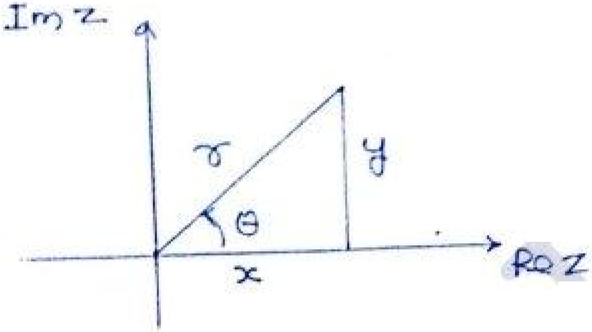
Complex Variable

$z = x + iy$



$\cos\theta = \frac{x}{r}$ $\qquad$ $\sin\theta = \frac{y}{r}$

$x = r\cos\theta$ $\qquad$ $y = r\sin\theta$

$\Rightarrow \quad z = r\cos\theta + i r\sin\theta$

$z = r(\cos\theta + i\sin\theta)$

$z = re^{i\theta}$

$r = |z| = \sqrt{x^2 + y^2}$

$\theta = \arg(z) = \tan^{-1}(y/x)$

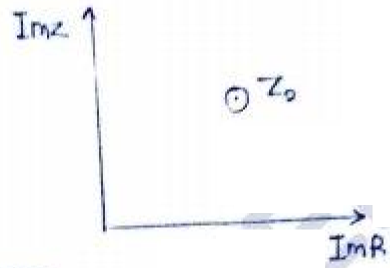
Note:- If two complex numbers z_1 & z_2 lies on the right half plane (or) both lies on imaginary axis then

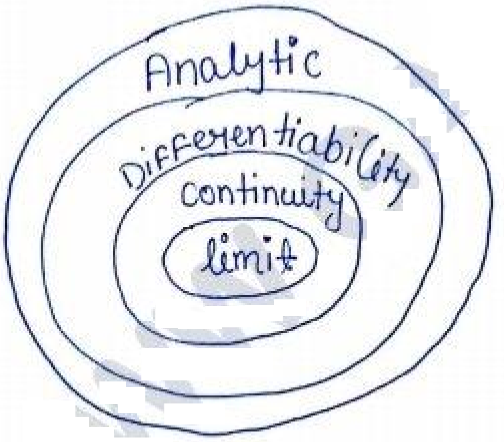
(i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

(ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Analytic function:-

A function $f(z)$ is said to be anglytic at a point z_0 if it is not only differtiable at the point z_0 but also differentiable in the neighbourhood point of z_0.





* The complex function $f(z) = u(x,y) + i\,v(x,y)$ is to be an analytic function if it satisfies the following two conditions of Cauchy - Riemann theorem.

(i) $u_x = v_y$ & $u_y = -v_x$ (C-R eq^{n}s)

(ii) u_x, u_y, v_x, v_y are continuous function of x & y

* If $f(z) = u(x,y) + iv(x,y)$ is analytic then ($u(x,y)$ & $v(x,y)$) are orthogonal to each other and $u(x,y), v(x,y)$ are homogeneous fu^n of x & y

i.e. $\boxed{u_{xx} + u_{yy} = 0}$; $\boxed{v_{xx} + v_{yy} = 0}$

~~Question~~ eg $f(z) = z^2$ analytic every where.

$$\underset{u+iv}{\downarrow} = (x+iy)^2 = x^2-y^2+i(2xy)$$

$u = x^2-y^2$ $\qquad$ $v = 2xy$

$u_x = 2x$, $u_y = -2y$ $\qquad$ $v_x = 2y$, $v_y = 2x$

clearly u_x, v_y, u_y, v_x are continuous

$u_x = v_y$ & $u_y = v_x$ C-R eq^n are satisfies at all points of z

$\therefore$ $f(z) = z^2$ is analytic every where.

eg $f(z) = \bar{z}$ is no where analytic

$\downarrow$

$u+iv = x-iy$

$u = x$ $\qquad$ $v = -y$

$u_x = 1$, $u_y = 0$ $\qquad$ $v_x = 0$, $v_y = -1$

$u_x \neq v_y$ Not satisfies on any point

$\therefore$ $f(z) = \bar{z}$ is not analytic at any point.

Milne-thomson method

when real part of analytic fcn $u(x,y)$ is given

1. Find u_x, u_y
2. we know that $f'(z) = u_x + iv_x$
$$= u_x - iu_y$$
3. Put $x=z$, $y=0$ in $f'(z)$ and then integrate wrt z to get $f(z)$.

Eg if $f(z) = u + iv$ is analytic & $u = \sin x \cosh y$ then $f(z)$ is ?

Soln

$$u = \sin x \cosh y$$

$$u_x = \cosh y \cdot \cos x \qquad u_y = \sin x \sinh y$$

we know that $f'(z) = u_x + i v_x$

$$= u_x - i u_y$$

$$f'(z) = \cosh y \cdot \cos x - i(\sin x \cdot \sinh y)$$

Put $x = z$, $y = 0$ in $f'(z)$

$$f'(z) = \cos z \cosh 0 - i \sin z \sinh 0$$

$$f'(z) = \cos z$$

$$\int f'(z) = \int \cos z \, dz$$

$$f(z) = \sin z$$

$$f(z) = \sin z$$

$$u + iv = \sin(x+iy)$$

$$= \sin x \cos(iy) + \cos x \sin(iy)$$

$$\sin(i\theta) = i \sinh\theta$$

$$\cos(i\theta) = \cosh\theta$$

$$\therefore u + iv = \sin x \cosh y + i \cos x \sinh y$$

$$v = \cos x \sinh y.$$

when Imaginary part of an analytic fuⁿ $v(x,y)$ is given

① find v_x, v_y

② we know that $f'(z) = u_x + i v_x$

$$= v_y + i v_x$$

③ put $x = z$, $y = 0$ in $f'(z)$ and then Integrate w.r.t. z to get $f(z)$.

Eg If $f(z) = u + iv$ is analytic & $v = 3x^2y - y^3$ then $f(z) =$ ___

Solⁿ $v = 3x^2y - y^3$

$$v_x = 6xy \qquad v_y = 3x^2 - 3y^2$$

we know that $f'(z) = U_x + iV_x$

$= V_y + iV_x$

$f'(z) = 3x^2 - 3y^2 + i(6xy)$

Put $x = z$, & $y = 0$ in $f'(z)$

$f'(z) = 3z^2.$

$\int f'(z) = \int 3z^2$

$F(z) = z^3$

Ques(1) WB $z = \ln(\sqrt{i})$

$z = \ln\left((0+i)^{1/2}\right) = \ln\left(\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2}\right)$

$= \ln\left(\left(e^{i\pi/2}\right)^{1/2}\right) = \ln\left(e^{i\pi/4}\right)$

$= i\pi/4$ (c) An

Ques $z = i^i = (0+i)^i = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^i$

$z = \left(e^{i\pi/2}\right)^i = e^{-\pi/2}$

Q.2 $f(z) = u + iv$ $\left\{ \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array} \right.$

$u = x^2 - y^2 + xy$

harmonic conjugate

$u_x = 2x + y$ $\qquad u_y = -2y + x$

$f'(z) = \cancel{u_x + iv_y}\ u_x + iv_x = u_x - iu_y$

$= \cancel{u_x + iv_x}$

$f'(z) = 2x + y - i(-2y + x)$

$x = z,\ y = 0$

$f'(z) = 2z - i(z)$

$\int f'(z) = \int z(2-i)$

$f(z) = \frac{z^2}{2}(2-i) = \frac{1}{2}(x^2 - y^2 + ixy)(2-i)$

$f(z) = x^2 - y^2 + xy + i\left(2xy + \frac{1}{2}(x^2 - y^2)\right)$

Q.4 $u = 2x(1-y)$

$u_x = 2(1-y) = v_y$, $u_y = -2x = -v_x$

✓ (a) $x^2 - (y-1)^2 \rightarrow v_y = 0 - 2(y-1)$, $v_x = -2x$

✗ (b) $(x-1)^2 + y^2 \rightarrow v_y = 2y$

✗ (c) $(x-1)^2 - y^2 \rightarrow u_y = -2y$

✗ (D) $x^2 + (y-1)^2$ $v_y = 2(y-1)$

$u_x = 2x\,0$

W.A Q 3 $\quad F(z) = z^2$ maps first quardrant onto

-,+	+,+
-,-	+,-

$= (x+iy)^2$

$= x^2-y^2+i(2xy)$

$z = x+iy \qquad z^2 = x^2-y^2+i(2xy)$

$$\left.\begin{matrix} 1+i \\ 1+2i \\ 2+i \\ 1+3i \end{matrix}\right\} \mathrm{I} \qquad \left.\begin{matrix} 0+2i \\ -3+4i \\ 3+4i \\ -8+6i \end{matrix}\right\}$$

upper half

I & II

Q.5 $\quad x^3 = 1 \rightarrow 1, \omega, \omega^2$

$i^3 = 1$

$\omega^3 = 1$

$(\omega^2)^3 = 1$

$(x-1)^3 = -8 =$

$(x-1) = -2, -2\omega, -2\omega^2$

$x = -1, 1-2\omega, 1-2\omega^2$

$(x-1)^3 = -8 \qquad (x-1)^3 = -8 \qquad (x-1)^3 = -8$

$(x-1)^3 = (-2)^3 \qquad (x-1)^3 = (-2\omega)^3 \qquad (x-1)^3 = (-2\omega^2)^3$

$x-1 = -2 \qquad x-1 = -2\omega \qquad x-1 = -2\omega^2$

$x = -1, \quad x = 1-2\omega, \quad x = 1-2\omega^2$

Q.6
W B.

$f(z) = x^2 + iy^2$

$u = x^2 \qquad v = y^2$

$u_x = 2x \qquad v_x = 0$

$u_y = 0 \qquad v_y = 2y$

at $x + iy = 0 + i0$ } satisfies
$x = 0,\ y = 0$

$u_x \neq u_y$ C-R
C-R eqn are satifies at only $z = 0$ but not neighbourhood pt
$\therefore f(z)$ not analytic

(d)

$g(z) = x^2 + y^2 + ixy$

$u = x^2 + y^2 \qquad v = xy$

$u_x = 2x \qquad v_x = y$

$u_y = 2y \qquad v_y = x$

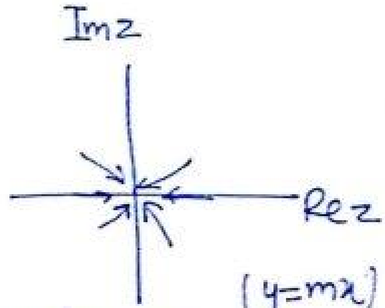
$u_x \neq u_y \quad (u_y \neq (-v_x))$

Not analytic
Similarly

Q.7

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x - iy}{x + iy}$$

$$\lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{x^2 - y^2}{x^2 + y^2} \right) - i \left(\frac{2xy}{x^2 + y^2} \right)$$



there are too many path
at every path limit should be equal
$y = mx$

$$\lim_{x \to 0} \left(\frac{x^2 - m^2x^2}{x^2 + m^2y^2} \right) - i \left(\frac{2x \cdot mx}{x^2 + m^2y^2} \right)$$

$$\lim_{x \to 0} \left(\frac{1 - m^2}{1 + m^2} \right) - i \left(\frac{2m}{1 + m^2} \right)$$

slope	path	limit
$m = 1$	$y = x$	$l = -i$
$m = 2$	$y = 2x$	$l = -\frac{3}{4} - i\frac{4}{5}$

limit does not exist

lt at $m = 1 \neq$ lt at $m = 2$

Zero of the function :- A point at which functional value is zero is known as zero of the function.

Pole of the function: A point at which function value is infinite is known as pole of the fu^n.

Singular Point :- A point at which function is not analytic is called a ~~sign~~ singular point

Eg

$$f(z) = \frac{z-1}{(z-3)(z-5)}$$

$z = 1$ is zero of the function

$z = 3, 5$ pole of the function

Taylor series :- If $f(z)$ is analytic at a point z_0 then $f(z)$ can be expressed as-

$$f(z) = f(z_0) + (z-z_0) f'(z) + \frac{(z-z_0)^2}{2!} f''(z) + \frac{(z-z_0)^3}{3!} f'''(z) + \dots$$

Eg:-The T.S.E. of $f(z) = \sin z$ at $z = \pi/4$ is - - -

$f(z) = \sin z$, $f'(z) = \cos z$, $f''(z) = -\sin z$, $f'''(z) = -\cos z$

$f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$, $f'(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$, $f''(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$, $f'''(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$

$$f(\pi/4) = f(\pi/4) + (z-\pi/4) f'(\pi/4) + (z-\frac{\pi}{4})^2 f''(\frac{\pi}{2})$$

$$= \frac{1}{\sqrt{2}} + \left(z-\frac{\pi}{4}\right)\frac{1}{\sqrt{2}} - \left(z-\frac{\pi}{4}\right)^2 \frac{1}{\sqrt{2}} - \left(z-\frac{\pi}{4}\right)^3 \left(\frac{1}{\sqrt{2}}\right) - -$$

Ques The T.S.E. of $f(z) = \frac{1}{z+4}$ at $z = 2$

$\downarrow$

let $z - 2 = t$ $\quad\quad$ $z - 2 = 0$

$z = t + 2$

$$f = \frac{1}{t+2+4} = \frac{1}{6(1+t/6)} = \frac{1}{6}\left(1+t/6\right)^{-1}$$

$$= \frac{1}{6}\left(1 - \frac{t}{6} + \frac{t^2}{6} - \frac{t^3}{6} + \cdots\right.$$

$$= \frac{1}{6}\left(1 - \frac{(z-2)}{6} + \frac{(z-2)^2}{6} - \frac{(z-2)^2}{6} + \cdots\right)$$

Question

The laurant Series Expansion $f(z) = \frac{1}{(z+1)(z+3)}$ in valid region $1 < |z| < 3$

$1 < |z| < 3$

$1 < |z|$ & $|z| < 3$

$\frac{1}{|z|} < 1$ & $\frac{|z|}{3} < 1$ $\leftarrow$ $a + ar + ar^2 - - - = \frac{a}{1-r}$

when $|r| < 1$

$$f(z) = \frac{1/2}{z+1} - \frac{1/2}{z+3}$$

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$$f(z) = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} \cdots\right)$$

$$- \frac{1}{6}\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 - - - \right)$$

Ans

Repeat if $|z| > 3$

$$3 < |z|$$

$$\frac{3}{|z|} < 1$$

$$f(z) = \frac{1/2}{(z+1)} - \frac{1/2}{z+3}$$

$$= \frac{1}{2z(1+1/z)} - \frac{1}{2(1+3/z)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + -\right) - \frac{1}{2z}\left(1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - -\right)$$

Singularities

In laurant series

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{Analytic part}} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{principal part.}}$$

(i) If there are no terms in the principal part of f(z) then z_0 is called Removable singular point.

eg $f(z) = \frac{z - \sin z}{z^3} = \frac{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} - \cdots\right)}{z^3}$

$$f(z) = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} \cdots$$

No negative power of z

$\therefore$ $z = 0$ is Removable singular.

(ii) If there are Infinite number of terms in the principal part of f(z) then z_0 is called Essential singular point.

Ex $f(z) = e^{1/z} = 1 + \frac{1/z}{1!} + \frac{1/z^2}{2!} + \frac{1/z^3}{3!} + \cdots$

Infinite numbers of -ve power

$\therefore$ $z = 0$ is essential singular point.

(iii) If there are finite Number of term in the principal part of $f(z)$ then z_0 is called a pole.

Eg: $f(z) = \frac{\sin z}{z^4} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} - \cdots}{z^4}$

$$f(z) = \underbrace{\frac{1}{z^3} - \frac{1}{3!\,z}} + \frac{z}{5!} - \frac{z^3}{7!} - -$$

finite number of $-$ve power of z

$\therefore$ $z = 0$ is a pole.

Residue:- In laurant series expansion, the coefficient of $\frac{1}{z-z_0}$ is called Residue of $f(z)$ at $z = z_0$.

eg $f(z) = \frac{z+1}{(z-2)(z-4)} = \frac{-3/2}{z-2} + \frac{5/2}{z-4}$

$$Res(z)_{at\, z=2} = -3/2$$

$$Res(z)_{at\, z=4} = \frac{5}{2}$$

eg $f(z) = e^{1/z} = 1 + \frac{1/z}{1!} + \frac{1/z^2}{2!} + \frac{1/z^3}{3!} + \cdots$

$$Res(z)_{at\, z=0} = \frac{1}{1!} = 1$$

eg $F(z) = \dfrac{\sin z}{z^4} = \dfrac{1}{z^3} - \dfrac{1}{3!\,z} + \dfrac{z}{5!} - \dfrac{z^3}{7!}$

$\underset{\text{at } z=0}{\operatorname{Res}(z)} = \dfrac{-1}{3!} = -1/6$

* If $z = z_0$ is a simple pole of $F(z)$ then

$$\boxed{\underset{z=z_0}{\operatorname{Res}(f(z))} = \lim_{z\to z_0} (z-z_0)\,f(z)}$$

eg $f(z) = \dfrac{z+1}{(z-2)(z-4)}$ poles are $= 2, 4$ simple poles

$\underset{\text{at } z=2}{\operatorname{Res} f(z)} = \lim_{z\to 2} \dfrac{\cancel{(z-2)}\,(2+1)}{\cancel{(z-2)}\,(2-4)} = -3/2$

$\underset{\text{at } z=4}{\operatorname{Res} f(z)} = \lim_{z\to 4} \dfrac{\cancel{(z-4)}\,(z+1)}{(z-2)\cancel{(z-4)}} = \lim_{z\to 4} \dfrac{z+1}{z-2} = \dfrac{4+1}{4-2} = 5/2$

* If $z = z_0$ is a pole of order m then

$$\boxed{\underset{z=z_0}{\operatorname{Res} f(z)} = \lim_{z\to z_0} \dfrac{1}{(m-1)!}\,\dfrac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m . f(z)\right)}$$

eg $f(z) = \dfrac{e^{3z}}{(z-1)^5}$; $z = 1$ is a pole of order $= 5$

$$\underset{\text{at } z=1}{\text{Res}(z)} = \lim_{z \to 1} \frac{1}{4!} \frac{d^4}{dz^4}\left(\frac{(z-1)^5 e^{3z}}{(z-1)^5}\right) = \lim_{z \to 1} \frac{1}{24} \frac{d^4 (e^{3z})}{dz^4}$$

$$\underset{\text{at } z=1}{\text{Res}(z)} = \lim_{z \to 1} \frac{81}{24} e^{3z} = \frac{81}{24} e^{3}$$

$$\underset{\text{at } z=1}{\text{Res}(z)} = \frac{27}{8} e^{3}$$

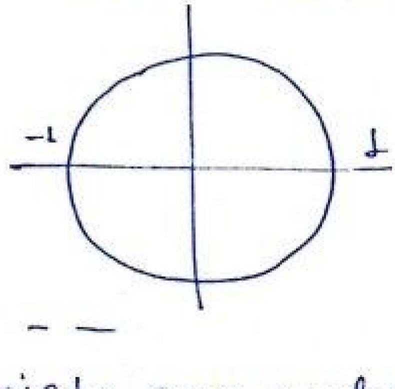
Cauchys Integral theorem

If $f(z)$ is analytic and $f'(z)$ is continuous ~~whin~~ within and on the boundary of a simple closed curve C then

$$\oint_z f(z)\,dz = 0$$

eg The value of $\int_C \sec z \, dz$ where C is $|z|$

$$\int f(z)dz = \int_C \sec z = \int_C \frac{1}{\cos z}$$



$$\pm \frac{3.14}{2} = \pm 1.07$$

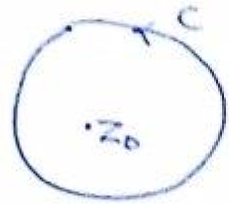
Singular point $= \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$ - -

all the above singular points are outside C so function is analytic inside & boundary of C → $|z|$ ∴ $\int_C \sec z \, dz = 0$

Cauchy's Integral formula :-

If $f(z)$ is analytic at all points within and on the the boundary of a simple closed curve C and z_0 is any point inside C (except z_0) then

$$\boxed{\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)}$$



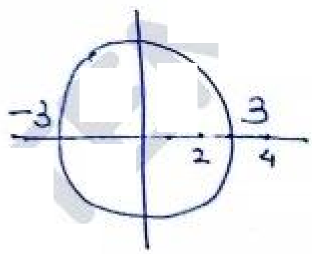
Note:- $\boxed{\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!}\, f^{(n)}(z_0)}$ where $f^{n}(z_0) = \left.\frac{d^n}{dz^n}\big((f(z))\big)\right|_{z=z_0}$

Question The value of $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-4)}$ where $C : |z| = 3$

Sol^n Singular point

$z = 2, 4$

2 → inside, 4 → outside



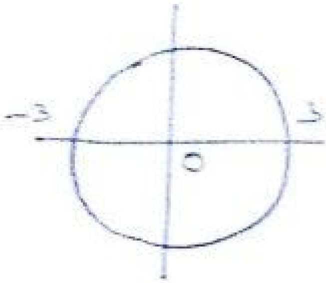
$$\oint_C \frac{\frac{\sin \pi z^2 + \cos \pi z^2}{(z-4)}}{(z-2)}$$

So $\oint f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-4)}$

$$\oint \frac{f(z)}{(z-2)} = 2\pi i\, f(2) = \frac{2\pi i \{\sin 4\pi + \cos 4\pi)}{(2-4)} = \frac{2\pi i(0+1)}{-2}$$

$$\oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-4)} = -\pi i$$

Ques The value of $\int_C \frac{e^{2z}}{(z+1)^4}\, dz$ $C \Rightarrow |z| = 3$



Pole = $-1 = z$

$$\int \frac{f(z)}{(z+1)^{n+1}} = \frac{2\pi i}{n!} f^{n}(z_0)$$

$$\int \frac{e^{2z}}{(z+1)^{3+1}} = \frac{2\pi i}{3!} f'''(z_0) = \frac{2}{6}\pi i\, f'''(-1)$$

$$f(z) = e^{2z}$$

$$f'(z) = 2e^{2z},\ f''(z) = 4e^{2z},\ f'''(z) = 8e^{2z}$$

$$\int_C \frac{e^{2z}}{(z+1)^4}\, dz = \frac{1}{3}\pi i\ 8 e^{-2} = \frac{8\pi}{3} i\, e^{-2}$$

Cauchy Integral formula :-

If $f(z)$ is analytic at all point within and on boundary of a simple closed curve C and z_0 is any point inside C then.

Cauchy Residue Theorem: —

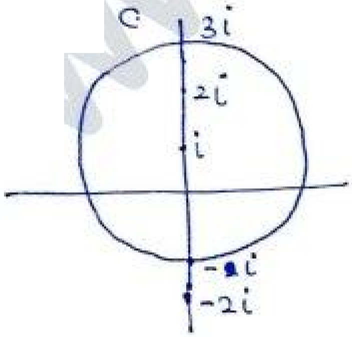
If $f(z)$ is analytic at all points inside and on the boundary of a simple closed curve C except at finite no. of poles lies inside C then

$$\oint_C f(z)\,dz = 2\pi i \left\{ \begin{array}{l} \text{Sum of residues of } f(z) \text{ at all} \\ \text{it poles inside } C \end{array} \right\}$$

(not even on boundary

$-2\pi i$ (sum)
↓
for clockwise

eg $\displaystyle\int_C \frac{1}{z^2+4}\,dz \qquad C = |z-i| = 2$



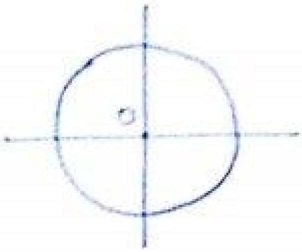
$z^2 + 4 = 0$

$z = \pm 2i$ poles.

$$\operatorname*{Res}_{z=2i} f(z) = \lim_{z \to 2i} \frac{\cancel{(z-2i)}\;1}{\cancel{(z-2i)}(z+2i)} = \frac{1}{2i+2i} = \frac{1}{4i}$$

By CRT so $\displaystyle\oint \frac{1}{z^2+4}\,dz = 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{2}$

eg Value of $\int_c e^{1/z} dz$ where c is $|z| = 1$



singular point $z = 0$

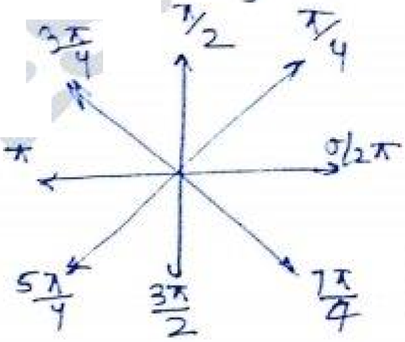
Residue of $e^{1/z}$ when $z = 0$, coefficient of $\frac{1}{z}$ in

$$e^{1/z} = 1 + \frac{1/z}{1!} + \frac{1/z^2}{2!} + \frac{1/z^3}{3!} \dots$$

$$\text{coeff} = \frac{1}{1!} = 1$$

So by CRT $\oint e^{1/z} dz = 2\pi i(1) = 2\pi i$

eg The value of $\int_{-\infty}^{\infty} \frac{1}{x^4+1}$ by Complex integral is



By Jordan's theorem $\int_{-\infty}^{\infty} \frac{1}{x^4+1} = \int \frac{1}{z^4+1} dz$

Poles are $z = (-1)^{1/4}$

$$z = \underbrace{e^{i\pi/4}, e^{i3\pi/4}}_{\text{upper half}}, \underbrace{e^{i5\pi/4}, e^{i7\pi/4}}_{\text{lower half.}}$$

Consider upper half; neglect lower half

$$\underset{z=e^{i\pi/4}}{\text{Res}}(f(z)) = \underset{z\to e^{i\pi/4}}{Lt} \left(z - e^{i\pi/4}\right)\frac{1}{z^4+1} \overset{(0/0)}{=} \underset{z\to e^{i\pi/4}}{Lt} \frac{1}{4} e^{-i3\pi/4} = \frac{1}{4}\left(\frac{-1}{2} - \frac{i}{\sqrt{2}}\right)$$

$$\underset{z=e^{i3\pi/4}}{\text{Res}}(f(z)) = \underset{z\to e^{i3\pi/4}}{Lt} \left(\cancel{z}\ \frac{1}{4} e^{-9\pi/4} = \frac{1}{4}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right.$$

by CRT

$$I = \int \frac{1}{z^4+1}\,dz = 2\pi i \times \frac{1}{4}\left\{\cancel{-\frac{1}{\sqrt{2}}}-\frac{i}{\sqrt{2}}+\cancel{\frac{1}{\sqrt{2}}}-\frac{i}{\sqrt{2}}\right\}$$

$$= \frac{-2\pi\, i \times i}{4 \times \sqrt{2}} = \frac{\pi}{\sqrt{2}} \text{ Ans}$$

Question The value $\int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+2}$ by Complex Integral.

$$\text{I.P of } \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+2x+2}\,dx = \text{I.P of } \int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+2z+2}\,dz$$

I.P- Imaginary part

Poles $z^2+2z+2 = 0$

$$z = -1 \pm i$$
$$= \underbrace{-1+i}_{\text{II qr}\ (\checkmark)},\ \underbrace{-1-i}_{(\times)} \text{ III quard.}$$

$$\underset{z\to -1+i}{\text{Res}} = \lim_{z=-1+i} \frac{\cancel{(z-(-1+i))}\, e^{iz}}{\cancel{(z-(-1+i))}(z-(-1-i))} = \frac{e^{i(-1+i)}}{(-1+i-(-1-i))}$$

$$\underset{z\to -1+i}{\text{Res}} = \frac{e^{-1-i}}{2i}$$

By CRT

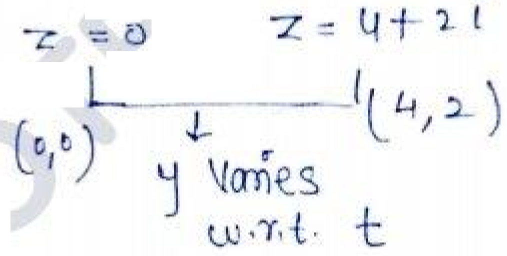
$$\int I = \text{I.P. of } \left\{2\pi i \left(\frac{e^{-i}e^{-1}}{2i}\right)\right.$$

$$= \text{I.P. of } \left\{ \pi e^{-1}\cos 1 - i\pi e^{-1}\sin 1\right.$$

$$I = -\pi e^{-1}\sin 1$$

$$* \quad I = \int_{-\infty}^{\infty} \frac{\cos x}{x^2+2x+2} = \pi e^{-1} \cos 1$$

Q.9 (W.B.) $\int_C \bar{z}\, dz$ where $C = z = t^2 + it$



$$= \int_0^2 (t^2 - it)(2t + i)\, dt$$

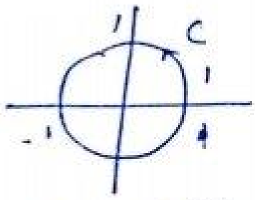
$$= \int_0^2 (2t^3 + it^2 - 2t^2 i + t)\, dt$$

$$= \int_0^2 (2t^3 + t)\, dt - i \int_0^2 t^2\, dt$$

$$= \left[\frac{2t^4}{4} + \frac{t^2}{2}\right]_0^2 - i\left[\frac{t^3}{3}\right]_0^2$$

$$= 10 - \frac{8i}{3}$$

Q.19 $\oint \frac{1}{z^2}\, dz$



$z = 0$ pole

$$\operatorname{Res}_{z=0}(z) = \lim_{z \to 0} \frac{1}{1!} \frac{d}{dz}\left\{(z-0)^2 \times \frac{1}{z^2}\right\} = 0$$

By CRT $\oint \frac{1}{z^2}\, dz = 2\pi i(0) = 0$

Q.34 $x \quad , \quad x \ln x \quad , \quad x^2$

$$y = c_1 x + c_2 x \ln x + c_3 x^2$$

$$= (c_1 + c_2 \ln x) x + c_3 x^2$$

$$y = (c_1 + c_2 z) e^z + c_3 e^{2z}$$

root $\rightarrow$ 1, 1, 2

D.e. $(\theta - 1)(\theta - 1)(\theta - 2) y = 0$ (c)