HOTS (Higher Order Thinking Skills)

Que 1. Show that there is no positive integer n for which $\sqrt{\mathbf{n-1}} + \sqrt{\mathbf{n+1}}$ is rational.

Sol. Let there be a positive integer n for which $\sqrt{\mathrm{n}-1} + \sqrt{\mathrm{n}+1}$ be rational number.

$\sqrt{n-1} + \sqrt{n+1} = \frac{p}{q}$; Where p, q are integer and q ≠ 0 ……(i)

$$\Rightarrow \frac{1}{\sqrt{n-1}+\sqrt{n+1}} = \frac{q}{p} \qquad \Rightarrow \frac{\sqrt{n-1}-\sqrt{n+1}}{\left(\sqrt{n-1}+\sqrt{n+1}\right)\times\left(\sqrt{n-1}-\sqrt{n+1}\right)}\frac{q}{p}$$

$$\Rightarrow \frac{\sqrt{n-1}-\sqrt{n+1}}{(n-1)-(n+1)} = \frac{q}{p} \qquad \Rightarrow \frac{\sqrt{n-1}-\sqrt{n+1}}{n-1-n-1} = \frac{q}{p}$$

$$\Rightarrow \frac{\sqrt{n+1}-\sqrt{n-1}}{2} = \frac{q}{p} \qquad \Rightarrow \sqrt{n+1} - \sqrt{n-1} = \frac{2q}{p} \text{ …..(ii)}$$

Adding (i) and (ii), we get

$$\sqrt{n-1} + \sqrt{n+1} + \sqrt{n+1} - \sqrt{n-1} = \frac{p}{q} + \frac{2q}{p}$$

$$\Rightarrow 2\sqrt{n+1} = \frac{p^2+2q^2}{pq} \qquad \Rightarrow \sqrt{n+1} = \frac{p^2+2q^2}{2pq}$$

⇒ $\sqrt{n+1}$ is rational number as $\frac{p^2+2q^2}{2pq}$ is rational.

⇒ n + 1 is perfect square of positive integer …(A)

Again subtracting (ii) from (i), we get

$$\sqrt{n-1} + \sqrt{n+1} - \sqrt{n+1} + \sqrt{n-1} = \frac{p}{q} - \frac{2q}{p} \qquad \Rightarrow \qquad 2\sqrt{n-1} = \frac{p^2-2q^2}{pq}$$

⇒ $\sqrt{n-1}$ is rational number as $\frac{p^2-2q^2}{2pq}$ is rational.

⇒ $\sqrt{n-1}$ is also perfect square of positive integer. ….(B)

From (A) and (B)

$\sqrt{n+1}\ and\ \sqrt{n-1}$ are perfect squares of positive integer. It contradict the fact that two perfect differ at least by 3.

Hence, there is no positive integer n for which $\sqrt{n-1} + \sqrt{n+1}$ is rational.

Que 2. Let a, b, c, k be rational numbers such that k is not a perfect cube. If a + bk$^{1/3}$ + ck$^{2/3}$ = 0, then prove that a = b = c = 0.

Sol. Given, $a + bk^{1/3} + ck^{2/3} = 0$ …(i)

Multiplying both sides by k$^{1/3}$, we have

$$ak^{1/3} + bk^{2/3} + ck = 0$$

Multiplying (i) by b and (ii) by c and then subtracting, we have

$$\left(ab + b^2k^{1/3} + bck^{2/3}\right) - \left(ack^{1/3} + bck^{2/3} + c^2k\right) = 0$$

$\Rightarrow \quad (b^2 - ac)k^{1/3} + ab - c^2k = 0$
$\Rightarrow \quad b^2 - ac = 0 \quad and \quad ab - c^2 = 0$ [Since $k^{1/3}$ is irrational]
$\Rightarrow \quad b^2 = ac \quad and \quad ab = c^2k$
$\Rightarrow \quad b^2 = ac \quad and \quad a^2b^2 = c^4k^2$
$\Rightarrow \quad a^2(ac) = c^4k^2$ [By putting b^2 = ac in $a^2 b^2 = c^4 k^2$]
$\Rightarrow \quad a^3c - k^2c^4 = 0 \quad \Rightarrow \quad (a^3 - k^2c^3)\, c = 0$
$\Rightarrow \quad a^3 - k^2c^3 = 0, or \ c = 0$

Now, $a^3 - k^2c^3 = 0 \quad \Rightarrow \quad k^2 = \frac{a^3}{c^3}$

$$\Rightarrow \quad (k^2)^{1/3} = \left(\frac{a^3}{c^3}\right)^{1/3} \quad \Rightarrow \quad k^{2/3} = \frac{a}{c}$$

This is impossible as $k^{2/3}$ is irrational and $\frac{a}{c}$ is rational.

$\therefore \quad a^3 - k^2c^3 \neq 0$
Hence, c = 0
Subtracting c = 0 in b^2 – ac = 0, we get b = 0
Substituting b = 0 and c = 0 in a + $bk^{1/3}$ + $ck^{2/3}$ = 0, we get a = 0
Hence, a = b = c = 0.

Que 3. Find the largest positive integer that will divide 398, 436 and 542 leaving remainders 7, 11 and 15 respectively.

Sol. It is given that on dividing 398 by the required number, there is a remainder of 7. This means that 398 – 7 = 391 is exactly divisible by the required number. In other words, required number is a factor of 391.
Similarly required positive integer is a factor of 436 – 11 = 425 and 542 – 15 = 527.
Clearly, required number is the HCF of 391, 425 and 527.
Using the factor tree we get the prime factorisations of 391, 425 and 425 and 527 as follows: 391 = 17 × 23, 425 = 5^2 × 17 and 527 = 17 × 31
∴ HCF of 391, 425 and 527 is 17.
Hence, required number = 17