

4. Differential Equations

Differential Equations

An equation involving independent variable x , dependent variable y and the differential coefficients $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, is called differential equation.

Examples : (i) $\frac{dy}{dx} = 1 + x + y$ (ii) $\frac{dy}{dx} + xy = \cot x$

(iii) $\left(\frac{d^4y}{dx^4}\right)^3 - 4\frac{dy}{dx} + 4y = 5 \cos 3x$ (iv) $x^2 \frac{d^2y}{dx^2} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 0$

(1) Order of a differential equation:

The order of a differential equation is the order of the highest derivative occurring in the differential equation. For example, the order of above differential equations are 1, 1, 4 and 2 respectively.

The order of a differential equation is a positive integer. To determine the order of a differential equation, it is not needed to make the equation free from radicals.

(2) Degree of a differential equation:

The degree of a differential equation is the degree of the highest order derivative, when differential coefficients are made free from radicals and fractions. The degree of above differential equations are 1, 1, 3 and 2 respectively.

Formation of differential equation

Formulating a differential equation from a given equation representing a family of curves means finding a differential equation whose solution is the given equation. The equation so obtained is the differential equation of order n for the family of given curves.

Algorithm for formation of differential equations

Step (i): Write the given equation involving independent variable x (say), dependent variable y (say) and the arbitrary constants.

Step (ii): Obtain the number of arbitrary constants in step (i). Let there be n arbitrary constants.

Step (iii): Differentiate the relation in step (i) n times with respect to x .

Step (iv): Eliminate arbitrary constants with the help of n equations involving differential coefficients obtained in step (iii) and an equation in step (i). The equation so obtained is the desired differential equation.

Variable separable type differential equation

(1) Equations in variable separable form:

If the differential equation of the form $f_1(x) dx = f_2(y) dy$ (i)

where f_1 and f_2 being functions of x and y only. Then we say that the variables are separable in the differential equation.

Thus, integrating both sides of (i), we get its solution as $\int f_1(x) dx = \int f_2(y) dy + c$, where c is an arbitrary constant.

There is no need of introducing arbitrary constants to both sides as they can be combined together to give just one.

(2) Equations reducible to variable separable form:

(i) Differential equations of the form $\frac{dy}{dx} = f(ax + by + c)$ can be reduced to variable separable form by the substitution $ax + by + c = Z$.

$$\therefore a + b \frac{dy}{dx} = \frac{dZ}{dx} ; \therefore \left(\frac{dZ}{dx} - a \right) \frac{1}{b} = f(Z) \Rightarrow \frac{dZ}{dx} = a + bf(Z).$$

(ii) Differential equation of the form

This is variable separable form and can be solved.

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}, \text{ where } \frac{a}{A} = \frac{b}{B} = K \text{ (say)}$$

$$\therefore \frac{dy}{dx} = \frac{K(Ax + By) + c}{Ax + By + C}$$

$$\text{Put } Ax + By = Z \Rightarrow A + B \frac{dy}{dx} = \frac{dZ}{dx}$$

$$\therefore \left[\frac{dZ}{dx} - A \right] \frac{1}{B} = \frac{KZ + c}{Z + C} \Rightarrow \frac{dZ}{dx} = A + B \frac{KZ + c}{Z + C}$$

Homogeneous differential equation

Exact differential equation

(1) Exact differential equation:

If M and N are functions of x and y , the equation $Mdx + Ndy = 0$ is called exact when there exists a function $f(x, y)$ of x and y such that

$$d[f(x, y)] = Mdx + Ndy \text{ i.e., } \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = Mdx + Ndy$$

where $\frac{\partial f}{\partial x}$ = Partial derivative of $f(x, y)$ with respect to x (keeping y constant). $\frac{\partial f}{\partial y}$ = Partial derivative of $f(x, y)$ with respect to y (treating x as constant)

The necessary and sufficient condition for the differential condition

$$Mdx + Ndy = 0 \text{ to be exact is } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

An exact differential equation can always be derived from its general solution directly by differentiation without any subsequent multiplication, elimination etc.

(2) Integrating factor:

If an equation of the form $Mdx + Ndy = 0$ is not exact, it can always be made exact by multiplying by some function of x and y . Such a multiplier is called an integrating factor.

(3) Working rule for solving an exact differential equation:

Step (i): Compare the given equation with $Mdx + Ndy = 0$ and find out M and N .

Then find out $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

Step (ii): Integrate M with respect to x treating y as a constant.

Step (iii): Integrate N with respect to y treating x as constant and omit those terms which have been already obtained by integrating M .

Step (iv): On adding the terms obtained in steps (ii) and (iii) and equating to an arbitrary constant, we get the required solution.

In other words, solution of an exact differential equation is

$$\int Mdx + \int Ndy = c$$

Regarding y as constant Only those terms not containing x

Solution by inspection

If we can write the differential equation in the form $f_1(x)y' + f_2(y)y' + \dots = 0$, then each term can be easily integrated separately. For this the following results must be memorized.

$$(i) \quad d(x + y) = dx + dy$$

$$(ii) \quad d(xy) = xdy + ydx$$

$$(iii) \quad d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$

$$(iv) \quad d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$$

$$(v) \quad d\left(\frac{x^2}{y}\right) = \frac{2xydx - x^2dy}{y^2}$$

$$(vi) \quad d\left(\frac{y^2}{x}\right) = \frac{2xydy - y^2dx}{x^2}$$

$$(vii) \quad d\left(\frac{x^2}{y^2}\right) = \frac{2xy^2dx - 2x^2ydy}{y^4}$$

$$(viii) \quad d\left(\frac{y^2}{x^2}\right) = \frac{2x^2ydy - 2xy^2dx}{x^4}$$

$$(ix) \quad d\left(\tan^{-1} \frac{x}{y}\right) = \frac{ydx - xdy}{x^2 + y^2}$$

$$(x) \quad d\left(\tan^{-1} \frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$$

$$(xi) \quad d[\ln(xy)] = \frac{xdy + ydx}{xy}$$

$$(xii) \quad d\left(\ln\left(\frac{x}{y}\right)\right) = \frac{ydx - xdy}{xy}$$

$$(xiii) \quad d\left[\frac{1}{2} \ln(x^2 + y^2)\right] = \frac{xdx + ydy}{x^2 + y^2}$$

$$(xiv) \quad d\left[\ln\left(\frac{y}{x}\right)\right] = \frac{xdy - ydx}{xy}$$

$$(xv) \ d\left(-\frac{1}{xy}\right) = \frac{xdy + ydx}{x^2y^2}$$

$$(xvi) \ d\left(\frac{e^x}{y}\right) = \frac{ye^x dx - e^x dy}{y^2}$$

$$(xvii) \ d\left(\frac{e^y}{x}\right) = \frac{xe^y dy - e^y dx}{x^2}$$

$$(xviii) \ d(x^m y^n) = x^{m-1} y^{n-1} (mydx + nxdy)$$

$$(xix) \ d\left(\sqrt{x^2 + y^2}\right) = \frac{xdx + ydy}{\sqrt{x^2 + y^2}}$$

$$(xx) \ d\left(\frac{1}{2} \log \frac{x+y}{x-y}\right) = \frac{xdy - ydx}{x^2 - y^2}$$

$$(xxi) \ \frac{d[f(x,y)]^{1-n}}{1-n} = \frac{f'(x,y)}{(f(x,y))^n}$$

Linear differential equation

Application of differential equation

Differential equation is applied in various practical fields of life. It is used to define various physical laws and quantities. It is widely used in physics, chemistry, engineering etc.

Some important fields of application are:

(i) Rate of change (ii) Geometrical problems etc.

Miscellaneous differential equation

A special type of second order differential equation:

$$\frac{d^2y}{dx^2} = f(x) \quad \dots(i)$$

Equation (i) may be re-written as $\frac{d}{dx} \left(\frac{dy}{dx} \right) = f(x)$

$$\Rightarrow d \left(\frac{dy}{dx} \right) = f(x) dx$$

$$\text{Integrating, } \frac{dy}{dx} = \int f(x) dx + c_1 \text{ i.e. } \frac{dy}{dx} = F(x) + c_1 \quad \dots(ii)$$

$$\text{Where } F(x) = \int f(x) dx + c_1 dx$$

$$\text{From (ii), } dy = f(x) dx + c_1 dx$$

$$\text{Integrating, } y = \int F(x) dx + c_1 x + c_2$$

$$\therefore y = H(x) + c_1 x + c_2$$

where $H(x) = \int F(x) dx$ c_1 and c_2 are arbitrary constants.