CHAPTER 9
Probability and Categorical Data

9-1. Permutations and Combinations

Fundamental Counting Principles

If an event can occur in m different ways and another event can occur in n different ways, then there are $m \times n$ total ways that both events can occur.

A **factorial** is the product of a whole number n, and all the whole numbers less than n and greater than or equal to 1.

$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$

For example, $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

When a group of objects or people is arranged in a certain order, the arrangement is called a **permutation**.

The number of permutations of n distinct objects taken r at a time is defined as ${}_nP_r = \dfrac{n!}{(n-r)!}$.

When a group of objects or people is selected, and the order is not important, the selection is called a **combination**.

The number of combinations of n distinct objects taken r at a time is defined as ${}_nC_r = \dfrac{n!}{r!(n-r)!}$.

The basic difference between a permutation and a combination is that **order is considered** in a permutation and **order is not considered** in a combination.

Example 1 □ There are 4 roads from Town A to Town B, and 3 roads from Town B to Town C. If a person travels from Town A to Town C and back, passing through Town B in both directions, how many different routes for the trip are possible?

Solution □ $A \to B$ 4 different routes
$B \to C$ 3 different routes
$C \to B$ 3 different routes
$B \to A$ 4 different routes

Therefore, there are $4 \times 3 \times 3 \times 4$, or 144, different possible routes for the trip.

Example 2 □ a. There are 2 chairs in a row. If there are 5 students, how many ways can the seats be filled?

b. How many different groups of 2 students can be formed from 5 students?

Solution □ a. In this arrangement, order is considered. AB and BA are not the same.

Use ${}_nP_r$

$${}_5P_2 = \frac{5!}{(5-2)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 20$$

There are 20 possible arrangements.

b. In this arrangement, order is not considered. Therefore, if AB is taken, then BA is excluded.

Use ${}_nC_r$.

$${}_5C_2 = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = 10$$

There are 10 possible combinations.

Exercise - Permutations and Combinations

Questions 1 and 2 refer to the following information.

A hiker is going to hike a mountain where there are four trails to the top of the mountain.

1

In how many different ways can he hike up and down the mountain?

2

If the hiker does not want to take the same trail both ways, in how many different ways can he hike up and down the mountain?

Questions 4 and 5 refer to the following information.

Sixteen players participated in a tennis tournament. Three players will be awarded for first, second, and third prize.

4

In how many different ways can the first, second, and third prizes be awarded?

5

How many different groups of 3 people can get prizes?

3

In how many ways can the letters of the word SUNDAY be arranged using only 3 of the letters at a time?

6

How many different four-letter patterns can be formed from the word MATH if the letters cannot be used more than once?

9-2. Rules of Probability

An **outcome** is one of the possible results that can occur as a result of a trial.
The **probability** of event A, denoted $P(A)$, is:

$$P(A) = \frac{\text{the number of times the desired outcome occurs}}{\text{the total number of outcomes}}.$$

If the outcome of one event does not influence the outcome of the second event, then the events are said to be **independent**. When drawing at random with replacement, the draws are independent. Without replacement, the draws are **dependent**.

If two events A and B are independent, then the probability that both events will occur is
$P(A \text{ and } B) = P(A) \cdot P(B)$.
If two events A and B are dependent, then the probability that both events will occur is
$P(A \text{ and } B) = P(A) \cdot P(B \text{ following } A)$.
If two events A and B cannot occur at the same time, then the probability that either A or B occurs is
$P(A \text{ or } B) = P(A) + P(B)$.

Example 1 □ A bag contains 6 red balls and 4 blue balls. Two balls are selected one at a time.

a. Find the probability of selecting a red ball then a blue ball, if the first ball is *replaced* before the second one is drawn.

b. Find the probability of selecting a red ball then a blue ball, if the first ball is *not replaced* before the second one is drawn.

c. Find the probability that both balls are the same color, if the first ball is *not replaced* before the second one is drawn.

Solution □ a. Since the first ball is replaced, the selection of the first ball does not influence the selection of the second ball. The event is independent.

$$P(r \text{ and } b) = P(r) \cdot P(b) = \frac{6}{10} \cdot \frac{4}{10} = \frac{6}{25}$$

b. Since the first ball is not replaced, the selection of the first ball influence the selection of the second ball. The event is dependent.

$$P(r \text{ and } b) = P(r) \cdot P(b \text{ following } r)$$
$$= \frac{6}{10} \cdot \frac{4}{9} = \frac{4}{15}$$

The conditional probability $P(b \text{ following } r)$ means that the second ball is blue on the condition that the first ball is red.

c. The event is dependent.

$$P(r \text{ and } r) = P(r) \cdot P(r \text{ following } r)$$
$$= \frac{6}{10} \cdot \frac{5}{9} = \frac{30}{90}$$

$P(r \text{ following } r) = \frac{5}{9}$ ← 5 red marbles are left ← 9 total marbles are left

$$P(b \text{ and } b) = P(b) \cdot P(b \text{ following } b)$$
$$= \frac{4}{10} \cdot \frac{3}{9} = \frac{12}{90}$$

$P(b \text{ following } b) = \frac{3}{9}$ ← 3 blue marbles are left ← 9 total marbles are left

$$P(\text{both same color})$$
$$= P(r \text{ and } r) + P(b \text{ and } b)$$
$$= \frac{30}{90} + \frac{12}{90} = \frac{42}{90} = \frac{7}{15}$$

$P(A \text{ or } B) = P(A) + P(B)$

Exercise - Rules of Probability

Questions 1 and 2 refer to the following information.

A bag contains 15 balls, numbered 1 through 15.

1

What is the probability of selecting a number that is odd or a multiple of 5?

2

A ball is selected at random then replaced in the bag. A second selection is then made. What is the probability that the first number is a prime number and the second number is a multiple of 3?

3

$$S = \{-5, -2, -1,\ 4\} \qquad T = \{-2,\ 3,\ 7\}$$

Product $p = s \cdot t$ is formed from the two sets above, in which s is a number from set S and t is a number from set T. What is the probability that the product $s \cdot t$ will be a positive number?

Questions 4 and 5 refer to the following information.

Janis is making a flight reservation for her business trip. The travel agent informs that the probability that her flight to Phoenix will arrive on schedule is 90% and the probability that her flight from Phoenix to Atlanta will arrive on schedule is 80%.

4

What is the probability that both flights arrive on schedule?

5

What is the probability that her flight to Phoenix is on schedule but her flight from Phoenix to Atlanta is not?

6

In a box of 12 headlamps 3 are defective. If you choose two headlamps without replacement, what is the probability that both headlamps are defective?

9-3. Categorical Data and Conditional Probabilities

The probability of event A, given that event B occurred, is called the **conditional probability** of A given B and is denoted by $P(A|B)$. [Note: The vertical bar between A and B is read "given."]

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{\text{population of } A \text{ in } B}{\text{population of } B}$$

Two-way **contingency table** is a type of table in a matrix format that displays relationships between two or more categorical variables.

The table on below left shows 300 randomly selected voters from a large city, categorized by age and voting preferences. The age of voters is the row variable, and the name of parties is the column variable.
The totals are placed in the right and bottom margins of the table and thus are called **marginal frequencies**.
The total of the marginal frequencies is the grand total, which is the size of the sample.
The frequencies in the contingency table are often expressed as percentages or relative frequencies.
The table on below right shows the contingency table expressed as percentages of the grand total.

	Democrat	Republican	Total
Under 40	84	63	147
40 or over	60	93	153
Total	144	156	300

	Democrat	Republican	Total
Under 40	28%	21%	49%
40 or over	20%	31%	51%
Total	48%	52%	100%

Example 1 □ The table below shows the results of a survey regarding Proposition A from two regions of a large city. Four hundred registered voters were surveyed.

	Voted For Proposition A	Voted Against Proposition A	No Opinion for Proposition A	Total
East	90	75	17	182
West	104	96	18	218
Total	194	171	35	400

a. What is the probability that a randomly chosen person has no opinion on Proposition A?

b. What is the probability that a randomly chosen person is from East given that he or she voted for Proposition A?

c. What is the probability that a randomly chosen person voted against Proposition A and is from West?

Solution □ a. Since 35 of the 400 voters have no opinion on Proposition A,

$$P(\text{No opinion for Proposition } A) = \frac{35}{400} \text{ or } \frac{7}{80}.$$

b. This is a conditional probability question. There are 194 voters who voted for Proposition A, and 90 of them are from East. Thus,

$$P(\text{population of East in the population Voted For Proposition } A) = \frac{90}{194} \text{ or } \frac{45}{97}.$$

c. $P(\text{against Proposition } A \text{ and from West}) = \dfrac{96}{400} = \dfrac{6}{25}$

Exercise - Categorical Data and Conditional Probabilities

Questions 1- 5 refer to the following information.

The table below shows the number of college faculty members in three departments: biological sciences, education, and social sciences.

	Biological Sciences	Education	Social Sciences	Total
Male	10	26	19	55
Female	15	21	17	53
Total	25	47	36	108

1

What is the probability that a randomly chosen faculty member is a female given that she is from Biological Sciences?

2

What is the probability that a randomly chosen faculty member is a male or from Social Sciences?

3

What is the probability that a randomly chosen faculty member is a female from Education department or a male from Social Sciences?

4

What is the probability that a randomly chosen faculty member is from Biological Sciences given that the faculty member is a male?

5

For Biological Science and Education faculties combined, $\frac{1}{6}$ of the female and $\frac{1}{4}$ of the male faculty members are associate professors. If a person is randomly chosen from these two departments, what is the probability that a faculty member is an associate professor?

Chapter 9 Practice Test

Questions 1-4 refer to the following information.

	Economics	History	Music
Male	24	20	19
Female	18	22	17

The table above shows the distribution of a group of 120 college students by gender and major.

1

If one student is randomly selected from the group, what is the probability that the student is a History major?

A) $\frac{36}{120}$

B) $\frac{40}{120}$

C) $\frac{42}{120}$

D) $\frac{46}{120}$

2

If a male student is selected at random, which of the following is closest to the probability that he is a Music major?

A) 0.270

B) 0.302

C) 0.317

D) 0.381

3

If one student is randomly selected from the group what is the probability that the student is a male Economics major?

A) $\frac{24}{120}$

B) $\frac{42}{120}$

C) $\frac{24}{42}$

D) $\frac{24}{63}$

4

If a Music major is selected at random, which of the following is closest to the probability that the student is a female?

A) 0.298

B) 0.315

C) 0.386

D) 0.472

Questions 5 and 6 refer to the following information.

	Under 30	30 or older	Total
Male	3		12
Female			20
Total	8	24	32

The incomplete table above shows the distribution of age and gender for 32 people who entered a tennis tournament.

5

If a tennis player is chosen at random, what is the probability that the player will be either a male under age 30 or a female aged 30 or older?

A) $\frac{15}{32}$

B) $\frac{18}{32}$

C) $\frac{20}{32}$

D) $\frac{24}{32}$

6

If a person is selected at random from the 30 or older player group, what is the probability that the person is a female?

A) $\frac{5}{20}$

B) $\frac{15}{20}$

C) $\frac{9}{24}$

D) $\frac{15}{24}$

Questions 7 and 8 refer to the following information.

Number of Visits to Movie Theaters by Students

	None	1 to 2	3 or more
Juniors	x	$2x$	$\frac{1}{2}x$
Seniors	y	$\frac{5}{2}y$	$\frac{1}{2}y$

The table above summarizes the number of visits to movie theaters by 168 juniors and 152 seniors during summer vacation.

7

If a student is selected at random from those who visited movie theaters at least once, what is the probability that the student is a junior?

A) $\frac{16}{39}$

B) $\frac{18}{39}$

C) $\frac{20}{39}$

D) $\frac{22}{39}$

8

If a student is selected at random, which of the following is closest to the probability that the student is a senior and visited movie theaters 1 or 2 times?

A) 0.156

B) 0.205

C) 0.297

D) 0.324

Answer Key

Section 9-1

1. 16 2. 12 3. 120 4. 3360 5. 560
6. 24

Section 9-2

1. $\frac{3}{5}$ 2. $\frac{2}{15}$ 3. $\frac{5}{12}$ 4. 0.72 5. 0.18
6. $\frac{1}{22}$

Section 9-3

1. $\frac{3}{5}$ 2. $\frac{2}{3}$ 3. $\frac{10}{27}$ 4. $\frac{2}{11}$ 5. $\frac{5}{24}$

Chapter 9 Practice Test

1. C 2. B 3. A 4. D 5. B
6. D 7. C 8. C

Answers and Explanations

Section 9-1

1. 16

There are 4 different ways of going up and 4 different ways of going down, so there are 4×4, or 16 different ways he can hike up and down.

2. 12

If the hiker does not want to take the same trail both ways, then there are 4 different ways of going up and 3 different ways of going down, so there are 4×3, or 12 different ways he can hike up and down.

3. 120

Use ${}_nP_r$, since order is considered.

$${}_6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6\cdot 5\cdot 4\cdot \cancel{3\cdot 2\cdot 1}}{\cancel{3\cdot 2\cdot 1}} = 120$$

Therefore, there are 120 ways the 6 letters in SUNDAY can be arranged when the letters are taken 3 at a time.

4. 3360

16	15	14

The first choice can be any one of the 16 players, the second choice can be any one of the 15 remaining players, and the third choice can be any one of the 14 remaining players. Therefore, there are $16\times 15\times 14$, or 3,360 different ways the first, second, and third prizes are awarded.

5. 560

In this case, order does not matter. We must find the combination of 16 people taken 3 at a time.

$${}_{16}C_3 = \frac{16!}{3!(16-3)!} = \frac{16!}{3!\times 13!} = \frac{16\cdot 15\cdot 14\cdot 13!}{3\cdot 2\cdot 1\cdot 13!} = 560$$

6. 24

4	3	2	1

The first choice can be any one of the 4 letters, the second choice can be any one of the 3 remaining letters, the third choice can be any one of the 2 remaining letters and the fourth choice is the 1 remaining letter. Therefore, there are $4\times 3\times 2\times 1$, or 24 different ways the 4 letters in MATH can be arranged.

Section 9-2

1. $\frac{3}{5}$

There are 8 odd numbers: 1, 3, 5, 7, 9, 11, 13, and 15. There are 3 multiples of 5: 5, 10, and 15. Therefore, there are 9 numbers which are either odd or a multiple of 5.

$$P(\text{odd or a multiple of 5}) = \frac{9}{15} = \frac{3}{5}$$

2. $\frac{2}{15}$

There are 6 prime numbers: 2, 3, 5, 7, 11, and 13.

Therefore, $P(\text{prime}) = \frac{6}{15}$.

There are 5 multiples of three: 3, 6, 9, 12, and 15.

Therefore, $P(\text{multiples of 3}) = \frac{5}{15}$.

The probability that the first number is a prime number and the second number is a multiple of 3 is $\frac{6}{15}\times\frac{5}{15}$, or $\frac{2}{15}$.

3. $\frac{5}{12}$

$S = \{-5, -2, -1, 4\}$ $T = \{-2, 3, 7\}$

Make a table of the possible products of $p = s \cdot t$.

$(-5)\cdot(-2) = 10$, $(-5)\cdot 3 = -15$, $(-5)\cdot 7 = -35$

$(-2)\cdot(-2) = 4$, $(-2)\cdot 3 = -6$, $(-2)\cdot 7 = -14$

$(-1)\cdot(-2) = 2$, $(-1)\cdot 3 = -3$, $(-1)\cdot 7 = -7$

$4\cdot(-2) = -8$, $4\cdot 3 = 12$, $4\cdot 7 = 28$

There are 12 products, 5 of which are positive numbers. Therefore,

$P(\text{product is a positive number}) = \frac{5}{12}$

4. 0.72

The probability that both flights arrive on schedule is $0.9 \times 0.8 = 0.72$.

5. 0.18

The probability that her flight to Phoenix is on schedule but her flight from Phoenix to Atlanta is not on schedule is $0.9 \times (1 - 0.8) = 0.18$.

6. $\frac{1}{22}$

If your first selection is a defective headlamp, 11 headlamps will be left and 2 of them will be defective. The probability that both headlamps are defective is $\frac{3}{12} \times \frac{2}{11} = \frac{1}{22}$.

Section 9-3

1. $\frac{3}{5}$

	Biological Sciences	Education	Social Sciences	Total
Male	10	26	19	55
Female	15	21	17	53
Total	25	47	36	108

There are 25 faculty members in Biological Sciences, and 15 of them are female. Therefore, the probability that the person chosen is a female given that she is from Biological Sciences is $\frac{15}{25} = \frac{3}{5}$.

2. $\frac{2}{3}$

There are 55 male faculty members and 36 Social Science faculty members. Since the male faculty in Social Sciences are counted twice you must subtract 19, from the sum of 55 and 36. Therefore, the probability that a randomly chosen faculty member is a male or from Social Sciences is $\frac{55}{108} + \frac{36}{108} - \frac{19}{108}$, or $\frac{2}{3}$.

3. $\frac{10}{27}$ or 0.37

There are 21 females from Education Department and 19 males from Social Sciences. Therefore, the probability that a randomly chosen faculty member is a female from Education Department or a male from the Social Sciences is $\frac{21}{108} + \frac{19}{108}$, or $\frac{10}{27}$.

4. $\frac{2}{11}$

There are 10 male faculty members in Biological Sciences out of 55 males. Therefore, the probability that a randomly chosen faculty member is from the Biological Sciences given that he is a male is $\frac{10}{55} = \frac{2}{11}$.

5. $\frac{5}{24}$

The number of females in both departments combined is $15 + 21$, or 36, and the number of males in both departments combined is $10 + 26$, or 36. The number of female associate professors in both department combined are $36 \times \frac{1}{6} = 6$ and the number of male associate professors in both department combined are $36 \times \frac{1}{4} = 9$.

There are $25 + 47$, or 72, faculty members in both departments combined and $6 + 9$, or 15, associate professors in both departments combined. Therefore, if a person is randomly chosen from these two departments, the probability that a faculty member is an associate professor is $\frac{15}{72} = \frac{5}{24}$.

Chapter 9 Practice Test

1. C

	Economics	History	Music
Male	24	20	19
Female	18	22	17

There are 120 student total and 42 students are History majors. Therefore, the probability that the student is a History major is $\frac{42}{120}$.

2. B

There are $24+20+19=63$ male students. If a male student is selected at random, the probability that he is a Music major is $\frac{19}{63}\approx 0.302$.

3. A

The probability that the student is a male Economics major is $\frac{24}{120}$.

4. D

There are $19+17$, or 36, Music majors. The probability that a Music major selected at random is a female is $\frac{17}{36}\approx 0.472$.

5. B

	Under 30	30 or older	Total
Male	3		12
Female			20
Total	8	24	32

There are 3 males under age of 30. The number of males 30 years or older is $12-3=9$. Therefore, the number of females 30 years or older is $24-9=15$. The probability that the player will be either a male under age 30 or a female aged 30 or older is $\frac{3+15}{32}=\frac{18}{32}$.

6. D

There are 15 females who are aged 30 or older. If a person is selected at random from the 30 or older player group, the probability that the person is a female is $\frac{15}{24}$.

7. C

Number of Visits to Movie Theaters by Students

	None	1 to 2	3 or more
Juniors	x	$2x$	$\frac{1}{2}x$
Seniors	y	$\frac{5}{2}y$	$\frac{1}{2}y$

There are 168 juniors and 152 seniors. Therefore, $x+2x+\frac{1}{2}x=168$, and $y+\frac{5}{2}y+\frac{1}{2}y=152$.
Solving the equations give $x=48$ and $y=38$.
There are $2x+\frac{1}{2}x=\frac{5}{2}x=\frac{5}{2}(48)=120$ juniors and $\frac{5}{2}y+\frac{1}{2}y=3y=3(38)=114$ seniors who visited movie theaters at least once.
If a student is selected at random from those who visited movie theaters at least once, the probability that the student is a junior is $\frac{120}{120+114}$, or $\frac{20}{39}$.

8. C

Seniors who visited movie theaters 1 or 2 times is $\frac{5}{2}y=\frac{5}{2}(38)=95$.
The probability that the student is a senior and visited movie theaters 1 or 2 times is
$\frac{95}{320}\approx 0.297$