Fill Ups of Mathematical Induction and Binomial Theorem

Q.1. The larger of $99^{50} + 100^{50}$ and 101^{50} is .... (1982 - 2 Marks)

Ans. $(101)^{50}$

Sol. Consider $(101)^{50} - \{(99)^{50} + (100)^{50}\} = (100 + 1)^{50} - (100 - 1)^{50} - (100)^{50}$

$= (100)^{50} [(1+ 0.0 1)^{50} - (1- 0.01)^{50} - 1]$

$= (100)^{50} [2 (^{50}C_1(0.01) + {}^{50}C_3(0.01)^3 + ...) - 1]$

$= (100)^{50} [2 (^{50}C_3(0.01)^3 + ...)] > 0$

$\therefore (101)^{50} > (99)^{50} + (100)^{50}$

$\therefore$ $(101)^{50}$ is greater.

Q.2. The sum of the coefficients of the plynomial $(1 + x - 3x^2)^{2163}$ is .... (1982 - 2 Marks)

Ans. -1

Sol. If we put $x = 1$ in the expansion of $(1+ x - 3x2)^{2163} = A_0 + A_1x + A_2x^2 + ...$
we will get the sum of coefficients of given polynomial, which clearly comes to be – 1.

Q.3. If $(1 + ax)^n = 1 + 8x + 24x^2 + ...$ then a = ... and n = ... (1983 - 2 Marks)

Ans. a = 2, n = 4

Sol. $(1 + ax)^n = 1 + 8x + 24x^2 + ...$

$\Rightarrow (1 + ax)^n = 1 + nxa + \frac{n(n-1)}{2!} a^2x^2 + ...$

$= 1 + 8 x + 24x 2+ ...$

Comparing like powers of x we get

$nax = 8x \Rightarrow na = 8$ ....(1)

$\frac{n(n-1)a^2}{2} = 24 \quad \Rightarrow \quad n(n-1)a^2 = 48$ ....(2)

Solving (1) and (2), $n = 4$, $a = 2$

Q.4. Let n be positive integer. If the coefficients of 2nd, 3rd, and 4th terms in the expansion of $(1 + x)^n$ are in A.P., then the value of n is .... (1994 - 2 Marks)

Ans. 7

Sol. We know that for a +ve integer n $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ...+ {}^nC_n x^n$
ATQ coefficients of 2^{nd}, 3^{nd}, and 4^{th} terms are in A.P. i.e.${}^nC_1$, ${}^nC_2$, ${}^nC_3$ are in A.P.
$\Rightarrow 2.{}^nC_2 = {}^nC_1 + {}^nC_3$

$$\Rightarrow \quad 2 \times \frac{n(n-1)}{2} = n + \frac{n(n-1)(n-2)}{3!}$$

$$\Rightarrow \quad n - 1 = 1 + \frac{n^2 - 3n + 2}{6} \Rightarrow n^2 - 9n + 14 = 0$$

$\Rightarrow (n - 7)(n - 2) = 0 \Rightarrow n = 7$ or 2
But for the existance of 4^{th} term, $n = 7$.

Q.5. The sum of the rational terms in th e expansion of $(\sqrt{2} + 3^{1/5})^{10}$ is .... (1997 - 2 Marks)

Ans. 41

Sol. Let T_{r+1} be the general term in the expansion of

$(\sqrt{2} + 3^{1/5})^{10}$

$\therefore \quad T_{r+1} = {}^{10}C_r (\sqrt{2})^{10-r}.(3^{1/5})^r.(0 \le r \le 10)$

$$= \frac{10!}{r!(10-r)!}.2^{5-r/2}.3^{r/5}$$

Let T_{r+1} will be rational if $2^{5-r/2}$ and $3^{r/5}$ are rational numbers.

$\Rightarrow 5-\frac{r}{2}$ and $\frac{r}{5}$ are integers.

$\Rightarrow r = 0$ and $r = 10 \Rightarrow T_1$ and T_{11} are rational terms.

$\Rightarrow$ Sum of T_1 and $T_{11} = {}^{10}C_0 25 - 0.30 + {}^{10}C_{10} 2^{5-5}.3^2$

$= 1.32.1 + 1.1.9 = 32 + 9 = 41$

Subjective questions of Mathematical Induction and Binomial Theorem

Q. 1. Given that (1979) $C_1 + {}^2C_2x + {}^3C_3x^2 + \ldots\ldots\ldots\ldots + 2n\, C_{2n}x^{2n-1} = 2n\,(1 + x)^{2n-1}$

where Cr = $\frac{(2n)!}{r!(2n-r)!}$ r = 0, 1, 2, ..., 2n

Prove that

$C_1^2 - 2C_2^2 + 3C_3^2 - \ldots\ldots\ldots\ldots - 2nC_{2n}^2 = (-1)^n n\, C_n.$

Ans. Sol. Given that

$C_1 + 2C_2x + 3C_3x^2 + \ldots + 2nC_{2n}x^{2n-1} = 2n\,(1 + x)^{2n-1}$ ....(1)

where $C_r = \frac{2n!}{r!(2n-r)!}$

$[C_1x + C_2x^2 + C_3x^3 + \ldots + C_{2n}x^{2n}]_0^x = [(1+x)^{2n}]_0^x$

$\Rightarrow C_1x + C_2x^2 + C_3x^3 + \ldots + C_{2n}x^{2n} = (1+x)^{2n-1}$

$\Rightarrow C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_{2n}x^{2n} = (1+x)^{2n}$ ....(2)

Changing x by $-\frac{1}{x}$, we get

$$\Rightarrow C_0 - \frac{C_1}{x} + \frac{C_2}{x^2} - \frac{C_3}{x^3} + \ldots + (-1)^{2n}\frac{C_{2n}}{x^{2n}} = \left(1 - \frac{1}{x}\right)^{2n}$$

$$\Rightarrow C_0x^{2n} - C_1x^{2n-1} + C_2x^{2n-2} - C_3x^{2n-3} + \ldots + C_{2n} = (x-1)^{2n} \quad \ldots(3)$$

Multiplying eqn. (1) and (3) and equating the coefficients of x^{2n-1} on both sides, we get

$-C_1^2 + 2C_2^2 - 3C_3^2 + \ldots + 2n\,C_{2n}^2$

= coeff.of x^{2n-1} in $2n(x - 1)(x^2 - 1)^{2n-1}$

= 2n [coeff. of x^{2n-2} in $(x^2-1)^{2n-1}$ – coeff. of x^{2n-1} in $(x^2-1)^{2n-1}$]

$= 2n\,[{}^{2n-1}C_{n-1}(-1)^{n-1} - 0]$
$= (-1)^{n-1}\,.\,2n\,{}^{2n-1}C_{n-1}$

$\Rightarrow\; C_1^2 - 2C_2^2 + 3C_3^2 + ... + 2n\,C_{2n}{}^2$

$= (-1)^n.\,2n\,{}^{2n-1}C_{n-1} = (-1)^n n.\left(\frac{2n}{n}.{}^{2n-1}C_{n-1}\right)$

$= (-1)^n n.\,{}^{2n}C_n \;=\; (-1)^n n.C_n.\; (\because {}^{2n}C_n = C_n)$

Hence Proved.

Q.2. Prove that $7^{2n} + (2^{3n-3})(3^{n-1})$ is divisible by 25 for any natural number n. (1982 - 5 Marks)

Ans.

Sol. P(n) : $7^{2n} + 2^{3n-3}, 3^{n-1}$ is divisible by 25 $\forall n \in N$.

Let us prove it by Mathematical Induction :

P(1) : $7^2 + 2^0.3^0 = 49 + 1 = 50$ which is divisible by 25.

$\therefore$ P (1) is true.

Let P(k) be true that is $7^{2k} + 2^{3k-3}, 3^{k-1}$ is divisible by 25.

$\Rightarrow 7^{2k} + 2^{3k-3}.\,3^{k-1} = 25m$ where $m \in Z$.

$\Rightarrow 2^{3k-3}\,.\,3^{k-1} = 25m - 72k$ ....(1)

Consider P(k + 1) : $7^{2(k+1)} + 2^{3(k+1)} - 3\,.3^{k+1-1}$

$= 7^{2k}.7^2 + 2^{3k}.\,3^k = 49.\,7^{2k} + 2^3\,.\,3.2^{3k-3}\,.\,3^{k-1}$

$= 49.\,7^{2k} + 24\,(25m - 7^{2k})$ (Using IH eq. (1))

$= 49.\,7^{2k} + 24 \times 25m - 24 \times 7^{2k}$

$= 25.\,7^{2k} + 24 \times 25m = 25\,(7^{2k} + 24\,m)$

$= 25 \times$ some integral value which is divisible by 25.

∴ P(k + 1) is also true.

Hence by the principle of mathematical induction

P(n) is true ∀ n∈Z .

Q.3. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ then show that the sum of the products of the C_i's taken two at a time,represented by $\sum_{0\le i<j\le n}\sum C_iC_j$ is equal to

$2^{2n-1} - \frac{(2n)!}{2(n!)^2}$ **(1983 - 3 Marks)**

Ans.

Sol. $S = \sum\sum C_iC_j$

$0 \le i < j \le n$ **NOTE THIS STEP**

$\Rightarrow S = C_0 (C_1 + C_2 + C_3 + \ldots + C_n) + C_1 (C_2 + C_3 + \ldots + C_n) + C_2 (C_3 + C_4 + C_5 + \ldots C_n)$

$+ \ldots C_{n-1}(C_n)$

$\Rightarrow S = C_0 (2^n - C_0) + C_1 (2^n - C_0 - C_1) + C_2(2^n - C_0 - C_1 - C_2) + \ldots + C_{n-1}(2^n - C_0 -$

$C_1 \ldots C_{n-1}) + C_n (2^n - C_0 - C_1 \ldots C_n)$

$\Rightarrow S = 2^n (C_0 + C_1 + C_2 + \ldots + C_{n-1} + C_n)$

$-(C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2) - S$

$$\Rightarrow \quad 2S = 2^n.2^n - \frac{2n!}{(n!)^2} = 2^{2n} - \frac{2n!}{(n!)^2}$$

$$\Rightarrow \quad S = 2^{2n-1} - \frac{2n!}{2(n!)^2}$$

Q.4. Use mathematical Induction to prove : If n is an y odd positive integer, then $n(n^2 - 1)$ is divisible by 24. (1983 - 2 Marks)

Ans. Sol. P(n) : $n(n^2 - 1)$ is divisible by 24 for n odd +ve integer.

For $n = 2m - 1$, it can be restated as P(m) : $(2m - 1)(4m^2 - 4m) = 4m(m - 1)(2m - 1)$ is divisible by 24 ∀ m ∈ N

$\Rightarrow$ P(m) : m (m – 1) (2m – 1) is divisible by 6 $\forall$ m $\in$N .

Here P(1) = 0, divisible by 6.

$\therefore$ P(1) is true.

Let it be true for m = k, i.e.,

k (k–1) (2k–1) = 6p

$\Rightarrow 2k^3 - 3k^2 + k = 6p$ ...(1)

Consider P(k + 1) : $k(k + 1)(2k + 1) = 2k^3 + 3k^2 + k$

$= 6p + 3k^2 + 3k^2$ (Using (1)

$= 6(p + k^2)$

$\Rightarrow$ divisible by 6

$\therefore$ P (k + 1) is also true.

Hence P(m) is true $\forall$ m $\in$N .

Q.5. If p be a natural number then prove that $p^{n+1} + (p + 1)^{2n-1}$ is divisible by $p^2 + p + 1$ for every positive integer n. (1984 - 4 Marks)

Ans. Sol. P(n) : $P^{n+1} + (p + 1)^{2n-1}$ is divisible by $p^2 + p + 1$

For n = 1, P(1) : $p^2 + p + 1$

which is divisible by $p^2 + p + 1$.

$\therefore$ P(1) is true.

Let P(k) be true, i.e., $p^{k+1} + (p + 1)2^{k-1}$ is divisible by $p^2 + p + 1$

$\Rightarrow p^{k+1} + (p + 1)^{2k-1}$

$= (p^2 + p + 1)$ m ....(1)

Consider P(k + 1) : $p^{k+2} + (p + 1)^{2k+1}$

$= p \cdot p^{k+1} + (p + 1)^{2k-1}. (p + 1)^2$

$= p [m (p^2 + p + 1) - (p + 1)^{2k-1}] + (p + 1)^{2k-1}(p + 1)^2$

$= p (p^2 + p + 1)m - p (p + 1)^{2k-1} + (p + 1)^{2k-1} (p^2 + 2p + 1)$

$= p (p^2 + p + 1)m + (p + 1)^{2k-1}(p^2 + p + 1)$

$= (p^2 + p + 1) [mp + (p + 1)^{2k-1}] = (p^2 + p + 1)$

some integral value

$\therefore$ divisible by $p^2 + p + 1$

$\therefore$ P (k + 1) is also true.

Hence by principle of mathematical induction P(n) is true $\forall$ $n \in N$.

Q.6. Given $s_n = 1 + q + q^2 + \ldots\ldots + q^n$;

$S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + \ldots + \left(\frac{q+1}{2}\right)^n, q \neq 1$ **Prove that**

${}^{n+1}C_1 + {}^{n+1}C_2 s_1 + {}^{n+1}C_3 s_2 + \ldots + {}^{n+1}C_n s_n = 2^n S_n$ **(1984 - 4 Marks)**

Ans. Sol.

We have $s_n = \frac{1-q^{n+1}}{1-q}$ ....(1)

and $S_n = \frac{1-\left(\frac{q+1}{2}\right)^{n+1}}{1-\left(\frac{q+1}{2}\right)} = \frac{2^{n+1}-(q+1)^{n+1}}{2^n(1-q)}$ ....(2)

Now, ${}^{n+1}C_1 + {}^{n+1}C_2 s_1 + {}^{n+1}C_3 s_2 + \ldots + {}^{n+1}C_{n+1} s_n$

$= \frac{1}{1-q}[{}^{n+1}C_1(1-q) + {}^{n+1}C_2(1-q^2) + {}^{n+1}C_3(1-q^3) + \ldots +$

$+ \ldots + {}^{n+1}C_n(1-q^{n+1})]$ Using (1)

$$= \frac{1}{1-q}\left[\left({}^{n+1}C_1 + {}^{n+1}C_2 + \ldots + {}^{n+1}C_{n+1}\right)\right.$$

$$\left. - \left({}^{n+1}C_1 q + {}^{n+1}C_2 q^2 + \ldots + {}^{n+1}C_{n+1} q^{n+1}\right)\right]$$

$$= \frac{1}{1-q}\left[2^{n+1} - 1) - \{(1+q)^{n+1} - 1\}\right]$$

$$= \frac{2^{n+1} - (1+q)^{n+1}}{(1-q)} = 2^n S_n \text{ [Using eq. (2)]}$$

Q.7. Use meth od of mathematical induction $2.7^n + 3.5^n - 5$ is divisible by 24 for all n > 0 (1985 - 5 Marks)

Ans.

Sol. Let $A_n = 2.7^n + 3.5^n - 5$

Then $A_1 = 2.7 + 3.5 - 5 = 14 + 15 - 5 = 24$.

Hence A_1 is divisible by 24.

Now assume that Am is divisible by 24 so that we may write

$A_m = 2.7^m + 3.5^m - 5 = 24k$, $k \in N$ ....(1)

Then $A_{m+1} - A_m$

$= 2(7^{m+1} - 7^m) + 3(5^{m+1} - 5^m) - 5 + 5$

$= 2.7^m(7-1) + 3.5^m(5-1) = 12.(7^m + 5^m)$

Since 7^m and 5^m are odd integers $\forall\ m \in N$, their sum must be an even integer, say $7^m + 5^m = 2p$, $p \in N$.

Hence $A_{m+1} - A_m = 12.2p = 24p$

or $A_{m+1} = A_m + 24p = 24k + 24p$ [by (1)]

Hence A_{m+1} is divisible by 24.

It follows by mathematical induction that A_n is divisible by 24 for all $n \in N$.

Q.8. Prove by mathematical induction that – (1987 - 3 Marks)

$\frac{(2n)!}{2^{2n}(n!)^2} \le \frac{1}{(3n+1)^{1/2}}$ **for all positive Integers n.**

Ans. Sol.

Let $P(n): \frac{(2n)!}{2^{2n}(n!)^2} \le \frac{1}{(3n+1)^{1/2}}$

For n =1, $P(1): \frac{2!}{2^2(1!)^2} \le \frac{1}{(3+1)^{1/2}} \Rightarrow \frac{1}{4} \le \frac{1}{2}$

$\Rightarrow \frac{1}{2} \le \frac{1}{2}$ which is true for n =1

Assume that P(k) is true, then

$P(k): \frac{(2k)!}{2^{2k}(k!)^2} \le \frac{1}{(3k+1)^{1/2}}$ ....(1)

For n = k +1,

$$\frac{[2(k+1)]!}{2^{2(k+1)}[(k+1)!]^2} = \frac{(2k+2)!}{2^{2k+2}[(k+1)!]^2}$$

$$= \frac{(2k+2)(2k+1)(2k)!}{4.2^{2k}(k+1)^2(k!)^2}$$

$$\le \frac{(2k+2)\ (2k+1)}{4(k+1)^2} \cdot \frac{1}{(3k+1)^{1/2}}$$

[Using Induction hypothesis (1)]

$$= \frac{(2k+1)}{2.(k+1)\ (3k+1)^{1/2}}$$

Thus, $\frac{[2(k+1)]!}{2^{2(k+1)}\ [(k+1)!]^2} \le \frac{(2k+1)}{2(k+1)\ (3k+1)^{1/2}}$ ....(2)

In order to prove P(k + 1), it is sufficient to prove that

$$\frac{(2k+1)}{2(k+1)(3k+1)^{1/2}} \le \frac{1}{(3k+4)^{1/2}} \quad ....(3)$$

Squaring eq. (3), we get

$$\frac{(2k+1)^2}{4(k+1)^2(3k+1)} \le \frac{1}{3k+4}$$

$\Rightarrow (2k + 1)^2 (3k + 4) - 4 (k + 1)^2 (3k + 1) \le 0$

$\Rightarrow (4k^2 + 4k + 1) (3k + 4) - 4 (k^2 + 2k +1) (3k +1) \le 0$

$\Rightarrow (12k^3 + 28k^2 + 19k + 4) - (12k^3 + 28k^2 + 20k + 4) \le 0$

$\Rightarrow -k \le 0$

which is true.

Hence from (2) and (3), we get

$$\frac{(2k+2)!}{2^{2k+2}[(k+1)!]^2} \le \frac{1}{(3k+4)1/2}$$

Hence the above inequation is true for n = k +1 and by the principle of induction it is true for all $n \in N$.

Q.9. Let R = $(5\sqrt{5}+11)^{2n+1}$ and f = R – [R], where [ ] denotes the greatest integer function. Prove that Rf = 4^{2n+4} . (1988 - 5 Marks)

Ans. Sol. We have $5\sqrt{5}-11 = \frac{4}{5\sqrt{5}+11} < 1$

Therefore $0 < 5\sqrt{5}-11 < 1$

This gives us $0 < (5\sqrt{5}-11)^{2n+1} < 1$ for every positive integer n.

Also $(5\sqrt{5}+11)^{2n+1} - (5\sqrt{5}-11)^{2n+1}$

$= 2[{}^{2n+1}C_1(5\sqrt{5})^{2n}.11 + {}^{2n+1}C_3(5\sqrt{5})^{2n-2}.11^3 +$

$....+ {}^{2n+1}C_{2n+1}11^{2n+1}]$

$= 2[{}^{2n+1}C_1(125)^n.11 + {}^{2n+1}C_3(125)^{n-1}.11^3 +$

$....+ {}^{2n+1}C_{2n+1}11^{2n+1}]$

= 2k

....(1)

where k is some positive integer.

Let F $= (5\sqrt{5} - 11)^{2n+1}$

Then equation (1) becomes R – F = 2k

$\Rightarrow$ [R] + R – [R]– F = 2k $\Rightarrow$ [R] + f – F = 2k

$\Rightarrow$ f – F = 2k – [R] $\Rightarrow$ f – F is an integer..

But $0 \le f < 1$ and $0 < F < 1$

Therefore $-1 < f - F < 1$

Since f – F is an integer, we must have f – F = 0

$\Rightarrow$ f = F.

Now, Rf = RF = $(5\sqrt{5}+11)^{2n+1}(5\sqrt{5}-11)^{2n+1}$

$= [(5\sqrt{5})^2 - 12]^{2n+1} = 4^{2n+1}$

Q.10. Using mathematical induction, prove that(1989 - 3 Marks) ${}^mC_0\,{}^nC_k + {}^mC_1\,{}^nC_{k-1} + ... {}^mC_k\,{}^nC_0 = {}^{(m+n)}C_k,$ **where m, n, k are positive integers, and** ${}^pC_q = 0$ **for** $p < q$.

Ans.

Sol. Let the given statement be

$P(m,n): {}^mC_0\,{}^nC_k + {}^mC_1\,{}^nC_{k-1} + \ldots + {}^mC_k\,{}^nC_0 = {}^{m+n}C_k$

where m, n, k ∈ N and ${}^pC_q = 0$ for $p<q$.

As k is a positive integer and ${}^pC_q = 0$ for $p<q$.

∴ k must be a positive integer less than or equal to the smaller of m and n,

We have k = 1, when m = n = 1

$\therefore \quad P(1,1) \text{ is } {}^1C_0\,{}^1C_1 + {}^1C_1\,{}^1C_0 = {}^2C_1 \Rightarrow 1 + 1 = 2.$

Thus P (1, 1) is true.

Now let us assume that P(m, n) holds good for any fixed value of m and n i.e.

$${}^mC_0\,{}^nC_k + {}^mC_1\,{}^nC_{k-1} + \ldots + {}^mC_k\,{}^nC_0 = {}^{m+n}C_k \quad \ldots(1)$$

$${}^{m+1}C_0\,{}^{n+1}C_k + {}^{m+1}C_1\,{}^{n+1}C_{k-1} + \ldots + {}^{m+1}C_k\,{}^{n+1}C_0$$

$$= {}^{m+n+2}C_k \quad \ldots(2)$$

Consider LHS

$$= {}^{m+1}C_0\,{}^{n+1}C_k + {}^{m+1}C_1\,{}^{n+1}C_{k-1} + \ldots + {}^{m+1}C_k\,{}^{n+1}C_0$$

$$= 1.({}^nC_{k-1} + {}^nC_k) + ({}^mC_0 + {}^mC_1)({}^nC_{k-2} + {}^nC_{k-1})$$

$$+ \ ({}^mC_1 + {}^mC_2)({}^nC_{k-3} + {}^nC_{k-2}) + \ldots + ({}^mC_{k-1} + {}^mC_k).1$$

$$= \ ({}^nC_{k-1} + {}^mC_1\,{}^nC_{k-2} + {}^mC_2\,{}^nC_{k-3} + \ldots + {}^mC_{k-1}\,{}^nC_0)$$

$$+ ({}^nC_k + {}^mC_1\,{}^nC_{k-1} + {}^mC_2\,{}^nC_{k-2} + \ldots + {}^mC_{k-1}\,{}^nC_1 + {}^mC_k)$$

$$+ ({}^mC_0\,{}^nC_{k-2} + {}^mC_1\,{}^nC_{k-3} + \ldots + {}^mC_{k-2}\,{}^nC_0)$$

$$+ ({}^mC_0\,{}^nC_{k-1} + {}^mC_1\,{}^nC_{k-2} + {}^mC_2\,{}^nC_{k-3}$$

$$+ \ldots + {}^mC_{k-2}\,{}^nC_1 + {}^mC_{k-1})$$

$= {}^{m+n}C_{k-1} + {}^{m+n}C_k + {}^{m+n}C_{k-2} + {}^{m+n}C_{k-1}$ [Using (1)]

$= {}^{m+n+1}C_k + {}^{m+n+1}C_{k-1} = {}^{m+n+2}C_k$

Hence the theorem holds for the next integers m + 1 and n +1. Then by mathematical induction the statement P (m, n) holds for all positive integral values of m and n.

Q.11. Prove that

$C_0 - 2^2C_1 + 3^2C_2 - .... + (-1)^n (n+1)^2C_n = 0, n > 2$, where $C_r = {}^nC_r$.

Ans. Sol. We know that $(1 – x)^n = C_0 – C_1x + C_2x^2 – C_3x^3 +....+ (– 1)^n C_nx^n$

Multiplying both sides by x, we get

$x(1 – x)^n = C_0 x – C_1x^2 + C_2x^3 – C_3x^4 +....+ (– 1)^n C_nx^{n+1}$

Differentiating both sides w.r. to x, we get

$(1 – x)^n – nx (1 – x)^n – 1 = C_0 – 2C_1 x + 3C_2 x^2 – 4C_3x^3 +....+ (– 1)^n (n + 1) C_nx^n$

Again multiplying both sides by x, we get

$x (1 – x)^n – nx^2 (1 – x)n – 1 = C_0x – 2C_1x^2 + 3C_2x^3 – 4C_3x^4 +....+ (– 1)^n (n + 1) C_nx^{n + 1}$

Differentiating above with respect to x, we get

$(1 – x)^n – nx (1– x)^{n – 1}– 2nx (1– x)^{n – 1} + nx^2 (n – 1) (1 – x)^{n – 2}$

$= C_0 – 2^2 C_1 x + 32 C_2 x^2 – 4^2 C_3 x^3+....+ (– 1)^n (n + 1)^2 C_n x^n$

Substituting x = 1, in above, we get

$0 = C_0 – 2^2C_1+ 3^2C_2– 4^2C_3 +....+ (– 1)^n (n + 1)^2C_n$

Hence Proved.

Q.12. Prove that $\frac{n^7}{7}+\frac{n^5}{5}+\frac{2n^3}{3}-\frac{n}{105}$ is an integer for every positive integer n. (1990 - 2 Marks)

Ans. Sol. We have

P(n) : $\frac{n^7}{7}+\frac{n^5}{5}+\frac{2n^3}{3}-\frac{n}{105}$ is an integer, ∀ n∈N

P(1) : $\frac{1}{7}+\frac{1}{5}+\frac{2}{2}-\frac{1}{105}$

$=\frac{15+21+70-1}{105}=\frac{105}{105}$ = 1 an integer

∴ P(1) is true

Let P(k) be true i.e.

$\frac{k^7}{7}+\frac{k^5}{5}+\frac{2k^3}{3}-\frac{k}{105}$ is an integer

$\Rightarrow \frac{k^7}{7}+\frac{k^5}{5}+\frac{2k^3}{3}-\frac{k}{105}$ = m, (say)

m ∈N ....(1)

Consider P(k + 1) :

$=\frac{(k+1)^7}{7}+\frac{(k+1)^5}{5}+\frac{2(k+1)^3}{3}-\frac{(k+1)}{105}$

$=\left(\frac{k^7+7k^6+21k^5+35k^4+35k^3+21k^2+7k+1}{7}\right)$

$+\left(\frac{k^5+5k^4+10k^3+10k^2+5k+1}{5}\right)$

$+2\left(\frac{k^3+3k^2+3k+1}{3}\right)-\left(\frac{k+1}{105}\right)$

$=\left(\frac{k^7}{7}+\frac{k^5}{5}+\frac{2k^3}{3}-\frac{k}{105}\right)$

$+[k^6+3k^5+5k^4+5k^3+3k^2+k+k^4$

$+2k^3+2k^2+k+2k^2+2k]+\left(\frac{1}{7}+\frac{1}{5}+\frac{2}{3}-\frac{1}{105}\right)$

= m + some integral value + 1

= some integral value

∴ P (k + 1) is also true.

Hence P (n) is true ∀ n ∈N , (by the Principle of Mathematical Induction.)

Q.13. Using in duction or otherwise, prove that for any nonnegative integers m, n, r and k, (1991 - 4 Marks)

$$\sum_{m=0}^{k}(n-m)\frac{(r+m)!}{m!} = \frac{(r+k+1)!}{k!}\left[\frac{n}{r+1}-\frac{k}{r+2}\right]$$

Ans. Sol. Let $P(k)=\sum_{m=0}^{k}\frac{(n-m)(r+m)!}{m!} = \frac{(r+k+1)!}{k!}\left[\frac{n}{r+1}-\frac{k}{r+2}\right]$

For k = 1, we will have two terms, on LHS, in sigma for m = 0 and m = 1, so that

LHS $=(n-0)\frac{r!}{0!}+(n-1)\frac{(r+1)!}{1!}$

and RSH $=\frac{(r+2)!}{1!}\left[\frac{n}{r+1}-\frac{1}{r+2}\right]$

Hence LHS = RHS for k = 1.

Now let the formula holds for k = s, that is let

$$\sum_{m=0}^{s}\frac{(n-m)\,(r+m)!}{m!} = \frac{(r+s+1)!}{s!}\left(\frac{n}{r+1}-\frac{s}{r+2}\right)\ldots(1)$$

Let us add next term corresponding to m = s + 1 i.e.

adding $\frac{(n-s-1)(r+s+1)!}{(s+1)!}$ to both sides, we get

$$\sum_{m=0}^{s+1}\frac{(n-m)\,(r+m)!}{m!} = \frac{(r+s+1)!}{s!}\left[\frac{n}{r+1}-\frac{s}{r+2}\right]$$

$$+\frac{(n-s-1)(r+s+1)!}{(s+1)!}$$

$$= \frac{(r+s+1)!}{(s+1)!}\left[\frac{(s+1)n}{r+1}-\frac{s(s+1)}{r+2}+n-s-1\right]$$

$$= \frac{(r+s+1)!}{(s+1)!}\left[n\left\{\frac{s+1}{r+1}+1\right\}-(s+1)\left\{\frac{s}{r+2}+1\right\}\right]$$

$$= \frac{(r+s+2)\,(r+s+1)!}{(s+1)!}\left[\frac{n}{r+1}-\frac{s+1}{r+2}\right]$$

Hence the formula holds for k = s + 1 and so by the induction principle, the formula holds for all natural numbers k.

Q.14. If $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$ **and ak = 1 for all k ≥ n, then show that** $b_n = {}^{2n+1}C_{n+1}$ **(1992 - 6 Marks)**

Ans.

Sol. Given that

$$\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r \ldots.(1)$$

and $a_k = 1, \forall k \geq n$

To prove $b_n = {}^{2n+1}C_{n+1}$ In the given equation (1) let us put x –3 = y so that x – 2 = y + 1 and we get

$$\sum_{r=0}^{2n} a_r(1+y)^r = \sum_{r=0}^{2n} b_r(y)^r$$

$$\Rightarrow \quad a_0 + a_1(1+y) + \ldots a_{n-1} + (1+y)^{n-1}(1+y)^n + (1+y)^{n+1} + \ldots (1+y)^{2n}$$

$$= \sum_{r=0}^{2n} b_r y^r \text{ [Using } a_k = 1, \forall\, k \geq n]$$

NOTE THIS STEP :

$\Rightarrow {}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \ldots + {}^{2n}C_n = b_n$

$\Rightarrow ({}^{n+1}C_{n+1} + {}^{n+1}C_n) + {}^{n+2}C_n + \ldots + {}^{2n}C_n = b_n$

[Using ${}^nC_n = {}^{n+1}C_{n+1} = 1$]

$\Rightarrow b_n = {}^{n+2}C_{n+1} + {}^{n+2}C_n + ... + {}^{2n}C_n$

[Using ${}^mC_r + {}^mC_{r-1} = {}^{m+1}C_r$]

Combining the terms in similar way, we get

$\Rightarrow b_n = {}^{2n}C_{n+1} + {}^{2n}C_n \Rightarrow b_n = {}^{2n+1}C_{n+1}$

Hence Proved

Q.15. Let $p \geq 3$ be an integer and α, β be the roots of $x^2 - (p + 1)x + 1 = 0$ using mathematical induction show that $\alpha^n + \beta^n$.
(i) is an integer and (ii) is not divisible by p (1992 - 6 Marks)

Ans.

Sol. Since α, β are the roots of $x^2 - (p + 1) x + 1 = 0$
$\therefore \alpha + \beta = p + 1$; $\alpha\beta = 1$
Here $p \geq 3$ and $p \in Z$

(i) To prove that $\alpha^n + \beta^n$ is an integer.

Let us consider the statement, "$\alpha^n + \beta^n$ is an integer."

Then for $n = 1, \alpha + \beta = p + 1$ which is an integer, p being an integer.

$\therefore$Statement is true for $n = 1$

Let the statement be true for $n \leq k$, i.e., $\alpha^k + \beta^k$ is an integer Then ,

$\alpha^{k+1} + \beta^{k+1} = \alpha^k . \alpha + \beta^k . \beta$

$= \alpha(\alpha^k + \beta^k) + \beta(\alpha^k + \beta^k) - \alpha\beta^k - \alpha^k \beta$

$= (\alpha + \beta)(\alpha^k + \beta^k) - \alpha\beta(\alpha^{k-1} + \beta^{k-1}$

$= (\alpha + \beta)(\alpha^k + \beta^k) - (\alpha^{k-1} + \beta^{k-1})$ .....(1) [as $\alpha\beta = 1$]

= difference of two integers = some integral value

$\Rightarrow$ Statement is true for $n = k + 1$.

$\therefore$By the principle of mathematical induction the given statement is true for $\forall\ n \in N$.

(ii) Let R_n be the remainder of $\alpha^n + \beta^n$ when divided by p where $0 \leq R_n \leq p-1$
Since $\alpha + \beta = p + 1$ $\quad \therefore R_1$

$= 1$ Also $\alpha^2 + \beta^2 = (\alpha + b)^2 - 2\alpha\beta = (p + 1)^2 - 2$

$= p^2 + 2p - 1 = p(p + 1) + p - 1$

$\therefore R_2 = p - 1$

Also from equation (1) of previous part

(i), we have $\alpha^{n+1} + \beta^{n+1} = (p + 1)(\alpha^n + \beta^n) - (\alpha^{n-1} + \beta^{n-1})$ $=$

$p(\alpha^n + \beta^n) + (\alpha n + \beta n) - (\alpha^{n-1} + \beta^{n-1})$

$\Rightarrow R_{n+1}$ is the remainder of $R_n - R_{n-1}$ when divided by p

$\therefore$We observe that $R_2 - R_1 = p - 1 - 1$

$\therefore R_3 = p - 2$

Similarly, R_4 is the remainder when $R_3 - R_2$ is divided by p

where $R_3 - R_2 = p - 2 - p + 1 = -1 = -p + (p - 1)$ $\therefore R_4 = p - 1$

$R_4 - R_3 = p - 1 - p + 1 = 1$ $\qquad \therefore R_5 = 1$

$R_5 - R_4 = 1 - p + 1 = -p + 2$ $\qquad \therefore R_6 = p - 2$

It is evident for above that the remainder is either 1 or $p - 1$ or $p - 2$.

Since $p \geq 3$, so none is divisible by p.

Q.16. Using mathematical induction, prove that $\tan^{-1}(1/3) + \tan^{-1}(1/7) + \ldots\ \tan^{-1}\{1/(n^2+n+1)\} = \tan^{-1}\{n/(n+2)\}$ (1993 - 5 Marks)

Ans. Sol. To prove

$$P(n):\ \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \ldots \tan^{-1}\left(\frac{1}{n^2+n+1}\right)$$

$$= \tan^{-1}\left(\frac{n}{n+2}\right)$$

For n = 1, LHS $= \tan^{-1}\frac{1}{3}$;

RHS= $\tan^{-1}\frac{1}{3}$ $\Rightarrow$ LHS = RHS.

∴P(1) is true.

Let P(k) be true, i.e.

$$\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\ldots \ldots\tan^{-1}\left(\frac{1}{k^2+k+1}\right)=\tan^{-1}\left(\frac{k}{k+2}\right)$$

Consider P (k + 1)

$$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}+\ldots\tan^{-1}\left(\frac{1}{k^2+k+1}\right)$$

$$+\tan^{-1}\left(\frac{1}{(k+1)^2+(k+1)+1}\right)$$

$$=\tan^{-1}\left[\frac{k+1}{(k+1)+2}\right]$$

LHS = $\tan^{-1}\left[\frac{k}{(k+2}\right]+\tan^{-1}\left(\frac{1}{(k^2+3k+3}\right)$ [Using equation (1)]

$$=\tan^{-1}\left[\frac{\frac{k}{(k+2}+\frac{1}{k^2+3k+3}}{1-\left(\frac{k}{k+2}\right)\left(\frac{1}{k^2+3k+3}\right)}\right]$$

$$=\tan^{-1}\left[\frac{(k+1)(k^2+2k+2)}{(k+3)(k^2+2k+2)}\right] = \tan^{-1}\left(\frac{k+1}{k+3}\right)=RHS$$

∴P(k + 1) is also true.

Hence by the principle of mathematical induction P(n) is true for every natural number.

Q. 17. Prove that $\sum_{r=1}^{k}(-3)^{r-1}\ {}^{3n}C_{2r-1} = 0$ = 0, where k = (3n) /2 and n is an even positive integer. (1993 - 5 Marks)

Ans. Sol.

To evaluate $\sum_{r=1}^{k}(-3)^{r-1}\ {}^{3n}C_{2r-1}$ where $k=\frac{3n}{2}$

and n is +ve even interger.

Let n = 2m, where m ∈ z^+ $\therefore$ $k=\frac{3(2m)}{2}=3m$

$$\therefore \quad \sum_{r=1}^{k}(-3)^{r-1}\ {}^{3n}C_{2r-1}=\sum_{r=1}^{3m}(-3)^{r-1}\ {}^{6m}C_{2r-1}$$

$$= {}^{6m}C_1-3.{}^{6m}C_3+3^2\ {}^{6m}C_5-\ldots\ldots \quad \ldots(1)$$

Now we know that

$$(1+a)^{6m}-(1-a)^{6m} = 2[{}^{6m}C_1a+{}^{6m}C_3a^3+{}^{6m}C_5a^5+\ldots..] \quad \ldots(2)$$

Keeping in mind the form of RHS in equation (1) and in equation (2)

We put $a=i\sqrt{3}$ in equation (2) to get

$$(1+i\sqrt{3})^{6m}-(1+i\sqrt{3})^{6m}$$

$$=2[{}^{6m}C_1i\sqrt{3}-{}^{6m}C_3i3\sqrt{3}+{}^{6m}C_5i3^2\sqrt{3}\ldots.]$$

$$\Rightarrow \quad (1+i\sqrt{3})^{6m}-(1-i\sqrt{3})^{6m}$$

$$= 2\sqrt{3}i[{}^{6m}C_1-3.{}^{6m}C_3+3^2\ {}^{6m}C_5\ldots.]\ldots.(3)$$

But $1+i\sqrt{3}=2(\cos\pi/3+i\sin\pi/3)$

$$\therefore \quad (1+i\sqrt{3})^{6m}=2^{6m}(\cos\pi/3+i\sin\pi/3)^{6m}$$

NOTE THIS STEP

$= 2^{6m}\left(\cos\frac{6m\pi}{3}+i\sin\frac{6m\pi}{3}\right)$ [Using D' Moivre's thm.]

Similarly,

$$(1-i\sqrt{3})^{6m} = 2^{6m}\left(\cos\frac{6m\pi}{3} - i\sin\frac{6m\pi}{3}\right)$$

$$\therefore \quad (1+i\sqrt{3})^{6m} - (1-i\sqrt{3})^{6m} = 2^{6m}.2\sin 2m\pi = 0$$

Substituting the above in equation (3) we get

$${}^{6m}C_1 - 3.{}^{6m}C_3 + 3^2\ {}^{6m}C_5 - ..... = 0$$

$$\Rightarrow \sum_{r=1}^{k}(-3)^{r-1}\ {}^{3n}C_{2r-1} = 0.$$

Hence Proved

Q.18. If x is not an integral multiple of 2π use mathematical induction to prove that : (1994 - 4 Marks)

$$\cos x + \cos 2x + ... + \cos nx = \quad \cos\frac{n+1}{2}x\sin\frac{nx}{2}\text{cosec}\frac{x}{2}$$

Ans. Sol. Let P (n) : cos x + cos 2x +....+ cos nx

$$= \cos\frac{n+1}{2}x\sin\frac{nx}{2}\cos ec\ \frac{x}{2} \quad ....(1)$$

where x is not an integral multiple of 2 p .

For n = 1 P (1) : L.H.S. = cos x

R.H.S. $= \cos\frac{1+1}{2}x\sin\frac{x}{2}\cos ec\ \frac{x}{2} = \cos x$

L.H.S. = R.H.S.

⇒ P (1) is true.

Let P(k) be true i.e.

cos x + cos 2x + ...+ cos kx

$$= \cos\frac{k+1}{2}x\sin\frac{kx}{2}\ \text{cosec}\ \frac{x}{2} \quad ....(2)$$

Consider P(k + 1) :

cos x + cos 2x + ...+ cos kx + cos (k + 1) x

$$= \cos\left(\frac{k+2}{2}\right)x \sin\frac{(k+1)x}{2} \operatorname{cosec}\frac{x}{2}$$

.L.H.S. [cos x + cos 2x + ...+ cos kx + cos (k + 1) x

$$= \cos\left(\frac{k+1}{2}\right)x \sin \operatorname{cosec}\frac{kx}{2}\ \frac{x}{2} + \cos(k+1)x \text{ [Using (2)]}$$

$$= \left[\cos\left(\frac{k+1}{2}\right)x \sin\frac{kx}{2} + \cos\,(k+1)x \sin\frac{x}{2}\right]\operatorname{cosec}\frac{x}{2}$$

$$= \frac{1}{2}\left[2\cos\frac{(k+1)x}{2}\sin\frac{kx}{2} + 2\cos\,(k+1)x\sin\frac{x}{2}\right]\operatorname{cosec}\frac{x}{2}$$

$$= \frac{1}{2}[\sin\left(\frac{2k+1}{2}\right)x - \sin\frac{x}{2}$$

$$+\sin\left(xk+\frac{3x}{2}\right)-\sin\left(xk+\frac{x}{2}\right)]\operatorname{cosec}\frac{x}{2}$$

$$= \frac{1}{2}\left[\sin\left(xk+\frac{3x}{2}\right)-\sin\frac{x}{2}\right]\operatorname{cosec}\frac{x}{2}$$

$$= \frac{1}{2}\left[2\cos\frac{(k+2)x}{2}\sin\frac{(k+1)}{2}\right]\operatorname{cosec}\frac{x}{2} = \text{R.H.S.}$$

∴P (k + 1) is also true.

Hence by the principle of mathematical induction

P (n) is true ∀ n∈ N .

Q.19. Let n be a positive integer and (1994 - 5 Marks)

$(1 + x + x^2)^n = a_0 + a_1x + \ldots\ldots\ldots\ldots + a_{2n} x^{2n}$

Show that $a_0^2 – a_1^2 + a_2^2 \ldots\ldots\ldots\ldots + a_{2n}^2 = a_n$

Ans. Sol. Given that,

$(1 + x + x^2)^n = a_0 + a_1x + \ldots + a_{2n}x^{2n}$ ....(1)

where n is a +ve integer.

Replacing x by $-\frac{1}{x}$ in eq n (1), we get

$$\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \frac{a_3}{x^3} + ... + \frac{a_{2n}}{x^{2n}} \quad ....(2)$$

Multiplying eq.'s (1) and (2) :

$$\frac{(1+x+x^2)^n (x^2-x+1)^n}{x^{2n}}$$

$$= (a_0 + a_1 x + ... + a_{2n}x^{2n})\left(a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} + ... + \frac{a_n}{x^{2n}}\right)$$

Equating the constant terms on both sides we get

$a_0^2 - a_1^2 + a_2^2 - a_3^2 + ... + a_{2n}^2$ = constant term in the expansion of

$$\frac{[(1+x+x^2)(1-x+x^2)]^n}{x^{2n}}$$

= Coeff. of x^{2n} in the expansion of $(1 + x^2 + x^4)^n$ But replacing x by x^2 in eq's (1), we have

$(1 + x^2 + x^4)^n = a_0 + a_1x^2 + ... + a_{2n} (x^2)^{2n}$

$\therefore$Coeff of $x^{2n} = a_n$

Hence we obtain, $a_0^2 - a_1^2 + a_2^2 - a_3^2 + ... + a_{2n}^2 = a_n$

Q.20. Using mathematical induction prove that for every integer $n \geq 1$, $(3^{2n}-1)$ is divisible by 2^{n+2} but not by 2^{n+3}. (1996 - 3 Marks)

Ans. Sol. For n = 1, $3^{2n} - 1 = 3^{21} - 1 = 9 - 1 = 8$ which is divisible by $2^{n+2} = 2^3 = 8$ but is not divisible by $2^{n+3} = 2^4 = 16$

Therefore, the result is true for n = 1.

Assume that the result is true for n = k.

That is, assume that $3^{2k} - 1$ is divisible by 2^{k+2} but is not divisible by 2^{k+3},

Since $3^{2k} - 1$ is divisble by 2^{k+2} but not by 2^{k+3}, we can write $3^{2^k} - 1 = (m)\, 2^{k+2}$ where m must be an odd positive integer, for otherwise $3^{2k} - 1$ will become divisible by 2^{k+3}.

For n = k +1, we have $3^{2^{k+1}} - 1 = 3^{2^k \cdot 2} - 1 = \left(3^{2^k}\right)^2 - 1$

$= (m.2^{k+2} + 1)^2 - 1$ [Using (1)]

$= m^2.(2^{k+2})^2 + 2m.2^{k+2} + 1 - 1$

$= m^2.2^{2k+4} + m.2^{k+3}$

$= 2^{k+3}(m^2.^{2k+1} + m.)$

$\Rightarrow 3^{2k+1} - 1$ is divisible by 2^{k+3} .

But $3^{2k+1} - 1$ is not divisible by 2^{k+4} for otherwise we must have 2 divides $m^2.\ 2^{k+1} + m$.

But this is not possible as m is odd.

Thus, the result is true for n = k + 1.

Q.21. Let $0 < A_i < p$ for i = 1, 2 ...., n. Use mathematical induction to prove that

$\sin A_1 + \sin A_2 ... + \sin A_n \leq n \sin$ $\left(\frac{A_1 + A_2 + ... + A_n}{n}\right)$

where ≥ 1 is a natural number. {You may use the fact that $p \sin x + (1-p) \sin y \leq \sin [px + (1-p)y]$,   where $0 \leq p \leq 1$ and $0 \leq x, y \leq \pi$} (1997 - 5 Marks)

Ans.

Sol. For n = 1, the inequalitity becomes $\sin A_1 \leq \sin A_1$, which is clearly true.

Assume that the inequality holds for n = k where k is some positive integer. That is, assume that

$\sin A_1 + \sin A_2 + ... + \sin A_k \leq k \sin \leq k \sin\left(\frac{A_1 + A_2 + ... + A_k}{k}\right)$ ....(1)

for same positive integer k.

We shall now show that the result holds for n = k + 1 that is, we show that sin A_1 + sin A_2 + ... + sin A_k+ sinA_{k+1}

$$\le (k+1)\sin\left(\frac{A_1 + A_2 + ... + A_{k+1}}{k+1}\right) ....(2)$$

L.H.S. of (2) = sin A_1 + sin A_2 + ... + sin A_k+ sinA_{k+1}

$$\le k\sin\left(\frac{A_1 + A_2 + ... + A_k}{k}\right) + \sin A_{k+1}$$

[Induction assumption]

$$= (k+1)\left[\frac{k}{k+1}\sin\alpha + \frac{1}{k+1}\sin A_{k+1}\right];$$

where $\alpha + \frac{A_1 + A_2 + ...A_k}{k}$

$$\therefore \text{L.H.S. of (2)} \le (k+1)\left[\left(1 - \frac{k}{k+1}\right)\sin\alpha + \frac{1}{k+1}\sin A_{k+1}\right]$$

$$\le (k+1)\sin\left\{\left(1 - \frac{k}{k+1}\right)\alpha + \frac{1}{k+1}A_{k+1}\right\}$$

[Using the fact p sin x + (1– p) sin y ≤ sin [px + (1– p)y]

$0 \le p \le 1, 0 \le x, y \le \pi]$

$$\left\{\frac{k}{k+1}\left(\frac{A_1 + A_2 + ...A_k}{k}\right) + \frac{1}{k+1}A_{k+1}\right\}$$

$$\left(\frac{A_1 + A_2 + ...A_{k+1}}{k+1}\right\}$$

Thus, the inequality holds for n = k + 1. Hence, by the principle of mathematical induction the inequality holds for all n∈ N.

Q.22. Let p be a prime and m a positive integer. By mathematical induction on m, or otherwise, prove that whenever r is an integer such that p does not divide r, p divides ${}^{mp}C_r$, (1998 - 8 Marks)

[Hint: You may use the fact that $(1+x)^{(m+1)}p = (1 + x)^p (1 + x)^{mp}$]

Ans. Sol. We know that $^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$

$$\therefore \quad {}^{mp}C_r = \frac{mp}{r}\,{}^{mp-1}C_{r-1}$$

$$= \left[\frac{m.{}^{mp-1}C_{r-1}}{r}\right]p$$

Now, L.H.S is an integer

$\Rightarrow$ RHS must be an integer

But p and r are coprime (given)

$\therefore$r must divide m. $^{mp-1}C_{r-1}$

or $\frac{m.\ {}^{mp-1}C_{r-1}}{r}$ is an integer..

$\Rightarrow \frac{{}^{mp}C_r}{p}$is an integer or $^{mp}C_r$ is divisible by p.

Q.23. Let n be any positive integer. Prove that (1999 - 10 Marks)

$$\sum_{k=0}^{m}\frac{\binom{2n-k}{k}}{\binom{2n-k}{n}}\cdot\frac{(2n-4k+1)}{(2n-2k+1)}2^{n-2k} = \frac{\binom{n}{m}}{\binom{2n-2m}{n-m}}2^{n-2m}$$

for each non-be gatuve integer m ≤ n. $m \le n.$ $\left(\text{Here}\binom{p}{q} = {}^pC_q\right).$

Ans. Sol.

$$\text{Let} \quad P(m) = \sum_{k=0}^{m}\frac{\binom{2n-k}{k}^{(2n-4k+1)}}{\binom{2n-k}{n}^{(2n-2k+1)}}2^{n-2k}$$

$$= \frac{\binom{n}{m}}{\binom{2n-2m}{n-m}}.2^{n-2m} \quad ....(1)$$

For m = o, LHS = $\dfrac{\binom{2n}{0}}{\binom{2n}{n}} \cdot \dfrac{2n+1}{2n+1} \cdot 2^n = \dfrac{1}{\binom{2n}{n}} 2^n,$

R.H.S. = $\dfrac{\binom{n}{0}}{\binom{2n}{n}} \cdot 2^n = \dfrac{1}{\binom{2n}{n}} 2^n$ = LHS

[∵ m = o ⇒ k = o]

∴P(o) holds true. Now assuming P (m)

L.H.S. of P(m + 1) = L.H.S. of

$$P(m) + \frac{\binom{2n-m-1}{m+1}}{\binom{2n-m-1}{n}} \cdot \frac{(2n-4m-3)}{(2n-2m-1)} \cdot 2^{n-2m-2}$$

$$= \frac{n!(n-m)!}{m!(2n-2m)!} \cdot 2^{n-2m}$$

$$+ \frac{n!(n-m-1)!(2n-4m-3)}{(m+1)!(2n-2m-2)!(2n-2m-1)} \cdot 2^{n-2m-2}$$

$$= \frac{n!(n-m-1)! 2^{n-2m-2}}{(m+1)!(2n-2m-1)!}$$

$$\times \left\{ \frac{(n-m).4\,(m+1)}{(2n-2m)} + (2n-4m-3) \right\}$$

$$= \frac{n!(n-m-1)! 2^{n-2m-2}\,(2n-2m-1)}{(m+1)!(2n-2m-1)!}$$

$$= \frac{n!(n-m-1)! 2^{n-2m-2}}{(m+1)!(2n-2m-2)!} = \frac{\binom{n}{m+1}}{\binom{2n-2m-2}{n-m-1}} \cdot 2^{n-2m-2}$$

= R.H.S. of P(m + 1).

Hence by mathematical induction, result follows for all $0 \le m \le n$.

Q.24. For any positive integer m, n (with n ≥ m), let $\binom{n}{m} = {}^nC_m$.

Prove that $\binom{n}{m}+\binom{n-1}{m}+\binom{n-2}{m}+.......+\binom{m}{m}=\binom{n+1}{m+2}$

Hence or otherwise, prove that

$\binom{n}{m}+2\binom{n-1}{m}+3\binom{n-2}{m}+.....+(n-m+1)\binom{m}{m}=\binom{n+2}{m+2}$ **(2000 - 6 Marks)**

Ans. Sol. Given that for positive integers m and n such that n ≥m, then to prove that

${}^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + ... + {}^mC_m = {}^{n+1}C_{m+1}$

L.H.S. ${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + ... + {}^{n-1}C_m + {}^n C_m$

[writing L.H.S. in reverse order]

$= ({}^{m+1}C_{m+1} + {}^{m+1}C_m) + {}^{m+2}C_m + ... + {}^{n-1}C_m + {}^n C_m$

$[\because {}^mC_m = {}^{m+1}C_{m+1}]$

$= ({}^{m+2}C_{m+1} + {}^{m+2}C_m) + {}^{m+3}C_m + ... + {}^nC_m$

$[\because {}^nC_{r+1} + {}^nC_r = {}^{n+1}C_{r+1}]$

$= {}^{m+3}C_{m+1} + {}^{m+3}C_m + ... + {}^nC_m$

Combining in the same way we get

$= {}^nC_{m+1} + {}^nC_m = {}^{n+1}C_{m+1}$ = R.H.S.

Again we have to prove ${}^nC_{m+2}\ {}^{n-1}C_{m+3}\ {}^{n-2}C_m + ... + (n-m+1)\ {}^mC_m = {}^{n+2}C_{m+2}$

$= [{}^nC_m + {}^{n-1}C_{m+}\ {}^{n-2}C_m + ... + {}^mC_m] + [\ {}^{n-1}C_m + {}^{n-2}C_m + ... + {}^mC_m\ ] + [\ {}^{n-2}C_m + ... + {}^mC_m] + ... + [\ {}^mC_m]$

[n – m + 1 bracketed terms] $= {}^{n+1}C_{m+1} + {}^nC_{m+1}\ {}^{n-1}C_{m+1} .... + {}^{m+1}C_{m+1}$

[using previous result.]

$= {}^{n+2}C_{m+2}$

[Replacing n by n + 1 and m by m + 1 in the previous result.] = R.H.S.

Q.25. For every positive integer n, prove that $\sqrt{(4n+1)} < \sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}$ Hence or other wise,

prove that $[\sqrt{n} + \sqrt{(n+1)}] = [\sqrt{4n+1}]$ where [x] denotes the greatest integer not exceeding x. (2000 - 6 Marks)

Ans. Sol. For n > 0 $\sqrt{4n+1} > 0, \sqrt{n} + \sqrt{n+1} > 0$ *and* $\sqrt{4n+2} > 0$

Now, $\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}$ to be proved.

I. To prove $\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1}$

Squaring both sides in $\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1}$

$\Rightarrow$ 4n +1< n + n +1+ $2\sqrt{n(n+1)}$

$\Rightarrow 2n < 2\sqrt{n(n+1)} \Rightarrow n < \sqrt{n(n+1)}$ which is true.

II. To prove $\sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}$

Squaring both sides,

n + n + 1 + $2\sqrt{n(n+1)} < 4n+2$

$\Rightarrow \quad 2\sqrt{n(n+1)} < 2n+1$ Squaring again

4 [n (n + 1)] < $4n^2$ + 1+ 4n or 0 < 1 which is true

Hence $\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}$

Further to prove $[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+1}]$, we have to prove that there is no positive integer which lies between

$\sqrt{4n+1}$ and $\sqrt{4n+2}$ or $[\sqrt{4n+1}] = [\sqrt{4n+2}]$. Using Mathematical induction.

We have to check $[\sqrt{4n+1}] = [\sqrt{4n+2}]$ for $n = 1$

$[\sqrt{5}] = [\sqrt{6}] \Rightarrow 2 = 2$, which is true

Assume for n = k (arbitrary)

i.e., $[\sqrt{4k+1}]=[\sqrt{4k+2}]$ To prove for n = k +1

To check $[\sqrt{4k+5}]=[\sqrt{4k+6}]$ since $k \geq 0$

Here 4k + 5 is an odd number and 4k + 6 is even number.

Their greatest integer will be different iff 4k + 6 is a perfect square that is $4k + 6 = r^2$

$\Rightarrow \quad k=\frac{r^2}{4}-\frac{6}{4}, \frac{6}{4}$ is not integer. But k has to be integer..

So 4k + 6 cannot be perfect square.

$\Rightarrow \quad [\sqrt{4k+5}]=[\sqrt{4k+6}]$

By Sandwich theorem

$\Rightarrow \quad [\sqrt{n}+\sqrt{n+1}]=[\sqrt{4n+1}]$

Q.26. Let a, b, c be positive real numbers such that $b^2 - 4ac > 0$ and let $\alpha_1 = c$. Prove by induction that

$\alpha_{n+1}=\frac{a\alpha_n^2}{(b^2-2a(\alpha_1+\alpha_2+...+\alpha_n)}$ **is well - defined and**

$\alpha_{n+1}<\frac{\alpha_n}{2}$ for all $n = 1, 2, \ldots$ **(Here, 'well - defined' means that the denominator in the expression for α_{n+1} is not zero.) (2001 - 5 Marks)**

Ans. Sol. We have a, b, c the +ve real number s.t. $b^2 – 4ac > 0$; $\alpha_1 = c$.

$$P(n): \alpha_{n+1}=\frac{a\alpha_n^2}{b^2-2a(\alpha_1+\alpha_2+....\alpha_n)}$$

is well defined and $\alpha_{n+1}<\frac{\alpha_n}{2}, \forall n=1,2,....$

For n = 1, $\alpha_2=\frac{a\alpha_1^2}{b^2-2a\alpha_1}=\frac{ac^2}{b^2-2ac}$

Now, $b^2 – 4ac > 0 \Rightarrow b^2 – 2ac > 2ac > 0$

$\therefore$ α^2 is well defined (as denomination is not zero)

Also, $\begin{bmatrix} \because & b^2-2ac>2ac \\ \Rightarrow & \frac{1}{b^2-2ac}<\frac{1}{2ac} \end{bmatrix} \Rightarrow \frac{\alpha_2}{c}<\frac{1}{2} \Rightarrow \frac{\alpha_2}{\alpha_1}<\frac{1}{2}$

$\therefore$P(n) is true for n =1.

Let the statement be true for $1 \le n \le k$ *i.e.*,

$$\alpha_{k+1}=\frac{a\alpha_k^2}{b^2-2a(\alpha_1+\alpha_2+....\alpha_k)}$$ is well defined

and $\alpha_{k+1}<\frac{\alpha_k}{2}$

Now, we will prove that P(k + 1) is also true

i.e. $\alpha_{k+2}=\frac{a\alpha_{k+1}^2}{b^2-2a(\alpha_1+\alpha_2+....+\alpha_k+\alpha_{k+1})}$

well defined and $\alpha_{k+2}<\frac{\alpha_{k+1}}{2}$.

We have

$\alpha_1=c, \alpha_2<\frac{c}{2}, \alpha_3<\frac{\alpha_2}{2}<\frac{c}{2^2}, \ \alpha_4<\frac{\alpha_3}{2}<\frac{c}{2^3}, ...$ (by IH)

Now, $(\alpha_{1+}+\alpha_2+...\alpha_k+\alpha_{k+1}<c+\frac{c}{2}+\frac{c}{2^k}+....+\frac{c}{2^k}$

$$=\frac{c\left(1-\frac{1}{2^{k+1}}\right)}{1-1/2}=2c\left(1-\frac{1}{2^{k+1}}\right)<2c$$

$\therefore \quad \alpha_1+\alpha_2+...+\alpha_{k+1}<2c$

$\Rightarrow \quad -2a(\alpha_1+\alpha_2+......+\alpha_{k+1})>-4ac$

$\Rightarrow \quad b^2-2a(\alpha_1+\alpha_2+......+\alpha_{k+1})>b^2-4ac>0$

$\therefore \alpha_{k+2}$ is well defined. Again by IH we have

$$\alpha_{k+1} < \frac{\alpha_k}{2} \Rightarrow 2\alpha_{k+1} < \alpha_k$$

$$\Rightarrow 4\alpha_{k+1}^2 < \alpha_k^2 \; [As\,by\,def.\,\alpha_{k+1}, \alpha_k\; are + ve]$$

$$\Rightarrow 4\alpha_{k+1} < \frac{\alpha_k^2}{\alpha_{k+1}}$$

$$\Rightarrow 4\alpha_{k+1} < \frac{b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_k)}{a}$$

$$\Rightarrow 4a\alpha_{k+1} < b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_k)$$

$$\Rightarrow 2a\alpha_{k+1} < b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_k + \alpha_{k+1})$$

$$\Rightarrow \frac{a\alpha_{k+1}^2}{b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_{k+1})} < \frac{1}{2}$$

$$\Rightarrow \frac{a\alpha_{k+1}}{b^2 - 2a(\alpha_1 + \alpha_2 + ... + \alpha_{k+1})} < \frac{\alpha_{k+1}}{2}$$

$$\Rightarrow \alpha_{k+2} < \frac{\alpha_{k+1}}{2}$$

$\therefore$P(k+1) is also true.

Thus by the Principle of Mathematical Induction the Statement P(n) is true $\forall$ n$\in$ N .

Q.27. Use mathematical induction to show that $(25)^{n+1}$ – 24n + 5735 is divisible by $(24)^2$for all n = 1, 2, ... (2002 - 5 Marks)

Ans. Sol. Let P(n) : $(25)^{n+1}$ – 24n + 5735 For n = 1.

P(1) : 625 – 24 + 5735 = 6336 = (24)2 × (11),

which is divisible by 242.

Hence P(1) is true Let P(k) be true, where k ≥ 1

⇒ $(25)^{k+1}$ – 24k + 5735

$= (24)^2\lambda$ where $\lambda \in N$

For n = k + 1,

P (k + 1) : (25) k + 2 – 24 (k + 1) + 5735

= 25 [(25) k + 1 – 24k + 5735]

+ 25.24.k – (25) (5735) + 5735 – 24 (k + 1)

$= 25 (24)^2 \lambda + (24)^2 k - 5735 \times 24 - 24$

$= 25 (24)^2 \lambda + (24)^2 k - (24) (5736)$

$= 25 (24)^2 \lambda + (24)^2 k - (24)^2 (239),$

$= (24)^2 [25 \lambda + k - 239]$

which is divisible by (24)2.

Hence, by the method of mathematical induction result is true $\forall$ n$\in$ N .

Q.28. Prove that (2003 - 2 Marks)

$$2^k \binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1}$$

$$+ 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} - .....(-1)^k \binom{n}{k}\binom{n-k}{0} = \binom{n}{k}.$$

Ans. Sol. To prove that $2^k\, {}^nC_0\, {}^nC_k - 2^{k-1}\, {}^nC_1\, {}^{n-1}C_{k-1} + 2^{k-2}\, {}^nC_2\, {}^{n-2}C_{k-2}$

$-..... + (-1)^k\, {}^nC_k\, {}^{n-k}C_0 = {}^nC_k$

LHS of above equation can be written as

$$\sum_{r=0}^{k} (-1)^r 2^{k-r}\, {}^nC_r\, {}^{n-r}C_{k-r}$$

$$= \sum_{r=0}^{k} (-1)^r 2^{k-r} \frac{n!}{r!(n-r)!} \frac{(n-r)!}{(k-r)!(n-k)!}$$

$$= \sum_{r=0}^{k} (-1)^r 2^{k-r} \frac{n!k!}{r!k!(n-k)!(k-r)!}$$

$$= \sum_{r=0}^{k} (-1)^r \frac{2^k}{2^r} \cdot \frac{n!}{k!(n-k)!} \frac{k!}{r!(k-r)!}$$

$$= 2^k \, {}^nC_k \sum_{r=0}^{k} (-1/2)^r \frac{k!}{r!(k-r)!}$$

$$= 2^k \, {}^nC_k \sum_{r=0}^{k} {}^kC_r (-1/2)^r = 2^k \, {}^nC_k (1-1/2)^k$$

$$= 2^k \, {}^nC_k \frac{1}{2^k} = {}^nC_k \text{ R.H.S. Hence Proved}$$

Q.29. A coin has probability p of showing head when tossed. It is tossed n times. Let p_n denote the probability that no two (or more) consecutive heads occur. Prove that $p_1=1$, $p_2=1-p^2$ and $p_n=(1-p).\ p_{n-1} + p(1-p)\ p_{n-2}$ for all $n \geq 3$. Prove by induction on n, that $p_n = A\alpha^n + B\beta^n$ for all $n \geq 1$, where a and b are the roots of quadratic equation

$x^2 - (1-p)x - p(1-p) = 0$ and $A = \frac{p^2+\beta-1}{\alpha\beta-\alpha^2}, B = \frac{p^2+\alpha-1}{\alpha\beta-\beta^2}$.

Ans. Sol. We have $\alpha + \beta = 1 - p$ and $\alpha\beta = -p(1-p)$

For n = 1, $p_n = p_1 = 1$

Also, $A\alpha^n + B\beta^n = A\alpha + B\beta = \frac{(p^2+\beta-1)\alpha}{\alpha\beta-\alpha^2}$

$$+ \frac{(p^2+\alpha-1)\beta}{\alpha\beta-\beta^2} = \frac{p^2+\beta-1}{\beta-\alpha} + \frac{p^2+\alpha-1}{\alpha-\beta}$$

$$= \frac{p^2+\beta-1-p^2-\alpha+1}{\beta-\alpha} = \frac{\beta-\alpha}{\beta-\alpha} = 1$$

For n = 2, $p_2 = 1 - p^2$

Also, $A\alpha^n + B\beta^n = A\alpha^2 + B\beta^2$

$$= \frac{(p^2+\beta-1)\alpha^2}{\alpha\beta-\alpha^2} + \frac{(p^2+\alpha-1)\beta^2}{\alpha\beta-\alpha^2}$$

which is true for n = 2

Now let result is true for k < n where n ≥ 3.

$$P_n = (1-p)\,P_{n-1} + p(1-p)\,P_{n-2}$$

$$= (1-p)(A\alpha^{n-1} + B\beta^{n-1}) + p(1-p)(A\alpha^{n-2} + B\beta^{n-2})$$

$$= A\alpha^{n-2}\{(1-p)\alpha + p(1-p)\} + B\beta^{n-2}\{(1-p)\beta - p(1-p)$$

$$= A\alpha^{n-2}\{(\alpha+\beta)\alpha - \alpha\beta\}$$

$$+ B\beta^{n-2}\{(\alpha+\beta)\beta - \alpha\beta\} \text{ by (1)}$$

$$= A\alpha^{n-2}\{\alpha^2 + \beta\alpha - \alpha\beta\} + B\beta^{n-2}\{\alpha\beta + \beta^2 - \alpha\beta\}$$

$$= A\alpha^{n-2}(\alpha^2) + B\beta^{n-2}(\beta^2) = A\alpha^n + B\beta^n$$

This is true for n. Hence by principle of mathematical induction, the result holds good for all n ∈ N .

Integar Type ques of Mathematical Induction and Binomial Theorem

Q.1. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n = (JEE Adv. 2013)

Ans. (6)

Sol. Let the coefficients of three consecutive term s of $(1 + x)^{n+5}$

be ${}^{n+5}C_{r-1}$, ${}^{n+5}C_r$, ${}^{n+5}C_{r+1}$,

then we have ${}^{n+5}C_{r-1} : {}^{n+5}C_r : {}^{n+5}C_{r+1} = 5 : 10 : 14$

$$\frac{{}^{n+5}C_{r-1}}{{}^{n+5}C_r} = \frac{5}{10} \Rightarrow \frac{r}{n+6-r} = \frac{1}{2}$$

or $n - 3r + 6 = 0$ ...(1)

Also $\frac{{}^{n+1}C_r}{{}^{n+5}C_{r+1}} = \frac{10}{14} \Rightarrow \frac{r+1}{n-r+5} = \frac{5}{7}$

or $5n - 12r + 18 = 0$ (2)

Solving (1) and (2) we get n = 6.

Q. 2. Let m be th e sm allest posi tive i n teger such th at th e coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\,{}^{51}C_3$ for some positive integer n. Then the value of n is (JEE Adv. 2016)

Ans. (5)

Sol. $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + mx)^{50}$

$$= (1+x)^2\left[\frac{(1+x)^{48}-1}{(1+x)-1}\right] + (1+mx)^{50}$$

$$= \frac{1}{x}\left[(1+x)^{50} - (1+x)^2\right] + (1+mx)^{50}$$

Coeff. of x^2 in the above expansion = Coeff. of x^3 in $(1 + x)^{50}$ + Coeff. of x^2 in $(1 + mx)^{50}$

$\Rightarrow {}^{50}C_3 + {}^{50}C_2\, m^2$

$\therefore (3n + 1)\, {}^{51}C_3 = {}^{50}C_3 + {}^{50}C_2\, m^2$

$$\Rightarrow (3n+1) = \frac{{}^{50}C_3}{{}^{51}C_3} + \frac{{}^{50}C_2}{{}^{51}C_3} m^2$$

$$\Rightarrow 3n + 1 = \frac{16}{17} + \frac{1}{17} m^2 \Rightarrow n = \frac{m^2 - 1}{51}$$

Least positive integer m for which n is an integer is m = 16 and then n = 5