* Differential Calculas *

Limit of a function at a point:-

A function $f(x)$ is said to have a limit l when x tends to a if for a given $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - l| < \epsilon$ whenever $|x - a| < \delta$.

eg: $f(x) = x + 4$

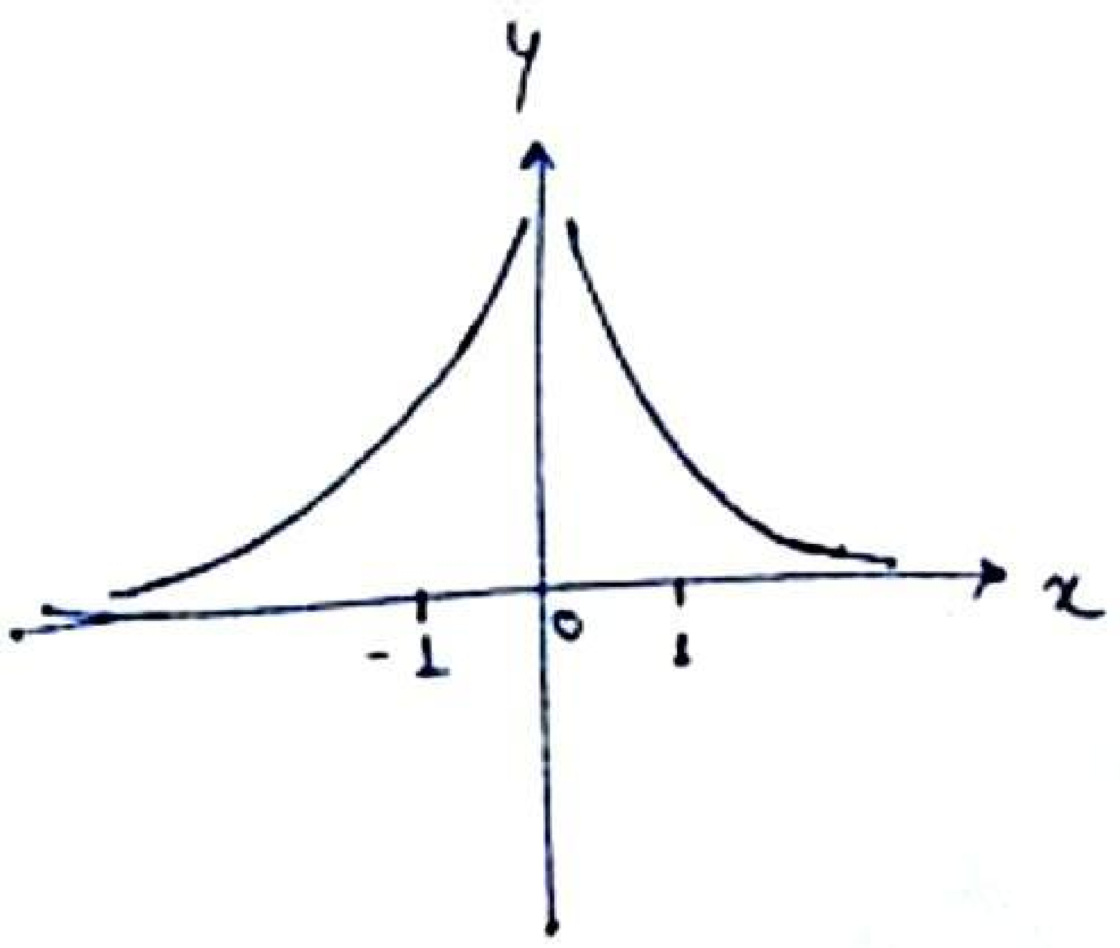
$x = 1.999$ $\quad f(x) = 5.999$

$x = 2.001$ $\quad f(x) = 6.001$

$x \to 2$ $\quad f(x) \to 6$

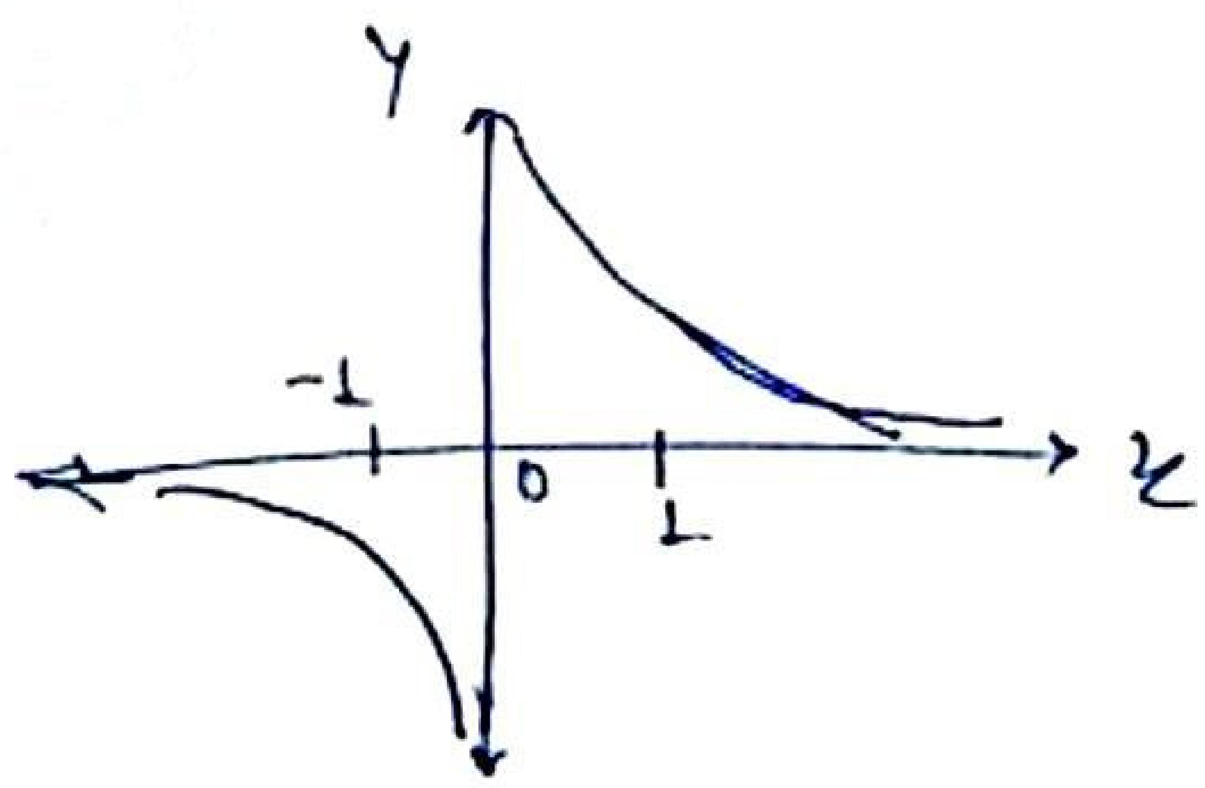
$$\lim_{x \to 2} (x+4) = 6$$

$f(x) = \frac{1}{x^2}$



$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

$f(x) = \frac{1}{x}$



$\lim_{x \to 0} \frac{1}{x}$ Doesn't exist.

$x \to 0^+$ $\quad y \to \infty$

$x \to 0^-$ $\quad y \to -\infty$

So limit doesn't exist.

Some standard limit :-

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to \infty} \frac{\sin x}{x} = 0$$

$$\lim_{x \to 0} \frac{\tan x}{x} = 1$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a \qquad \lim_{x \to \infty} \frac{\sin x}{x} = 0$$

$$\lim_{x \to 0} \frac{1 - \cos mx}{x^2} = \frac{m^2}{2}$$

$$\lim_{x \to 0} \frac{1 - \cos mx}{1 - \cos nx} = \frac{m^2}{n^2}$$

$$\lim_{x \to 0} (1 + x)^{1/x} = e$$

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{\frac{x}{a}} = e$$

Continuity of a function at a point:-

A Function $f(x)$ is said to Continuous at a point $x = a$ if $\underset{x \to a}{Lt} f(x) = f(a)$

i.e $\underset{x \to a^-}{Lt} f(x) = \underset{x \to a^+}{Lt} f(x) = f(a)$

L.H.L. = R.H.L. = $f(a)$.

* If f & g are continuos then
  (i) $f+g$, $f-g$, fg are also continuous
  (ii) f/g is Continuous only when $g(x) \neq 0$

* Every constant function is Continuous.
* Every polynomial is Continuous.
* $\sin x$, $\cos x$ are always Continuous.
* e^x, e^{-x} are always Continuous.

· Derivative of a function at a point:-

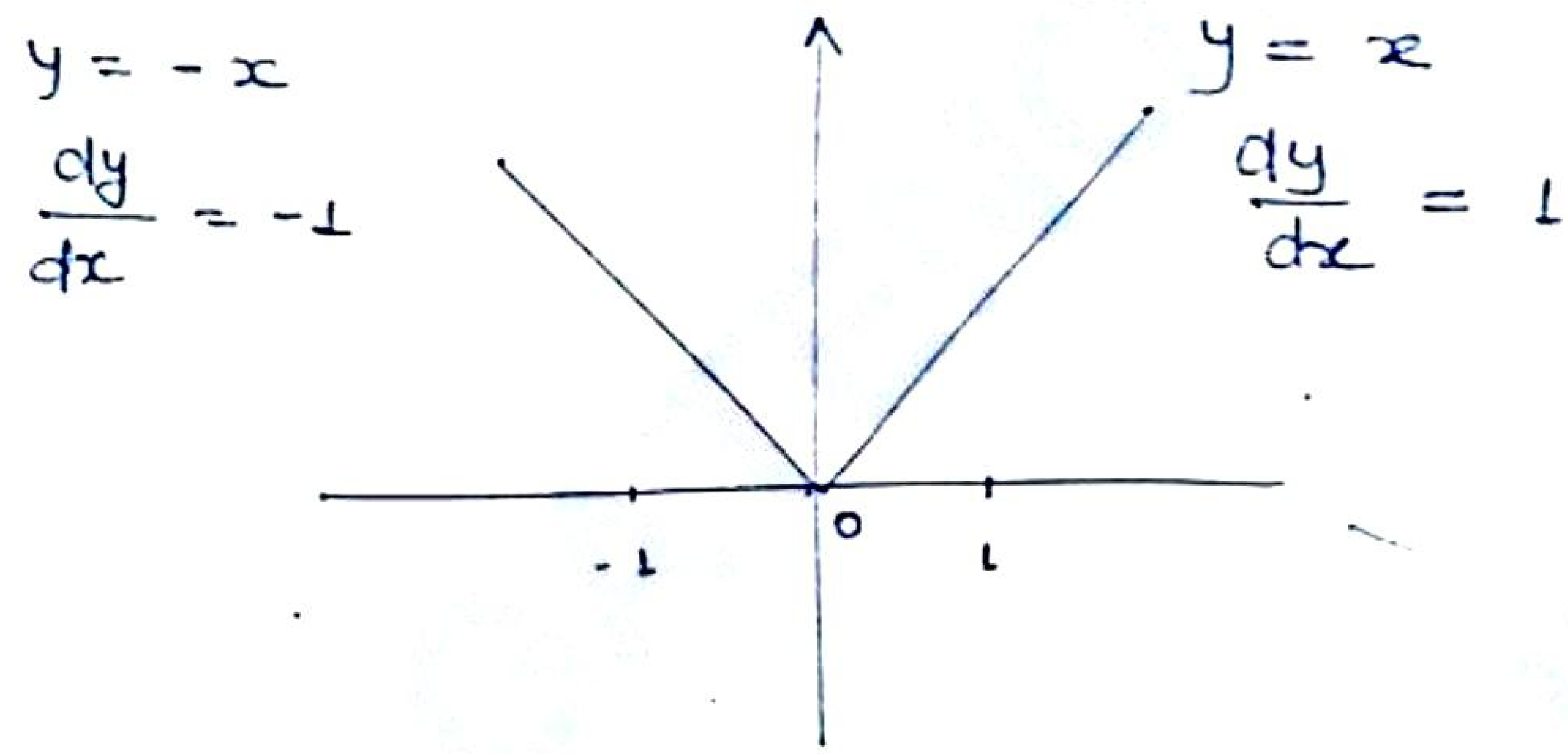
A Function $f(x)$ is said to be derivable at a point $x = a$ if $\underset{x \to a}{Lt} \frac{f(x)-f(a)}{x-a}$ exists finitely.

i.e. $\underset{x \to a^-}{Lt} \frac{f(x)-f(a)}{x-a} = \underset{x \to a^+}{Lt} \frac{f(x)-f(x)}{x-a}$ = finite value

LHD = RHD = finite value.

Note:- Every differential function is continuous but the converse need not true.

e.g. $f(x) = |x|$ is continuous for all values of x but it is not deriable at $x = 0$



At $x = 0$, L.H.D. $= -1$

R.H.D. $= 1$

LHD $\neq$ RHD

Quest.

$$\lim_{x \to 0} \frac{5^x \cdot 2^x - 2^x - 5^x + 1}{x \sin x}$$

$$\lim_{x \to 0} \frac{2^x(5^x - 1) - 1(5^x - 1)}{x \sin x}$$

$$\lim_{x \to 0} \frac{(5^x - 1)}{x} \cdot \frac{(2^x - 1)}{x} \cdot \frac{x}{\sin x}$$

$$\ln 5 \cdot \ln 2 \cdot 1$$

$$= (\ln 5)(\ln 2)$$

Question $\displaystyle \lim_{x \to 0} \frac{4^x - 3^x}{3^x - 2^x}$

$$\lim_{x \to 0} \frac{3^x\left(\left(\frac{4}{3}\right)^x - 1\right)}{3^x\left(1 - \left(\frac{2}{3}\right)^x\right)} = \frac{\left(\frac{4}{3}\right)^x - 1}{1 - \left(\frac{2}{3}\right)^x}$$

$$= \frac{\ln(4/3)}{-\ln(2/3)} = \frac{\ln(4/3)}{(3/2)}$$

*

$$\ln a + \ln b = \ln(ab)$$

$$\ln a - \ln b = \ln(a/b)$$

$$\ln a^b = b \ln a$$

$$\frac{\ln a}{\ln b} = \ln_b a$$

Ques $\displaystyle \lim_{n \to \infty} \frac{2^{n+1} + 3^{n+1}}{2^n + 3^n}$

$$\lim_{n \to \infty} \frac{2^n \cdot 2 + 3^n \cdot 3}{2^n + 3^n} = \frac{\cancel{3^n}\left(\left(\frac{2}{3}\right)^n 2 + 3\right)}{\cancel{3^n}\left(\left(\frac{2}{3}\right)^n + 1\right)}$$

$n \to \infty$

$\left(\frac{2}{3}\right)^n \to 0$

$$= \frac{0 \cdot 2 + 3}{0 + 1} = 3$$

Question:- $\underset{\theta \to \pi/2}{Lt} (1 - 5\cot\theta)^{\tan\theta}$

$$\lim_{\tan\theta \to \tan\frac{\pi}{2}} \left(1 - \frac{5}{\tan\theta}\right)^{\tan\theta}$$

$$\lim_{\tan\theta \to \infty} \left(1 - \frac{5}{\tan\theta}\right)^{\frac{\tan\theta}{(-5)}(-5)}$$

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{x/a} = e$$

$$\lim_{\tan\theta \to \infty} \left(\left(1 - \frac{5}{\tan\theta}\right)^{\frac{\tan\theta}{-5}}\right)^{-5}$$

$$\Rightarrow e^{-5}$$

Ques. $\lim_{x \to 0} (1 - \sin x \cos x)^{\operatorname{cosec} 2x}$

$$\lim_{x \to 0} \left(1 - \frac{\sin 2x}{2}\right)^{\frac{1}{\sin 2x}} \quad \therefore \lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$$

$$\lim_{x \to 0} \left(1 - \frac{\sin 2x}{2}\right)^{\frac{1}{\frac{\sin 2x}{(-2)}}(-2)} = \lim_{x \to 0} \left(\left(1 - \frac{\sin 2x}{2}\right)^{\frac{1}{\frac{\sin 2x}{-2}}}\right)^{-\frac{1}{2}}$$

$$= e^{-1/2}$$

Q.1
WB

$$f(x) = \begin{cases} 2x+1 & \leq 1 \\ ax^2+b & 1 < x < 3 \\ 5x+2a & x \geq 3 \end{cases}$$

L.H.L = R.H.L = f(a)

$3 = a + b$ —(1), $9a + b = 15$ —(2)

from (1) & (2)

$a = 2, \ b = 1$

Q.3

$$f(x) = \begin{cases} x^2+3x+a & x \leq 1 \\ bx+2, & x > 1 \end{cases}$$

L.H.D = R.H.D

$2x + 3 = b$

at $x = 1$ & $b = 5$

L.H.L = RHL

$1 + 3 + a = 5 + 2$

$a = 3$

$a = 3, \ b = 5$

$$f'(x) = \begin{cases} 2x+3 & x \leq 1 \\ b & x > 1 \end{cases}$$

L.H.D = R.H.D.

Q.4

$x = a(\theta - \sin\theta)$ $\qquad$ $y = a(1 - \sin\theta)$

$\frac{dx}{d\theta} = a(1 - \cos\theta)$ $\qquad$ $\frac{dy}{d\theta} = a\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\theta/2 \, \cos\theta/2}{2\sin^2\theta/2} = \cot(\theta/2)$$

$$\frac{dy}{dx} = \cot \theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\cot \theta/2\right)$$

$$= \frac{d}{d\theta}\left(\cot \theta/2\right) \cdot \frac{d\theta}{dx}$$

$$= -\operatorname{cosec}^2(\theta/2)\,\frac{1}{2} \cdot \frac{1}{a(1-\cos\theta)}$$

$$= -\operatorname{cosec}^2(\theta/2) \cdot \frac{1}{2}\,\frac{1}{a \cdot 2\sin^2\theta/2}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{4a}\operatorname{cosec}^4(\theta/2)$$

Q.6 W.B:

$$\lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3} = b$$

$$\lim_{x\to 0} \frac{2\cos 2x + a\cos x}{3x^2} = b$$

Given that b is finite it is possible only when numerator is also becomes zero.

i.e. $2\cos 0 + a\cos 0 = 0$

$2 + a = 0 \Rightarrow a = -2$

$$\lim_{x\to 0} \frac{-4\sin 2x + 2\sin x}{6x} = b$$

$$\underset{x \to 0}{Lt} \frac{-8\cos 2x + 2\cos x}{6} = b$$

$$\frac{-8+2}{6} = b$$

$$b = -1$$

Q.8 $$\underset{n \to \infty}{Lt} \left(\frac{n}{n^2} + \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} \cdots \frac{n}{n+(n-1)^2} \right)$$

$n = 1 \cdot 1, \frac{1}{2}, \frac{1}{3}$ -

$n = 100$ 100, 0.01, 0.0099

n increase
Gap decreasing
So series is continuous

$$\underset{n \to \infty}{Lt} \int_{x=0}^{n-1} \frac{n}{n^2+x^2} dx$$

$$\underset{n \to \infty}{Lt} n \cdot \frac{1}{n} \tan^{-1}\left(\frac{x}{n}\right) \Big|_0^{n-1}$$

$$\underset{n \to \infty}{Lt} \tan^{-1}\left(\frac{n-1}{n}\right) = \underset{n \to \infty}{Lt} \tan^{-1}\left(1 - \frac{1}{n}\right)$$

$$= \underset{n \to \infty}{Lt} \tan^{-1} 1$$

$$= \frac{\pi}{4}$$

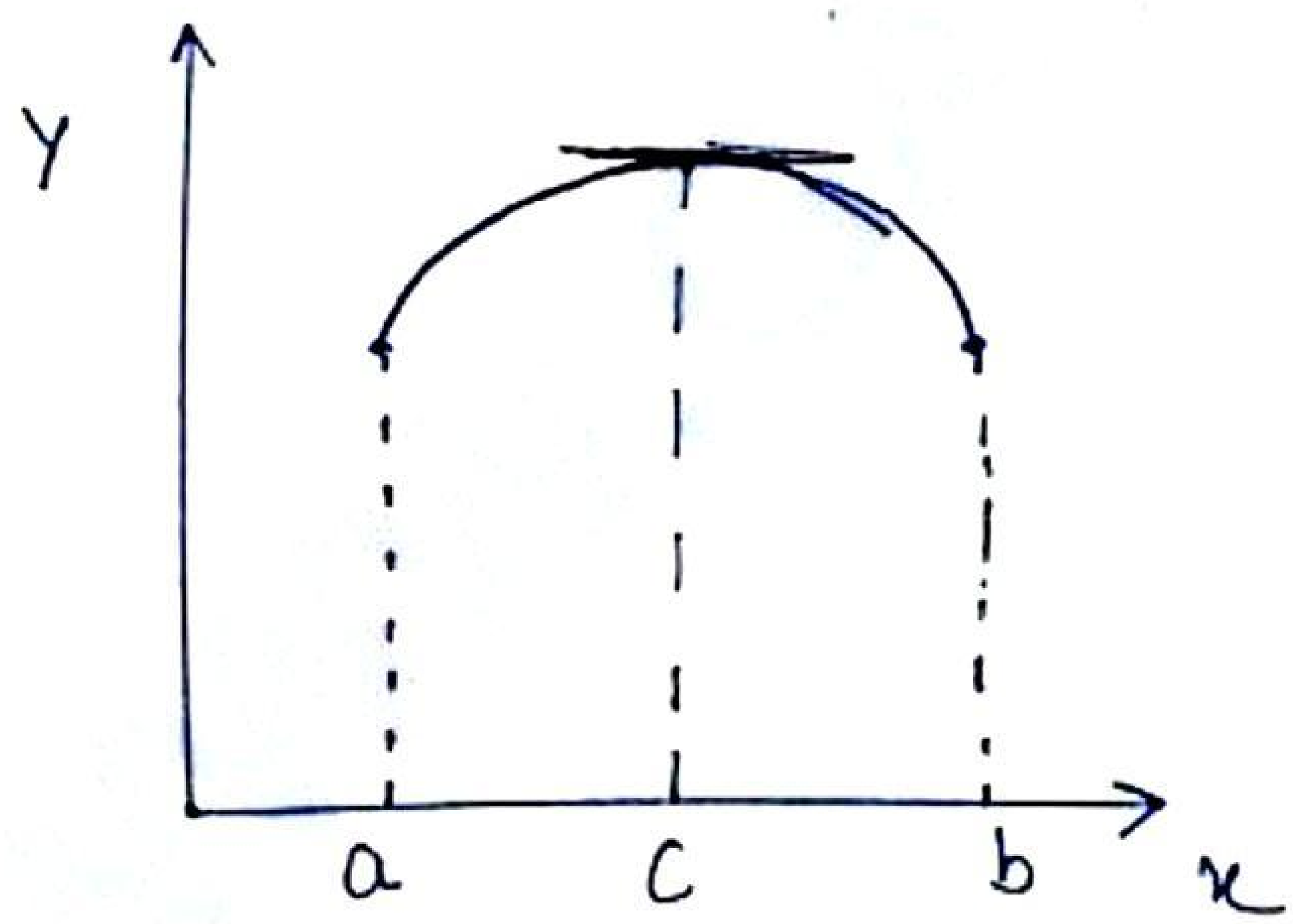
Rolles theorem:-

If $f(x)$ is a real valued function defined on $[a, b]$ and

(i) $f(x)$ is continuous on $[a, b]$

(ii) $f(x)$ is derivable on (a, b)

(iii) $f(a) = f(b)$ then there exists atleast on point $c \in (a,b)$ such that $f'(c) = 0$.



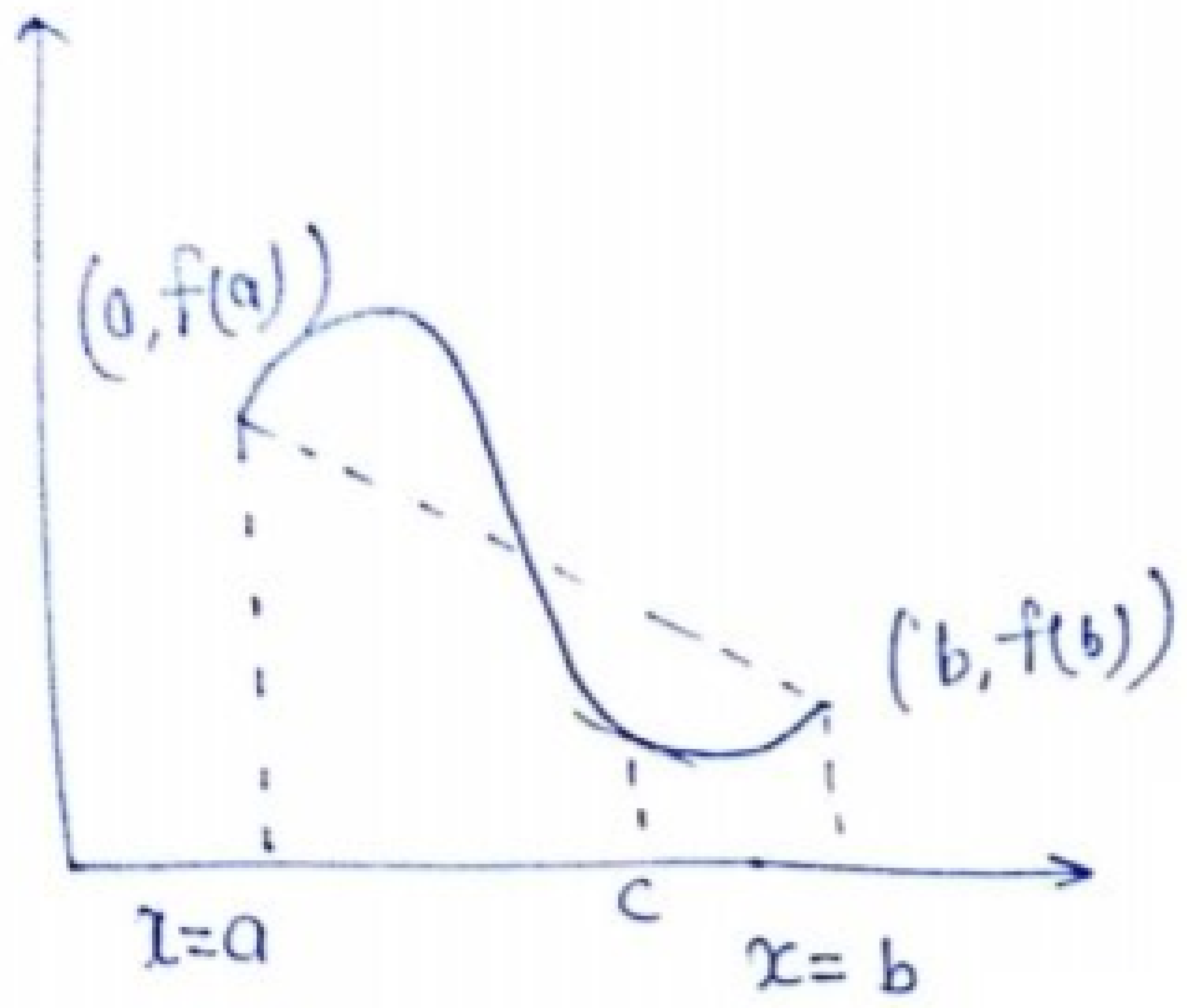
Lagrange's Mean Value theorem:-

If $f(x)$ is a real valued function defined on $[a,b]$ and (i) $f(x)$ is continuous on $[a, b]$

(ii) $f(x)$ is derivable on (a, b)

then there exists atleast on point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Cauchys mean value theorem:-

If $f(x)$, $g(x)$ are two real valued functions defined on $[a, b]$ and

(i) $f(x)$, $g(x)$ are continuous on $[a, b]$

(ii) $f(x)$, $g(x)$ are derivalble on (a, b)

(iii) $g'(x) \neq 0 \;\forall\; x\,(a, b)$ then there exists atleast on point $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Note:- Geometrical interpretation of Cauchy's mean value theorem & langrang mean value theorem are same.

Taylors Series $\boxed{\underset{\text{at } x=a}{f(x)} = \sum_{n=0}^{\infty} \frac{(x-a)^n f^n(a)}{n!}}$

If $f(x)$ is continuosly differentiable at a point $x = a$ then $f(x)$ can be expressed as

$$f(x) = f(a) + (x-a) f'(a) + (x-a)^2 \frac{f''(a)}{2!} + (x-a)^3 \frac{f'''(a)}{3!} + -$$

if $a = 0$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - -$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - - -$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + -$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + - -$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + - - .$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + ..$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

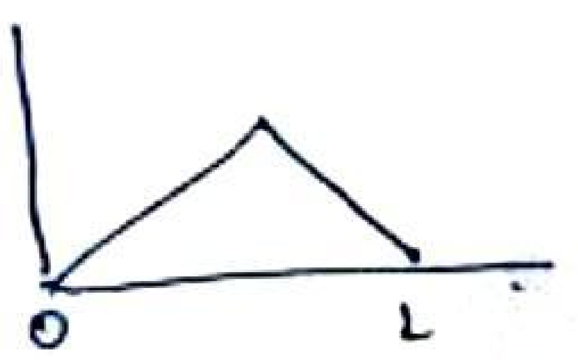
$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

Q.9 (a) $f(x) = \tan(\pi x)$

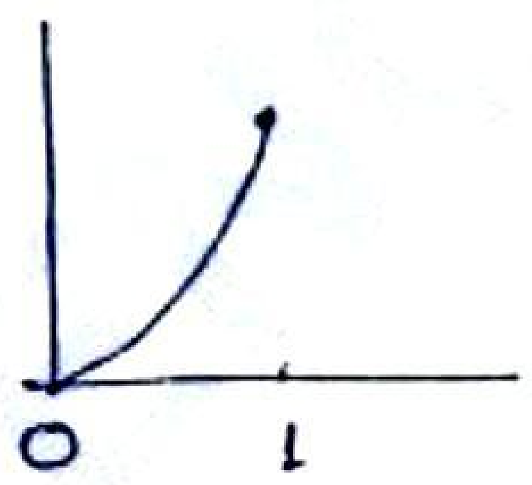
$x = \frac{1}{2}$, $f(\frac{1}{2}) = \tan\frac{\pi}{2} = \infty$

$f(x)$ is not cont. at $x = \frac{1}{2}$

(b) $f(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 1-x, & 0 \le x \le \frac{1}{2} \end{cases}$



(c)



$f(x) = x^2$ in $[0,1]$

$f(0) = 0$

$f(1) = 1$

$f(0) \neq f(1)$

$x \in [0,1]$

(d) $f(x) = \sqrt{x(1-x)}$

$f(0) = 0$

$f(1) = 0$

$f'(x) = \frac{1(1-2x)}{2\sqrt{x(1-x)}}$

is finite

Q.10 $f(x) = (1+x)\log_e(1+x)$

$f(0) = (1+0)\log_e(1)$

$f(0) = 0$

$f(1) = 2\log 2$

$f'(x) = \frac{(1+x)}{(1+x)} + \log_e(1+x)$

$f'(x) = 1 + \log(1+x)$

is finite in $x \in [0,1]$

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\frac{(1+c)}{(1+c)} + \log(1+c) = \frac{2\log 2 - 0}{1-0}$$

base = e

$$1 + \log(1+c) = 2\log 2$$

$$\log(1+c) = \log 4 - \log e$$

$$\log(1+c) = \log(4/e)$$

$$1+c = 4/e \qquad c = \frac{4}{e} - 1 \qquad \boxed{c = \frac{4-e}{e}}$$

Ans

Q.5

$$\lim_{x \to 0} e^x (\cos x)^{\frac{1}{\sin^2 x}}$$

$$\lim_{x \to 0} e^x \left(1 - 2\sin^2 \frac{x}{2}\right)^{\frac{1}{\sin^2 x}}$$

$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$

$\cos x = 1 - 2\sin^2 \frac{x}{2}$

$$\lim_{x \to 0} e^x \left(1 - 2\sin^2 \frac{x}{2}\right)^{\frac{1}{4\sin^2 \frac{x}{2} \cos^2 \frac{x}{2}}}$$

$\sin^2 x = 4\sin^2 \frac{x}{2} \cos^2 \frac{x}{2}$

$$\lim_{x \to 0} e^x \left(1 - 2\underbrace{\sin^2 \frac{x}{2}}\right)^{\left(\frac{1}{(-2\sin^2 \frac{x}{2})(-2)\underbrace{\cos^2 \frac{x}{2}}}\right)}$$

$$\cancel{\lim} \quad 1.(e)^{-\frac{1}{2}}$$

$$\lim_{x \to 0} e^x \left\{ \underbrace{\left(1 - 2\sin^2 \frac{x}{2}\right)^{\left(-\frac{1}{2\sin^2 \frac{x}{2}}\right)}} \right\}^{\left\{\frac{1}{(-2)\underbrace{\cos^2 \frac{x}{2}}}\right\}}$$

$$\cancel{\lim_{x \to 0}} = 1 \cdot (e)^{(-\frac{1}{2})}$$

$$= e^{-1/2}$$

(11)

$x = a\cos^3\theta \qquad \frac{dx}{d\theta} = 3a\cos^2\theta.(-\sin\theta)$

$y = a\sin^3\theta \qquad \frac{dy}{d\theta} = 3a\sin^2\theta\,(+\cos\theta)$

$$\frac{dy}{dx} = \frac{3a\sin^2\theta\ \cancel{\cos\theta}}{-3a\cos^2\theta\ \cancel{\sin\theta}} = \underbrace{-\tan\theta}$$

$$m. = \frac{a-0}{0-a} = -1$$

$\Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$x = \frac{a}{2\sqrt{2}}$

$y = \frac{a}{2\sqrt{2}}$

Q.12

$$\cancel{\sin x = x - \frac{x^3}{3!}} +$$

$$\sin x = \sin\left(\frac{\pi}{6}\right) + \left(x - \frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) + \left(x - \frac{\pi}{6}\right)^2\left(\frac{-\sin(\pi/6)}{2!}\right)$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \left(\frac{1}{2}\right)\left(x - \frac{\pi}{6}\right)^2\left(\frac{1}{2!}\right)$$

~~Q.13~~

Q.11 By CMVT

$$\frac{f'(\theta)}{g'(\theta)} = \frac{f(\pi/2) - f(0)}{g(\pi/2) - g(0)}$$

$$= \frac{a.\,3\cos^2\theta\,(-\sin\theta)}{a.3\sin^2\theta\,(\cos\theta)} = \frac{0-a}{a-0}$$

$$-\frac{\cos\theta}{\sin\theta} = -1$$

$$\tan\theta = 1 \qquad \theta = \pi/4$$

$$x = a\cos^3\pi/6 \qquad y = a\sin^3\pi/6$$

$$\left(\frac{a}{2\sqrt{2}},\ \frac{a}{2\sqrt{2}}\right)$$

Q.13 $f(x) = \dfrac{\sin x}{x - \pi}$

$$f(x) = \sin x = 0 + (x-\pi)(-1) + (x-\pi)^2 \frac{(-0)}{2!}$$

$$= \frac{0}{(x-\pi)} - 1 + \frac{(x-\pi)^2}{3!}(-1) + (x-\pi)^3 \left\{ \frac{+\cos(\pi}{3!}\right.$$

$$= -1 + \frac{(x-\pi)^2}{3!}$$

or $f(x) = \dfrac{\sin x}{x-\pi}$ at $x = \pi$

$x - \pi = 0$

let $x - \pi = t$

$x = \pi + t$

$$f(x) = \frac{\sin(\pi + t)}{t} = -\frac{\sin t}{t} \quad \text{at } t = 0$$

$$= \frac{-t + \frac{t^3}{3!} - \frac{t^5}{5!} + \cdots}{t}$$

$$f(x) = -1 + \frac{t^2}{3!} - \frac{t^4}{5!}$$

$$f(x) = -1 + \frac{(x-\pi)^2}{3!} - \frac{(x-\pi)^4}{5!}$$

Q.14

$$f(x) = \log\left(\frac{1+x}{1-x}\right)$$

$$f(x) = \log(1+x) - \log(1-x)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \cdots$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \cdots$$

$$f(x) = 2x + 2\frac{x^3}{2} + 2\frac{x^5}{5}$$

$$f(x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} \cdots\right)$$

Q.15 $\quad e^x$ at $x = 2$

$$(x-2) = 0$$

$$x - 2 = t$$

$$x = 2 + t$$

$$f(t) = e^{2+t} = e^2 \cdot e^t$$

$$f(x) = e^2\left\{1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} \cdots\right\}$$

$$\text{Coeff.} = e^2 \frac{(x-2)^4}{4!}$$

$$= \frac{e^2}{4!}$$

Q.16

$$y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \cdots}}}$$

$$y = \sqrt{\tan x + y}$$

$$y^2 = \tan x + y$$

$\cancel{\frac{2dy}{dx}}$ $$2y \cdot \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\sec^2 x}{(2y-1)}$$

Question

The coefficient of $\left(x - \frac{\pi}{4}\right)^3$ in T.S.E. of

$f(x) = 3\sin x \cos\left(x + \frac{\pi}{4}\right)$ about $\frac{\pi}{4}$ is

Sol

Coeff. of $\left(x - \frac{\pi}{4}\right)^3 = \dfrac{f'''(\pi/4)}{3!}$

$$f(x) = 3\sin x \cos(x + \pi/4)$$

$$= 3\sin x \left(\cos x \cos\frac{\pi}{4} - \sin x \sin\frac{\pi}{4}\right)$$

$$f(x) = \frac{3}{\sqrt{2}}\sin x \cos x - \frac{3}{\sqrt{2}}\sin^2 x$$

$$f(x) = \frac{3}{2\sqrt{2}}\sin 2x - \frac{3}{\sqrt{2}}\sin^2 x$$

$$f(x) = \frac{3}{2\sqrt{2}}\sin 2x - \frac{3}{2\sqrt{3}} + \frac{3}{2\sqrt{2}}\cos 2x$$

$$f'(x) = \frac{3}{2\sqrt{2}}(2\cos 2x) + \frac{3}{2\sqrt{2}}(-2\sin 2x)$$

$$f''(x) = \frac{3}{2\sqrt{2}}(-4\sin 2x) + \frac{3}{2\sqrt{2}}(-4\cos 2x)$$

$$f'''(x) = \frac{3}{2\sqrt{2}}(-8\cos 2x) + \frac{3}{2\sqrt{2}}(8\sin 2x)$$

$$f'''\left(\frac{\pi}{4}\right) = \frac{3}{2\sqrt{2}}\left(-8\cos\frac{\pi}{2}\right) + \frac{3}{2\sqrt{2}}\left(8\sin\frac{\pi}{2}\right)$$

$$f'''\left(\frac{\pi}{4}\right) = \frac{12}{\sqrt{2}}$$

$$\text{Coeff.} = \frac{f'''\left(\frac{\pi}{4}\right)}{3!} = \frac{12}{\sqrt{2}\times 6} = \sqrt{2} \quad \underline{\text{Ans}}$$

Maxima & Minima

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$$\frac{dy}{dx} = 0, \quad \frac{d^2y}{dx^2} > 0$$

point of minima

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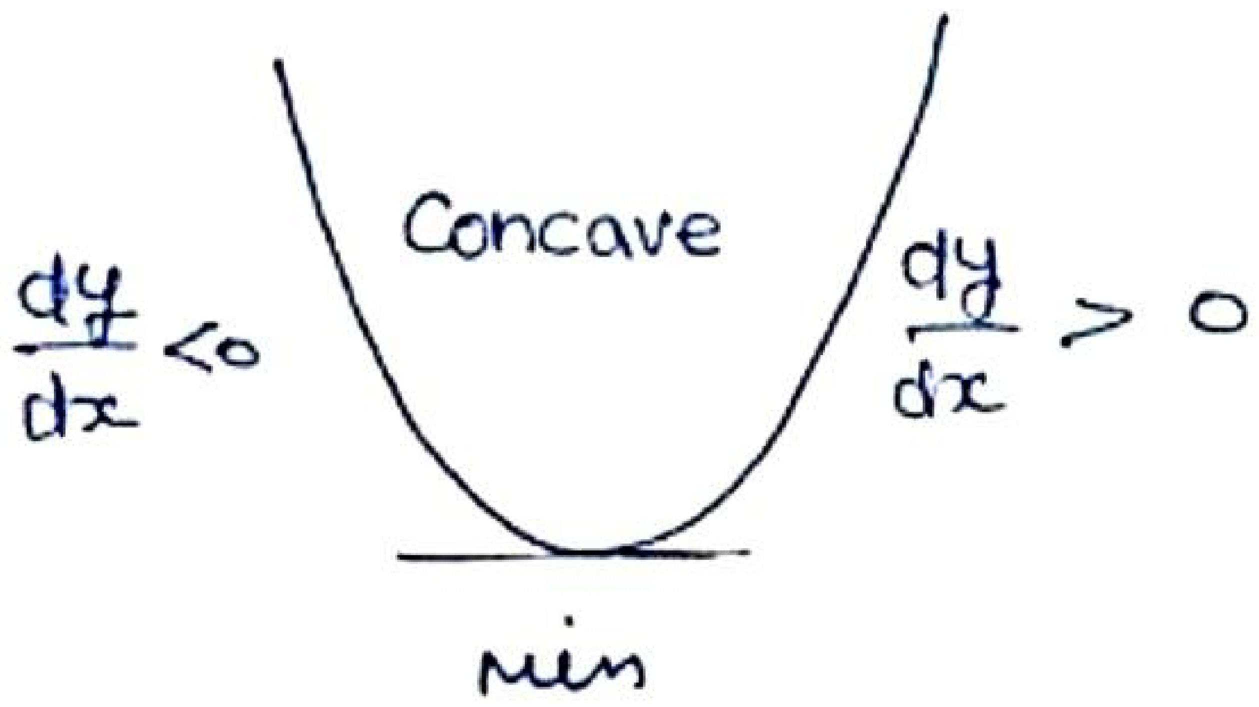
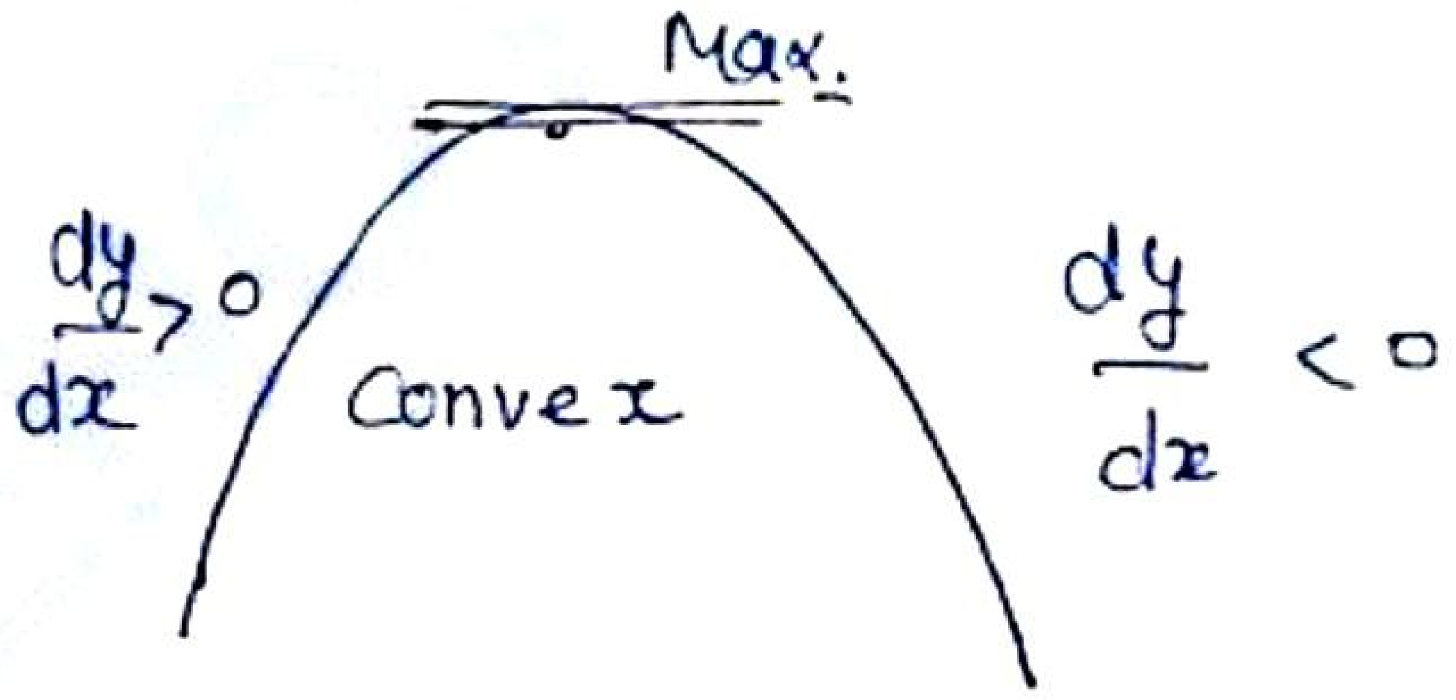
$$\frac{dy}{dx} = 0, \quad \frac{d^2y}{dx^2} < 0$$

point of maxima





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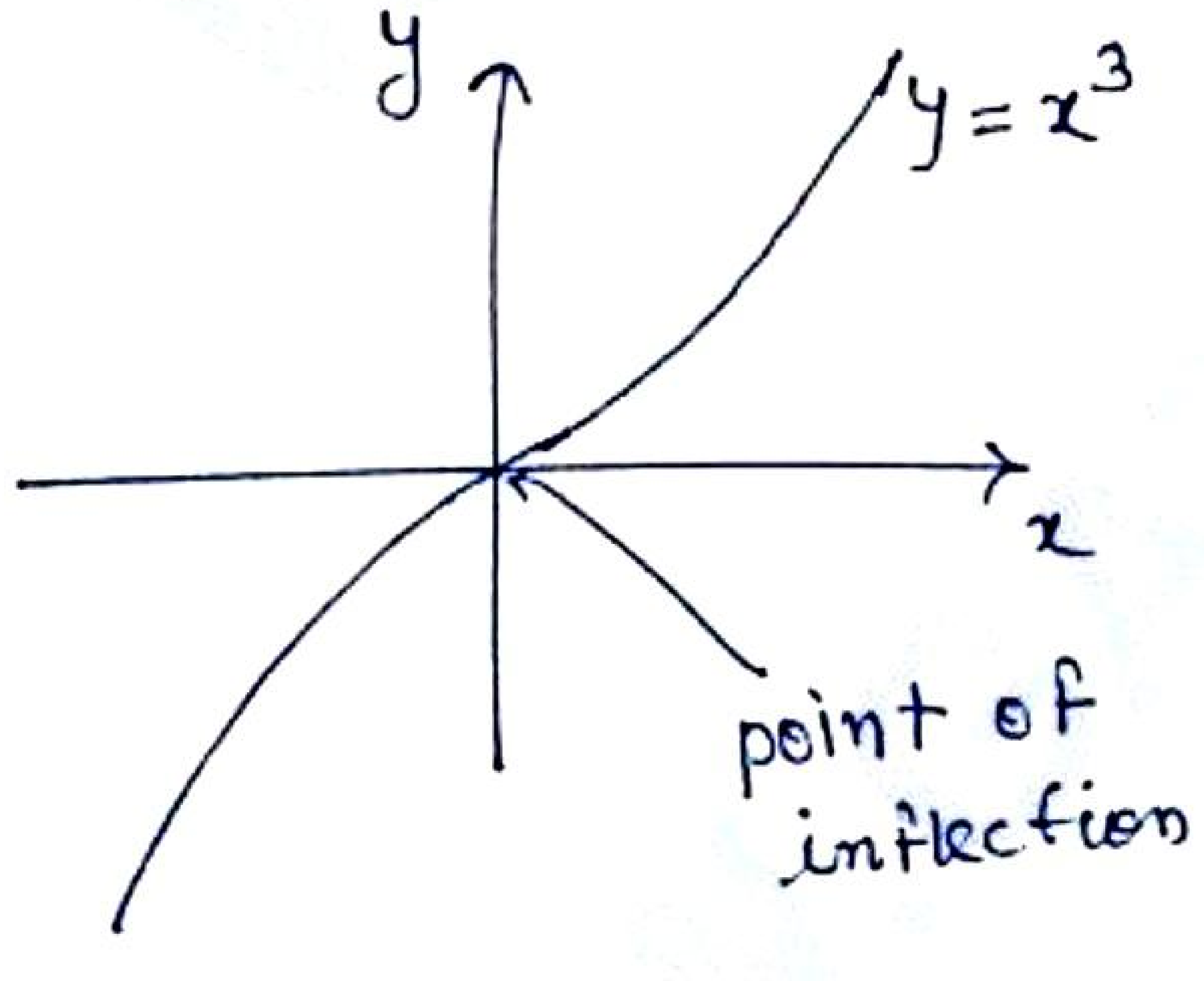
$$\frac{dy}{dx} = 0, \quad \frac{d^2y}{dx^2} = 0, \quad \frac{d^3y}{dx^3} \neq 0$$

point of inflection



* where curve changes from concave to convex

* The Absolute minima (or) Global minima of $f(x)$ in $[a, b]$ is defined as $\min\{f(a), f(b), \text{all its relative minima in } [a, b]\}$

* The Absolute maxima (or) Global maxima of $f(x)$ in $[a, b]$ is defined as $\text{Max.}\{f(a), f(b)$ all its relative maxima in $[a, b]\}$

* The maximum value of $a\cos x + b\sin x$ is given by $\sqrt{a^2+b^2}$.

* The minimum value of $a\cos x + b\sin x$ is given by $-\sqrt{a^2+b^2}$.

Question The function $f(x) = x^2 + \frac{250}{x}$ at $x = 5$ attains

$$f(x) = x^2 + \frac{250}{x}$$

$$f'(x) = 2x - \frac{250}{x^2} \quad \text{at } x=5,\ f'(5) = 10 - \frac{250}{25} = 0$$

$$f''(x) = 2 + \frac{500}{x^3}$$

$$\text{at } x = 5,\ f''(5) = 2 + \frac{500}{125} = 6 > 0$$

Relative minima.

Question The max. Value of $f(x) = \frac{e^{\sin x}}{e^{\cos x}}$, $x \in R$ is __

$$f(x) = e^{\sin x - \cos x}$$

$$\text{max. value } f(x) = e^{\sqrt{1^2+(-1)^2}} = e^2.$$

Question :-

The fn^n $f(x) = x^2 e^{-x}, x > 0$ attains maxima at $x = ?$

$$f(x) = x^2 e^{-x}$$

$$f'(x) = x^2(-e^{-x}) + e^{-x}(2x)$$

$$f'(x) = e^{-x}(-x^2 + 2x)$$

$$f'(x) = 0$$

$$-x^2 + 2x = 0$$

$$x = 0, \ x = 2$$

$$f''(x) = e^{-x}(-2x+2) + (-x^2+2x)(-e^{-x})$$

$$f''(x) = e^{-x}(-2x + 2 + x^2 - 2x)$$

$$= e^{-x}(x^2 - 4x + 2)$$

$f''(0) = 2 > 0$ min

$f''(2) = \frac{-2}{e^2} < 0$ Max.

Question $x^3 - 9x^2 + 24x + 5 = 0$ maximum Value?

$$f(x) = x^3 - 9x^2 + 24x + 5$$

$$f'(x) = 3x^2 - 18x + 24$$

$$f'(x) = 0 \Rightarrow 3(x^2 - 6x + 8) = 0$$

$$x = 2, 4$$

$$f''(x) = 6x - 18$$

$f''(2) = 12 - 18 < 0$ R.Min

$f''(4) = 24 - 18 > 0$ R.max.

max. Value of $f(x) = f(2)$

$= 8 - 36 + 48 + 5$

$= 25$ Ans

For an interval $[1, 6]$

max. value of $f(x)$ in $[1,6] = \text{Max}\{f(1), f(6), f(2)\}$

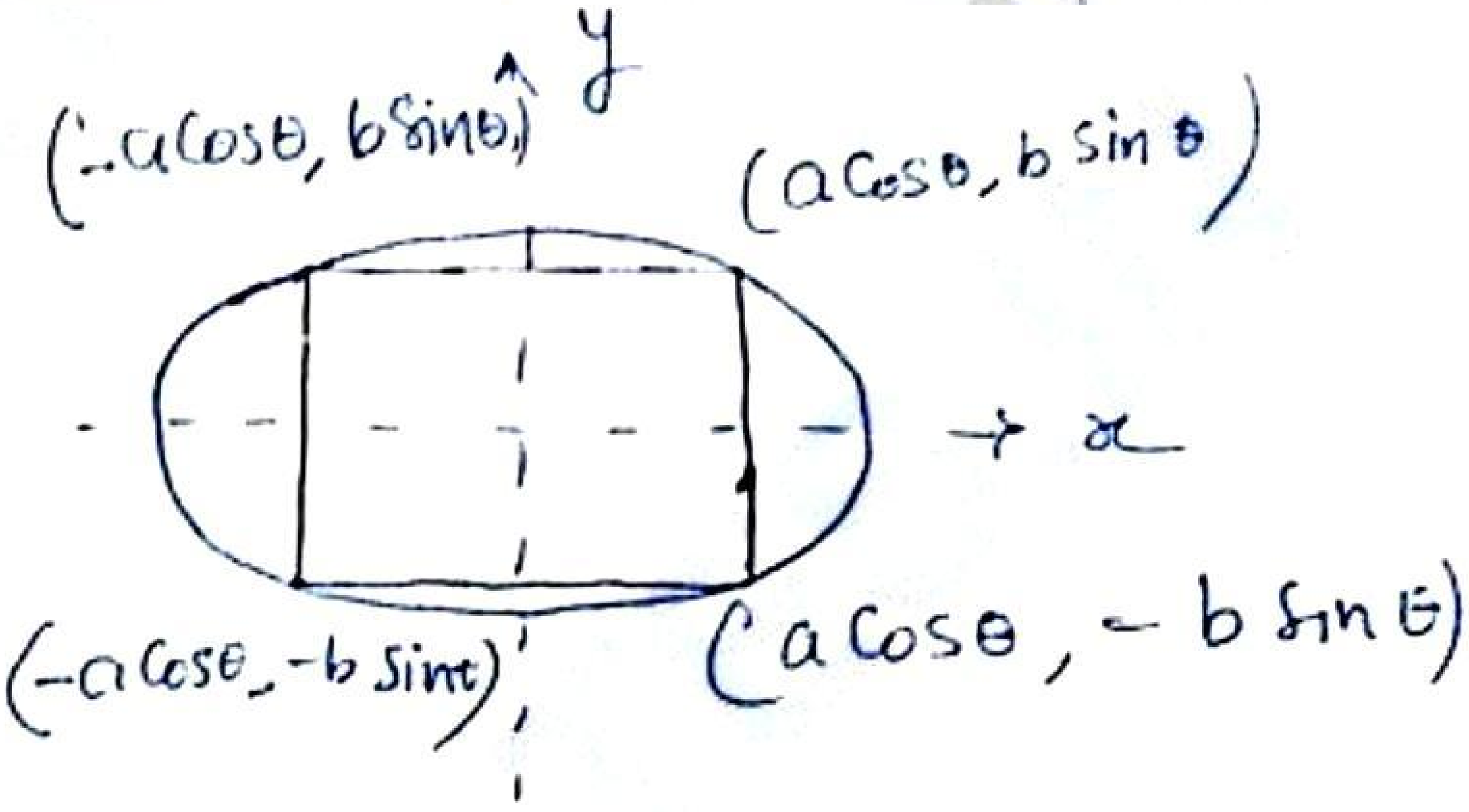
$= \text{Max}\{21, 41, 25\}$

got max. value in $[1,6] = \underline{41}$

Question: The maximum area of the rectangle whose ~~w~~ vertices ~~ha~~ lies on the ~~s~~ ellips. $x^2+4y^2=1$ is ___?

Sol^n

$x^2+4y^2=1$



$$\frac{x^2}{(1)^2}+\frac{y^2}{(1/2)^2}=1$$

$(a\cos\theta, b\sin\theta)$ $\quad a=1, \ b=\frac{1}{2}$

$A = l \cdot b$

$A = 2a\cos\theta \cdot 2b\sin\theta$

$A = 2ab\sin 2\theta$

are is max when $\sin 2\theta = 1$

$A = 2ab = 2\times 1\times \frac{1}{2}$

$A = 1$

Function of Two Variable $u = f(x,y)$

(1) Find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$

(2) Solve the eq^{n}s

$\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$ to Get stationary point

(3) Find the value of

$r = \frac{\partial^2 u}{\partial x^2}$, $s = \frac{\partial^2 u}{\partial x \partial y}$, $t = \frac{\partial^2 u}{\partial y^2}$

at each stationary point.

(i) If $rt - s^2 > 0$ & $r > 0$ at a stationary point then the point is called point of Relative minima.

(ii) If $rt - s^2 > 0$ & $r < 0$ at a stationary point then the point is called point of Relative maxima.

iii) If $rt - s^2 < 0$ at a stationary point then the point is called point of inflection.

(iv) If $rt - s^2 = 0$ at a stationary point then no conclusion can be drawn.

Q.24 $f(x,y) = 2x^4 + y^2 - x^2 - 2y$

$\frac{\partial f}{\partial x} = 8x^3 - 2x$; $\frac{\partial f}{\partial y} = 2y - 2$

$8x^3 - 2x = 0$ $\quad 2y - 2 = 0$

$2x(4x^2 - 1) = 0$ $\quad y = 1$

$x = 0, \pm \frac{1}{2}$

station points $(0,1), (\frac{1}{2},1), (-\frac{1}{2},1)$

$r = \frac{\partial^2 f}{\partial x^2} = 24x^2 - 2$; $s = \frac{\partial^2 f}{\partial x \partial y} = 0$; $t = \frac{\partial^2 f}{\partial y^2} = 2$

at $(0,1)$ $\quad r = -2, \; s = 0, \; t = 2$ $\quad \underline{rt - s^2 < 0}$ $\quad \underline{r < 0}$

at $(\frac{1}{2},1)$ $\quad r = \frac{24}{2} - 2$ $\quad \underline{rt - s^2 > 0}$ $\quad \underline{r > 0}$

$r = 10$; $s = 0$; $t = 2$

at $(-\frac{1}{2},1)$ $\quad r = 10$; $s = 0$; $t = 2$ $\quad \underline{rt - s^2 > 0}$, $\underline{r > 0}$

maxima = × $\quad$ Point of inflection = $(0,1)$

minima = $(\pm\frac{1}{2}, 1)$ $\quad$ saddle point

Ans $(\frac{1}{2}, 1)$. $\underline{(b)}$

Q.25 $f(x,y) = 4x^2 + 6y^2 - 8x - 4y + 8$

$\frac{\partial f}{\partial x} = 8x - 8$ $\qquad$ $\frac{\partial f}{\partial y} = 12y - 4$

$r = \frac{\partial^2 f}{\partial x^2} = 8$, $r = \frac{\partial^2 f}{\partial x \partial y} = 0$, $t = \frac{\partial^2 f}{\partial y^2} = 12$

$\frac{\partial f}{\partial x} = 0$ $\qquad$ $\frac{\partial f}{\partial y} = 12y - 4$

$8x - 8 = 0$ $\qquad$ $12y - 4 = 0$

$x = 1$ $\qquad$ $y = 1/3$

$r = 8$, $s = 0$ $t = 12$

$$\left.\begin{array}{l} rt - s^2 = 96 > 0 \\ r = 8 > 0 \end{array}\right\} \text{R. mimima.}$$

Mini value at $(1, \frac{1}{3})$, $f(1, \frac{1}{3}) = \underline{\underline{10/3}}$

Integration by Parts :- ILATE

$$\int u\,dv = uv - \int v\,du$$

→ Integers
→ logarithm
→ Algebra
→ Trignometry
→ Exponential.

eg. $\int \ln x\,dx = ?$

$u = \ln x$, $\quad dv = dx$

$v = \int dv$

$du = \frac{1}{x}dx$

$= \int dx$

$= x$

$$\int \ln x\,dx = x\ln x - \int x \frac{1}{x}\,dx$$

$$\int \ln x\,dx = x\ln x - x + c$$

eg $\int_1^e \sqrt{x}\ln x\,dx$

$dV = \sqrt{x}\,dx$

$u = \ln x$

$V = \int dv = \int \sqrt{x}\,dx$

$du = \frac{1}{x}dx$

$V = \frac{x^{3/2}}{3/2}$

$$\int_1^e \sqrt{x}\ln x = \left(\ln x \cdot \frac{x^{3/2}}{3/2}\right)\Big|_1^e - \int_1^e \frac{x^{3/2}}{3/2}\cdot\frac{1}{x}\,dx$$

$$= \left(\frac{2}{3}e^{3/2} - 0\right) - \frac{2}{3}\int_1^e x^{1/2}\,dx$$

$$= \frac{2}{3} e^{3/2} - \frac{2}{3}\left(\frac{x^{3/2}}{3/2}\right)\Big|_1^e$$

$$= 2/_3\, e^{3/2} - 4/_9\left(e^{3/2} - 1\right)$$

$$\int_1^e \sqrt{x}\ln x \, dx = 2/_9\, e^{3/2} + 4/_9 \quad \text{Ans}$$

trick

$$\int t^2 \sin t \, dt = -t^2\cos t + 2t\sin t + 2\cos t + c$$

for AT AE

Assign sign after derivative.

D → $+t^2$ $-2t$ $+2$ -0

I → $\sin t$ $-\cos t$ $-\sin t$ $\cos t$

* $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ even function} \\ 0 & \text{if } f(x) \text{ is odd function} \end{cases}$$

* $$\int_0^a f(x)\,dx = \int_0^a f(x-a)\,dx$$

* $$\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \frac{a}{2}$$

Question $\int_0^{\pi/2} \ln(\cot x)\,dx = ?$

$$I = \int_0^{\pi/2} \ln(\cot x)\,dx \quad \text{---} (1)$$

$$\int_0^{\pi/2} \ln\left(\cot\left(\frac{\pi}{2}-x\right)\right) = \int_0^{\pi/2} \ln(\tan x)\,dx \quad \text{---} (2)$$

$$I + I = \int_0^{\pi/2} \ln(\cot x)\,dx + \int_0^{\pi/2} \ln(\tan x)\,dx$$

$$2I = \int_0^{\pi/2} \ln(1)\,dx = 0$$

$$I = 0$$

Question $\int_0^{\pi/2} (a^2\cos^2 x + b^2\sin^2 x)\,dx$ $\quad (1)$

$$\int_0^{\pi/2} a^2\cos^2\left(\frac{\pi}{2}-x\right) + b^2\sin^2\left(\frac{\pi}{2}-x\right)\,dx = \int_0^{\pi/2} a^2\sin^2 x + b^2\cos^2 x\,dx$$

$$I + I = \int_0^{\pi/2} (a^2+b^2)\,dx$$

$$2I = \frac{\pi}{2}(a^2+b^2)$$

$$I = \frac{\pi}{4}(a^2+b^2)$$

Question $\int_0^{\pi/4} \log(1+\tan x)\,dx$

$$I = \int_0^{\pi/4} \log\left(1+\tan\left(\tfrac{\pi}{4}-x\right)\right)dx$$

$$I = \int_0^{\pi/4} \ln\left(1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right)$$

$$I = \int_0^{\pi/4} \ln\left(1+\frac{1-\tan x}{1+\tan x}\right)$$

$$I = \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan x}\right)dx$$

$$I = \int_0^{\pi/4} \ln 2\,dx - \int_0^{\pi/4} \ln(1+\tan x)\,dx$$

$$I = \frac{\pi}{4}\ln 2 - I$$

$$2I = \frac{\pi}{4}\ln 2 \Rightarrow I = \frac{\pi}{8}\ln 2$$

Question $\int_0^{\pi/2} \frac{1}{1+\sqrt{\cot x}}\,dx$

$$\int_0^{\pi/2} \frac{1}{1+\sqrt{\frac{\cos x}{\sin x}}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\sin\left(\frac{\pi}{2} - x\right)}} = \frac{\pi/2}{2} = \frac{\pi}{4}$$

<u>Property</u>

$$\ast \quad \int_0^{\pi/2} \cos^n x\,dx = \int_0^{\pi/2} \sin^n x\,dx = \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\frac{1}{2}\cdot\frac{\pi}{2} & \text{when } n \to \text{even} \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\frac{2}{3}\cdot 1 & \text{when } n \to \text{odd} \end{cases}$$

Ques $\int_0^{\pi/2} \cos^8 x\,dx = \frac{7.5.3.1}{8.6.4.2} \times \frac{\pi}{2}$

Ques $\int_0^{\pi/2} \cos^7 x\,dx = \frac{\cancel{6}^{2}.4.2}{7.5.\cancel{3}} \times 1 = \frac{16}{35}$

Ques $\int_0^{\pi/2} \sin^4 x\,dx = \frac{3.1}{4.2} \times \frac{\pi}{2}$

<u>Property</u>

$$\ast \quad \int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{(m-1)(m-3)(m-5)\cdots(2(\text{or})1)\,(n-1)(n-3)(n-5)\cdots(2(\text{or})1)}{(m+n)(m+n-2)(m+n-4)\cdots(2(\text{or})1)} \times k$$

$$\text{when } k = \begin{cases} \pi/2 & \text{when both } m \,\&\, n \text{ are even} \\ 1 & \text{otherwise.} \end{cases}$$

eg $\int_0^{\pi/2} \sin^8 x \cos^6 x \, dx$

$$= \frac{(7.5.3.1)(5.3.1)}{14.12.10.8.6.4.2} \times \frac{\pi}{2}$$

eg $\int_0^{\pi/2} \sin^6 x \cos^5 x \, dx$

$$= \frac{5.3.1.\ 4.2}{11.9.7.5.3.1} \times 1$$

eg $\int_0^{\pi/6} \cos^4(3\theta) \sin^3(6\theta) \, d\theta$

$x = 0$

$3\theta = x \qquad x = \pi/2$

$3d\theta = dx \qquad d\theta = dx/3$

$$= \int_0^{\pi/2} \cos^4(x) \sin^3(2x) \frac{dx}{3}$$

$$= \frac{1}{3}\int_0^{\pi} \cos^4(x) \left(2\sin x \cos x\right)^3 \cdot \frac{dx}{3}$$

$$= \frac{8}{3}\int_0^{\pi} \sin^3 x \cos^7 x \, dx$$

$$I = \frac{8}{3} \times \frac{2.\ 6.4.2}{10.8.6.4.2}(1) = \frac{1}{15}$$

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$$

Question
$$I = \int_0^{2\pi} \sin^5 x\,dx \cos^5 x\,dx$$

sin + ve, cosec	All + ve
tan + ve, cot	cos + ve, sec + ve

$$f(2\pi - x) = \sin^5(2\pi - x)\cos^5(2\pi - x)$$
$$= (-\sin x)^5 (\cos x)^5$$
$$= -\sin^5 x \cos^5 x = -f(x)$$

$$I = \underline{0}.$$

Question
$$\int_0^{\pi} \sin^5 x \cos^4 x\,dx$$

$$f(\pi - x) = \sin^5(\pi - x)\cos^4(\pi - x)$$
$$= (\sin^5 x)(-\cos x)^4$$
$$f(\pi - x) = \sin^5 x \cos^4 x = f(x)$$

So
$$2\int_0^{\pi/2} \sin^5 x \cos^4 x\,dx = 2 \cdot \frac{4 \cdot 2 \cdot 3 \cdot 1}{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}(1)$$

$$I = \frac{16}{315}$$

$$* \quad I = \int_0^{\pi} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}\,dx = 2\int_0^{\pi/2} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}\,dx$$

$$= 2\int^{\pi/2} \frac{\sec^2 x}{a^2 + b^2\tan^2 x}\,dx$$

$$= \frac{2}{b^2}\int_0^{\pi/2} \frac{\sec^2 x}{\frac{a^2}{b^2} + \tan^2 x}\,dx$$

Let $\tan x = t$ $\qquad x = 0,\ t = 0$

$\sec^2 x\,dx = dt$ $\qquad x = \pi/2,\ t = \infty$

$$= \frac{2}{b^2}\int_0^{\infty} \frac{dt}{\frac{a^2}{b^2} + t}$$

$$= \frac{2}{b^2}\left(\frac{1}{a/b}\tan^{-1}\left(\frac{t}{a/b}\right)\right)\Bigg|_0^{\infty}$$

$$= \frac{2}{b^2} \times \frac{b}{a}\left(\tan^{-1}\infty - \tan^{-1} 0\right)$$

$$I = \frac{2}{ab}\left(\pi/2 - 0\right)$$

$$\boxed{I = \frac{\pi}{ab}}$$

$\underline{\underline{*}}$ $\int_0^a x f(x)\, dx = \frac{a}{2}\int_0^a f(x)\, dx$ if $f(a-x) = f(x)$

$\underline{eg.}$ $\int_0^{\pi} x \sin^6 x \cos^4 x \, dx$

$f(x) = \sin^6 x \cos^4 x$

$f(\pi - a) = \sin^6 x (-\cos^4 x)$

$f(\pi - a) = \sin^6 x \cos^4 x$

$= \frac{\pi}{2}\int_0^{\pi} \sin^6 x \cos^4 x \, dx$

$= \frac{\pi}{2} \cdot 2\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$

$= \frac{\pi}{2} \cdot 2 \left(\frac{5 \cdot 3 \cdot 1 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \times \frac{\pi}{2}\right) = \frac{3\pi^2}{512}$

$\underline{eg}$ $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}$

$f(x) = \frac{\sin x}{1 + \cos^2 x}$

$f(\pi - x) = \frac{\sin x}{1 + \cos^2 x}$

$f(\pi - x) = f(x)$

$= \frac{\pi}{2} \cdot \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} = \frac{\pi}{2} \cdot 2 \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x}\, dx$

Let $\cos x = t$

$-\sin x \, dx = dt$

$\sin x \, dx = -dt$

$x = 0 \quad t = 1$

$x = \pi/2 \quad t = 0$

$$= \pi \int_1^0 \frac{-dt}{1+t^2} = \pi \int_0^1 \frac{dt}{1+t^2}$$

$$= \pi \left(\tan^{-1} t\right)_0^1$$

$$= \pi \left(\tan^{-1} 1 - \tan^{-1} 0\right)$$

$$I = \pi \left(\pi/4 - 0\right) = \frac{\pi^2}{4}$$

Gamma function:-

$$\boxed{\Gamma n = \int_0^\infty e^{-t}\, t^{n-1}\, dt}$$

$$\Gamma n+1 = n \Gamma n \text{ or } n!$$

$$\Gamma \tfrac{1}{2} = \sqrt{\pi}$$

* n is +ve integer

$$\boxed{\Gamma n+1 = n!}$$

$$\Gamma 4 = 3! = 6$$

$$\Gamma 5 = 4! = 24$$

* n is +ve rational fraction

$$\boxed{\Gamma n+1 = n \Gamma n}$$

$$\Gamma \tfrac{5}{2} = \Gamma \tfrac{3}{2}+1 = \tfrac{3}{2} \Gamma \tfrac{3}{2}$$

$$= \frac{3}{2} \Gamma \tfrac{1}{2}+1 = \frac{3}{2} \cdot \frac{1}{2} \Gamma \tfrac{1}{2}$$

$$= \frac{3\sqrt{\pi}}{4}$$

* n is -ve rational fraction.

$$\boxed{\Gamma n = \frac{\Gamma n+1}{n}}$$

$$\Gamma -\tfrac{1}{2} = \frac{\Gamma -\frac{1}{2}+1}{-\frac{1}{2}} = \frac{\Gamma \frac{1}{2}}{-\frac{1}{2}}$$

$$= \frac{\sqrt{\pi}}{-\frac{1}{2}} = -2\sqrt{\pi}$$

* n is -ve Integer

↓

Gamma function is not define.

Question $\displaystyle\int_0^\infty \frac{x^2}{2^x}\,dx$

Sol^n

$$= \int_0^\infty 2^{-x} x^2\,dx$$

$$2^{-x} = e^{t}$$

$$x \ln 2 = t$$

$$x = \frac{t}{\ln 2} \Rightarrow dx = \frac{dt}{\ln 2}$$

$x = 0$, $t = 0$

$x = \infty$, $t = \infty$

$$= \int_0^\infty e^{-t} \frac{t^2}{(\ln 2)^2} \cdot \frac{dt}{\ln 2} = \frac{1}{(\ln 2)^3} \int_0^\infty e^{-t}\, t^{3-1}\,dt$$

$$= \frac{\Gamma 3}{(\ln 2)^3} = \frac{2!}{(\ln 2)^3} = \frac{2}{(\ln 2)^3}$$

Question $\int_0^\infty e^{-y^3} y^{1/2} dy$

$y^3 = t \Rightarrow 3y^2 . dy = dt$

$\begin{cases} y=0, & t=0 \\ y=\infty, & t=\infty \end{cases} \Rightarrow y^{1/2} dy = \frac{dt}{3 . y^{3/2}} = \frac{dt}{3 . t^{1/2}}$

$$= \int_0^\infty e^{-t} \frac{dt}{3 . t^{1/2}}$$

$$= \frac{1}{3}\int_0^\infty e^{-t} t^{-1/2} dt$$

$$= \frac{1}{3}\int_0^\infty e^{-t} t^{1/2 - 1} dt$$

$$= \frac{1}{3}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{3}$$

Beta function :-

$$\boxed{\beta(m,n) = \int_0^t x^{m-1}(1-x)^{n-1}\,dx}$$

$$\boxed{\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}}$$

Question $\displaystyle\int_0^1 \frac{x^6}{\sqrt{1-x^2}}\,dx$

let $1-x^2 = t$

$x^2 = 1-t$

$x = \sqrt{1-t}$

$dx = \dfrac{1}{2\sqrt{1-t}}(-dt)$ $\qquad x=0 \quad t=1$

$\qquad\qquad\qquad\qquad\qquad x=1 \quad t=0$

$$= \int_1^0 \frac{(1-t)^3}{\sqrt{t}} \times \frac{1\,(-dt)}{2\sqrt{1-t}}$$

$$= \frac{1}{2}\int_0^1 t^{-1/2}(1-t)^{5/2}\,dt = \frac{1}{2}\int_0^1 t^{\frac{1}{2}-1}(1-t)^{\frac{7}{2}-1}\,dt$$

$$= \frac{1}{2}\beta\left(\frac{1}{2},\frac{7}{2}\right) = \frac{1}{2}\,\frac{\Gamma\frac{1}{2}\cdot\Gamma\frac{7}{2}}{\Gamma\frac{1}{2}+\frac{7}{2}} = \frac{5\pi}{32}$$

* The length of the arc $y = f(x)$ between $x = a$ & $x = b$ is given by $l = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

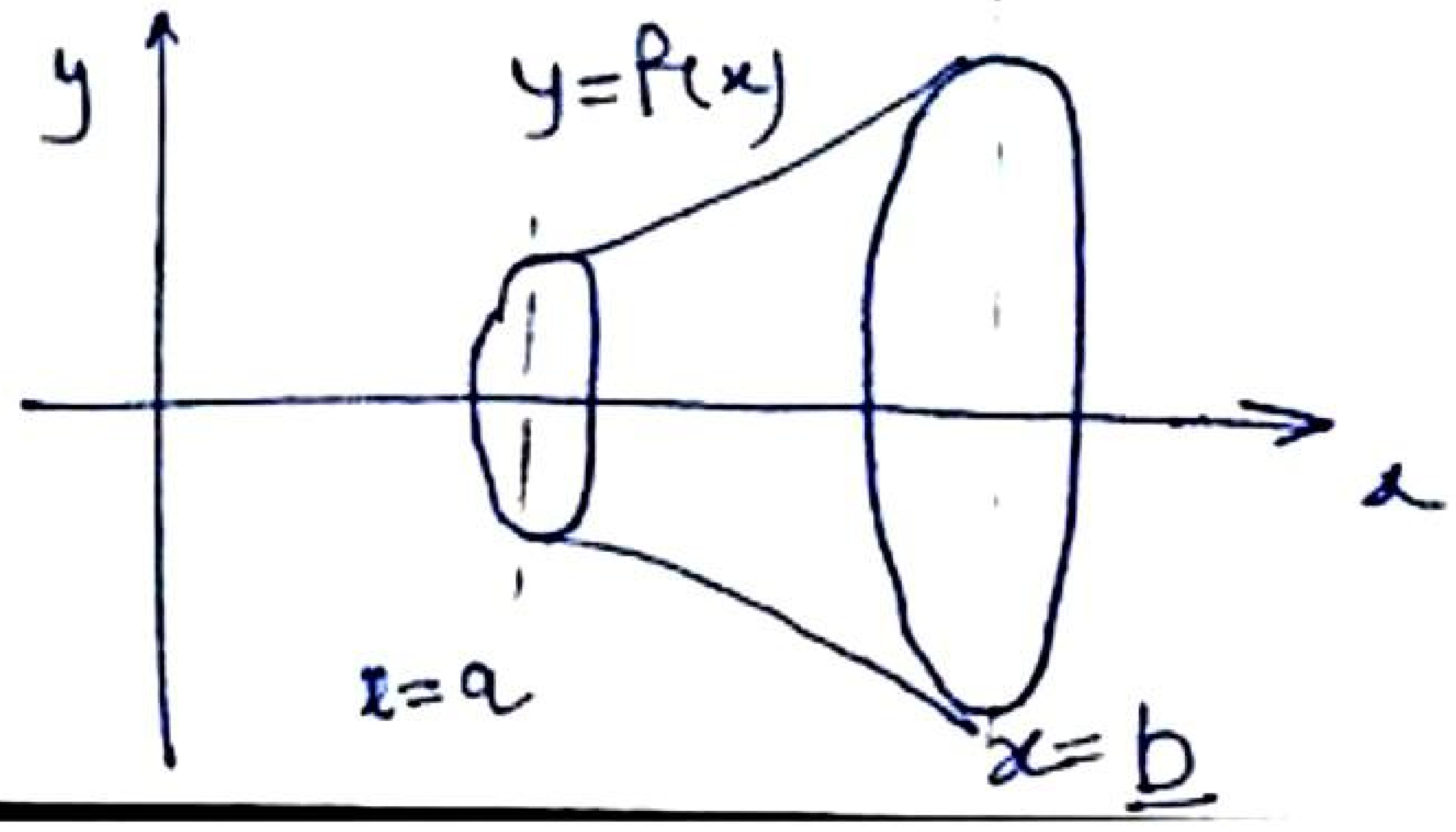
* The length of the arc $x = f(y)$ between $y = c$ & $x = d$ is given by $l = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

* The length of the arc $r = f(\theta)$ between $\theta = \theta_1$ & $\theta = \theta_2$ is given by $l = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

* The length of the arc $x = \phi(t)$, $y = \psi(t)$ between $t = t_1$ & $t = t_2$ is given by $l = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

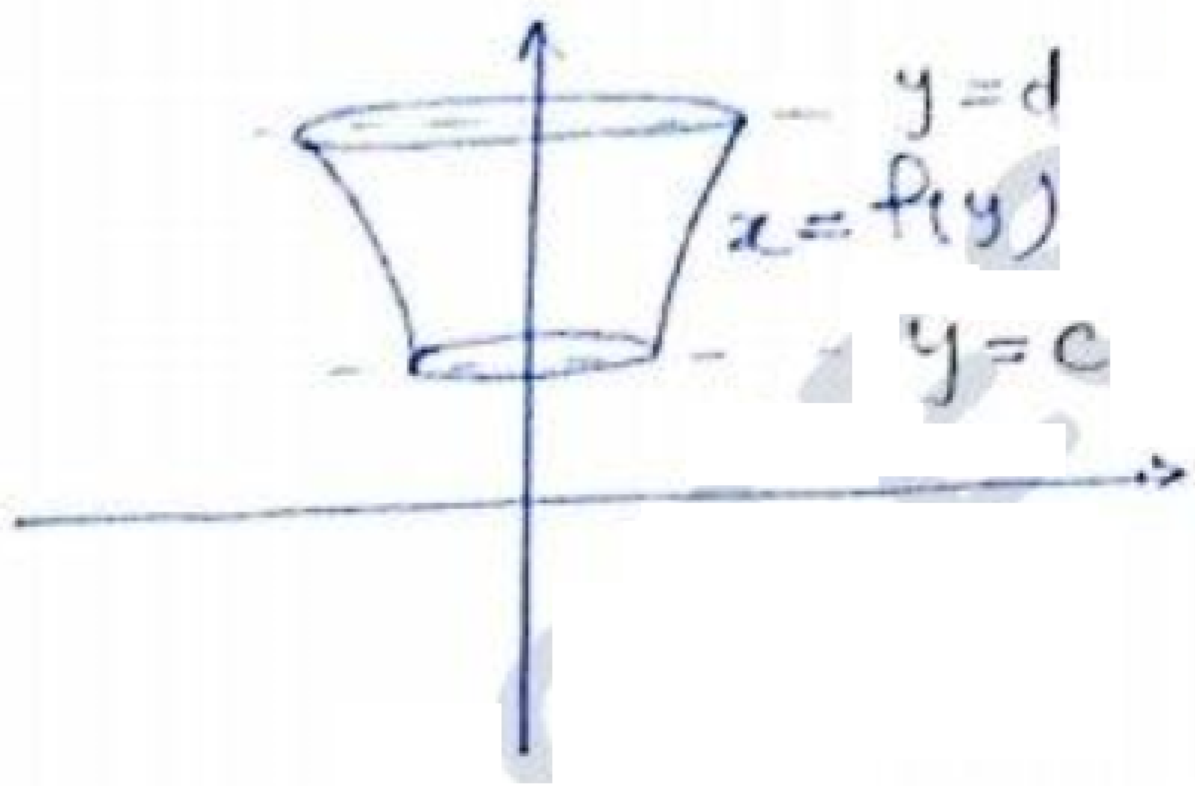
* The Volume of Solid revolution of the arc $y = f(x)$ around x axis between $x = a$ & $x = b$ is given by

$$\boxed{V = \int_a^b \pi y^2\, dx}$$



* The Volume of Solid revolution of the arc $x = f(y)$ around y axis between $y = c$ & $y = d$ is given by

$$\boxed{V = \int_c^d \pi x^2 \, dy}$$



* Question The length of the arc $y = \frac{2}{3} x^{3/2}$ b/w $x = 0$ & $x = 1$

Sol^n

$$y = \frac{2}{3} x^{3/2}$$

$$\frac{dy}{dx} = x^{1/2}$$

$$l = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_0^1 \sqrt{1 + (x^{1/2})^2} \, dx$$

$$l = (1 + x)^{1/2} \, dx$$

$$l = \left. \frac{(1+x)^{3/2}}{3/2} \right|_0^1 = \frac{2}{3} \left(2^{3/2} - 1\right)$$

$$l = 1.22 \text{ km}$$

Question If l is the length of portion of the curve b/w $y=1$, $y=3$ where $\frac{dx}{dy} = \frac{\sqrt{1+y^2+y^4}}{y}$

Sol^n

$$l = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$l = \int_1^3 \sqrt{1+\left(\frac{1+y^2+y^4}{y^2}\right)}\, dy$$

$$l = \int_1^3 \sqrt{\frac{1+2y^2+y^4}{y^2}}\, dy$$

$$l = \int_1^3 \sqrt{\left(\frac{y^2+1}{y}\right)^2}\, dy = \int_1^3 \left(y+\frac{1}{y}\right) dy$$

$$l = \ln y + \frac{y^2}{2} \Big|_1^3$$

$$l = \ln 3 + \frac{9}{2} - \ln 1 - \frac{1}{2}$$

$$l = 4 + \ln 3$$

$$l = 5.09.$$

Question The volume generated by revolving the area bounded by $y^2 = 8x$ & the line $x = 2$ about y-axis is.

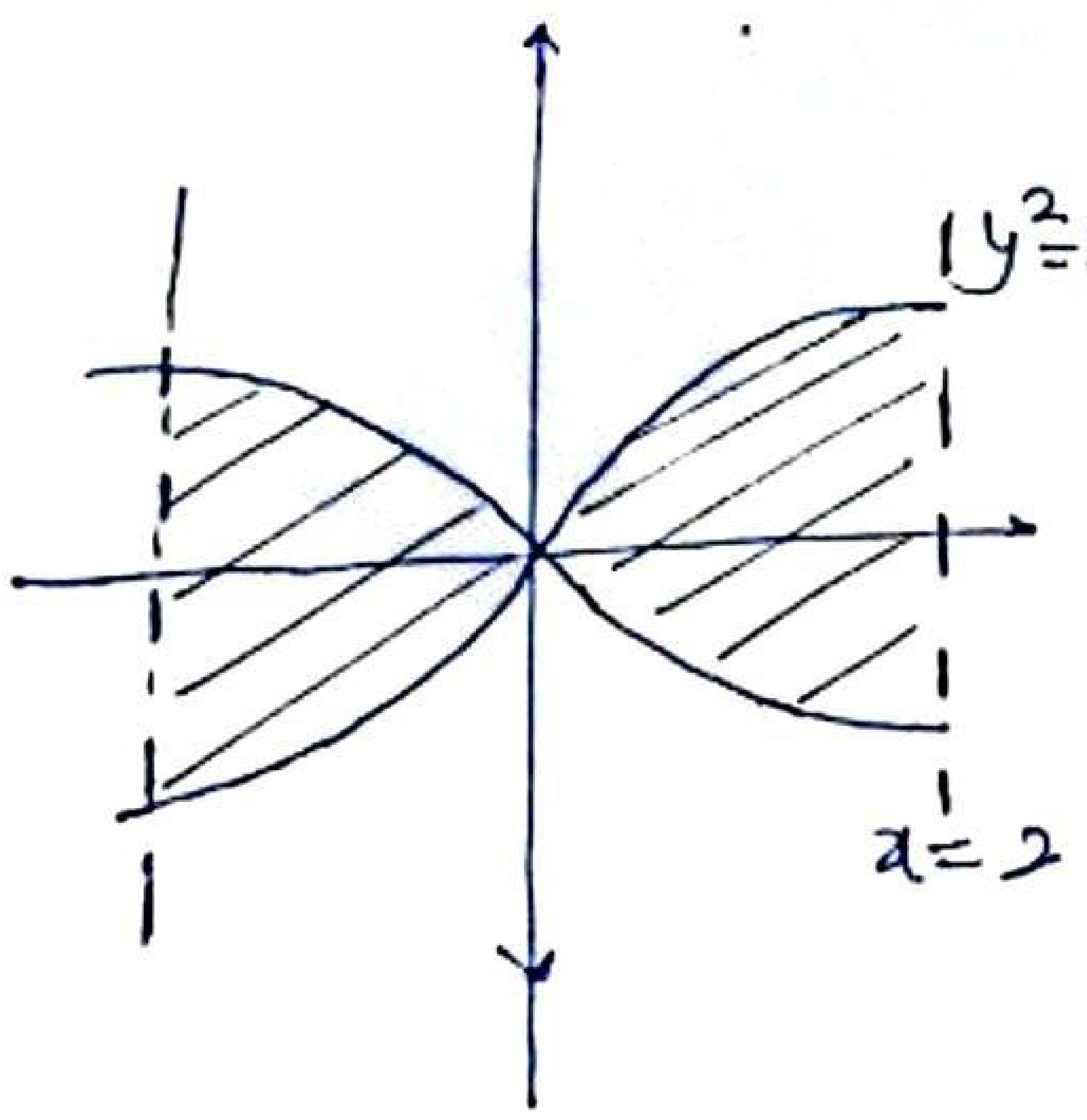
Soln

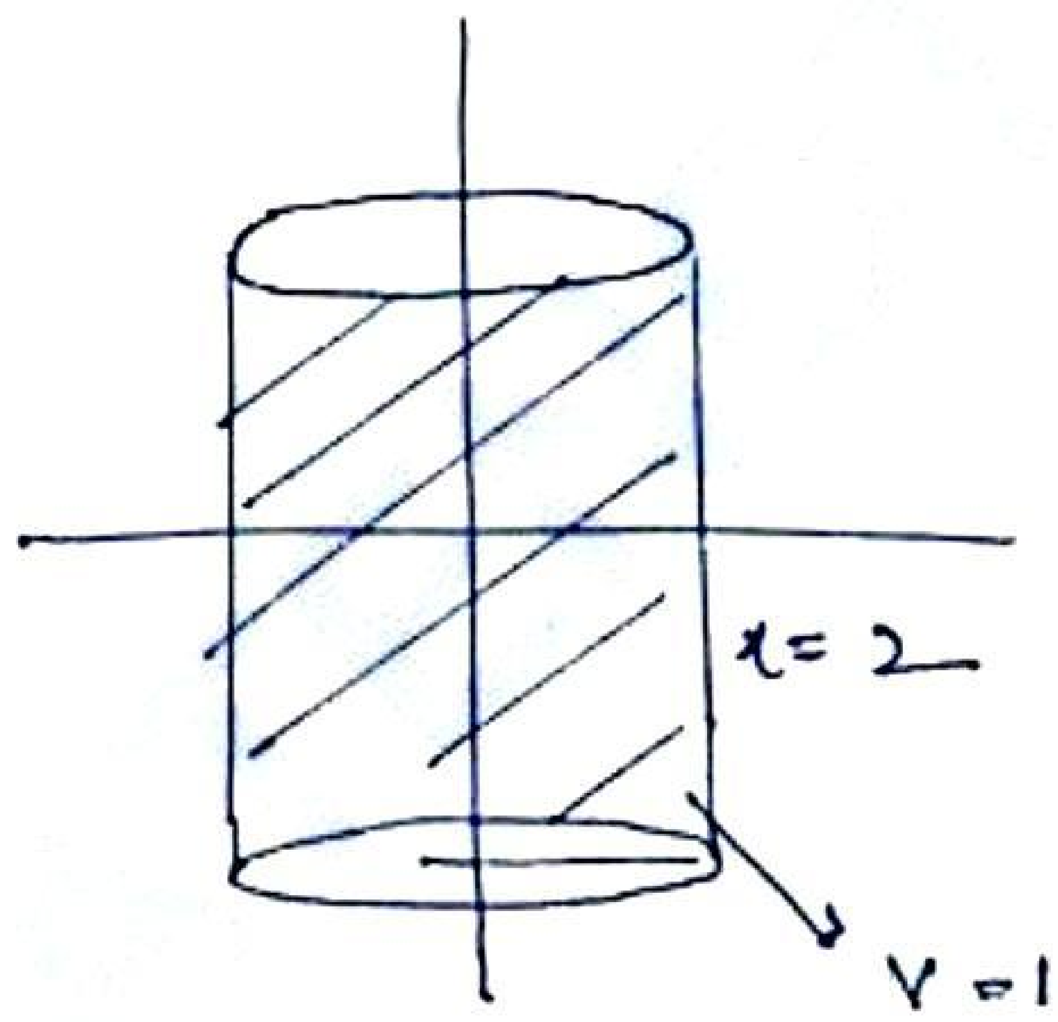
$$y^2 = 8x \qquad x = 0 \to 2$$

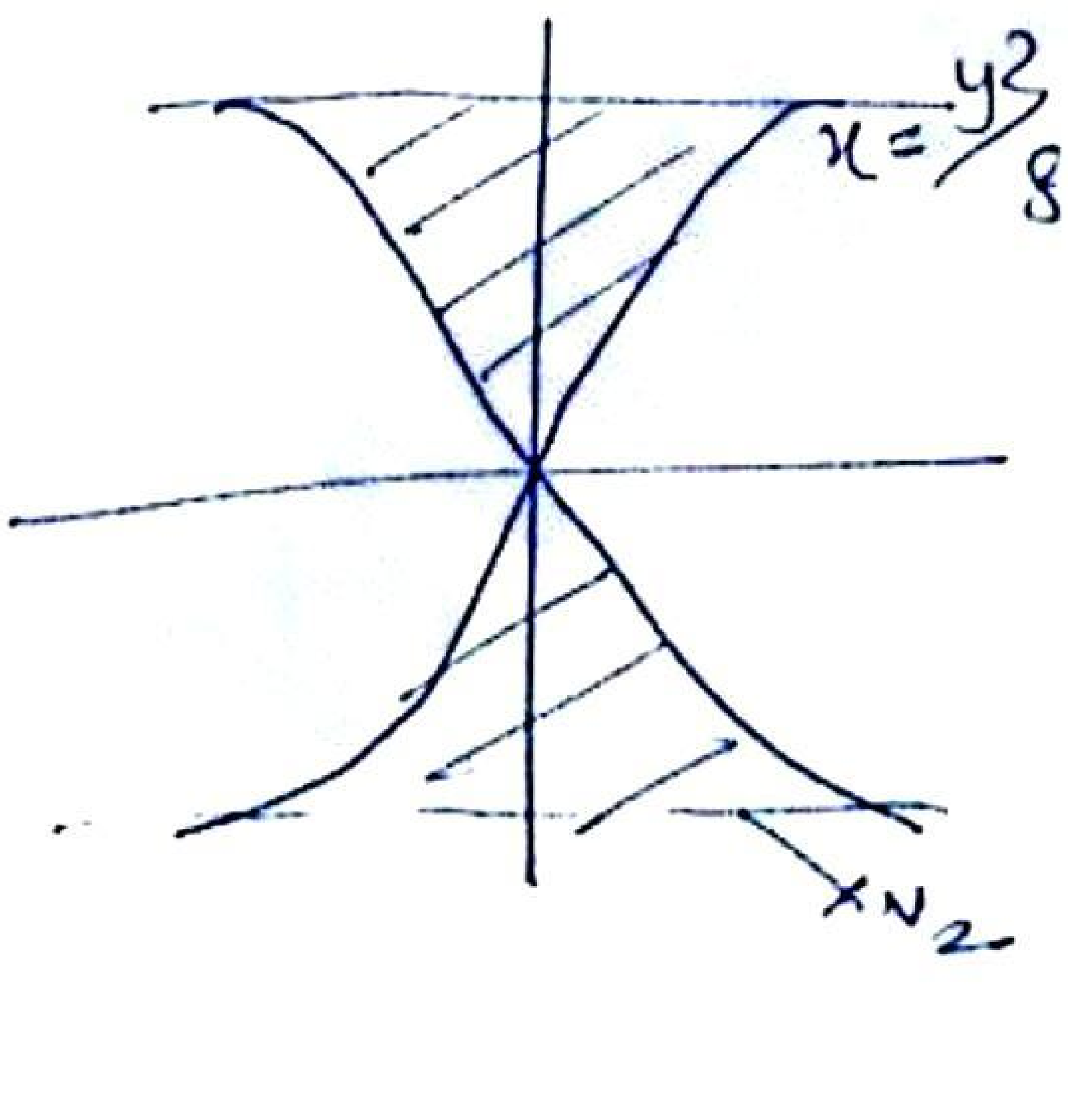
$$V = \int \pi x^2 \, dy$$

$$V = \int_0^2 \pi \frac{y}{4} \, dy = \frac{\pi}{4}\left[\frac{y^2}{2}\right]_0^2$$

$$V = \frac{\pi}{4} \times \frac{4}{2} = \frac{\pi}{2}$$







the Vol. of Solid revolution of the line $x = 2$ around y-axis b/w $y = -4$ to $y = 8$ is

$$V_1 = \int_c^d \pi (x)^2 dy$$

$$V_1 = \int_{-4}^{4} \pi (2)^2 dy = 32\pi$$

The Volume of solid revolution of ~~area~~ arc $x = y^2/8$ around y-axis b/w $y = -4$ to $y = 4$ is.

$$V_2 = \int_c^d \pi x^2 dy = \int_{-4}^{4} \pi \left(\frac{y^2}{8}\right)^2 dy$$

eue

$$V_2 = 2\int_0^4 \pi \frac{y^4}{64} dy$$

$$V_2 = \frac{32\pi}{5}$$

So volume $V_0 = V_1 - V_2 = \left(32 - \frac{32}{5}\right)\pi$

$$\text{Volume} = \frac{128}{5}\pi$$

Question The area bounded by $y = x^2+1$ & $x+y=3$ is —

Sol^n

$$y = x^2+1 \quad -(1)$$

$$x+y = 3 \quad -(2)$$

$$3 - x = x^2+1$$

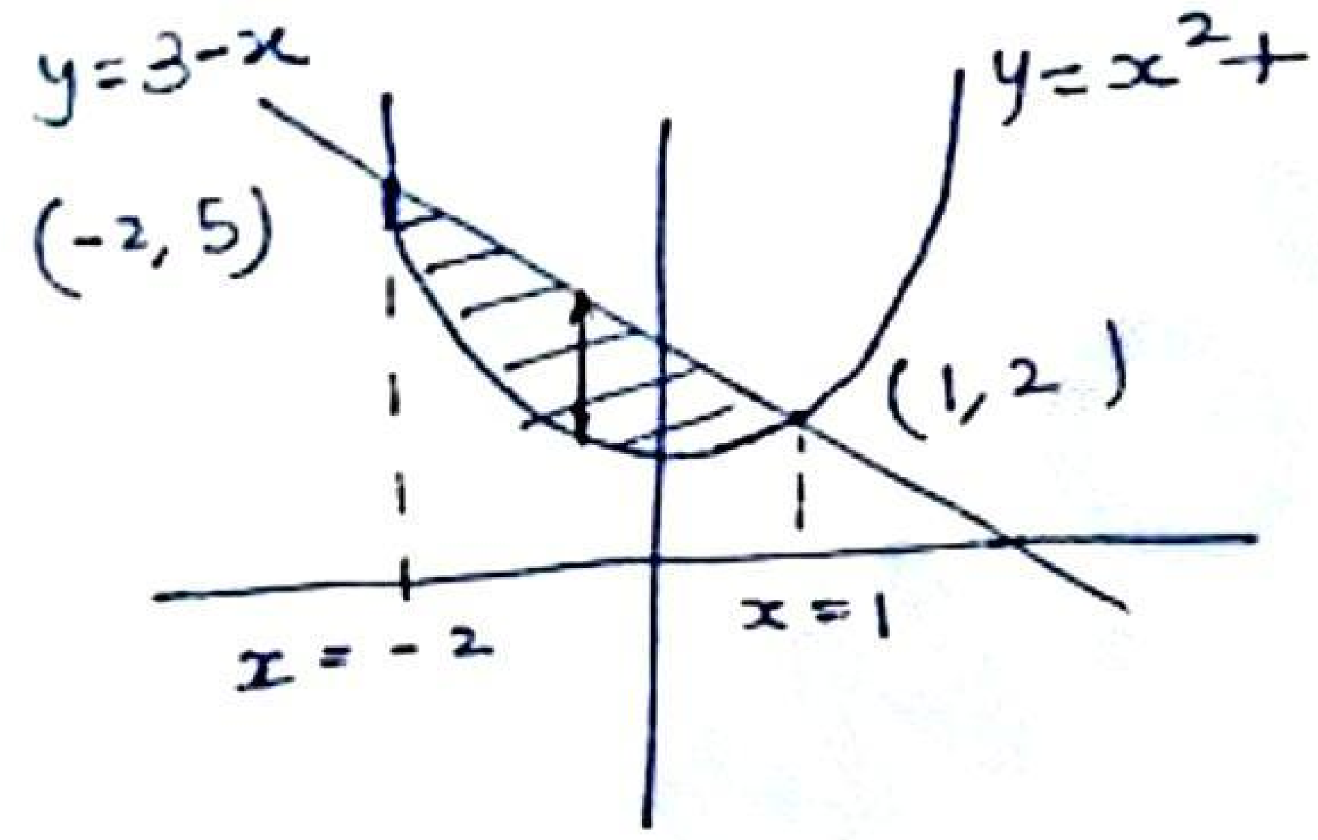
$$(x-1)(x+2) = 0$$

Point of intersection

$(-2,5)\ (1,2)$

$x = 1 \qquad x = -2$

$y = 2 \qquad y = 5$



$$\text{Area} = \int\left(\int dy\right) dx$$

$$A = \int_{-2}^{1}\int_{x^2+1}^{3-x} dy\, dx$$

$$A = \int_{-2}^{1}\left(y\Big|_{x^2+1}^{3-x}\right)dx = \int_{-2}^{1}(3-x-x^2-1)\,dx$$

$$A = \left(2x - \frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_{-2}^{1} = \left(2-\frac{1}{2}-\frac{1}{3}\right) - \left(-4-2+8/3\right)$$

$$A = \frac{7}{6} + \frac{10}{3} = \frac{9}{2}$$

Question The area bounded by $x = 6y^2 - 2$ & $x = 2y^2$ is.

Sol^n

$$x = 6y^2 - 2 \quad -(1)$$

$$x = 2y^2 \quad -(2)$$

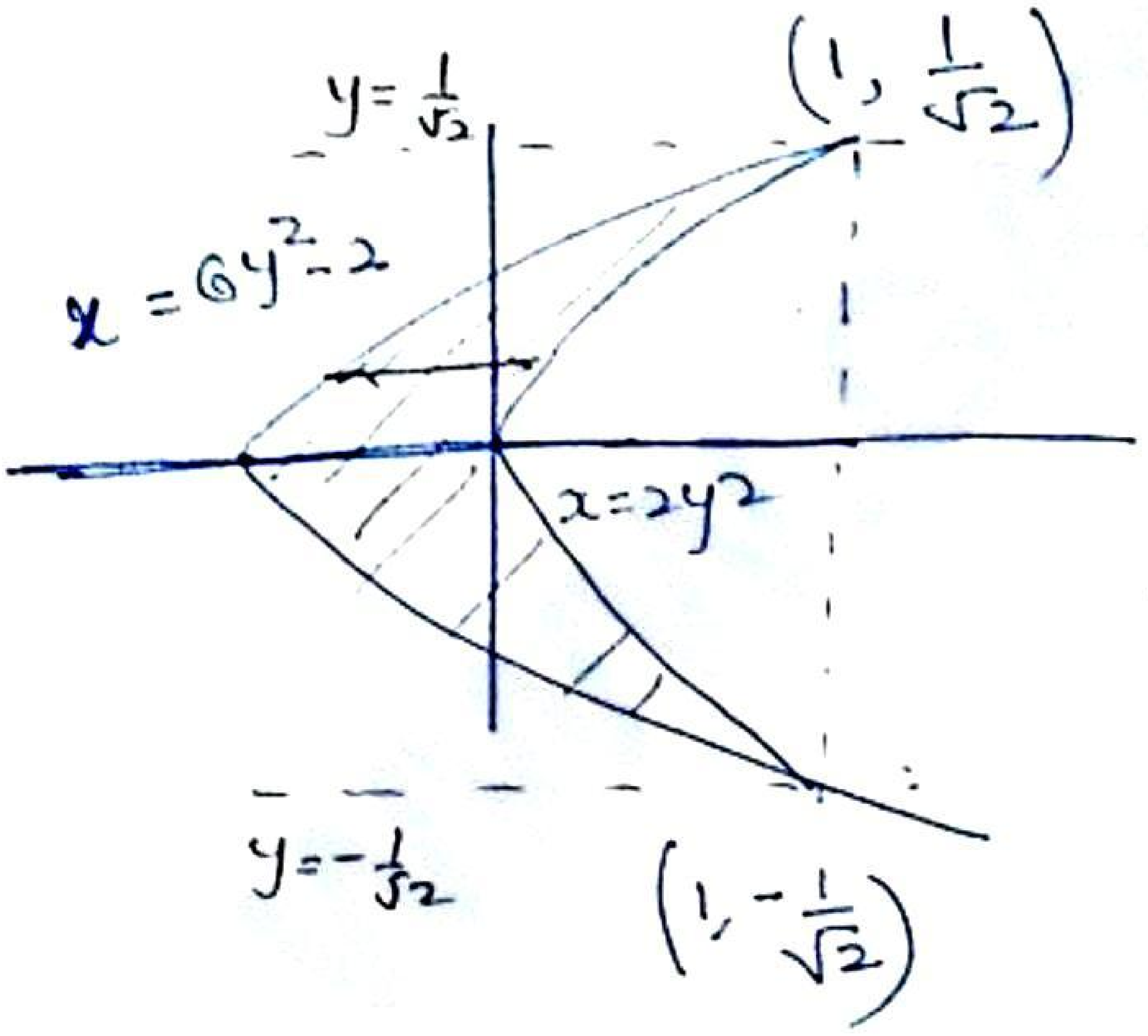
$$2y^2 = 6y^2 - 2$$

$$4y^2 = 1$$

$y = \frac{1}{\sqrt{2}}$ $\quad$ $y = -\frac{1}{\sqrt{2}}$

$x = 1$ $\quad$ $x = 1$

Point of intersection $\left(1, \frac{1}{\sqrt{2}}\right)$ & $\left(1, -\frac{1}{\sqrt{2}}\right)$



$$\text{Area} = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{6y^2-2}^{2y^2} dx\, dy$$

$$A = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left(2y^2 - 6y^2 + 2\right) dy$$

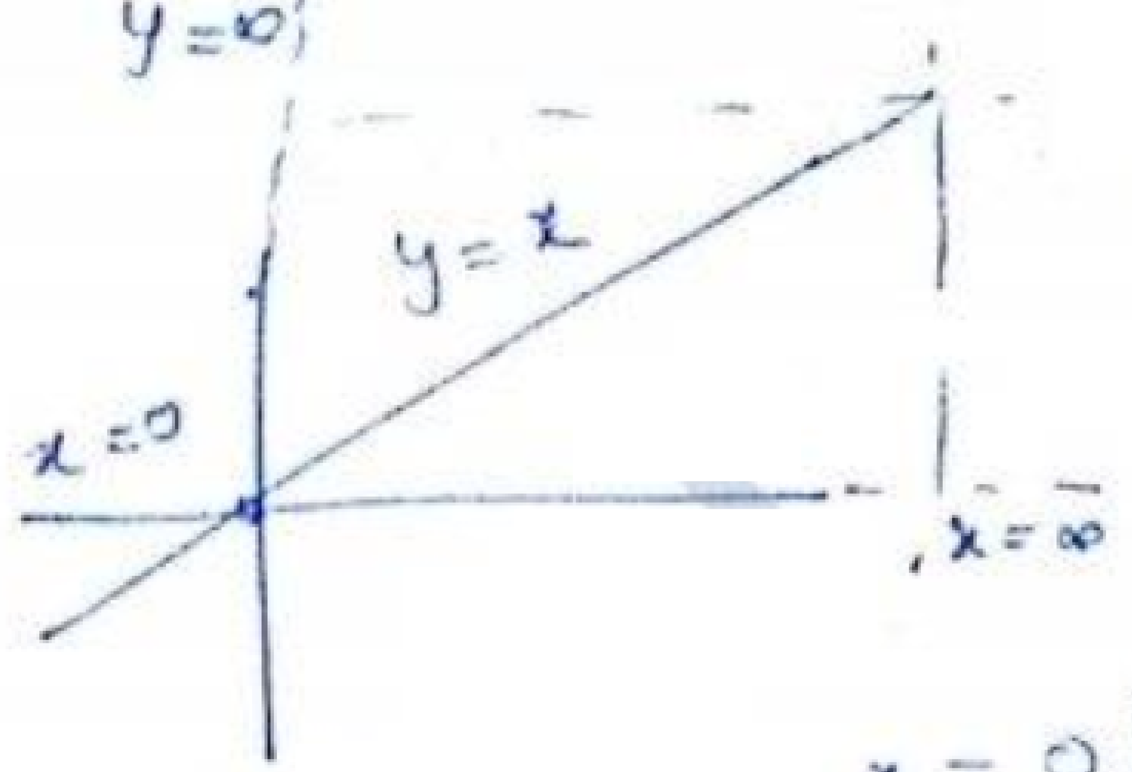
$$A = 2\int_0^{\frac{1}{\sqrt{2}}} \left(-4y^2 + 2\right) dy = 2\left[-\frac{4y^3}{3} + 2y\right]_0^{\frac{1}{\sqrt{2}}}$$

$$A = 2\left[-\frac{4}{3}\frac{1}{2\sqrt{2}} + \frac{2}{\sqrt{2}}\right]$$

$$A = \frac{2}{\sqrt{2}}\left[2 - \frac{4}{6}\right] = \frac{8}{3\sqrt{2}} = \frac{4\sqrt{2}}{3}$$

Ques

$$\int_0^\infty\left(\int_x^\infty \frac{1}{y} e^{-y/2}\, dy\right) dx$$



$$\Gamma n = \int e^{-x}(x)^{n-1}$$

$$\left\{\begin{array}{ll} y = x & x = 0 \\ y = \infty & x = \infty \end{array}\right\}$$

$$-y/2 = t \Rightarrow y = -2t$$

$$dy = -2\,dt$$

$$I = \int_0^\infty \int_0^y \frac{1}{y} e^{-y/2}\, dx\, dy$$

$$I = \int_0^\infty \frac{1}{y} e^{-y/2} \cdot x\Big|_0^y\, dy = \int_0^\infty e^{-y/2}\, dy$$

$$I = \int_0^\infty e^{-y/2}\, dy \qquad -y/2 = t$$

$$-\frac{dy}{2} = dt$$

$$dy = -2\,dt$$

$$I = \int_0^{-\infty} e^{-t}(-2)\, dt$$

$$I = -2\left[e^{-t}\right]_0^{-\infty}$$

$$I = -2\left[e^{-\infty} - e^{0}\right]$$

$$I = \underline{2}$$

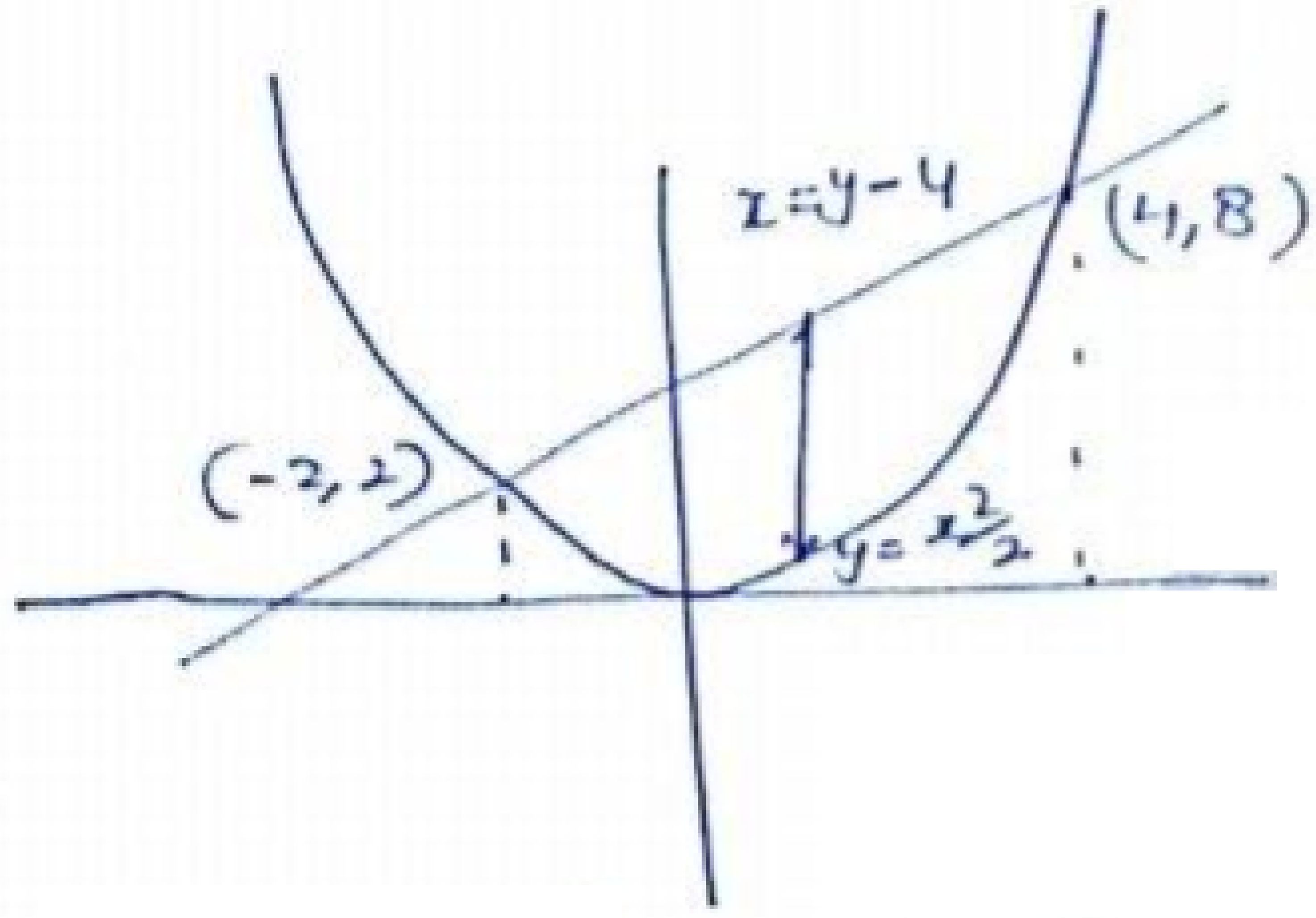
Question Area bounded by $2y = x^2$ & $x = y - 4$ is -

Sol^n

$$2y = x^2 \quad (1)$$
$$x = y - 4 \quad (2)$$

From (1) & (2) point of intersection

$$(-2, 2) \quad (4, 8)$$



$$\text{Area} = \int_{-2}^{4} \int_{\frac{x^2}{2}}^{x+4} dy\, dx$$

$$A = \int_{-2}^{4} \left(x + 4 - \frac{x^2}{2}\right) dx = \int_{-2}$$

$$A = \left[\frac{x^2}{2} + 4x - \frac{x^3}{6} \right]_{-2}^{4}$$

$$A = \left(8 + 16 - \frac{32}{3}\right) - \left(2 - 8 + \frac{4}{3}\right)$$

$$A = 30 - 12$$

$$A = 18$$

Q. The value of $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$

$$I = \int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$$

$$= \int_0^1 \left[x e^{y/x} \right]_0^{x^2} dx$$

$$I = \int_0^1 \left[x e^{x^2/x} - x \right]$$

$$I = \int_0^1 x(e^{x} - 1)\, dx$$

$$I = \int_0^1 (xe^{x} - x)\, dx$$

$$I = \left(xe^{x} - e^{x} - \frac{x^2}{2} \right)\Big|_0^1$$

$$I = \left(e - e - \frac{1}{2} \right) - (0 - 1 - 0)$$

$$I = -\frac{1}{2} + 1$$

$$I = \frac{1}{2}$$

Ques find $\int_0^1 \int_{2y}^2 e^{x^2}\,dx\,dy$

$$I = \int_0^1 \left(\int_{2y}^{2} e^{x^2} dx \right) dy$$

$x^2 = t$

$2x\,dx = dt$

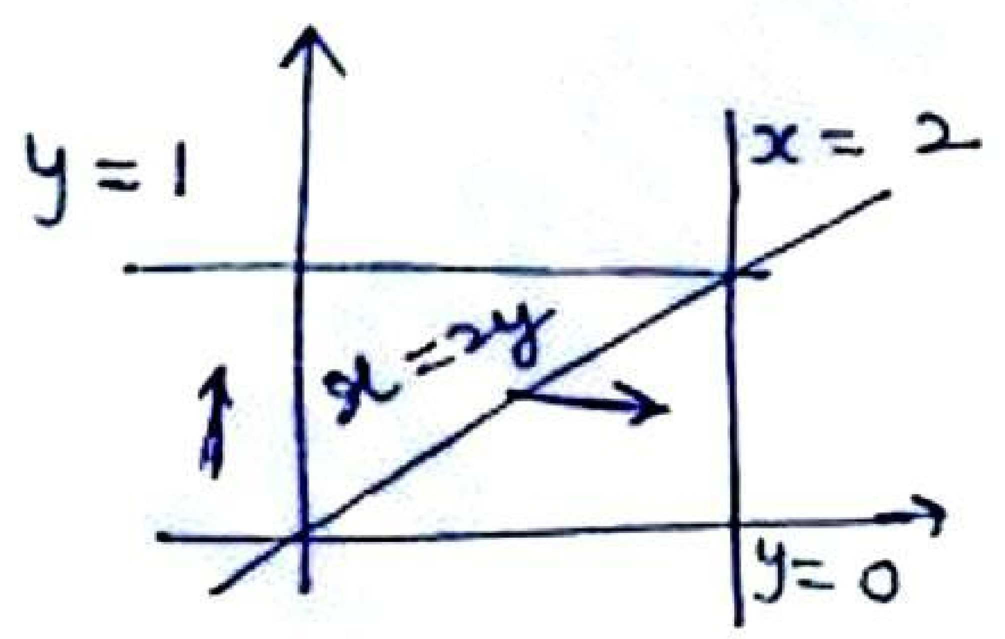
$dx = \frac{dt}{2\sqrt{t}}$

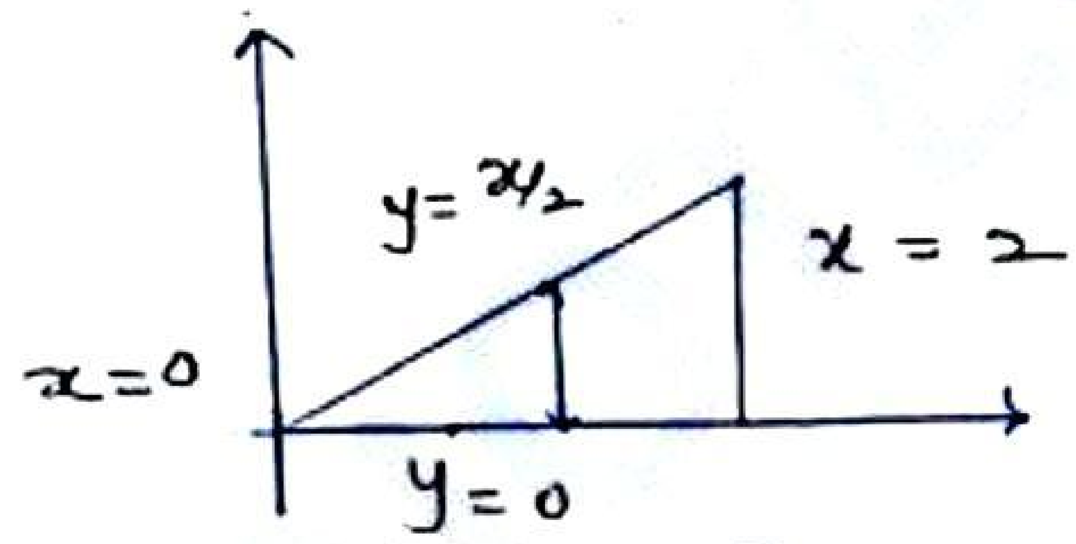
$$I = \int_0^1 \int_{4y^2}^{4} \left(\frac{e^t}{2\sqrt{t}}\,dt \right) dy$$

integration becomes difficult change the order i.e. change limits

$\Rightarrow$ $x = 2y$ $\quad$ $x = 2$

$y = 0$ $\quad$ $y = 1$





$$I = \int_0^2 \int_0^{x/2} e^{x^2} dy\,dx = \int_0^{x} e^{x^2}\, y\Big|_0^{x/2} dx$$

$$I = \int_0^2 e^{x^2} \frac{x}{2}\,dx$$

$x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$x = 0 \quad t = 0$

$x = 2 \quad t = 4$

$$I = \int_0^4 e^t \frac{dt}{4}$$

$$I = \frac{1}{4}\left(e^4 - e^0\right) = \frac{e^4 - 1}{4}$$

Question

$$I = \int_0^{\pi}\int_x^{\pi}\int_0^{2} \frac{\sin y}{y}\, dz\, dy\, dx$$

$$I = \int_0^{\pi}\int_x^{\pi}\left(\int_0^{2} \frac{\sin y}{y}\, dz\right) dy\, dx$$

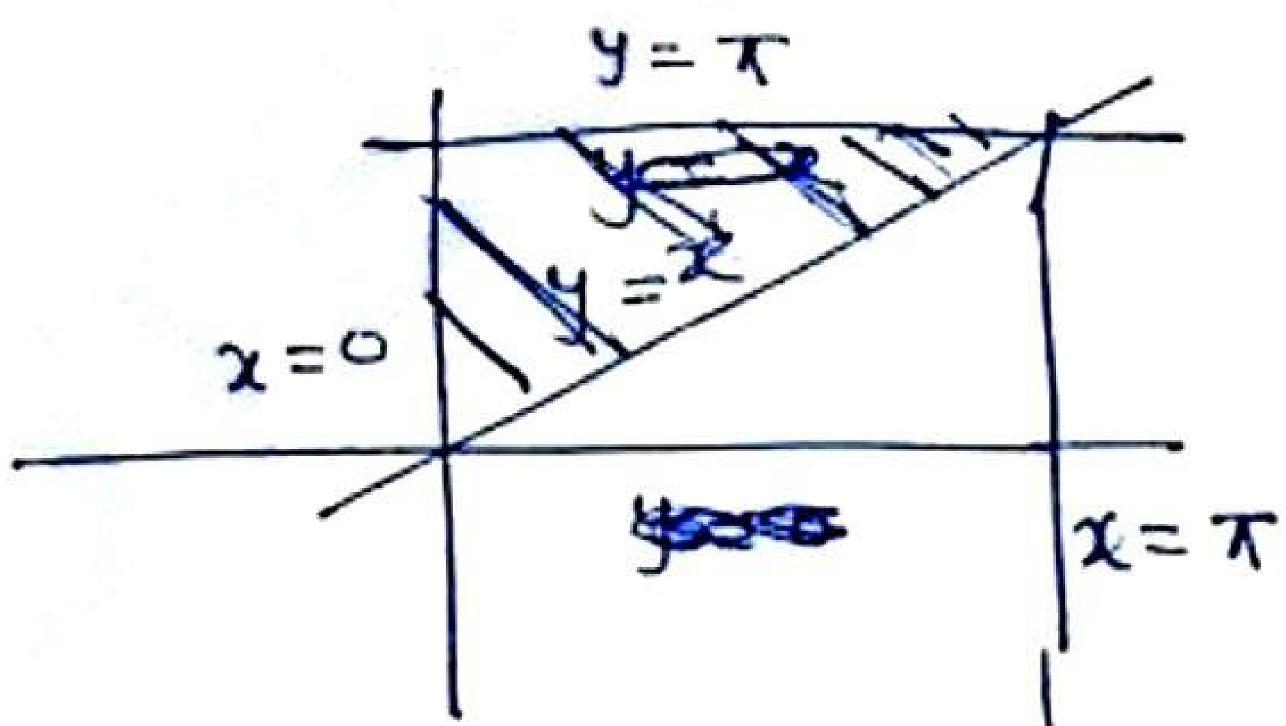
$$I = \int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, z\Big|_0^2\, dy\, dx$$

$$I = \int_0^{\pi}\left(\int_x^{\pi} 2\frac{\sin y}{y}\, dy\right) dx$$

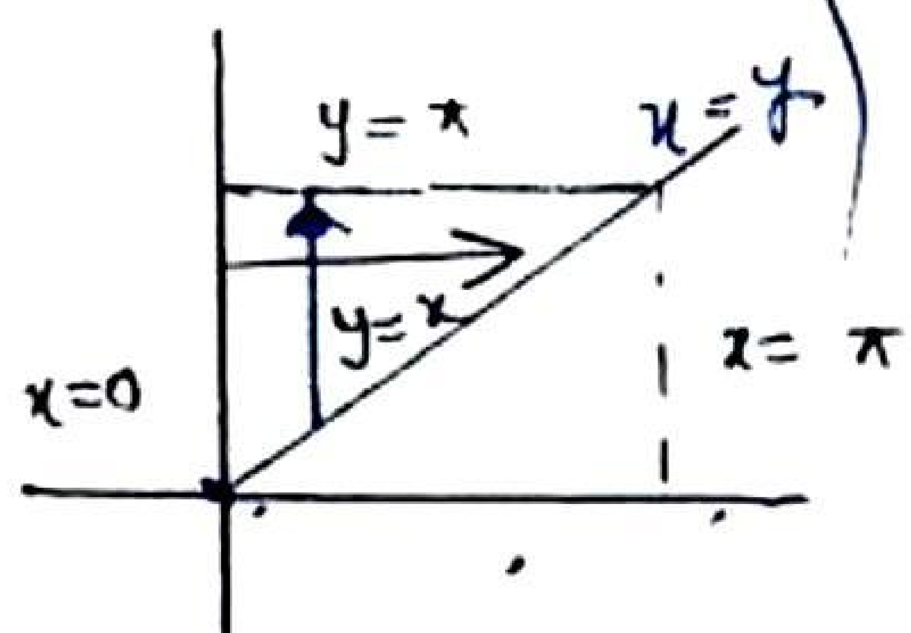
$y = x \qquad x = 0$

$y = \pi \qquad x = \pi$

$$\cancel{I = \int_0^{x}\left(\int_0^{y} \frac{2\sin y}{y}\, dx\right) dy}$$



$$I = \int_0^{\pi}\int_0^{y} 2\frac{\sin y}{y}\, dx\, dy$$



$$I = \int_0^{\pi} 2\frac{\sin y}{y}\,(x)\Big|_0^{y}\, dy$$

$$I = \int_0^{\pi} 2\frac{\sin y}{y}\, y\, dy$$

$$I = 2\left(-\cos y\right)_0^{\pi} = -2(-1-1)$$

$$I = 4$$

Change of Variable

Jacobian:- if $x = \phi(u,v)$, $y = \psi(u,v)$ then the Jacobian of x & y w.r.t. u & v is given by

$$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix}$$

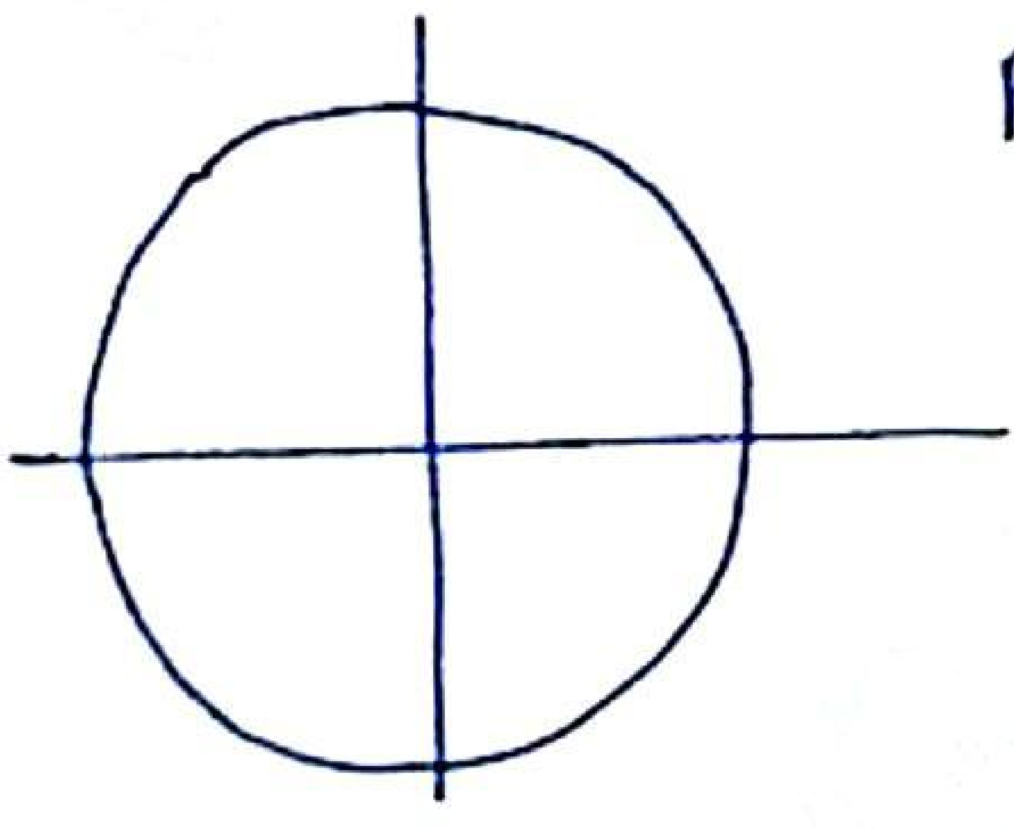
Cartesian to Circle(Polar)

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$J\left(\frac{x,y}{r,\theta}\right) = r\cos^2\theta + r\sin^2\theta = r$$

eg



Area of $x^2+y^2=a^2$

Cartesian	polar form
$y = -\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$	$r \to 0$ to a
$x \to -a$ to a	$\theta \to 0$ to 2π

$$\text{Area} = \iint dy\,dx = \iint |J|\,dr\,d\theta$$

$$\text{Area} = \int_0^{2\pi}\int_0^{a} r\,dr\,d\theta = \int_0^{2\pi} \frac{a^2}{2}\,d\theta = \frac{a^2}{2}(2\pi) =$$

$$\text{Area} = \pi a^2$$

Cartesion to Cylinderical form

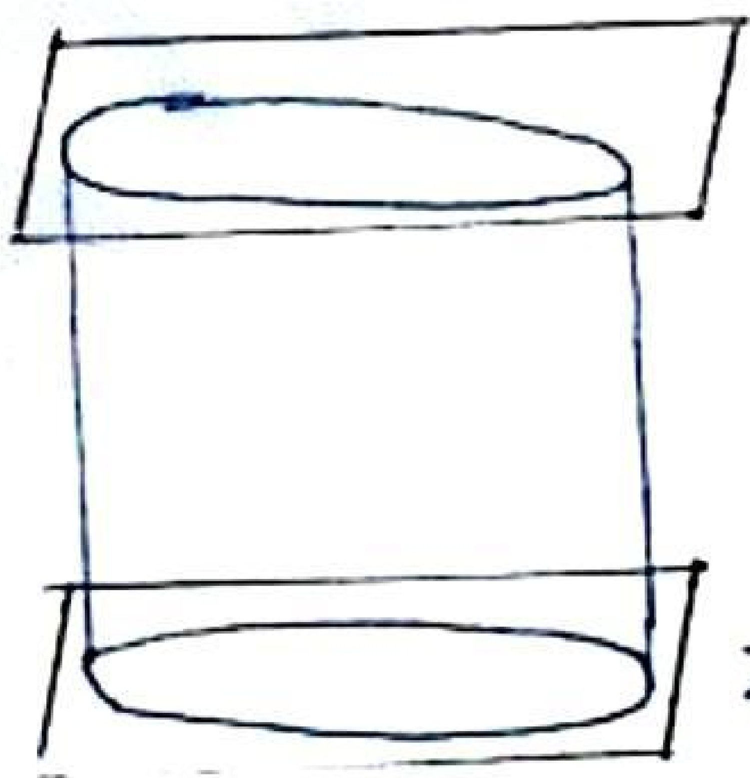
$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$

$$J\left(\frac{x,y,z}{r,\theta,z}\right) = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$J\left(\frac{x,y,z}{r,\theta,z}\right) = r\cos^2\theta + r\sin^2\theta = r$$

eg $x^2+y^2 = a^2$

$z = 0 \rightarrow z = b$



Cartesion

$z \rightarrow 0$ to b

$y \rightarrow -\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

$x \rightarrow -a$ to a

Cylinderical form

$z \rightarrow 0$ to b

$r \rightarrow 0$ to a

$\theta \rightarrow 0$ to 2π

$$\text{Volume } V = \iiint dz\,dy\,dx = \int_0^b\int_0^{2\pi}\int_0^a r\,dr\,d\theta\,dz$$

$$V = \int_0^b\int_0^{2\pi}\left(\frac{a^2}{2}\right)d\theta\,dz = \int_0^b \pi a^2 dz = \pi a^2[z]_0^b$$

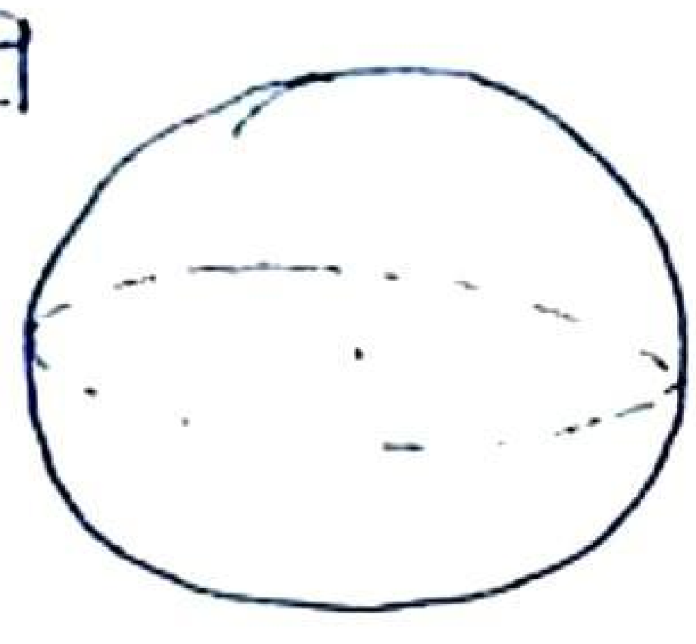
$$\text{Volume} = \pi a^2 b$$

Cartesion to Spherical form

$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$

$$J\left(\frac{x, y, z}{\rho, \theta, \phi}\right) = \begin{vmatrix} x_\rho & x_\theta & x_\phi \\ y_\rho & y_\theta & y_\phi \\ z_\rho & z_\theta & z_\phi \end{vmatrix} = \rho^2 \sin\phi$$

eg



$x^2+y^2+z^2=a^2$

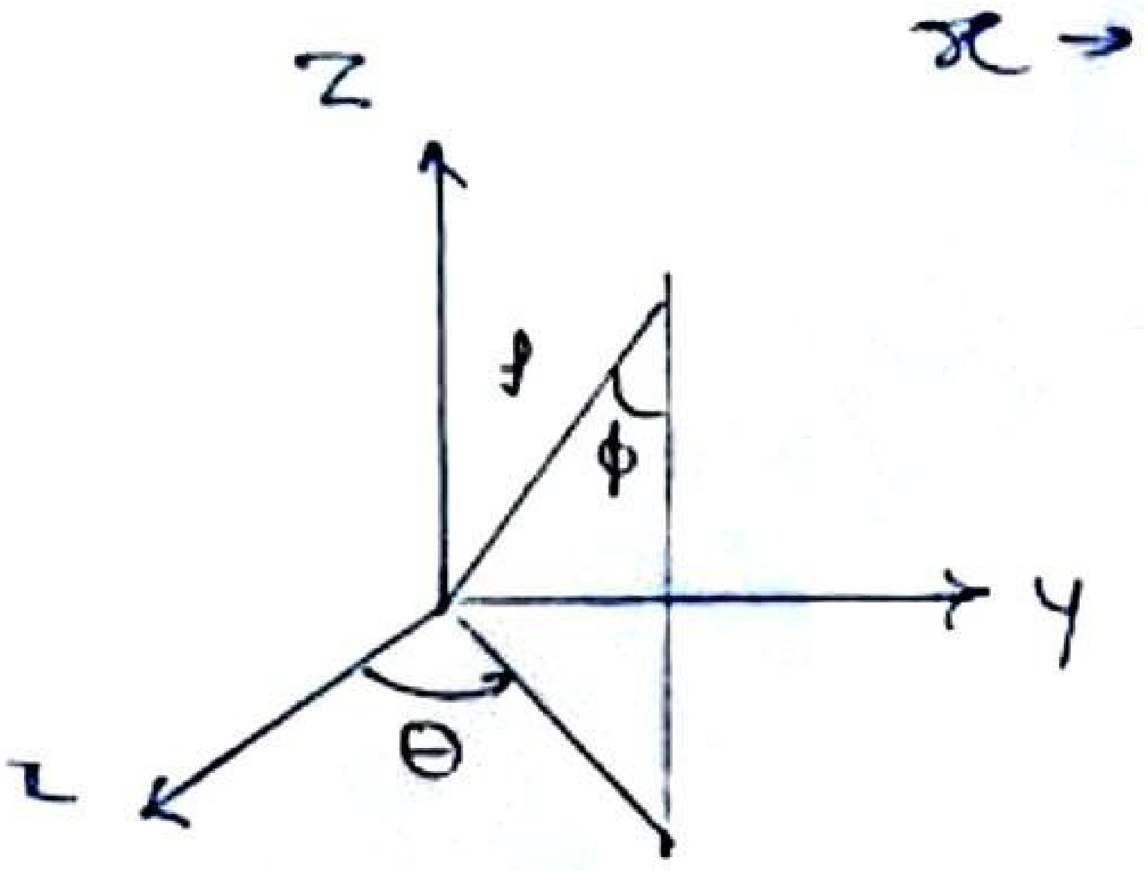
Cartesion form

Spherical form

$z \to -\sqrt{a^2-x^2-y^2}$ to $\sqrt{a^2-x^2-y^2}$

$y \to -\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

$x \to -a$ to a



Spherical form

$\rho \to 0$ to a

$\theta \to 0$ to 2π

$\phi \to 0$ to π

$$\text{Volume} = \iiint dz\,dy\,dx = \int_0^{a}\int_0^{2\pi}\int_0^{\pi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$\text{Vol.} = \int_0^{\pi}\int_0^{2\pi} \sin\phi \left(\frac{\rho^3}{3}\right)\Big|_0^{a} d\theta\, d\phi = \int_0^{\pi}\int_0^{2\pi}\left(\frac{a^3}{3}\sin\phi\, d\theta\right) d\phi$$

$$\text{Vol.} = \int_0^{\pi} \frac{a^3 \sin\phi}{3}\,\theta\Big|_0^{2\pi} d\phi = \int_0^{\pi} \frac{2\pi a^3}{3} \sin\phi \, d\phi = \left[\frac{2\pi}{3} a^3 (-\cos\theta)\right]_0^{\pi}$$

$$\text{Vol} = -\frac{2a^3}{3}\pi(-1-1) \Rightarrow \text{Vol.} = \frac{4\pi a^3}{3}$$

Homogeneous function:-

A function $f(x,y)$ is said to be homogeneous of degree n if $f(kx,ky) = k^n f(x,y)$

Eg $f(x,y) = \dfrac{x^4+y^4}{x-y}$ is homogeneous of degree 3

$$f(kx,ky) = \frac{k^4x^4+k^4y^4}{kx-ky} = \frac{k^4(x^4+y^4)}{k(x-y)} = k^3\left(\frac{x^4+y^4}{x-y}\right)$$

$k^3 \rightarrow$ Degree $\underline{\underline{3}}$

Eg $f(x,y) = x^8 \sin^{-1}\left(\frac{x}{y}\right)$ is homogeneous of degree 8.

$$f(kx,ky) = k^8x^8 \sin^{-1}\left(\frac{kx}{ky}\right) = k^8 x^8 \sin^{-1}\left(\frac{x}{y}\right)$$

Eg $f(x,y) = x^4 \sin^{-1}\left(\frac{x}{y}\right) + y^{-4}\cos^{-1}\left(\frac{x}{y}\right) \rightarrow$ Not homogeneous

Eg $f(x,y) = \sin^{-1}(x^6+y^6) \rightarrow$ Not homogeneous.

Euler's theorem:-

* if $u = f(x,y)$ is homogeneous of degree n than

(i) $x\frac{du}{dx} + y\frac{du}{dy} = nu$

(ii) $x^2\frac{d^2u}{dx^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

eg $u = \frac{x^4y^4}{x+y}$ is a homogeneous function of degree 7.

(i) $x\frac{du}{dx} + y\frac{du}{dy} = 7u$

(ii) $x^2\frac{d^2u}{dx^2} + 2xy\frac{d^2u}{dxdy} + y^2\frac{d^2u}{dy^2} = 7\times6\times u = 42u.$

* If $u = f(x,y) + g(x,y)$ where f & g are homogeneous of degree m, n respectively then

(i) $x\frac{du}{dx} + y\frac{du}{dy} = mf + ng$

(ii) $x^2\frac{\partial^2u}{\partial x^2} + 2xy\frac{\partial^2u}{\partial x\partial y} + y^2\frac{\partial^2u}{\partial y^2} = m(m-1)f + n(n-1)g$

eg $u = \underbrace{x^n f_1(x/y)}_{\text{degree } n} + \underbrace{y^{-n} f_2(x/y)}_{\text{degree } -n}$

(i) $= nf + (-n)g$

(ii) $= n(n-1)f + (-n)(-n-1)g = (n^2-n)f + (n^2+n)g.$

* If $u = f(x,y)$ is not homogeneous but $F(u)$ is homogeneous of degree n then

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{F(u)}{F'(u)} = g(u)$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\left(g'(u) - 1\right)$

eg $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ is not homogeneous

but $\tan u = \frac{x^3+y^3}{x+y}$ is homogeneous of degree 2

(i) $= 2\frac{\tan u}{\sec^2 u} = 2 . \frac{\sin u}{\cos u} . \cos^2 u = \underbrace{\sin(2u)}_{g(u)}$

ii) $= g(u)\left(g'(u) - 1\right)$

$= \sin 2u\left(2\cos 2u - 1\right)$

Total derivative :-

* If $u = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$ then total derivative of u w.r.t. t is given by

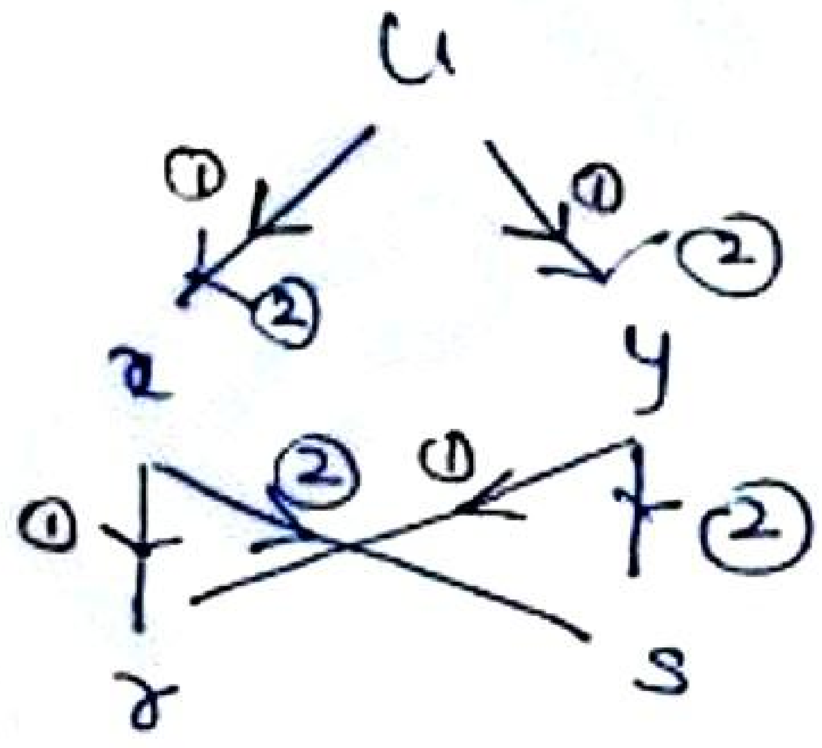
$$\frac{du}{dt} = \frac{du}{dx} \cdot \frac{dx}{dt} + \frac{du}{dy} \cdot \frac{dy}{dt}$$

Chain Rule

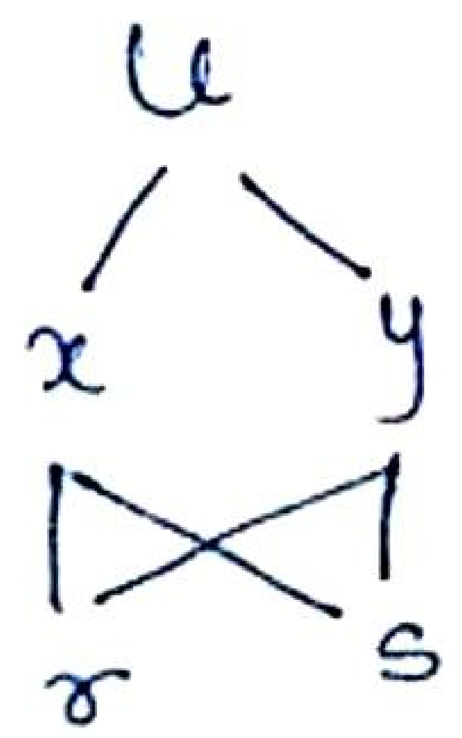
* If $u = \phi(x, y)$ where $x = \phi(r, s)$, $y = \psi(r, s)$ then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$



Question If $u = f(x, y)$ where $x = r + s$, $y = r - s$ then $u_r + u_s = ?$



$$u_r = u_x x_r + u_y y_r \qquad u_s = u_x x_s + u_y y_s$$

$$u_r = u_x(1) + u_y(1) \qquad u_s = u_x(1) + u_y(-1)$$

$$u_r = u_x + u_y \qquad u_s = u_x - u_y$$

$$u_r + u_s = u_x + u_y + u_x - u_y = 2u_x$$