

18. $\dfrac{1}{x[6(\log x)^2+7(\log x)+2]}$

19. $\dfrac{\sin 2x\cos 2x}{\sqrt{4-\sin^4 2x}}$

20. $\dfrac{\sin x+\cos x}{9+16\sin 2x}$

21. $\dfrac{3x-1}{(x-2)^2}$

22. $\int \dfrac{1-\cos 2x}{1+\cos 2x}dx=$

(a) $\tan x+x+c$ (b) $\cot x+x+c$ (c) $\tan x-x+c$ (d) $\cot x-x+c$

23. $\int \dfrac{1}{\sqrt{32-2x^2}}dx=$

(a) $\sin^{-1}(x/4)+c$ (b) $\dfrac{1}{\sqrt{2}}\sin^{-1}(x/4)+c$ (c) $\sin^{-1}\left(\dfrac{\sqrt{2}x}{4}\right)+c$ (d) $\cos^{-1}(x/4)+c$

24. $\int \log x\, dx=$

(a) $x\log(xe)+c$ (b) $x\log x+c$ (c) $x\log(x/e)+c$ (d) $\log x/e$

25. $\int \dfrac{1}{x(x+1)}\ dx$

(a) $\log\left(\dfrac{x}{x+1}\right)+c$ (b) $\log\left(\dfrac{x+1}{x}\right)+c$ (c) $\dfrac{1}{2}\log\left(\dfrac{x}{x+1}\right)+c$ (d) $\dfrac{1}{2}\log\left(\dfrac{x+1}{x}\right)+c$

IMPORTANT POINTS

1. If given function is $f(x)$ and its integral is $F(x)$ then by definition of integration $\dfrac{d}{dx}F(x)=f(x)$.
2. Integration is called antiderivative or primitive, it is a inverse process of differentiation.
3. For a constant k, $\int k\, f(x)dx=k\int f(x)\,dx$
4. $\int[f_1(x)\pm f_2(x)]dx=\int f_1(x)dx\pm\int f_2(x)dx$
5. Some standard formulae for integration

(i) $\int x^n dx=\dfrac{x^{n+1}}{n+1}+c,\quad n\neq -1$

(ii) $\int \dfrac{1}{x}dx=\log|x|+c$

(iii) $\int e^x dx=e^x+c$

(iv) $\int a^x dx=\dfrac{a^x}{\log a}+c$

(v) $\int \sin x dx = -\cos x + c$

(vi) $\int \cos x dx = \sin x + c$

(vii) $\int \sec^2 x\, dx = \tan x + c$

(viii) $\int \cos ec^2 x\, dx = -\cot x + c$

(ix) $\int \sec x \tan x\, dx = \sec x + c$

(x) $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$

(xi) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c = -\cos^{-1} x + c$

(xii) $\int \frac{1}{1+x^2} = \tan^{-1} x + \mathrm{c} = -\cot^{-1} x + c$

(xiii) $\int \frac{1}{x\sqrt{x^2-1}} = \sec^{-1} x + c = -\operatorname{cosec}^{-1} x + c$

(xiv) $\int \frac{|x|}{x} dx = |x| + c, \quad x \neq o$

(xv) $\int dx = x + c$

(xvi) $\int o\, dx = c$

6. Integration by substitution

(i) $\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c$

(ii) $\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$

(iii) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

(iv) $\int \frac{1}{ax+b} dx = \frac{1}{a} \log |ax+b| + c$

(v) $\int e^{ax+b} dx = \frac{e^{ax+b}}{a} + c$

(vi) $\int \sin(ax+b) dx = \frac{-\cos(ax+b)}{a} + c$

(vii) $\int \cos(ax+b) dx = \frac{\sin(ax+b)}{a} + c$

7. Use of substitituon method in standard formulae

(i) $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

(ii) $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} x/a + c$

(iii) $\int \frac{1}{\sqrt{x^2+a^2}} dx = \log |x + \sqrt{x^2+a^2}| + c$

(iv) $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log |x + \sqrt{x^2-a^2}| + c$

8. Standard Integrals

(i) $\int \frac{1}{x^2-a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$

(ii) $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$

(iii) $\int \sqrt{a^2-x^2} dx = \frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$

(iv) $\int \sqrt{a^2+x^2} dx = \frac{x}{2} \sqrt{a^2+x^2} + \frac{a^2}{2} \log |x + \sqrt{a^2+x^2}| + c$

(v) $\int \sqrt{x^2-a^2} dx = \frac{x}{2} \sqrt{x^2-a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2-a^2}| + c$

(vi) $\int \tan x dx = \log|\sec x| + c$ (vii) $\int \cot x dx = \log|\sin x| + c$

(viii) $\int \sec x\, dx = \log|\sec x + \tan x| + c = \log\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + c$

(ix) $\int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + c = \log|\tan x/2| + c$

9. Integration by parts:

(i) The integral of the product of two functions = (first function) × ∫ second function dx − ∫ (differential coefficient of first function) × ∫ integral of second function dx.

i.e. $\int \underset{\text{I}}{u}\,\underset{\text{II}}{v}\, dx = u\int v dx - \int\left[\frac{du}{dx}\times\int v dx\right]dx$

(ii) $\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx] + c = \frac{e^{ax}}{a^2+b^2}\sin[bx - \tan^{-1} b/a] + c$

(iii) $\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx] + c = \frac{e^{ax}}{a^2+b^2}\cos\left[bx - \tan^{-1} b/a\right] + c$

(iv) $\int e^{x}[f(x)+f'(x)]dx = e^{x} f(x) + c$

(v) $\int [x f'(x) + f(x)]dx = x f(x) + c$

(vi) $\int [f(\log x) + f'(\log x)]dx = xf(\log x) + c$

Answer

Exercise 9.1

1. (i) $\frac{3}{5}\cdot x^{5/3} + c$ (ii) $\frac{e^{3x}}{3} + c$ (iii) $\frac{(1/2)^x}{(\log 1/2)} + c$ (iv) $\frac{x^3}{3} + c$

2. $5\sin x + 3\cos x + 2\tan x + c$

3. $x^2/2 + 1/x + c$

4. $\tan x - \cot x + c$

5. $2/3\cdot x^{3/2} + 2/5\cdot x^{5/2} + c$

6. $\frac{a^{x+1}}{x+1} + c$

7. $x - \tan^{-1} x + c$

8. $x + \cos x + c$

9. $\tan x + \sec x + c$

10. $(\pi/2)x + c$

11. $x - 2\tan^{-1} x + c$

12. $\tan x - x + c$

13. $-\cot x - x + c$

14. $\frac{2}{3}(1+x)^{3/2} + \frac{2}{3}x^{3/2} + c$

15. $\tan x + \cot x + c$

16. $x - \tan x + \sec x + c$

17. $-\cot x - \cot x \operatorname{cosec} x + c$

18. $x + \tan^{-1} x + 3sec^{-1}x + \frac{2^x}{\log 2} + c$

19. $x + \operatorname{cosec} x + c$

20. $x^2/2 + \log|x| + 2x + c$

21. $x + c$

22. $\sqrt{2}\sin x + c$

23. $-\cot x - \tan x + c$

24. $-3\operatorname{cosec} x - 4\cot x + c$

Exercise 9.2

1. (i) $(-1/2)\cos x^2 + c$ (ii) $\frac{1}{3}(x^2+1)^{3/2} + c$

2. (i) $\log|e^x + \cos x| + c$ (ii) $2\sqrt{1+e^x} + c$

3. (i) $2\sqrt{e^x+1} + \log\left|\frac{e^x}{e^x+2}\right| + c$ (ii) $2\sin(e^{\sqrt{x}}) + c$

4. (i) $\log|1+\log x| + c$ (ii) $\frac{1}{4}(1+\log x)^4 + c$

5. (i) $\frac{e^{m\tan^{-1}x}}{m} + c$ (ii) $\frac{(\tan x)^{p+1}}{p+1} + c$

6. (i) $\frac{1}{\sqrt{2}}\log|\sec x + \tan x| + c$; (ii) $\log|\operatorname{cosec} 2x - \cot 2x| + \log|\operatorname{cosec} x - \cot x| + c$

7. (i) $\frac{1}{2}\left[\sin x - \frac{1}{5}\sin 5x\right] + c$ (ii) $\pm 2(\sin x/2 + \cos x/2) + c$

8. (i) $\frac{1}{8}\left[3x + 2\sin 2x + \frac{1}{2}\sin 4x\right] + c$; (ii) $\frac{-3}{4}\cos x - \frac{1}{12}\cos 3x + c$

9. (i) $\log|\tan x| + \frac{1}{2}\tan^2 x + c$; (ii) $\tan(xe^x) + c$

10. (i) $\frac{1}{2}[x + \log|\sin x - \cos x|] + c$; (ii) $\frac{1}{2}[x + \log|\sin x + \cos x|] + c$

11. (i) $2\sqrt{\tan x} + \frac{2}{3}\tan^{5/2} x + c$ (ii) $\log|\sin x + \cos x| + c$

12. (i) $x\cos 2a + \sin 2a.\log|\sin(x-a)| + c$; (ii) $x\cos a + \sin a.\log|\sin(x-a)| + c$

13. (i) $\frac{1}{3}\log|\sin 3x| - \frac{1}{5}\log|\sin 5x| + c$; (ii) $\log|\sin(x+\pi/6)\sin(x-\pi/6)| + c$

14. (i) $\frac{1}{5}\log\left|\tan\left(\frac{x+\tan^{-1}(4/3)}{2}\right)\right| + c$; (ii) $\operatorname{cosec}(a-b)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + c$

15. (i) $\frac{1}{2(b-a)}\log(a\cos^2 x + b\sin^2 x) + c$; (ii) $\sqrt{2}\sec\alpha\sqrt{\tan x\cos\alpha + \sin\alpha} + c$

16. (i) $\frac{2}{\cos a}\sqrt{\tan x\cos a + \sin a} + c$; (ii) $2[\sin x + x\cos\alpha] + c$

Exercise 9.3

1. (i) $\frac{1}{10}\tan^{-1}\frac{x}{5} + c$; (ii) $\frac{1}{\sqrt{2}}\sin^{-1}\frac{x}{4} + c$

2. (i) $\log|1-\sqrt{1-e^{2x}}| + c$; (ii) $\frac{1}{2}\log\left[2x + \sqrt{4x^2+1}\right] + c$

3. (i) $\frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right)+c$; (ii) $-\log|(2-x)+\sqrt{x^2-4x+5}|+c$

4. (i) $\frac{1}{3}\log|x^3+\sqrt{x^6+4}|+c$; (ii) $\frac{1}{5}\sin^{-1}(x^5)+c$

5. (i) $\tan^{-1}(x+3)+c$; (ii) $\frac{1}{\sqrt{2}}\log\left|(x-1/4)+\sqrt{x^2-1/2x+1}\right|+c$

6. (i) $\frac{1}{\sin\propto}\tan^{-1}\left(\frac{e^x+\cos\propto}{\sin\propto}\right)+c$; (ii) $\log|\tan x+\sqrt{\tan^2 x+3}|+c$

7. (i) $\sin^{-1}(2x-3)+c$; (ii) $\frac{1}{\sqrt{5}}\sin^{-1}\left(\frac{5x-4}{6}\right)+c$

8. (i) $\sin^{-1}(\sin x-\cos x)+c$; (ii) $\log|(x+a)+\sqrt{x^2+2xa+b^2}|+c$

9. (i) $a\sin^{-1}\sqrt{x/a}+\sqrt{x}\sqrt{a-x}+c$; (ii) $-a\cos^{-1}x/a-\sqrt{a^2-x^2}+c$

10. (i) $\frac{2}{3}\sin^{-1}(x/a)^{3/2}+c$; (ii) $\frac{1}{a^2}\cdot\frac{x}{\sqrt{x^2+a^2}}+c$

11. (i) $\frac{x}{\sqrt{1-x^2}}+c$; (ii) $\sqrt{x^2+1}+\log(x+\sqrt{x^2+1})+c$

12. (i) $2\sin^{-1}\left(\frac{x-\propto}{\beta-x}\right)+c$; (ii) $\sin^{-1}(x-1)+c$

13. (i) $\log\left|(x-3/2)+\sqrt{x^2-3x+2}\right|+c$; (ii) $\sin^{-1}\left(\frac{\sin x}{2}\right)+c$

Exercise 9.4

1. $\frac{1}{24}\log\left|\frac{4x-3}{4x+3}\right|+c$

2. $\frac{1}{12}\log\left|\frac{x-6}{x+6}\right|+c$

3. $\log|x+1|+2\log|x-2|+c$

4. $\frac{11}{4}\log\left|\frac{x+1}{x+3}\right|+\frac{5}{2}\log\left|\frac{1}{x+1}\right|+c$

5. $-\frac{1}{6}\log|x+1|+\frac{4}{5}\log|x-2|+\frac{9}{10}\log|x+3|+c$

6. $\frac{1}{7}\log\left|\frac{x-2}{x+2}\right|+\frac{\sqrt{3}}{7}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+c$

7. $\frac{1}{4}\log\left|\frac{x+1}{x-1}\right|-\frac{1}{2(x-1)}+c$

8. $x+\frac{1}{3}\log\frac{(x-2)^4}{|x+1|}+c$

9. $\frac{1}{a^2-b^2}[a\tan^{-1}(x/a)-b\tan^{-1}(x/b)]+c$

10. $-\frac{1}{6}\log|x|+\frac{3}{10}\log|x-2|-\frac{2}{15}\log|x+3|+c$

11. $-\log|x|+3\log|x-2|-\log|x+2|+c$

12. $\frac{1}{9}\log\left|\frac{x+2}{x-1}\right|-\frac{1}{3(x-1)}+c$

13. $\log\frac{(1+x)^2}{1+x^2}-\tan^{-1}x+c$

14. $\log|x|-\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+c$

15. $x+3\log|x+2|-\log|x+1|+c$

16. $\log\frac{\sqrt{x^2+1}}{|x+1|}+c$

17. $\frac{1}{2}\log\left|\frac{e^x-1}{e^x+1}\right|+c$

18. $\log\left|\frac{e^x}{e^x-1}\right|-\frac{1}{e^x-1}+c$

19. $\log\left|\frac{2+e^x}{3+e^x}\right|+c$

20. $\log\left|\left(\frac{2+\tan x}{3+\tan x}\right)\right|+c$

21. $\log|x|-\frac{1}{5}\log|x^5+1|+c$

22. $\frac{1}{a^n}\log\left(\frac{x^n}{a+bx^n}\right)+c$

23. $\log|x+2|-\frac{1}{2}\log(x^2+4)+\tan^{-1}(x/2)+c$

24. $\log|\sec x+\tan x|-2\tan(x/2)+c$

Exercise 9.5

1. $\frac{1}{3}\tan^{-1}\left(\frac{x^2+1}{2}\right)+c$

2. $\frac{1}{3}\log\left|\frac{2x-1}{2x+2}\right|+c$

3. $\frac{1}{6}\tan^{-1}\left(\frac{3x-2}{2}\right)+c$

4. $\frac{1}{4}\log\left|\frac{x+1}{3-x}\right|+c$

5. $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+c$

6. $\tan^{-1}[\sin(x+2)]+c$

7. $\frac{1}{2}\log|x^2+2x-4|-\frac{2}{\sqrt{5}}\log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right|+c$

8. $\frac{3}{4}\log|2x^2-2x+3|+\frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+c$

9. $\frac{1}{2}\log|x^2+4x+5|-\tan^{-1}(x+2)+c$

10. $3\log|2-\sin x|+\frac{4}{2-\sin x}+c$

11. $-\frac{1}{2}|e^{-2x}+3e^{-x}+2|+\frac{3}{2}\log\left|\frac{e^{-x}+1}{e^{-x}+2}\right|+c$

12. $\frac{1}{2}\log\left|\left(x-5/8\right)+\sqrt{x^2-5x/4+1/4}\right|+c$

13. $\sin^{-1}(2x-5)+c$

14. $\sin^{-1}\left|\frac{2x+1}{\sqrt{5}}\right|+c$

15. $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4x-3}{\sqrt{41}}\right)+c$

16. $\sqrt{x^2-2x+4}+3\log|(x-1)+\sqrt{x^2-2x+4}|+c$

17. $\sqrt{x^2-x+1}+\frac{3}{2}\log|(x-1/2)+\sqrt{x^2-x+1}|+c$

18. $\sqrt{x^2+2x+2}+2\log|(x+1)+\sqrt{x^2+2x+2}|+c$

19. $-\log|(\cos x+1/2)+\sqrt{\cos^2 x+\cos x}|+c$

20. $-\cos\propto\sin^{-1}\left(\frac{\cos x}{\cos\alpha}\right)-\sin\alpha.\log|\sin x+\sqrt{\sin^2 x-\sin^2\alpha}|+c$

21. $\frac{1}{2}x^2-x+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$

22. $\frac{1}{4}\log\left|\frac{e^x+1}{e^x+5}\right|+c$

Exercise 9.6

1. (i) $x\sin x+\cos x+c$; (ii) $x\tan x-\log\sec x+c$

2. (i) $-e^{-x}(x^3+3x^2+6x+6)+c$; (ii) $-x^3\cos x+3x^2\sin x+6x\cos x-6\sin x+c$

3. (i) $\frac{x^4}{4}\left[(\log x)^2-\frac{1}{2}\log x+\frac{1}{8}\right]+c$; (ii) $\frac{1}{2}e^{x^2}(x^2-1)+c$

4. (i) $(e^x-1)e^{e^x}+c$; (ii) $x(\log x)^2-2x\log x+2x+c$

5. (i) $x\cos^{-1}x-\sqrt{1-x^2}+c$; (ii) $(x+a)\tan^{-1}\sqrt{x/a}-\sqrt{ax}+c$

6. (i) $3x\sin^{-1}x+3\sqrt{1-x^2}+c$; (ii) $x\tan x/2-2\log|\sec x/2|+c$

7. (i) $\frac{1}{2}\left[x\cos^{-1}x-\sqrt{1-x^2}\right]+c$; (ii) $2\left[\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x}\right]+c$

8. (i) $\frac{-x(1-\sin x)}{\cos x}+\log(1+\sin x)+c$; (ii) $\frac{x^3}{3}\tan^{-1}x-\frac{x^6}{6}+\frac{1}{6}\log(1+x^2)+c$

9. (i) $-\sin^{-1}x.\cos(\sin^{-1}x)+x+c$

10. $\frac{-\tan^{-1}x}{\sqrt{1+x^2}}+\frac{x}{\sqrt{1+x^2}}+c$

11. $e^x\log\sin x+c$

12. $x\tan x+c$

13. $-e^x\cot x/2+c$

14. $e^x(\log x-1/x)+c$

15. $e^x\log|\sec x+\tan x|+c$

16. $e^x\sec x+c$

17. $\frac{e^x}{x^2}+c$

18. $\frac{e^x}{1+x^2}+c$

19. $\frac{1}{2}\sin 2\theta\log\left|\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right|+\frac{1}{2}\log(\cos 2\theta)+c$

20. $\frac{x\sin x+\cos x}{x\cos x-\sin x}+c$

21. $x\sec^{-1}x-\log[x+\sqrt{x^2-1}]+c$

22. $x(\sin^{-1}x)^2+2\sqrt{1-x^2}(\sin^{-1}x)-2x+c$

Exercise 9.7

1. $\frac{e^{2x}}{5}[2\cos x+\sin x]+c$

2. $\frac{1}{2}x[\sin(\log x)-\cos(\log x)]+c$

3. $\frac{e^{\tan^{-1}x}}{1+a^2}\left[\frac{a+x}{\sqrt{1+x^2}}\right]+c$

4. $\frac{2}{3}e^{x/\sqrt{2}}\left[\frac{1}{\sqrt{2}}\cos(x+\propto)+\sin(x+\propto)\right]+c$

5. $\frac{e^x}{2}-\frac{e^x}{10}[\cos 2x+2\sin 2x]+c$

6. $\frac{e^{a\sin^{-1}x}}{1+a^2}[x+a\sqrt{1-x^2}]+c$

7. $\frac{x}{1+b^2}[\cos(b\log x/a)+b\sin(b\log x/a)]+c$

8. $\frac{e^{4x}}{8}\left[\frac{1}{13}(4\cos 6x+6\sin 6x)+\frac{1}{5}(4\cos 2x+2\sin 2x)\right]+c$

9. $\frac{x-1}{2}\sqrt{2x-x^2}+\frac{1}{2}\sin^{-1}(x-1)+c$

10. $\frac{x+2}{2}\sqrt{x^2+4x+6}+\log|(x+2)+\sqrt{x^2+4x+6}|+c$

11. $\dfrac{(x+3)\sqrt{x^2+6x-4}}{2}+\dfrac{13}{2}\log|(x-2)+\sqrt{x^2+6x-4}|+c$

12. $\dfrac{4x+3}{8}\sqrt{2x^2+3x+4}+\dfrac{23}{16\sqrt{2}}\log\left(\dfrac{4x+3}{4}\right)+\sqrt{(x^2+\dfrac{3}{2}x+2)}+c$

13. $\dfrac{1}{3}x^3\sqrt{a^2-x^6}+\dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x^3}{a}\right)+c$

14. $\dfrac{1}{3}\left(x^2+1\right)^{3/2}+\dfrac{x}{2}\sqrt{x^2+1}+\dfrac{1}{2}\log|x+\sqrt{x^2+1}|+c$

15. $\dfrac{5}{2}\sin^{-1}\left(\dfrac{x+2}{\sqrt{5}}\right)+\dfrac{x+2}{2}\sqrt{1-4x-x^2}+c$

16. $\dfrac{(4x+3)}{8}\sqrt{4-3x-2x^2}+\dfrac{41\sqrt{2}}{32}\sin^{-1}\left(\dfrac{4x+3}{\sqrt{41}}\right)+c$

Miscellaneous Exercise – 9

1. $2(\tan x+\sec x)-x+c$

2. $\dfrac{e^x}{30}[\sin 3x-3\cos 3x+20\sin x-20\cos x]+c$

3. $\dfrac{x^3}{3}\log|1-x^2|-\dfrac{2}{3}\left(x+\dfrac{x^3}{3}\right)+\dfrac{1}{3}\log\left|\dfrac{1+x}{1-x}\right|+c$

4. $\sqrt{x^2+ax}-2\sqrt{ax+a^2}+a\log\left(\sqrt{a+x}-\sqrt{x}\right)+c$

5. $\dfrac{-\sin 2x}{2}+c$

6. $x(\tan x-\sec x)-\log|\sec x|+\log|\sec x+\tan x|+c$

7. $\dfrac{1}{2}[\sin^{-1}(x/a)+\log|x+\sqrt{a^2-x^2}|]+c$

8. $2\log|(1+x)|+\dfrac{2}{1+x}+c$

9. $\dfrac{1}{2}\cos ec2\alpha\cdot\log\left|\dfrac{(x-\alpha)}{(x+\alpha)}\right|+c$

10. $2x\tan^{-1}x-\log(1+x^2)+c$

11. $-\log|(\sin x+\cos x)+\sqrt{\sin 2x}|+c$

12. $\tan^{-1}(\tan^2 x)+c$

13. $\log|x+2|+\dfrac{1}{2+x}+c$

14. $\tan x-\cot x-3x+c$

15. $\dfrac{-\tan^{-1}x}{x}-\dfrac{(\tan^{-1}x)^2}{2}+\log\left(\dfrac{|x|}{\sqrt{1+x^2}}\right)+c$

16. $\log\left|\dfrac{\tan x}{\tan x+2}\right|+c$

17. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{2x-1}{\sqrt{2}}\right)+c$

18. $\log\left|\dfrac{2\log x+1}{3\log x+2}\right|+c$

19. $\dfrac{1}{4}\sin^{-1}\left[\dfrac{\sin^2 2x}{2}\right]+c$

20. $\dfrac{1}{40}\log\left|\dfrac{5+4(\sin x-\cos x)}{5-4(\sin x-\cos x)}\right|+c$

21. $3\log|x-2|-\dfrac{5}{x-2}+c$

22 (c)

23. (b)

24. (c)

25. (a)