02

Real Numbers

2.01 Introduction

In previous classes, we have studied about use of operations on Natural numbers, Integers, Rational and Irrational numbers. Now, we will study about Rational numbers and related fundamental principles and proof of rational and irrational numbers, nature of terminating and non-terminating numbers.

We know that, any positive integer can be expressed as product of two or more than two numbers. Also, we know about division of numbers, that remainder obtained form divison of numbers, that remainder obtained form division of two positive numbers is less than denominator. This important fact is basic fundamental theorem of Arithmetic. In this chapter, we will use these concepts to prove the irrationality of numbers $\sqrt{2}, \sqrt{3}, \sqrt{5}$ etc. and also study about decimal expansion of rational numbers.

2.02 Euclid's Division Lemma

Euclid was a Greak Mathematician and known by his work for geometry and number system. He had established the principles related to division of real numbers. In number system, Euclid's Division Algorithm is based on division lemma which is established by him.

Let a be any integer $(a \neq 0)$ and b and c are two whole numbers such that $\frac{b}{a} = c$ then number b is called dividend, a as divisor and c as quotient. For division, following properties should be kept in mind.

(i) Any non zero integer can be divided by $\pm$ 1.

(ii) 0 can be divided by any number.

(iii) No number can be divided by 0.

(iv) If none of a and b is zero then division algorithm can be applied.

(v) If a and b are non zero integers and q and r are other integers such that $a = bq + r$. In previous classes we have studied the division algorith. We know that when a positive integer (say a) is divided by another positive integer (say b) we get quotient (say q) and remainder (say r). Now we consider the following pairs of integers :

(i) 56, 16 (ii) 10, 2 (iii) 5, 7

Relations for these pairs can be written as follows :

(i) $56 = 16 \times 3 + 8$ (16 on dividing 56 by 16 quotient is 3 and remainder is 8)

(ii) $10 = 5 \times 2 + 0$ (5 on dividing 10 by 2 quotient is 5 and remainder is 0)

(iii) $5 = 7 \times 0 + 5$ (This relation also holds since 7 is larger than 5)

Therefore, form above examples, it is clear that for each pair of positive integers a and b, we have found whole numbers q and r, satisfying the relation :

$$a = bq + r, \text{ where } 0 \leq r < b$$

This result is known as Euclid's divison Lemma in Arithmetics and formally can be expressed by the following way :

Theorem 2.1 (Euclid's Division Lemma)

If a and b are two positive integers then there exist unique integers q and r satisfying

$a = bq + r$, where $0 \le r < b$.

Note : Above lemma can be used for all integers (except zero) and it is kept in mind that q and r may be zero.

Applications of above euclid's division and method can be understood by the following examples :

Example 1. Show that any positive integer can be written in the form $3q$ or $3q + 1$ or, $3q + 2$, where q is some integer.

Solution : Let a is any positive integer and $b = 3$.

Applying division algorithm with a and b.

$a = 3q + r$, where $0 \le r < 3$ and q is any integer. Putting $r = 0, 1, 2$

$a = 3q + 0$, $a = 3q + 1$ $a = 3q + 2$

$\therefore$ $a = 3q$, $a = 3q + 1$ $a = 3q + 2$

Therefore, any integer can be written in the form of $3q$, $3q + 1$, $3q + 2$.

Example 2. Show that every positive even integer is of the form $2q$ and that every positive odd integer is of the from $2q + 1$, where q is any integer.

Solution : Let a be any positive integer and $b = 2$

Then by Euclid's algorithm,

$a = 2q + r$, where $0 \le r < 2$ and q is any integer

By putting $r = 0, 1$

$a = 2q + 0$, $a = 2q + 1$ ($\because$ r is an integer)

$a = 2q$, $a = 2q + 1$

Since q is an integer and $a = 2q$ then a is an even integer.

We know that any integer can be even or odd. So, if a is an even integer then $a + 1$ i.e. $2q + 1$ will be odd integer.

Example 3. Use Euclid's division lemma to show that the square of any positive integer is either of the form $3m$ or $3m + 1$ for some integer m.

Solution : Let a be any positive integer.

We know that it will be of the form of any positive integer

$a = 3q$ or, $a = 3q + 1$ or, $a = 3q + 2$

(i) If $a = 3q$ then, $a^2 = (3q)^2 = 9q^2 = 3(3q) = 3m$, where $m = 3q$

(ii) If $a = 3q + 1$ then $a^2 = (3q + 1)^2 = 9q^2 + 6q + 1$

$$= 3q(3q + 2) + 1$$

$$= 3m + 1$$

where $m = q(3q + 2)$

(iii) If $a = 3q + 2$, then

$$a^2 = (3q+2)^2 = 9q^2 + 12q + 4$$

$$= 9q^2 + 12q + 3 + 1 = 3(3q^2 + 4q + 1) + 1$$

$$\Rightarrow \quad = 3m + 1$$

where $m = (3q^2 + 4q + 1)$

from above (i), (ii) and (iii) it is clear that square of any integer a is of the form 3 m or $3m$ +1.

2.03. Euclid's Division Algorithm (Method)

Here we will study the application of Euclid's division algorithm based on Euclid's division lemma. The word algorithm comes form the name of the 9th century Persian mathematician Al-Khwarizmi. This algorithm is a series of well defined steps which gives a procedure for solving a type of problem.

Euclid's division algorithm is a technique to compute the highest common factor (HCF) of two positive integers. HCF of two positive integers a and b is the largest positive integer that divides both a and b.

To find, highest common factor by Euclid's division algorithm, Euclid's division lemma is used in the following steps.

Step 1 : Let a and b (where a > b) are two positive integers, then apply Euclid's division lemma to a and b so, we find whole numbers q and r such that : $a = bq + r, \; 0 \le r < b$

Step 2 : If $r = 0$, then b is highest common factor of a and b and if $r \ne 0$, then apply the division lemma to b and a to obtained integers q_1 and r_1 such that $b = rq_1 + r_1$

Step 3 : Now, if $r_1 = 0$ then r will be HCF of a and b and if $r_1 \ne 0$ then apply Euclid's division lemma for r and r_1

Step 4 : Continue this process till the remainder becomes zero. In the situation when remainder becomes zero, the divisor at this stage will be the required HCF. This method can easily be understood by the following examples.

Example 1. Use Euclid's algorithm to find the HCF of 81 and 237.

Solution :

Step 1 : Here given integers 81 and 237 are such that 237 > 81, we apply the Euclid's division lemma to these integers, to get

$$237 = 81 \times 2 + 75 \qquad \text{. . . (i)}$$

Step 2 : Since remainder $75 \ne 0$, we apply the Euclid's division lemma to 81 and 75 to get

$$81 = 75 \times 1 + 6 \qquad \text{. . . (ii)}$$

Step 3 : From equation (ii) it is clear that, remainder $6 \ne 0$ so again apply Euclid's division lemma to 75 and 6, to get

$$75 = 6 \times 12 + 3 \qquad \text{. . . (iii)}$$

Step 4 : We will continue this process till remainder becomes zero, here remainder $3 \ne 0$, we apply the division lemma to 6 and 3, to get

$$6 = 3 \times 2 + 0 \qquad \text{. . . (iv)}$$

from equation (iv) it is clear that remainder is zero so process is completed. Since the divisor at this stage is 3, hence the HCF of 81 and 237 is 3. In brief this division process can be understood as follows

81 | 237 | 2
162
75 | 81 | 1
75
6 | 75 | 12
72
HCF = 3 | 6 | 2
6
0 = Remainder

Example 2. An army contigent of 616 members is to march behind an army band of 32 members in a parade. The two groups are to march in the same number of columns. What is the maximum number of columns in which they can march ?

Solution : The members of an army contingent and an army band are to march in the same number of columns. Therefore a maximum number of columns where two groups can march will be HCF of 616 and 32 . So we apply Euclid's division lemma to find HCF of 616 and 32. So

$$616 = 32 \times 19 + 8 \qquad \text{. . . (i)}$$

Here remainder $8 \neq 0$, we apply Euclid's division lemma to 32 and 8,

$$32 = 8 \times 4 + 0 \qquad \text{. . . (ii)}$$

Now, remainder = 0, so HCF of 616 and 32 is 8. In this way two groups can march in 8 columns.

In brief, this division algorithm can be understood as follows

32 | 616 | 19
608
HCF = 8 | 32 | 4
32
0 = Remainder

Example 3. Find the greatest number which divides 245 and 2053 such that it leaves remainder 5 in each case.

Solution : Given that required number divides 245 and 2053 such that it leaves remainder 5 in each case

$\therefore$ $245 - 5 = 240$ and $2053 - 5 = 2048$

i.e., 240 and 2048 can be completely divisible by the required number. This is possible only when there exist a common factor between them.

It is given that required number is greatest number in common factors. So required number will be HCF of 240 and 2048. Applying Euclid's division algorithm step by step, we get

$$2048 = 240 \times 8 + 128$$
$$240 = 128 \times 1 + 112$$
$$128 = 112 \times 1 + 16$$
$$112 = 16 \times 7 + 0$$

It is clear that last remainder = 0. So required HCF is 16.

In brief, this division algorithm can be understood as follows

240 | 2048 | 8
1920
128 | 240 | 1

$$
\begin{array}{l}
\underline{128} \\
112 \mid 128 \mid 1 \\
\quad\ \ \underline{112} \\
\text{HCF} = \quad 16 \mid 112 \mid 7 \\
\qquad\qquad \underline{112} \\
\qquad\qquad\ 0 = \text{Remainder}
\end{array}
$$

Hence

HCF = 16

Exercise 2.1

1. Show that the square of any positive odd integer is of the form $4q + 1$, where q is any integer.
2. Use Euclid's division lemma to show that the cube of any positive integer is of the form $9q$ or $9q + 1$ or $9q + 8$, where q is some integer.
3. Show that any positive odd integer is of the form $6q + 1$, or $6q + 3$, or $6q + 5$, where q is some posetive integer.
4. Use Euclid's division algorithm to find the HCF of :
   (i) 210, 55 (ii) 420, 130 (iii) 75, 243
   (iv) 135, 225 (v) 196, 38220 (vi) 867, 255
5. If HCF of number 408 and 1032 is expressed in the form of $1032x - 408 \times 5$, then find the value of x.

2.04. Fundamental Theroem of Arithmetic

In previous classes, we have studied about prime and composite numbers. We known that any positive prime number is divisible by 1 or itself *i.e.,* factors of a prime number p will be of the form $1 \times p$

Now, we consider any positive integer and express it into factor form. For example

$$5313 = 3 \times 7 \times 11 \times 23$$

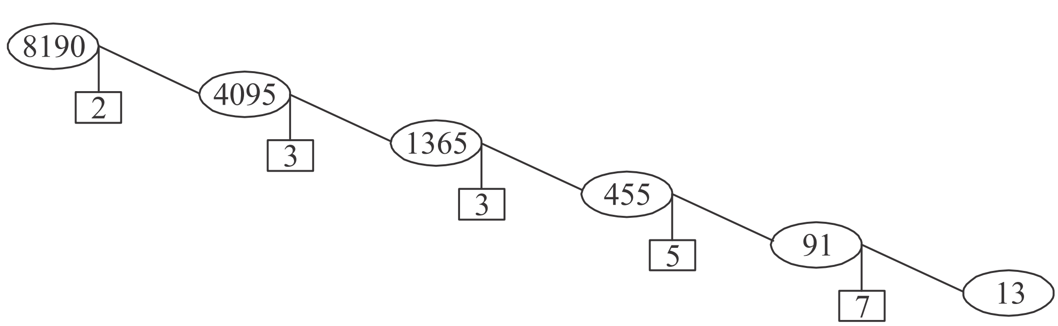
or $$140 = 4 \times 5 \times 7 \quad \text{etc.}$$

From the above example it is clear that each factor will be either prime integer or a composite integer. If any factor is composite integer then it can be further factorized, till we get all prime factors. For example, factors of 140 will be as follows :

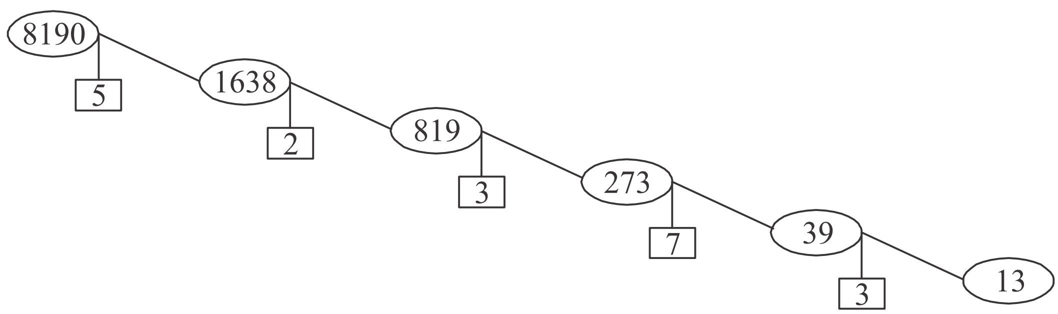
$$140 = 2 \times 2 \times 5 \times 7$$

or $$140 = 2^2 \times 5 \times 7$$

Now, we concentrate on factor tree of any positive integer. Let us take some number say, 8190 and factorize it as shown.



i.e., $$8190 = 2 \times 3 \times 3 \times 5 \times 7 \times 13 \qquad \ldots \text{(i)}$$



i.e., $8190 = 5 \times 2 \times 3 \times 7 \times 3 \times 13$ . . . (ii)

or in this example 8190 is factorized without considering the order of prime numbers in which they are appearing.

So it is clear that every composite number can be expressed (factorised) as a product of primes and this factorisation is unique, apart form the order in which the prime factors occur. If we write factors in ascending order and same prime numbers in the power form then for number 8190, following conjecture is obtained.

$$8190 = 2 \times 3^2 \times 5 \times 7 \times 13$$

This is known as basic or fundamental Theorem of Arithmetic.

Let us now formally state this theorem

Theorem 2.2. (Fundamental Theorem of Arithmetic)

Every composite number can be expressed (factorised) as a product of primes and this factorisation is unique, apart from the order in which the prime factors occur.

This theroem can be understand by the following examples.

Example 1. Examine whether there is any value of n for which 6^n ends with the digit zero ?

Solution : We know that if the number 6^n, for any n were end with the digit zero, then it would be divisible by 5 i.e., the prime factorisation of 6^n would contain the prime factor 5. Here for any n, number 6^n is positive integer which ends with zero. So on factorising, we get

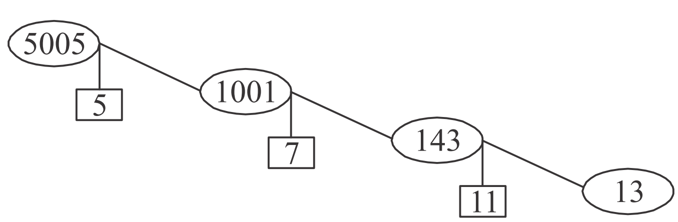
$$6^n = (2 \times 3)^n = 2^n \times 3^n$$

So in the factorisation of 6^n, there is no other prime factor except 2 and 3 i.e., number 5 does not occur in factors. Therefore, there is no natural number n for which 6^n ends with the digit zero.

Example 2. Express following positive integers as a product of its prime factors.

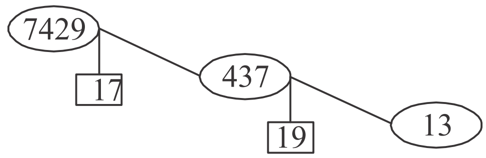
(i) 5005 (ii) 7429

Solution : (i) Factor tree of number 5005 is



So $5005 = 5\times7\times11\times13$ is prime factorisation

(ii) Following will be factor tree of number 7429



So $7429 = 17\times19\times13$ is prime factorisation.

We have already studied how to find the HCF and LCM of two positive integers by prime factorisation method. Here we will find HCF and LCM by fundamental theorem of Arithmetic. This can be understood by the following example. Consider the pair of integers (26 and 91)

Here $26 = 2^1\times13^1$

and $91 = 7^1\times13^1$ are prime factors

$\therefore$ HCF (26, 91) = 13^1

= Product of the smallest power of each common prime factor in the numbers.

and $\text{LCM}(26, 91) = 2^1\times7^1\times13^1$

= Product of the greatest power of each prime factor involved in the numbers.

Here, we see, $\text{HCF}(26, 91)\times\text{LCM}(26, 91) = 26\times91$

Thus, on the basis of fundamental theorem of Arithmetic we concludes that for any two positve integers *a* and *b*.

$$\text{HCF}(a, b)\times\text{LCM}(a, b) = a\times b$$

We can use this result to find the LCM of two positive integers, if we have already found the HCF of the two positive integers.

Example 3. Find the HCF and LCM of 144, 180 and 192 using the prime factorisation method.

Solution : The prime factorisation of 144, 180 and 192 are as follows

$$144 = 2\times2\times2\times2\times3\times3 = 2^4\times3^2$$

$$180 = 2\times2\times3\times3\times5 = 2^2\times3^2\times5^1$$

$$192 = 2\times2\times2\times2\times2\times2\times3 = 2^6\times3^1$$

To find HCF, we write smallest power of each common prime factor as follows

Common factor	Smallest power
2	2
3	1

Here 2^2 and 3^1 are the smallest powers of the common fators 2 and 3 respectively.

$\therefore$ $\text{HCF} = 2^2\times3^1 = 4\times3 = 12$

To find LCM, we write greatest power of each prime factor as follows

Prime factors	Greatest power
2	6
3	2
5	1

$\therefore \quad LCM = 2^6 \times 3^2 \times 5^1 = 64 \times 9 \times 5 = 2880$

Example 4. Find the HCFand LCM of pair of integers (510, 92) and verify whether product of two numbers of pair = HCF × LCM.

Solution : The prime factorisation of 510 and 92 can be written as follows

$$510 = 2 \times 3 \times 5 \times 17 = 2^1 \times 3^1 \times 5^1 \times 17^1$$

$$92 = 2 \times 2 \times 23 = 2^2 \times 23^1$$

To find HCF, we write smallest power of each common prime factor as follows

Common factor	Smallest factor
2	1

$\therefore \quad HCF = 2^1 = 2$

To find LCM, we write greatest power of each factor as follows

Prime factor	Greatest power
2	2
3	1
5	1
17	1
23	1

$\therefore \quad LCM = 2^2 \times 3^1 \times 5^1 \times 17^1 \times 23^1 = 23460$

Verification : Product of two numbers of pair $= 510 \times 92 = 46920$ . . . (i)

and HCF × LCM = 2 × 23460 = 46920 . . . (ii)

from (i) and (ii), we can say that

Product of two numbers = HCF × LCM

Exercise 2.2

1. Express each number as a product of its prime factors :
   (i) 468 (ii) 945 (iii) 140
   (iv) 3825 (v) 20570
2. Find the LCM and HCF of the following pairs of integers and verify that HCF × LCM = Product of the two numbers :
   (i) 96 and 404 (ii) 336 and 54 (iii) 90 and 144
3. Find the HCF and LCM of the following integers by applying the prime factorisation method
   (i) 12, 15 and 21 (ii) 24, 15 and 36 (iii) 17, 23 and 29 (iv) 6, 72 and 120
   (v) 40, 36 and 126 (vi) 8, 9 and 25

4. There is a circular path around a sports field. Raman takes 18 minutes to complete one round of the circular path, while Anupriya takes 12 minutes for the same. Suppose both of them start from the same point and at the same time and go in the same direction. After how many minutes will they meet again at the starting point?

5. In a seminar number of participants in Hindi, English and Mathematics are 60, 84 and 108 respectively. If equal number of participants of same subject are sitting in each room then find the least number of rooms required.

2.05 Proof of irrationality of Numbers

In previous classes, we have studied about irrational numbers in brief. We have also studied the existance of irrational numbers and there representation on number line. Generally, irrational numbers are expressed in the form $\sqrt{p}$, where p is positive prime number. We know that any irrational number cannot be written in the following p/q , Here p and q are integers and $q \neq 0$

For example : $\sqrt{2}, \sqrt{3}, \sqrt{5}, 7\sqrt{5}$ etc. are irrational numbers. In earlier classes we have study properties of irrational numbers that sum and difference of rational and irrational numbers are also irrational numbers and it is also true that multiplication and division of non zero rational and irrational number are also an irrational number.

Here, we will establish the proof of irrational numbers $\sqrt{2}, \sqrt{3}$ and $\sqrt{5}$ *i.e.,* we will prove $\sqrt{2}, \sqrt{3}$ and $\sqrt{5}$ as irrational numbers. We will use the following theroem and proof by contradiction to prove irrationality of these numbers.

Theorem 2.3 Let *p* is a prime number and *'a'* is a positive integer if *p* dividies a^2 then *p* also divides *a*.

Proof: This theorem is obvious result of fundamental theorem of Arithmetic discussed in the preceeding section.

Let the prime factorisation of a be as follows:

$a \ ?\ p_1 p_2 \ldots p_n$ where $p_1, p_2, \ldots, p_n$ are primes, not necessarily distinct.

Now $a^2 \ ?\ (p_1 p_2 \ldots p_n)(p_1 p_2 \ldots p_n) \ ?\ p_1^2 p_2^2 \ldots p_n^2$

Here, it is given that p is a prime number which divides a^2. However, using the uniqueness part of the fundamental theorem of Arithmetic, we can say that the prime factors of a^2 are $p_1, p_2, \ldots, p_n$. So p is one of the $p_1, p_2, \ldots, p_n$. Now, since $a \ ?\ p_1 p_2 \ldots p_n$, p divides a.

Theorem 2.4. Prove that $\sqrt{2}$ is irrational.

Proof : Let $\sqrt{2}$ is a rational number.

So, for integers a and b we can write as

$$\sqrt{2} = \frac{a}{b}, \ b \neq 0$$

where a and b are coprime i.e., there is no common factor in a and b other than 1.

$\therefore$ $$\sqrt{2}b = a$$

Squaring on both sides, we get

$$2b^2 = a^2 \qquad \ldots \text{(i)}$$

$\because$ $2b^2$ is divided by 2. So, we can say 2 divides a^2

But from theorem 2.3 it is clear that 2 divides *a* so we obtained the first result as 2 devides *a*.

Hence, integer *a* can be written as *a* = *2c*, where *c* is an integer.

$\therefore$ $a^2 = 4c^2 \qquad \ldots \text{(ii)}$

Substituting the value of a^2 from (i) in (ii), we get

$$2b^2 = 4c^2$$

$$\Rightarrow \qquad b^2 = 2c^2$$

Since $2c^2$ is divisible by 2. This means that 2 divides b^2 also is divides *b*.

Hence again from theorem (2.3) we can say that 2 divides *b* so we obtained the second result as 2 divides b.

Therefore from last two results it is evident that *a* and *b* have at least 2 as a common factor.

But this contradicts the fact that *a* and *b* have no common factors other than 1. If means that our hypothesis i.e., $\sqrt{2}$ is rational is wrong. So,we can conclude that $\sqrt{2}$ is irrational.

Theroem 2.5 Prove that $\sqrt{3}$ is irrational.

Proof : Let $\sqrt{3}$ is rational,then for integers *a*and *b* we can write

$$\sqrt{3} = \frac{a}{b}, \; b \neq 0$$

where *a* and *b* are co-prime numnber,s means have no common factor other than 1

$$\therefore \qquad \sqrt{3}b = a$$

Squaring on both sides, we get

$$3b^2 = a^2 \qquad \ldots \text{(i)}$$

$\therefore$ $3b^2$ is divisibley 3. So, we can say a^2 is divisible by 3 and by theorem 2.3, it is clear that a is also divisible by 3.

So we can write *a* = *3c* where *c* is an integer.

$$\therefore \qquad a^2 = 9c^2 \qquad \ldots \text{(ii)}$$

From equation (i) and (ii), we get

$$3b^2 = 9c^2$$

i.e., $b^2 = 3c^2$

Since $3c^2$ is devisible by 3 this means that b^2 is divisible by 3, and so using by theorem (2.3) *b* is also divisible by 3.

$\therefore$ Thus from these two results it is evident that *a* and *b* have atleast 3 as a common factor.

But this contradicts the fact that *a* and *b* are co-prime numbers.

This means that out hypothesis is wrong. So, we can conclude that $\sqrt{3}$ is irratiuonal.

Theorem, 2.6. Show the $\sqrt{5}$ is irrational

Proof : Let us assumethat $\sqrt{5}$ is rational, then for two integers *a* and *b* we can write as

$$\sqrt{5} = \frac{a}{b},\ b \neq 0$$

where *a* and *b* are co-prime numbers.It means they have no common factor other than 1.

$\therefore \qquad \sqrt{5}\, b = a$

$\Rightarrow \qquad 5b^2 = a^2 \qquad \text{. . . (i)}$

$\because \quad 5b^2$ is divisible by 5 so a^2 will also be divisible by 5.

By Theorem 2.3, we can say that 5 will divide *a* so integer *a* can be written as *a* = 5 *c*, where *c* is some integer

$\Rightarrow \qquad a^2 = 25c^2 \qquad \text{. . . (ii)}$

From equation (i) and (ii) we get

$$5b^2 = 25c^2$$

$\Rightarrow \qquad b^2 = 5c^2$

$5c^2$ is devisible by 5 so it is clear that b^2 will also be divisible by 5.

By using theorem 2.3 we can say that 5 will divide *b*.

$\therefore$ Thus from the above discussion it is evident that *a* and *b* have at least 5 as acommon factor but this contradicts the fact that *a* and *b* are co-primes.

This means our hypothesis is wrong. So we can conclude that $\sqrt{5}$ is irrational.

The sum or difference or multiplication and division of rational and irrational numbers can easily be understood by the following examples.

Example 1. Prove that $7\sqrt{5}$ is irrational.

Solution : Let $7\sqrt{5}$ is a rational number

$\therefore \qquad 7\sqrt{5} = \frac{a}{b},\ b \neq 0$, where *a*, *b* are co-prime numbers

or $\qquad \sqrt{5} = \frac{a}{7b} \qquad \text{. . . (i)}$

since, *a*, *b* are integers, so $\frac{a}{7b}$ is rational and so $\sqrt{5}$ is rational but this contradicts the fact that $\sqrt{5}$ is irrational, so our hypothesis is wrong, so we can conclude that $7\sqrt{5}$ is irrational.

Example 2. Prove that $3+2\sqrt{5}$ is irrational.

Solution : Let $3+2\sqrt{5}$ is rational

$$\therefore \qquad 3+2\sqrt{5}=\frac{a}{b},\ b\neq 0 \qquad \ldots \text{(i)}$$

where a, b are co-prime. By equation (i) we can say

$$\therefore \qquad 2\sqrt{5}=\frac{a}{b}-3$$

$$\text{or} \qquad \sqrt{5}=\frac{a-3b}{2b} \qquad \ldots \text{(ii)}$$

$\because$ a and b are integers so $\frac{a-3b}{2b}$ will be rational. So from equation (ii) we obtain the result that $\sqrt{5}$ is rational but this contradicts the fact that $\sqrt{5}$ is irrational. So our hypothesis is wrong. So, we can conclude that $3+2\sqrt{5}$ is irrational.

Example 3. Show that $\sqrt{2}+\sqrt{5}$ is irrational.

Solution : Let $\sqrt{2}+\sqrt{5}$ is rational.

$$\therefore \qquad \sqrt{2}+\sqrt{5}=\frac{a}{b},\ b\neq 0 \qquad \ldots \text{(i)}$$

where integers a, b are co-primes.
Equation (i) can be written as

$$\sqrt{5}=\frac{a}{b}-\sqrt{2}$$

Squaring on both sides, we get

$$5=\left(\frac{a}{b}-\sqrt{2}\right)^2$$

$$\Rightarrow \qquad 5=\frac{a^2}{b^2}+2-2\sqrt{2}\frac{a}{b}$$

$$\Rightarrow \qquad 2\sqrt{2}\frac{a}{b}=\frac{a^2}{b^2}-3$$

$$\Rightarrow \qquad \sqrt{2}=\frac{a^2-3b^2}{2ab} \qquad \ldots \text{(ii)}$$

$\because$ a, b are integers so $\frac{a^2-3b^2}{2ab}$ will be rational so from euation (ii) $\sqrt{2}$ is rational but this contradicts the fact that $\sqrt{2}$ is irrational.So, our hypothesis is wrong. So, we conclude that $\sqrt{2}+\sqrt{5}$ is irrational.

Exericse 2.3

1. Prove that $5-\sqrt{3}$ is irratioanl.
2. Prove that following numbers are irrational

   (i) $\frac{1}{\sqrt{2}}$ (ii) $6+\sqrt{2}$ (iii) $3\sqrt{2}$

3. If p and q are positive prime numbers then prove that $\sqrt{p}+\sqrt{q}$ is irrational.

2.06. Decimal Expansion of Rational Numbers

We know that $\frac{p}{q}, q \neq 0$ is rational, where p and q are co-prime numbers. In previous classes we have studied about decimal expansion of rational numbers. We know that rational numbers have either a terminating decimal expansion or a non-terminating repeating decimal expansion. In this section, we are going to consider a rational number, and explore exactly when the decimal expansion of numbers is terminating and when it is non-terminating. Let us consider the following rational numbers to understand the nature of decimal expansion:

(i) 0.375 (ii) 1.512 (iii) 0.01764 (iv) 23.3408

By changing above decimal numbers in fraction form

(i) $0.375 = \frac{375}{1000} = \frac{375}{10^3}$ (ii) $1.512 = \frac{1512}{1000} = \frac{1512}{10^3}$

(iii) $0.01764 = \frac{1764}{100000} = \frac{1764}{10^5}$ (iv) $23.3408 = \frac{233408}{10000} = \frac{233408}{10^4}$

Denominators of all these numbers are powers of 10. This decimal expansion is of terminating nature. We know that powers of 10 can only have powers of 2 and 5 as factors.So cancelling out the common factors between the numerator and the denominator, we find the following results :

(i) $0.375 = \frac{375}{10^3} = \frac{3\times 5^3}{2^3\times 5^3} = \frac{3}{2^3} = \frac{3}{2^3\times 5^0}$

(ii) $1.512 = \frac{1512}{10^3} = \frac{2^3\times 3^3\times 7}{2^3\times 5^3} = \frac{3^3\times 7}{5^3} = \frac{189}{2^0\times 5^3}$

(iii) $0.01764 = \frac{1764}{10^5} = \frac{2^2\times 3^2\times 7^2}{2^5\times 5^5} = \frac{3^2\times 7^2}{2^3\times 5^5} = \frac{441}{2^3\times 5^5}$

(iv) $23.3408 = \frac{233408}{10^4} = \frac{2^6\times 7\times 521}{2^4\times 5^4} = \frac{14588}{2^0\times 5^4}$

From the above, it is clear that prime factorisation of denominator of rational number is of the form $2^m\times 5^n$ where m, n are some non-negative integers.

Theorem 4. Let x be a rational number whose decimal expansion terminates. Then x can be expressed in the form $\frac{p}{q}$, q ? 0, where p and q are co-prime and the prime factorisation of q is of the form $2^m\times 5^n$, where m, n are non-negative integers. Decimal expansion of the rational number $a/b,\ b \neq 0$, where a and b are prime integers is possible if b is some power of 10. Now, consider whether the converse of this theorem will be true?

For example:

(i) $\frac{3}{8} = \frac{3 \times 5^3}{2^3 \times 5^3} = \frac{375}{10^3} \Rightarrow \frac{3}{8} = 0.375$

(ii) $\frac{189}{125} = \frac{3^3 \times 7 \times 2^3}{5^3 \times 2^3} = \frac{1512}{10^3} \Rightarrow \frac{189}{125} = 1.512$

(iii) $\frac{441}{25000} = \frac{3^2 \times 7^2 \times 2^2}{2^3 \times 5^5 \times 2^2} = \frac{1764}{10^5} \Rightarrow \frac{441}{25000} = 0.01764$

From above exmaple it is clear that we can convert a rational number of the form $p/q,\ q \neq 0$ where q is of the form $2^m \times 5^n$ to an equivalent rational number of the form $\frac{a}{b}$, where b is some power of 10. Therefore, the decimal expansion of such rational number terminates. Let us write down this result as the following theorem.

Theorem 5. Let $x = \frac{p}{q},\ q \neq 0$ be a rational number, such that the prime factorisation of q is of the form $2^m \times 5^n$, where m, n are non-negative intergers.Then decimal expansion of x terminates.

Now we consider the rational numbers whose decimal expansions are non-terminating recurring.

For example :

Consider the following rational numbers :

(i) $\frac{5}{3}$ (ii) $\frac{29}{343}$ (iii) $\frac{77}{210}$

(i) $\frac{5}{3} = 1.6666...$ (ii) $\frac{29}{343} = 0.0845481...$ (iii) $\frac{77}{210} = 0.36666...$

In the above rational numbers, denominator is not of the form $2^m \times 5^n$ and on dividing the numerator by the denominator the remainder 0 will not be obtained.

It means that for these rational numbers their decimal expansions are non-terminating recurring.

This statement can be expressed in the form of statement of a theorem.

Theorem 6. Let $x = \frac{p}{q},\ q \neq 0$ is rational number such that the prime factorisation of q is not of the form $2^m \times 5^n$, where m, n are non-negative integers, then x has a decimal expansion which is non-terminating recurring.

Example 1. Without actually performing the long division method, state whether the following rational numbers will have a terminating decimal expansion or a non-terminating recurring decimal expasnion

(i) $\frac{17}{8}$ (ii) $\frac{64}{455}$ (iii) $\frac{125}{441}$

Solution : (i) Here, $\frac{17}{8} = \frac{17}{2^3 \times 5^0}$

It is clear that denominator 8 is of the form $2^m \times 5^n$. So $\frac{17}{8}$ has terminating decimal expansion.

(ii) Here $\frac{64}{455} = \frac{64}{5 \times 7 \times 13}$

It is clear that denominator 455 is not of the form $2^m \times 5^n$. So $\frac{64}{455}$ has non-terminating recurring decimal expansion.

(iii) Here $\frac{125}{441} = \frac{5^3}{3^2 \times 7^2}$

It is clear denominator 441 is not of the form $2^m \times 5^n$. So $\frac{125}{441}$ has non-terminating recurring decimal expansion.

Exericse 2.4

1. Without actually performing the long division method, state whether the following rational numbers will have a terminating decimal expansion or a non-terminating recurring decimal expansion.

   (i) $\frac{15}{1600}$ (ii) $\frac{13}{3125}$ (iii) $\frac{23}{2^3 \times 5^2}$ (iv) $\frac{17}{6}$

   (v) $\frac{129}{2^2 \times 5^7 \times 7^5}$ (vi) $\frac{35}{50}$ (vii) $\frac{7}{80}$

2. Write down the decimal expansion of the following rational numbers and show whether these are terminating.

   (i) $\frac{13}{125}$ (ii) $\frac{14588}{625}$ (iii) $\frac{49}{500}$

3. For the followig decimal expansions, decide whether they are rational or not. If they are rational then write the note on prime factors of its denominator.

   (i) 0.120120012000120000 . . . (ii) 43.123456789 (iii) $27.\overline{142857}$

Miscellaneous Exercise -2

1. Sum of powers of prime factors of 196 is :

   (a) 1 (b) 2 (c) 4 (d) 6

2. If two numbers are wirtten in the form $m = pq^3$ and $n = p^3q^2$ then HCF of m, n whereas p, q are prime numbers is :

   (a) pq (b) pq^2 (c) p^2q^2 (d) p^3q^3

3. HCF of 95 and 152 is :

   (a) 1 (b) 19 (c) 57 (d) 38

4. Product of two numbers is 1080 and their HCF is 30 then their LCM is:

(a) 5 (b) 16 (c) 36 (d) 108

5. Decimal expansion of the number $\frac{441}{2^2 \times 5^7 \times 7^2}$ is

(a) Terminating (b) Non-terminating recurring

(c) Terminating and non-terminating both (d) Number is not a rational number

6. In the decimal expansion of number $\frac{43}{2^2 \times 5^3}$ after how many decimal places will terminate?

(a) 1 (b) 2 (c) 3 (d) 4

7. The lowest number, when multiplied by $\sqrt{27}$ gives a natural number, will be

(a) 3 (b) $\sqrt{3}$ (c) 9 (d) $3\sqrt{3}$

8. If HCF = LCM for two rational numbers then numbers should be

(a) Composite (b) Equal (c) Prime (d) Co-prime

9. If LCM of a and 18 is 36 and HCF of a and 18 is 2 then value of a is

(a) 1 (b) 2 (c) 5 (d) 4

10. If n is a natural number, then unit digit in $6^n - 5^n$ is :

(a) 1 (b) 6 (c) 5 (d) 9

11. If $\frac{p}{q} (q \neq 0)$ is a rational number then what condition apply for q whereas $\frac{p}{q}$ is a terminating decimal?

12. Simplify $\frac{2\sqrt{45} + 3\sqrt{20}}{2\sqrt{5}}$ and find whether it is rational or irrational number.

13. Prove that any positve odd integer is of the form $4q + 1$ or $4q + 3$ where q is any integer.

14. Prove that product of two consecutive positive integers is divisible by 2.

15. Find the largest number which is divided by 2053 and 967, left remainder as 5 and 7 respectively.

16. Describe, why $7 \times 11 \times 13 + 13$ and $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 5$ are composite numbers?

17. If HCF of two numbers 306 and 657 is 9, then find their LCM.

18. A rectangular varandah is 18 m 72 cm long and 13 m × 20 cm wide. Square tiles of same dimensions are used to cover it.Find the least number of such tiles.

19. Prove that following numbers are irrational numbers :

(i) $5\sqrt{2}$ (ii) $\frac{2}{\sqrt{7}}$ (iii) $\frac{3}{2\sqrt{5}}$ (iv) $4 + \sqrt{2}$

20. What can you say about the prime factors of denominator of the followign rational numbers.

(i) 34.12345 (ii) $43.\overline{123456789}$

Important Points

1. Euclid's division lemma : For two positive integers a and b there exist whole numbers q and r satisfying $a = bq + r,\ 0 \le r < b$. This result is also true for r = 0 and q = 0.
2. Euclid's division algorithm : According to this, the HCF of any two positive integers a and b, with $a > b$ is obtained as follows.

   Step 1 : Apply the division lemma to find q and r where $a = bq_1 + r_1,\ 0 \le r_1 < b$

   Step 2 : If $r_1 = 0$, the HCF of a and b is b

   Step 3 : If $r_1 \ne 0$, then apply Euclid's lemma to b and r_1 and obtained the integers q_2 and r_2 where as $b = q_2 r_1 + r_2$.

   Step 4 : If $r_2 = 0$ then r_1 is HCF of a, b.

   Step 5 : If r_2 ? 0 continue the process till the remainder r_n becomes zero.The divisor r_{n-1} at this stage will be HCF of a and b.
3. The fundamental theorem of Arithemtic : Every compositive number can be expressed (factorised) as a product of primes and this factorisation is unique, apart from the order in which the prime factors occur.
4. Each composit number can be uniquelly expressed in ascending and descending order of powers of prime factors.
5. For positive integer a ,prime number p is such that p divided a^2 then p will diivide a also.
6. If p is positive prime number, then $\sqrt{p}$ is an irrational number.
7. For any rational number p / q, its decimal expansion will be terminating if its denominator q can be written in the form of $2^m \times 5^n$, where m, n are non-negative integers. Here p and q are co-prime integers. If q cannot be expressed in the form $2^m \times 5^n$, then decimal expansion will be non-terminating recurring .

Answers

Exericse 2.1

4. (i) 5 (ii) 10 (iii) 3 (iv) 45 (v) 196 (vi) 51 5. 2

Exercise 2.2

1. (i) $2^2 \times 3^2 \times 13$ (ii) $3^3 \times 5 \times 7$ (iii) $2^2 \times 5 \times 7$ (iv) $3^2 \times 5^2 \times 17$ (v) $2 \times 5 \times 11^2 \times 17$
2. (i) HCF = 4, LCM = 9696 (ii) HCF = 6, LCM = 3024 (iii) HCF = 18, LCM = 720
3. (i) HCF = 3, LCM = 420 (ii) HCF = 3, LCM = 360 (iii) HCF = 1, LCM = 11339
   (iv) HCF = 6, LCM = 360 (v) HCF = 2, LCM = 2520 (vi) HCF = 1, LCM = 1800
4. 36 minute 5. 21

Exercise 2.4

1. (i) Terminating (ii) Terminating (iii) Terminating (iv) Non-terminating
   (v) Non-terminating (vi) Termianting (vii) Terminating

2. (i) 0.104 (ii) 23.3408 (iii) 0.098

3. (i) Irrational

(ii) Rational prime factor of denominator is of the form $2^m \times 5^n$ where *m*, *n* are non-negative intergers.

Miscellaneous Exericse-2

1. (c) 2. (b) 3. (b) 4. (c) 5. (a) 6. (d)

7. (b) 8. (b) 9. (d) 10. (a)

11. Prime factorisation of denominator q is of the form $2^m \times 5^n$ where *m*, *n* are non-negative integers.

12. Rational number 15. 64 17. 22338 18. 4290

20. (i) Since its decimal expansion is terminating so its denominator is of the form $2^m \times 5^n$,where *m*, *n* are non negative integers.

(ii) Since its decimal expansion is non-terminating repeating so its denominator is not of the form $2^m \times 5^n$ where *m*,*n* are nonnegative integers.