Probability

- **Complementary events**
  For an event E such that $0 \le P(E) \le 1$ of an experiment, the event $\overline{E}$ represents 'not E' , which is called the complement of the event E. We say, E and $\overline{E}$ are **complementary** events.
  $P(E) + P(\overline{E}) = 1$
  $\Rightarrow P(\overline{E}) = 1 - P(E)$

  Example: A pair of dice is thrown once. Find the probability of getting a different number on each die.

  Solution: When a pair of dice is thrown, the possible outcomes of the experiment can be listed as:

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

  The number of all possible outcomes = 6 × 6 = 36
  Let E be the event of getting the same number on each die.
  Then, $\overline{E}$ is the event of getting different numbers on each die.

  Now, the number of outcomes favourable to E is 6.
  $\therefore P(\overline{E}) = 1 - P(E) = 1 - \frac{6}{36} = \frac{5}{6}$
  Thus, the required probability is $\frac{5}{6}$.

- **Algebra of events**
  - **Complementary event:** For every event *A*, there corresponds another event *A′* called the complementary event to *A*. It is also called the event 'not *A*'.

    $A' = \{\omega : \omega \in S \text{ and } \omega \notin A\} = S - A$.

  - **The event '*A* or *B*':** When sets *A* and *B* are two events associated with a sample space, then the set $A \cup B$ is the event 'either *A* or *B* or both'.

That is, event '*A* or *B*' = $A \cup B$ = {ω: ω ∈ *A* or ω∈ *B*}

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  That is, event '*A* and *B*' = $A \cap B$ = {ω: ω ∈ *A* and ω∈ *B*}

- **The event '*A* but not *B*':** When sets *A* and *B* are two events associated with a sample space, then the set *A* - *B* is the event '*A* but not *B*'.

  That is, event '*A* but not *B*' = $A - B = A \cap B'$ = {ω: ω ∈ *A* and $\omega \notin$ *B*}

  Example: Consider the experiment of tossing 2 coins. Let *A* be the event 'getting at least one head' and *B* be the event 'getting exactly two heads'. Find the sets representing the events
  (i) complement of '*A* or *B*'
  (ii) *A* and *B*
  (iii) *A* but not *B*

  Solution:
  Here, *S* = {HH, HT, TH, TT}
  A = {HH, HT, TH}, *B* = {HH}
  (i) *A* or *B* = $A \cup B$ = {HH, HT, TH}
  Hence, complement of *A* or *B* = (*A* or *B*)' = $(A \cup B)'$ = U – $(A \cup B)$= {TT}
  (ii) *A* and *B* = $A \cap B$ = {HH}
  (iii) *A* but not *B* = *A* – *B* = {HT, TH}

- **Mutually Exclusive Events**

  Two events, *A* and *B*, are called mutually exclusive events if the occurrence of any one of them excludes the occurrence of the other event i.e., if they cannot occur simultaneously. In this case, sets *A* and *B* are disjoint i.e., $A \cap B = \emptyset$
  If $E_1, E_2, \ldots E_n$ are *n* events of a sample space *S*, and if
  $\bigcup_{i=1}^{n} E_i = S$, $E_1 \cup E_2 \cup \ldots \cup E_n = \cup_{i=1}^{n} E_i = S$ then
  $E_1, E_2, \ldots E_n$ are called mutually exclusive and exhaustive events.
  In other words, at least one of $E_1, E_2, \ldots E_n$ necessarily occurs whenever the experiment is performed.
  The events $E_1, E_2, \ldots E_n$, i.e., *n* events of a sample space (*S*) are called mutually exclusive and exhaustive events if
  $E_i \cap E_i = \phi$ for $i \neq j$ i.e., events E_i and E_j are pairwise disjoint, and
  $\cup_{i=1}^{n} = S$

Example: Consider the experiment of tossing a coin twice. Let A and B be the event of "getting at least one head" and "getting exactly two tails" respectively. Are the events A and B mutually exclusive and exhaustive?

Solution: Here, S = {HH, HT, TH, TT}

A = {HH, HT, TH}

B = {TT}

Now, $A \cap B = \emptyset$ and $A \cup B$ = {HH, HT, TH, TT} = S

Thus, A and B are mutually exclusive and exhaustive events.

- The number P (ω_i) i.e., the probability of the outcome ω_i, is such that
  - $0 \le P(\omega_i) \le 1$
  - $\sum P(\omega_i) = 1$ for all $\omega_i \in S$
  - For any event A, $P(A) = \sum P(\omega_i)$ *for all* $\omega_i \in A$
- For a finite sample space S, with equally likely outcomes, the probability of an event A is denoted as P (A) and it is given by

$$P(A) = \frac{n(A)}{n(S)},$$

- Where, $n(A)$ = Number of elements in set A and $n(S)$ = Number of elements in set S

If A and B are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

- If A is any event, then

$$P(A') = 1 - P(A)$$

Example: Consider the experiment of tossing a die. Let A be the event "getting an even number greater than 2" and B be the event "getting the number 4". Find the probability of
(i) getting an even number greater than 2 or the number 4
(ii) getting a number, which is not the number 4, on the top face of the die

Solution: Here, S = {1, 2, 3, 4, 5, 6}

A = {4, 6}, B = {4}

$A \cap B$ = {4}

$p(A) = \frac{2}{6}, \; p(B) = \frac{1}{6}, \; p(A \cap B) = \frac{1}{6}$

(i) Required probability = P $(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= \frac{2}{6} + \frac{1}{6} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

(ii) $P(B) = \frac{1}{6}$

$\therefore$ P (not B) = 1 – P (B) $= 1 - \frac{1}{6} = \frac{5}{6}$

Hence, the required probability of not getting number 4 on the top face of the die is $\frac{5}{6}$.

Example: 20 cards are selected at random from a deck of 52 cards. Find the probability of getting at least 12 diamonds.

Solution: 20 cards can be selected at random from a deck of 52 cards in $^{52}C_{20}$ ways. Hence, Total possible outcomes = $^{52}C_{20}$ P (at least 12 diamonds) = P (12 diamonds or 13 diamonds) = P (12 diamonds) + P (13 diamonds)

$$= \frac{^{13}C_{12} \times {}^{39}C_8}{^{52}C_{20}} + \frac{^{13}C_{13} \times {}^{39}C_7}{^{52}C_{20}}$$

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$$= \frac{13 \times {}^{39}C_8 + {}^{39}C_7}{^{52}C_{20}}$$

$$= \frac{13 \times \frac{39!}{31! \times 8!} + \frac{39!}{32! \times 7!}}{^{52}C_{20}}$$

$$= \frac{13 \times \frac{39!}{31! \times 8!} + \frac{39! \times 8}{32 \times 31! \times 7! \times 8}}{^{52}C_{20}}$$

$$= \frac{13 \times \frac{39!}{31! \times 8!} + \frac{8}{32} \times \frac{39!}{31! \times 8!}}{^{52}C_{20}}$$

$$= \frac{\frac{53}{4} \times \frac{39!}{31! \times 8!}}{^{52}C_{20}}$$

$$= \frac{53}{4} \times \frac{^{39}C_8}{^{52}C_{20}}$$