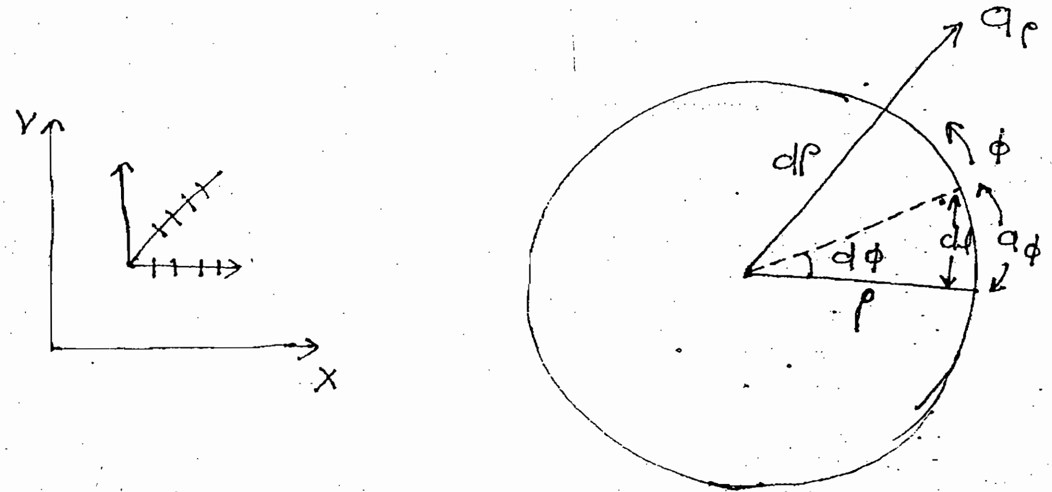
Lecture - 3

Line as a Vector :-

It has a magnitude equal to its length and a direction in which the length parameter increase

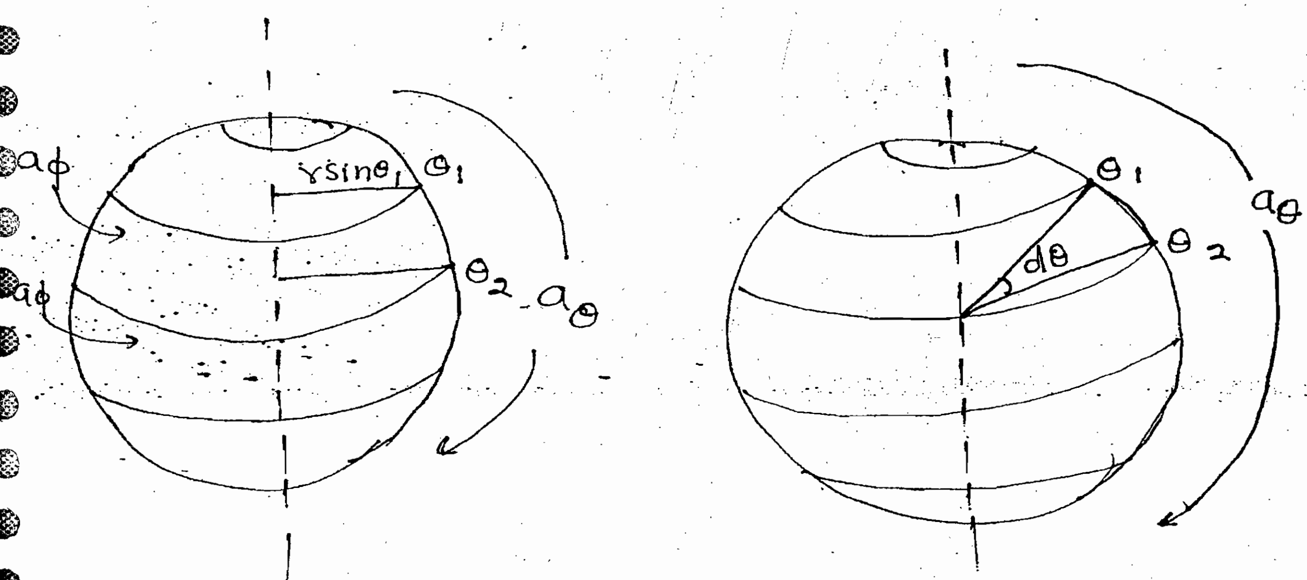
Cartesian :-



$$d\vec{l} = dx\, a_x + dy\, a_y + dz\, a_z$$

Spherical :-

$$d\vec{l} = dr\, a_r + r\, d\theta \cdot a_\theta + r \sin\theta \cdot d\phi\, a_\phi$$



In angular direction length is curvature

Note :-

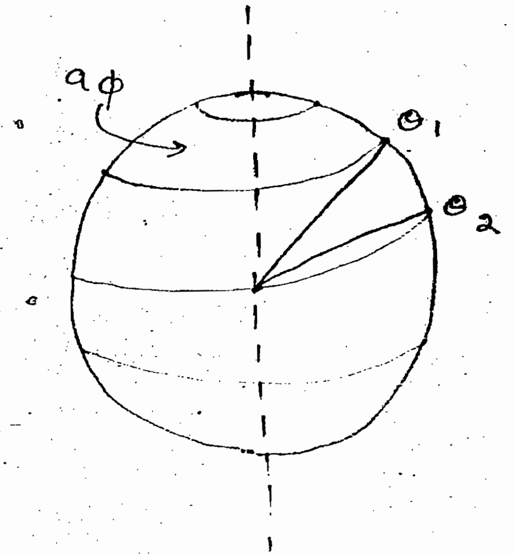
In angular directions length is a curvature not straight line or linear

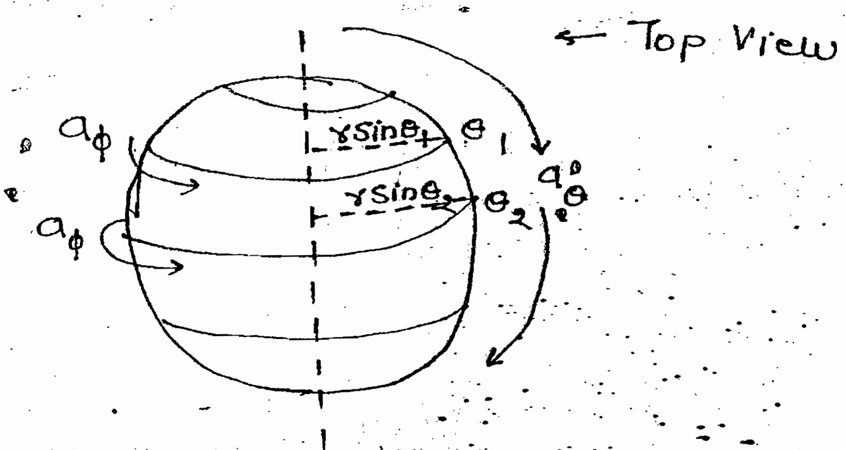
Curvature length = Radius x angle

Cartesian → $\vec{dl} = dx\, a_x + dy\, a_y + dz\, a_z$

Cylindrical → $\vec{dl} = d\rho\, a_\rho + \rho\, d\phi\, a_\phi + dz\, a_z$

Spherical → $\vec{dl} = dr\, a_r + r\, d\theta\, a_\theta + r\sin\theta\, d\phi\, a_\phi$





Note:-

In ϕ direction in spherical coordinating system curvature length is height on the sphere dependent i.e. it depends on the θ value of the circle .

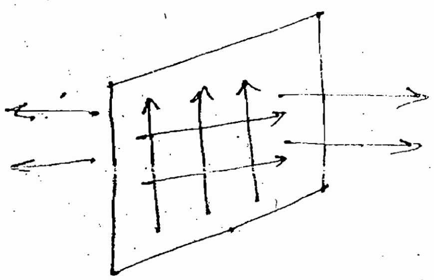
Summary :-

Parameters :-			Scaling Factors		
x	y	z	1	1	1
ρ	ϕ	z	1	ρ	1
r	θ	ϕ	1	r	$r\sin\theta$
u	v	w	h_1	h_2	h_3

$$\vec{dl} = h_1 du a_x + h_2 dv a_v + h_3 dw a_w$$

Surface as a vector :-

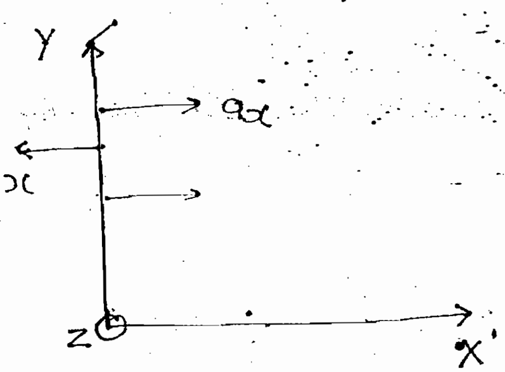
It has a magnitude equal to its area and a direction normal to the plane of surface

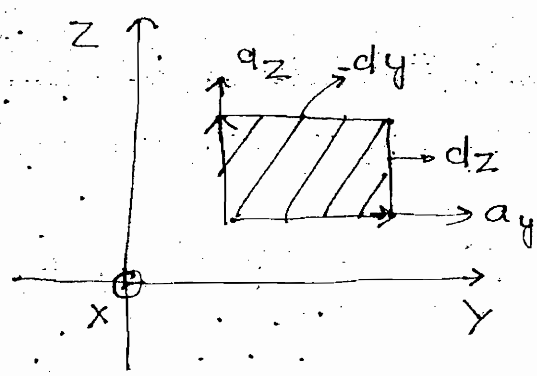


Direction is unique only when we take normal to the surface as there can be two tangential directions.

x = constant surface

$x = 0$ plane, YZ plane

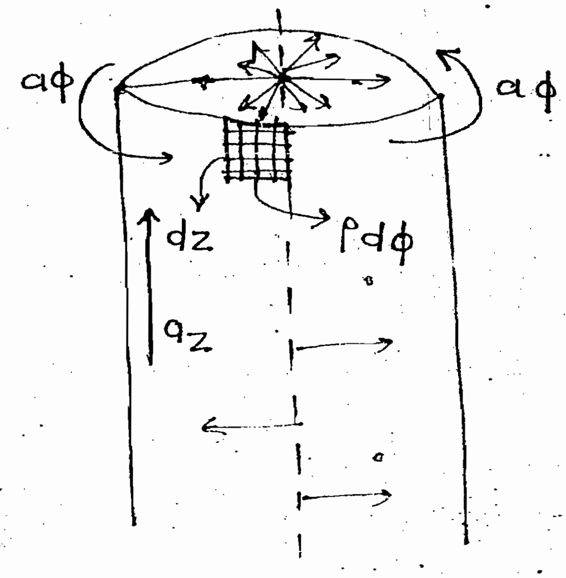


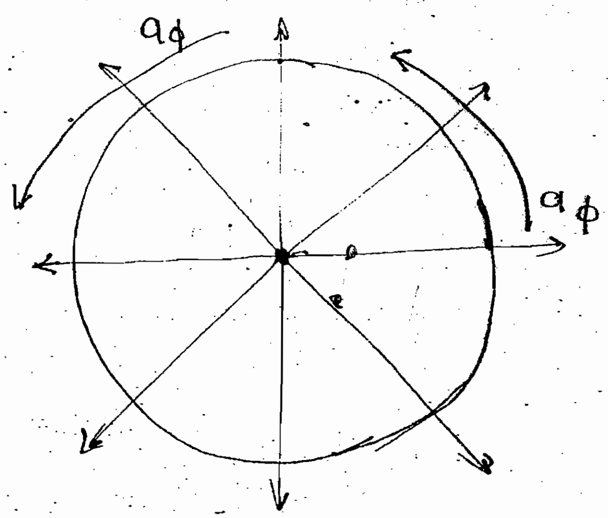


Cartesian :-

$$\boxed{\vec{ds} = \underbrace{dy\,dz}_{\text{mag.}}\ \underbrace{a_x}_{\text{direction}} + dz\,dx\,a_y + dx\,dy\,a_z}$$

Cylindrical Coordinate system :-





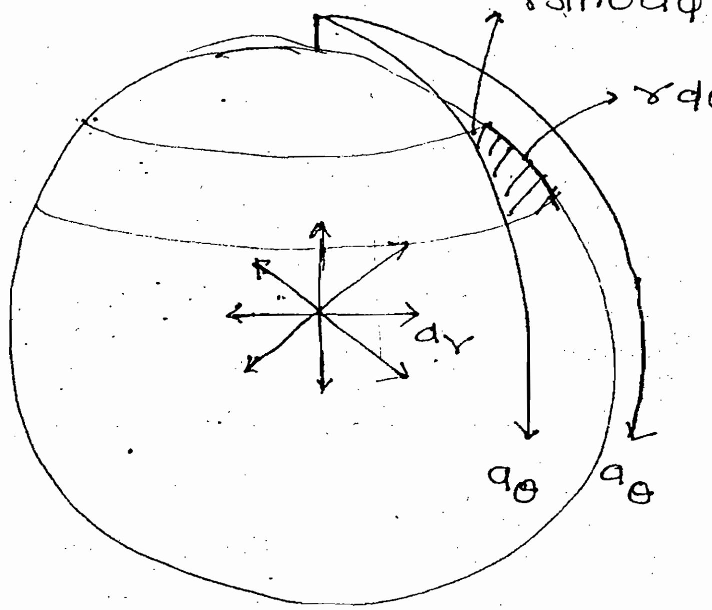
a_ρ = normal i.e. direction

a_ϕ , a_z = tangential → ∥ to surface

$$\vec{ds} = \rho\,d\phi\,dz\,a_\rho + dz\,d\rho\,a_\phi + \rho\,d\rho\,d\phi\,a_z$$

Spherical coordinate System :-

$r =$ constant , sphere



a_r = normal ↓ direction

a_θ , a_ϕ = tangential

$$\boxed{d\vec{s} = r^2 \sin\theta\, d\theta\, d\phi\, \underline{a_r} + r\sin\theta\, d\phi\, dr\, \underline{a_\theta} + r\, dr\, d\theta\, \underline{a_\phi}}$$

$$\boxed{ds = h_2 h_3\, dv\, dw\, a_u + h_3 h_1\, dw\, du\, a_v + h_1 h_2\, du\, dv\, a_w}$$

Summary :-

1 parameter = Constant
2 parameter = variable → surface

Surface direction = Constant direction
= UNIQUE

2 Parameter = Constant
1 Parameter = Variable → line

Lines direction = Variable's direction
= UNIQUE

3 Parameters = variable] → Volume

3 Parameters = constant] → Point

Volume as a scalar :-

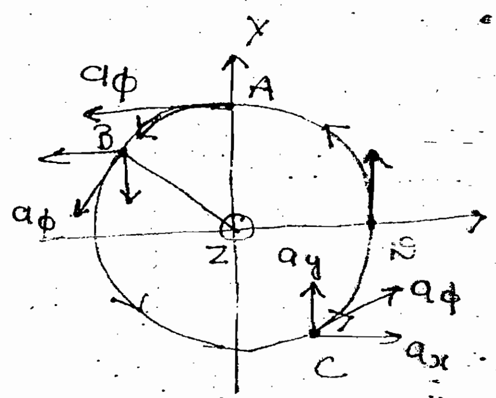
It has no unique direction & it is the scalar triple product of lengths in all three dimensions

$$dv = dx\,dy\,dz$$
$$= \rho\, d\rho\, d\phi\, dz$$
$$= r^2 \sin\theta\, dr\, d\theta\, d\phi$$
$$= h_1 h_2 h_3\, du\, dv\, dw$$

Workbook -1

2. At Point A

$a_\phi = -a_x$



divided by $\sqrt{2}$ to make magnitude = 1

At Point B

$$a_\phi = \frac{-a_x - a_y}{\sqrt{2}}$$

At Point C

$$a_\phi = \frac{a_x + a_y}{\sqrt{2}}$$

At point D

$a_\phi = a_y$

3. $\vec{D} = xy\, a_x + yz\, a_y + zx\, a_z$ C/m²

$y = 2 \qquad 0 < x < 4$

$\qquad\qquad 0 < z < 2$

Since only 2 variable are present. Hence it is not a closed surface

$$ds = dx\,dz\,a_y$$

$$\int \vartheta \cdot ds = \Psi_e$$

$$\Rightarrow \Psi_e = \int_{x=0}^{4}\int_{z=0}^{2} yz\, dx\, dz \Big|_{y=2}$$

$$= 2\,\frac{z^2}{2}\Big|_0^2 \cdot x\Big|_0^4 = 16\,C \longrightarrow \text{flux}$$

4. $\vec{D} = 5(\rho-3)^2 a_\rho$

$\rho = 4$, $0 < \phi < \pi$

$-5 < z < 5$

$$ds = \rho\, d\phi\, dz\, a_\rho$$

$$\Psi_e = \int_{\phi=0}^{\pi}\int_{z=-5}^{5} 5(\rho-3)^2 a_\rho \cdot \rho\, d\phi\, dz\, a_\rho \Big|_{\rho=4}$$

$$= 20\,\phi\Big|_0^{\pi} z\Big|_{-5}^{5}$$

$$= 200\pi$$

Divergence and curl of vector :-

$$\vec{A} = A_u a_u + A_v a_v + A_w a_w$$

$$\nabla\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}(h_2h_3A_u) + \frac{\partial}{\partial v}(h_3h_1A_v) + \frac{\partial}{\partial w}(h_1h_2A_w)\right]$$

Cartesian :-

$$\nabla\cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

cylindrical :-

$$\nabla\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\cdot A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Spherical :-

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Workbook - 1

$$\vec{D} = \rho \cdot z \cos^2\phi\, a_z$$

$$\rho_v = ? \quad \text{at } (1, \pi/4, 3)$$

$$\nabla \cdot D = \rho_v$$

$$\nabla \cdot D = \frac{\partial(\rho z \cos^2\phi)}{\partial z}$$

$$= \rho\cos^2\phi = 1 \cdot \frac{1}{2} = 0.5\ C/m^3$$

5. $$\vec{D} = \rho \cdot z \cdot \cos^2\phi\, a_\rho$$

$$\nabla \cdot D = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho \cdot \rho z \cos^2\phi)$$

$$= \frac{1}{\rho} z\cos^2\phi \cdot 2\rho = 2 \cdot 3 \cdot \frac{1}{2} = 3\ C/m^3$$

7. $$\vec{F} = \rho a_\rho + \rho \sin^2\phi\, a_\phi - z a_z$$

$$\nabla \cdot \vec{F} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho \cdot \rho) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(\rho \cdot \sin^2\phi) + \frac{\partial}{\partial z}(-z)$$

$$= 2 + 2\sin\phi\cos\phi - 1$$

$$= 1 + \sin 2\phi$$

Ans - D

Ques - GATE 2012

$$\vec{A} = k r^n a_r \qquad n = ?$$

If $\nabla \cdot A = 0$

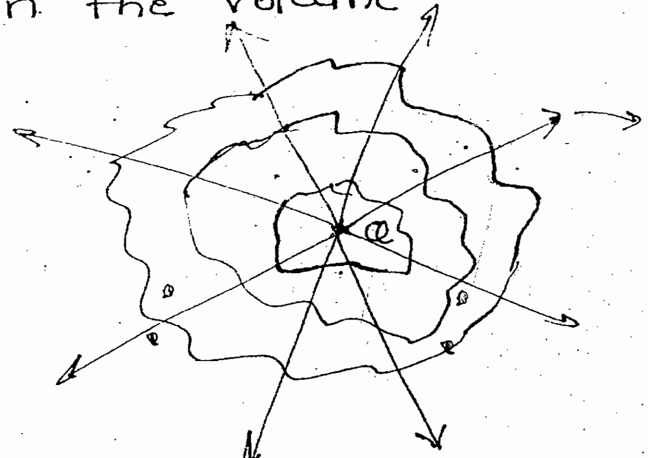
Soln:-
$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 k r^n)$$
$$= \frac{k}{r^2}(n+2) r^{n+1} = 0$$
$$\Rightarrow n = -2 \quad \text{, Ans}$$

Curl of a vector $\vec{A}$:-
$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 a_u & h_2 a_v & h_3 a_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1 A_v & h_2 A_v & h_3 A_w \end{vmatrix}$$

Static Electric Fields :-

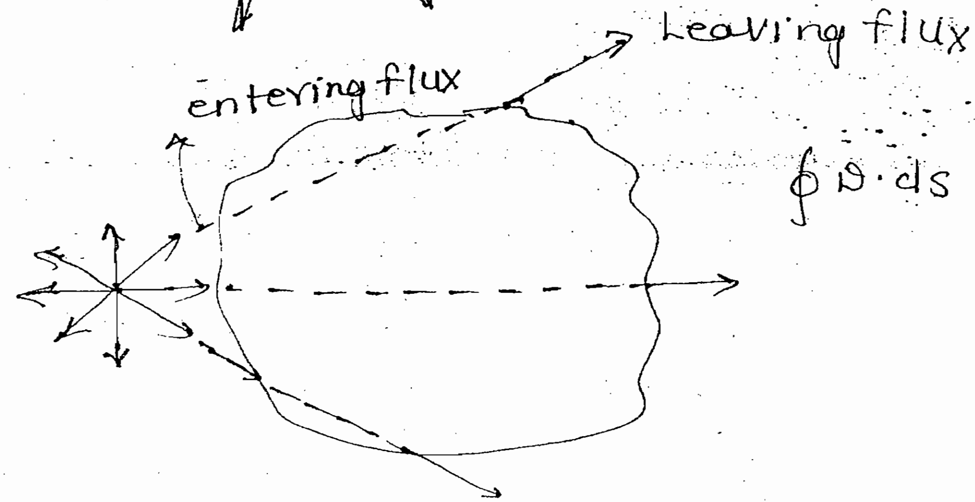
Gauss Law :- total effects

The net electric flux leaving any closed surface is always equal to the charge enclosed in the volume cause



Flux leaving

$$\oint D \cdot ds = Q.$$



$$\oint D \cdot ds = 0$$

12

The word electric flux means the attractive or repulsive force on any +ve test charge placed in the electric field. Hence electric field or electric flux or lines of force physically represent the direction in which a test charge moves away when placed in the field

→ When the surface is closed the total outflow or flux leaving is independent of density and area. i.e.

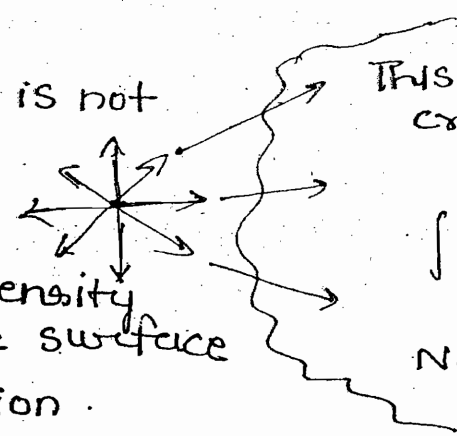
$$\oint D \cdot ds = Q$$

→ If the charge is outside the effects still exists but net effects is zero

$$\therefore \text{entering flux} = \text{leaving flux}$$

$$\oint D \cdot ds = 0$$

→ If the surface is not completely enclosing the flux crossing depends on the density of the flux and the surface area of consideration.



This is crossing flux

$$\int D \cdot ds = \Psi_c$$

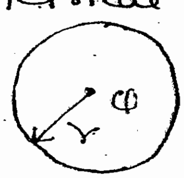
↓

Not Gauss Law

Application of Gauss Law! -

1) Electric field strength of a point charge! -

Consider a concentric symmetrical surface such that D is same everywhere



$$\oint D \cdot ds = Q$$

r = constant

a_r const. directed

$$\Rightarrow \quad D \cdot S = Q$$

$$\Rightarrow \quad D = \frac{Q}{S}$$

$$\Rightarrow \quad \boxed{D = \frac{Q}{4\pi r^2} a_r} \quad C/m^2$$

→ Coulomb had a different measure of field strength i.e. the force measured & he related this with charge measure using ϵ. This is called as intensity

$$\vec{E} = \frac{\vec{F}}{Q_t} = \frac{\vec{D}}{\epsilon} = \frac{Q}{4\pi\epsilon r^2} a_r$$

↓

Intensity (N/C)

$$\Rightarrow \quad \boxed{F = \frac{Q_1 Q_2}{4\pi\epsilon r^2}}$$

This is coulomb's law

Static Magnetic Fields :-

Biot - Savart's Law (Ampere's law for current elements) :-

It is derived from Ampere's law & consider a $d\vec{l}$ length I (DC current) carrying as a basic cause of H field

$I d\vec{l}$ – (Amp-m) is basic cause of H field

$$\vec{H} = \frac{I dl \times a_r}{4\pi r^2}$$

↓

Intensity (Amp/m).

→ $\vec{H}$ direction = I flow direction [cross] × Radial vector from current

→ Lorentz had a different measure of field strength i.e. density measure which is physically force per basic cause

$$\boxed{\vec{B} = \mu\vec{H} = \frac{\mu I dl \times a_r}{4\pi r^2} = \frac{\vec{F}}{\vec{Idl}} = \frac{\text{Newton}}{\text{Amp-m}}}$$

$$\vec{F} = q\,(V \times B)$$

$$\Rightarrow dF = dQ\left(\frac{dl}{dt} \cdot B\cos\theta\right) = I\,dl\,B$$

$$\Rightarrow B = \frac{F}{Idl}$$

Note:-

→ $D \cdot E = \frac{\text{col}}{m^2} \times \frac{\text{Newton}}{\text{col}} = \frac{\text{Newton}}{m^2}$

= Electric pressure at that point

→ $B \cdot H = \frac{\text{Newton}}{\text{Amp-m}} \times \frac{\text{Amp}}{m} = \frac{\text{Newton}}{m^2}$

= Magnetic Pressure.

→ $\frac{\text{Newton} \times m}{m^2 \times m} = \frac{\text{Joules}}{m^3}$ = Energy density

→ $\frac{1}{2} D \cdot E = \frac{1}{2} \epsilon E^2$ = Energy density in electric field

→ $\frac{1}{2} B \cdot H = \frac{1}{2} \mu H^2$ = Energy density in H field