04

Rational Exponents and Radicals

Exponential Form

The repeated multiplication of the same non-zero rational number *a* with itself in the form of a^n, i.e. $a \times a \times \ldots \times a$ (n terms) $= a^n$, where *a* is called the base and *n*, an integer called the exponent or index. This type of representation of a number is called the exponential form of the given number.

e.g. $6 \times 6 \times 6 = 6^3$

Here, 6 is the base and 3 is the exponent and we read it as '6 raised to the power of 3'.

When $n = 0$, then $(a)^0 = 1$, $a \neq 0$

Rational Exponents

A rational exponents represent both an integer exponent and nth root.

e.g. $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

Negative Integral Exponents

For any non-zero integer *a*, we have

$$a^{-n} = \frac{1}{a^n} \quad \text{or} \quad a^{-n} \times a^n = 1$$

So, a^{-n} is the multiplicative inverse or reciprocal of a^n and *vice-versa*.

e.g. $(5)^{-2} = \frac{1}{5^2}$

Example 1 The multiplicative inverse of 10^{-5} is

(a) 10^4 (b) 10^5
(c) 10^6 (d) None of these

Sol. (*b*) We haves, $10^{-5} = \frac{1}{10^5}$

Reciprocal of $\frac{1}{10^5} = 10^5$

$\therefore$ Multiplicative inverse of 10^{-5} is 10^5.

$[\because 10^{-5} \times 10^5 = 10^\circ = 1]$

Laws of Exponent

I. If a, b are any real numbers and m, n are positive integers, then

(i) $a^m \times a^n = a^{m+n}$ (ii) $a^m \div a^n = a^{m-n}$

(iii) $(a^m)^n = a^{mn}$

(iv) $(ab)^n = a^n b^n$

(v) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

II. If *a*, *b* are any real numbers and *m*, *n* are negative integers, then

(i) $a^{-m} \times a^{-n} = \frac{1}{a^m} \times \frac{1}{a^n}$

$$= \frac{1}{a^m \times a^n} = \frac{1}{a^{m+n}} = a^{-(m+n)}$$

(ii) $a^{-m} \div a^{-n} = \frac{1}{a^m} \div \frac{1}{a^n} = \left(\frac{1}{a^m} \times \frac{a^n}{1}\right)$

$$= \frac{a^n}{a^m} = a^{n-m} = a^{-m-(-n)}$$

(iii) $(a^{-m})^{-n} = \left[\frac{1}{(a^{-m})}\right]^n$

$$= (a^m)^n = a^{mn} = a^{(-m)(-n)}$$

(iv) $(ab)^{-n} = \frac{1}{(ab)^n} = \frac{1}{a^n \times b^n}$

$$= \frac{1}{a^n} \times \frac{1}{b^n} = a^{-n} \times b^{-n}$$

(v) $\left(\frac{a}{b}\right)^{-n} = \frac{1}{\left(\frac{a}{b}\right)^n} = \frac{1}{\frac{a^n}{b^n}} = \frac{b^n}{a^n} = \frac{a^{-n}}{b^{-n}}$

Example 2 The value of

$(7^{-1} - 8^{-1})^{-1} - (3^{-1} - 4^{-1})^{-1}$ is

(a) 44 (b) 56 (c) 68 (d) 12

Sol. (*a*) *Using law of exponents,* $a^{-m} = \frac{1}{a^m}$

[∵ a is non-zero integer]

$\therefore (7^{-1} - 8^{-1})^{-1} - (3^{-1} - 4^{-1})^{-1}$

$$= \left(\frac{1}{7} - \frac{1}{8}\right)^{-1} - \left(\frac{1}{3} - \frac{1}{4}\right)^{-1}$$

$$= \left(\frac{1}{56}\right)^{-1} - \left(\frac{1}{12}\right)^{-1}$$

$$= 56 - 12 = 44$$

Example 3 Find x, if $\left(\frac{2}{9}\right)^3 \times \left(\frac{2}{9}\right)^{-6} = \left(\frac{2}{9}\right)^{2x-1}$.

(a) 1 (b) −1
(c) 2 (d) −2

Sol. (*b*) Given, $\left(\frac{2}{9}\right)^3 \times \left(\frac{2}{9}\right)^{-6} = \left(\frac{2}{9}\right)^{2x-1}$

Using law of exponents, $a^m \times a^n = (a)^{m+n}$

[∵ a is non-zero integer]

Then, $\left(\frac{2}{9}\right)^{3-6} = \left(\frac{2}{9}\right)^{2x-1} \Rightarrow \left(\frac{2}{9}\right)^{-3} = \left(\frac{2}{9}\right)^{2x-1}$

On comparing, we get $-3 = 2x - 1$

$\Rightarrow \quad -2 = 2x$

$\Rightarrow \quad x = -1$

Radicals or Surds

If $\sqrt[n]{a}$ is irrational, where a is a rational number and n is a positive integer, then $\sqrt[n]{a}$ or $a^{1/n}$ is called a radical of order n and a is called the radicand.

A radical of order 2 is called a quadratic (or square) radical.

A radical of order 3 is called a cubic radical.

A radical of order 4 is called a biquadratic radical.

Let n be a positive integer and a be a real number.
- (i) If a is irrational, then $\sqrt[n]{a}$ is not a radical.
- (ii) If a is rational, then $\sqrt[n]{a}$ is a radical.

Types of Radical

There are two types of radical.

(i) **Pure Radical** A radical which has only irrational factor is called a pure radical factor. e.g. $\sqrt{21}$

(ii) **Mixed Radical** A radical which have both rational and irrational factors, is called mixed radical. e.g. $2\sqrt{3}$

Laws of Radical

As radical can be expressed with fractional exponent, the laws of indices are therefore, applicable to radical.

(i) $(\sqrt[n]{a})^n = a$ (ii) $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$

(iii) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$

(iv) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ (v) $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$

Example 4 The value of $\left[\left(\frac{25}{9}\right)^{5/2}\right]^{3/5}$ is

(a) $\frac{25}{27}$ (b) $\frac{125}{27}$

(c) $\frac{25}{9}$ (d) None of these

Sol. (*b*) $\left[\left(\frac{25}{9}\right)^{5/2}\right]^{3/5} = \left[\left\{\left(\frac{5}{3}\right)^2\right\}^{5/2}\right]^{3/5}$

$$= \left[\left(\frac{5}{3}\right)^5\right]^{3/5} = \left(\frac{5}{3}\right)^3 = \frac{125}{27}$$

Example 5 The value of $(0.000064)^{5/6}$ is

(a) $\frac{32}{100000}$ (b) $\frac{16}{10000}$

(c) $\frac{16}{100000}$ (d) None of these

Sol. (*a*) $(0.000064)^{5/6} = \left(\frac{64}{1000000}\right)^{5/6}$

$$= \left[\left\{\left(\frac{2}{10}\right)^6\right\}^{1/6}\right]^5 = \left(\frac{2}{10}\right)^5 = \frac{32}{100000}$$

Comparison of Two Radicals

If two radicals are of same order, then the radical whose radicand is greater than other, is greater radical. If two radicals of different order are to be compared, then first we reduce them into the same but smallest order and then compare the two radicals.

Example 6 Which is larger $\sqrt[3]{3}, \sqrt[4]{4}, \sqrt[3]{5}$ or $\sqrt[4]{5}$?

(a) $\sqrt[3]{3}$ (b) $\sqrt[4]{4}$

(c) $\sqrt[3]{5}$ (d) $\sqrt[4]{5}$

Sol. (*c*) Given radicals are of order 3 and 4 whose LCM is 12.

Now, we change each one into a radical of order 12.

$$\therefore \sqrt[3]{3} = 3^{1/3} = 3^{\left(\frac{1}{3}\times\frac{4}{4}\right)} = 3^{4/12} = (3^4)^{1/12} = (81)^{1/12}$$

$$\sqrt[4]{4} = 4^{1/4} = 4^{\left(\frac{1}{4}\times\frac{3}{3}\right)} = 4^{3/12} = (4^3)^{1/12} = (64)^{1/12}$$

$$\sqrt[3]{5} = (5)^{1/3} = 5^{\left(\frac{1}{3}\times\frac{4}{4}\right)} = (5^4)^{1/12} = (625)^{1/12}$$

and $\sqrt[4]{5} = 5^{1/4} = (5)^{\left(\frac{1}{4}\times\frac{3}{3}\right)} = (5^3)^{1/12} = (125)^{1/12}$

It is clear that larger value is $(625)^{1/12}$ i.e. $\sqrt[3]{5}$.

Practice Exercise

1. For a fixed base, if the exponent decreases by 1, the number becomes
(a) one-tenth of the previous number
(b) ten times of the previous number
(c) hundredth of the previous number
(d) hundred times of the previous number

2. The reciprocal of $\left(\frac{2}{5}\right)^{-1}$ is

(a) $\frac{2}{5}$ (b) $\frac{5}{2}$

(c) $-\frac{5}{2}$ (d) $-\frac{2}{5}$

3. The radical form of $\left(\frac{13}{25}\right)^{3/4}$ is

(a) $\sqrt[3]{\left(\frac{13}{25}\right)^4}$ (b) $\sqrt[4]{\left(\frac{13}{25}\right)^3}$

(c) $\sqrt[4]{\left(\frac{25}{13}\right)^3}$ (d) $\sqrt[3]{\left(\frac{25}{13}\right)^4}$

4. The value of $(-2)^{2\times 3-1}$ is

(a) 32 (b) 64 (c) −32 (d) −64

5. $(-9)^3 \div (-9)^8$ is equal to

(a) $(9)^5$ (b) $(9)^{-5}$

(c) $(-9)^5$ (d) $(-9)^{-5}$

6. Simplify $(4^{-1} + 3^{-1} + 6^{-2})^{-1}$.

(a) $\frac{18}{11}$ (b) $\frac{11}{18}$

(c) $\frac{19}{7}$ (d) None of these

7. The value of $\left(-\frac{1}{125}\right)^{-2/3}$ is

(a) 5 (b) 25

(c) − 25 (d) None of these

8. The value of $\frac{5}{(121)^{-1/2}}$ is

(a) −55 (b) $\frac{1}{55}$ (c) $-\frac{1}{55}$ (d) 55

9. The value of $(512)^{-3/9}$ is

(a) $\frac{1}{4}$ (b) 8

(c) $\frac{1}{8}$ (d) $-\frac{1}{8}$

10. The value of $3 \times 9^{-3/2} \times 9^{1/2}$ is

(a) $\frac{1}{3}$ (b) 3

(c) 27 (d) $-\frac{1}{3}$

11. The value of $(216^{2/3})^{1/2}$ is

(a) 3 (b) 9

(c) 12 (d) 6

12. The value of $27^{-1/3} \times (27^{2/3} \div 27^{1/3})$ is

(a) 4 (b) 3

(c) 2 (d) 1

13. The value of $(6.25)^{-1/2}$ is

(a) 0.25 (b) 0.5 (c) 0.4 (d) 2.5

14. Find the value of x, if

$\left(\frac{5}{3}\right)^{-2} \times \left(\frac{5}{3}\right)^{-14} = \left(\frac{5}{3}\right)^{8x}$.

(a) −3 (b) 2

(c) −2 (d) 1

15. The value of $(0.03125)^{-2/5}$ is

(a) 1 (b) 2

(c) 3 (d) 4

16. The value of $(x^{a-b})^c \times (x^{b-c})^a \times (x^{c-a})^b$ is

(a) 0 (b) 1

(c) x^{ab} (d) x^{bc}

17. The value of $(12^2 + 5^2)^{1/2}$ is

(a) 11 (b) 13 (c) 12 (d) 15

18. If $9\sqrt{x} = \sqrt{12} + \sqrt{147}$, the value of x is

(a) 1 (b) 2 (c) 3 (d) 4

19. Simplify $\dfrac{49 \times z^{-3}}{7^{-3} \times 10 \times z^{-5}}$, $(z \neq 0)$.

(a) $\frac{7^4}{10}z^2$ (b) $\frac{7^5}{10}z^2$

(c) $\frac{7^6}{10}z^2$ (d) None of these

20. The arrangements of the numbers $\sqrt[4]{3}, \sqrt[6]{10}, \sqrt[12]{25}$ in descending order will be

(a) $\sqrt[4]{3}, \sqrt[12]{25}, \sqrt[6]{10}$ (b) $\sqrt[6]{10}, \sqrt[4]{3}, \sqrt[12]{25}$

(c) $\sqrt[4]{3}, \sqrt[6]{10}, \sqrt[12]{25}$ (d) $\sqrt[6]{10}, \sqrt[12]{25}, \sqrt[4]{3}$

21. The value of

$\left(\frac{a^x}{a^y}\right)^{x+y} \times \left(\frac{a^y}{a^z}\right)^{y+z} \times \left(\frac{a^z}{a^x}\right)^{z+x}$ is

(a) 0 (b) $\frac{1}{y}$ (c) 1 (d) $\frac{1}{xyz}$

22. The value of the expression

$\dfrac{\left(p+\frac{1}{q}\right)^m \times \left(p-\frac{1}{q}\right)^m}{\left(q+\frac{1}{p}\right)^m \times \left(q-\frac{1}{p}\right)^m}$ is

(a) $\left(\frac{p}{q}\right)^{2m}$ (b) $\left(\frac{p}{q}\right)^{-2m}$ (c) $(pq)^{2m}$ (d) $\left(\frac{p}{q}\right)^{m}$

23. The value of $\dfrac{2^{10+n} \times 4^{3n-5}}{2^{4n+1} \times 2^{3n-1}}$ is

(a) 0 (b) 1 (c) 3 (d) 4

24. The value of $\dfrac{(81)^{1/3} \times (576)^{1/3}}{(64)^{2/3} \times (27)^{2/3}}$ is

(a) $\frac{1}{4}$ (b) $\frac{3}{4}$

(c) $\frac{5}{8}$ (d) None of these

Answers

1	(a)	2	(b)	3	(b)	4	(c)	5	(d)	6	(a)	7	(b)	8	(d)	9	(c)	10	(a)
11	(d)	12	(d)	13	(c)	14	(c)	15	(d)	16	(b)	17	(b)	18	(c)	19	(b)	20	(b)
21	(c)	22	(a)	23	(b)	24	(a)												

Hints & Solutions

1. (a) For a fixed base, if the exponent decreases by 1, the number becomes one-tenth of the previous number.

e.g. For 10^5, exponent decreases by 1.

i.e. $10^{5-1} = 10^4$

$\therefore \quad \frac{10^4}{10^5} = \frac{1}{10}$

2. (b) Using law of exponents, $a^{-m} = \frac{1}{a^m}$

[$\because$ a is non-zero integer]

$\therefore \quad \left(\frac{2}{5}\right)^{-1} = \frac{1}{\left(\frac{2}{5}\right)^1} = \frac{5}{2}$

3. (b) The radical form of $\left(\frac{13}{25}\right)^{3/4} = \sqrt[4]{\left(\frac{13}{25}\right)^3}$

4. (c) Given, $(-2)^{2\times 3-1} = (-2)^{6-1} = (-2)^5$

$= (-2)\times(-2)\times(-2)\times(-2)\times(-2) = -32$

[for $(-a)^m$, if m is odd, then $(-a)^m$ is negative]

5. (d) Given, $(-9)^3 \div (-9)^8$

Using law of exponents, $\quad a^m \div a^n = (a)^{m-n}$

[$\because$ a is non-zero integer]

$\therefore \quad (-9)^3 \div (-9)^8 = (-9)^{3-8} = (-9)^{-5}$

6. (a) Using law of exponents, $a^{-m} = \frac{1}{a^m}$

$\therefore \quad (4^{-1} + 3^{-1} + 6^{-2})^{-1}$

$= \left(\frac{1}{4} + \frac{1}{3} + \frac{1}{36}\right)^{-1} = \left(\frac{9+12+1}{36}\right)^{-1}$

[$\because$ LCM of 4, 3 and 36 = 36]

$= \left(\frac{22}{36}\right)^{-1} = \frac{36}{22}$

$= \frac{18}{11}$

7. (b) $\left(-\frac{1}{125}\right)^{-2/3}$

$= \left[\left\{\left(-\frac{1}{5}\right)\times\left(-\frac{1}{5}\right)\times\left(-\frac{1}{5}\right)\right\}^{-1/3}\right]^2$

$= (-5)^2 = 25$

8. (d) $\frac{5}{(121)^{-1/2}} = 5\times 121^{1/2}$

$= 5\times(11^2)^{1/2} = 5\times 11 = 55$

9. (c) $(512)^{-3/9} = (2^9)^{-3/9}$

$= 2^{9\times\left(-\frac{3}{9}\right)} = 2^{-3} = \left(\frac{1}{2}\right)^3$

$= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{8}$

10. (a) $3\times 9^{-3/2}\times 9^{1/2}$

$= 3\times\left(3^{2\times -\frac{3}{2}}\right)\times\left(3^{2\times\frac{1}{2}}\right) = 3\times(3)^{-3}\times 3$

$= 3\times\left(\frac{1}{3}\right)^3\times 3 = 3\times\frac{1}{27}\times 3 = \frac{1}{3}$

11. (d) $(216^{2/3})^{1/2} = \left(6^{3\times\frac{2}{3}}\right)^{1/2} = 6^{2\times\frac{1}{2}} = 6^1 = 6$

12. (d) $27^{-1/3}\times(27^{2/3}\div 27^{1/3})$

$= 3^{3\times -\frac{1}{3}}\times[(3^3)^{2/3}\div(3^3)^{1/3}] = 3^{-1}\times(3^2\div 3)$

$= \frac{1}{3}\times(9\div 3) = \frac{1}{3}\times 3 = 1$

13. (C) $(6.25)^{-1/2} = \left(\frac{625}{100}\right)^{-1/2}$

$= \left(\frac{25}{4}\right)^{-1/2} = \left(\frac{4}{25}\right)^{1/2} = \left(\frac{2}{5}\right)^{2\times\frac{1}{2}}$

$= \frac{2}{5}$ or 0.4

14. (c) We have, $\left(\frac{5}{3}\right)^{-2}\times\left(\frac{5}{3}\right)^{-14} = \left(\frac{5}{3}\right)^{8x}$

Using law of exponents, $a^m\times a^n = (a)^{m+n}$

[$\because$ a is non-zero integer]

Then, $\left(\frac{5}{3}\right)^{-2}\times\left(\frac{5}{3}\right)^{-14} = \left(\frac{5}{3}\right)^{8x}$

$\Rightarrow \left(\frac{5}{3}\right)^{-2-14} = \left(\frac{5}{3}\right)^{8x} \Rightarrow \left(\frac{5}{3}\right)^{-16} = \left(\frac{5}{3}\right)^{8x}$

On comparing both sides, we get $\ -16 = 8x$

$\Rightarrow \qquad x = -2$

15. (d) $(0.03125)^{-2/5} = \left(\frac{3125}{100000}\right)^{-2/5} = \left(\frac{100000}{3125}\right)^{2/5}$

$$= \left(\frac{10}{5}\right)^{5 \times \frac{2}{5}} = (2)^2 = 4$$

16. (b) $(x^{a-b})^c \times (x^{b-c})^a \times (x^{c-a})^b$

$= x^{ac-bc} \times x^{ba-ca} \times x^{cb-ab}$

$= x^{ac-bc+ba-ca+cb-ab} = x^0 = 1$

17. (b) $(12^2 + 5^2)^{1/2} = (144 + 25)^{1/2}$

$$= (169)^{1/2} = (13^2)^{1/2} = 13$$

18. (c) $9\sqrt{x} = \sqrt{12} + \sqrt{147}$

$= \sqrt{2 \times 2 \times 3} + \sqrt{3 \times 7 \times 7}$

$= 2\sqrt{3} + 7\sqrt{3} \Rightarrow 9\sqrt{x} = 9\sqrt{3}$

$\Rightarrow \quad x^{1/2} = 3^{1/2}$

On comparing both sides, we get x = 3.

19. (b) $\frac{49 \times z^{-3}}{7^{-3} \times 10 \times z^{-5}} = \frac{(7)^2 \times z^{-3}}{7^{-3} \times 10 \times z^{-5}}$

$$= \frac{(7)^{2+3} \times z^{-3+5}}{10}$$

$[\because a^m \div a^n = (a)^{m-n}]$

$$= \frac{(7)^5 z^2}{10} = \frac{7^5}{10} z^2$$

20. (b) LCM $(4, 6, 12) = 12$

Now, $\sqrt[4]{3} = \sqrt[12]{3^3} = \sqrt[12]{27}$

$\sqrt[6]{10} = \sqrt[12]{10^2} = \sqrt[12]{100}$

and $\sqrt[12]{25} = \sqrt[12]{25}$

Here, $\sqrt[12]{100} > \sqrt[12]{27} > \sqrt[12]{25}$

$\Rightarrow \quad \sqrt[6]{10} > \sqrt[4]{3} > \sqrt[12]{25}$

Hence, desending order of given numbers are $\sqrt[6]{10}$, $\sqrt[4]{3}$ and $\sqrt[12]{25}$.

21. (c) $\left(\frac{x^x}{a^y}\right)^{x+y} \times \left(\frac{a^y}{a^z}\right)^{y+z} \times \left(\frac{a^z}{a^x}\right)^{z+x}$

$= (a^{x-y})^{x+y} \times (a^{y-z})^{y+z} \times (a^{z-x})^{z+x}$

$= a^{x^2-y^2} \times a^{y^2-z^2} \times a^{z^2-x^2}$

$= a^{x^2-y^2+y^2-z^2+z^2-x^2}$

$= a^0 = 1$

22. (a) $\frac{\left(p + \frac{1}{q}\right)^m \times \left(p - \frac{1}{q}\right)^m}{\left(q + \frac{1}{p}\right)^m \times \left(q - \frac{1}{p}\right)^m}$

$= \frac{(pq+1)^m (pq-1)^m}{(q)^{2m}} \times \frac{p^{2m}}{(pq+1)^m (pq-1)^m}$

$= \left(\frac{p}{q}\right)^{2m}$

23. (b) $\frac{2^{10+n} \times 4^{3n-5}}{2^{4n+1} \times 2^{3n-1}} = \frac{2^{10+n} \times 2^{6n-10}}{2^{4n+1} \times 2^{3n-1}}$

$$= \frac{2^{10+n+6n-10}}{2^{4n+1+3n-1}} = \frac{2^{7n}}{2^{7n}} = 1$$

24. (a) $\frac{(81)^{1/3} \times (576)^{1/3}}{(64)^{2/3} \times (27)^{2/3}} = \frac{(3^4)^{1/3} \times (2^6 \times 3^2)^{1/3}}{(4^3)^{2/3} \times (3^3)^{2/3}}$

$$= \frac{3^{4/3} \times 2^2 \times 3^{2/3}}{4^2 \times 3^2} = \frac{3^2 \times 2^2}{4^2 \times 3^2}$$

$$= \frac{9 \times 4}{16 \times 9} = \frac{1}{4}$$