Exercise 17.4

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We assume there is a solution of the form

$$y=\sum_{n=0}^{\infty} c_n x^n$$

Then $\quad y'=\sum_{n=1}^{\infty} n c_n x^{n-1}$

$$=\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n$$

Substituting in differential equations $y'-y=0$, we get

$$\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n-\sum_{n=0}^{\infty}c_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)c_{n+1}-c_n\right]x^n=0$$

This equation is true if

$$(n+1)c_{n+1}-c_n=0$$

i.e. $\quad c_{n+1}=\dfrac{c_n}{n+1}$

Put $n=0,1,2,3,\ldots\ldots\ldots\ldots$, then

$$c_1=\frac{c_0}{1}, c_2=\frac{c_1}{2}=\frac{c_0}{1.2}=\frac{c_0}{2!}$$

$$c_3=\frac{c_2}{3}=\frac{c_0}{1.2.3}=\frac{c_0}{3!},\ c_4=\frac{c_3}{4}=\frac{c_0}{1.2.3.4}=\frac{c_0}{4!}$$

In general $c_n=\dfrac{c_0}{n!}$

The solution is $y=\sum_{n=0}^{\infty}\dfrac{c_0}{n!}x^n$

i.e. $\quad \boxed{y=c_0\sum_{n=0}^{\infty}\dfrac{x^n}{n!}=c_0e^x}$

Consider the differential equation is $y' = xy$.

Assume there is a solution of the form,

$$y = c_0x^0 + c_1x^1 + c_2x^2 + c_3x^3 + c_4x^4 + ...$$
$$= \sum_{n=0}^{\infty} c_n x^n$$

On differentiating both sides of the equation, $y = \sum_{n=0}^{\infty} c_n x^n$, we get

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$
$$= \sum_{n=1}^{\infty} n c_n x^{n-1}$$

Substitute in the given differential equation, we get

$$y' = xy$$
$$\sum_{n=1}^{\infty} n c_n x^{n-1} = x \sum_{n=0}^{\infty} c_n x^n$$
$$\sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} c_n x^{n+1}$$
$$\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n = \sum_{n=1}^{\infty} c_{n-1} x^n$$
$$c_1 + \sum_{n=1}^{\infty} (n+1) c_{n+1} - \sum_{n=1}^{\infty} c_{n-1} x^n = 0$$
$$c_1 + \sum_{n=1}^{\infty} \left[(n+1) c_{n+1} - c_{n-1} \right] x^n = 0$$

Equating coefficients on both sides

$$c_1 = 0$$

And $(n+1)c_{n+1} - c_{n-1} = 0$, $n = 1, 2, ...$

That is $c_{n+1} = \frac{c_{n-1}}{n+1}$, $n = 1, 2, ...$

Put n = 1, then

$$c_2 = \frac{c_0}{2}$$
$$= \frac{c_0}{2.1!}$$

Put n = 2, then

$$c_3 = \frac{c_1}{3}$$
$$= \frac{0}{3} \quad \text{(As } c_1 = 0\text{)}$$
$$= 0$$

Put n = 3, then

$$c_4 = \frac{c_2}{4}$$
$$= \frac{c_0}{2.4}$$
$$= \frac{c_0}{2^2(1.2)}$$
$$= \frac{c_0}{2^2.2!}$$

Put n = 4, then

$$c_5 = \frac{c_3}{5}$$
$$= 0$$

Put n = 5, then

$$c_6 = \frac{c_4}{6}$$
$$= \frac{c_0}{2.4.6}$$
$$= \frac{c_0}{2^3(1.2.3)}$$
$$= \frac{c_0}{2^3.3!}$$

Put n = 6, then

$$c_7 = \frac{c_5}{7}$$
$$= 0$$

Put n = 7, then

$$c_8 = \frac{c_6}{8}$$
$$= \frac{c_0}{2.4.6.8}$$
$$= \frac{c_0}{2^4(1.2.3.4)}$$
$$= \frac{c_0}{2^4.4!}$$

In this manner, we can find the other constants also.

Therefore the required solution is

$$y = c_0x^0 + c_1x^1 + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \ldots + c_nx^n + \ldots$$
$$= c_0x^0 + (0)x^1 + \left(\frac{c_0}{2\cdot 1!}\right)x^2 + (0)x^3 + \left(\frac{c_0}{2^2\cdot 2!}\right)x^4 + (0)x^5 + \left(\frac{c_0}{2^3\cdot 3!}\right)x^6 \ldots + \left(\frac{c_0}{2^n\cdot n!}\right)x^n + \ldots$$
$$= c_0(1) + \left(\frac{c_0}{2\cdot 1!}\right)x^2 + \left(\frac{c_0}{2^2\cdot 2!}\right)x^4 + \left(\frac{c_0}{2^3\cdot 3!}\right)x^6 \ldots + \left(\frac{c_0}{2^n\cdot n!}\right)x^n + \ldots$$
$$= c_0\left(1 + \frac{x^2}{2\cdot 1!} + \frac{x^4}{2^2\cdot 2!} + \frac{x^6}{2^3\cdot 3!} + \ldots + \frac{x^{2n}}{2^n\cdot n!} + \ldots\right)$$
$$= c_0\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n\cdot n!}$$

Therefore the power series solution is

$$y = c_0\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n.n!}$$
$$= c_0e^{\frac{x^2}{2}}$$

Hence the result is $\boxed{y = c_0e^{\frac{x^2}{2}}}$.

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The given equation is

$$y' = x^2y \quad \text{------------ (1)}$$

Let $y = \sum_{n=0}^{\infty} c_nx^n$ be the series solution of (1)

On differentiating $y' = \sum_{n=1}^{\infty} nc_nx^{n-1}$

Substituting for (1)

$$\sum_{n=1}^{\infty} n c_n x^{n-1} - x^2 \sum_{n=0}^{\infty} c_n x^n = 0$$

Or $$\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+2} = 0$$

Or $$\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=2}^{\infty} c_{n-2} x^n = 0$$

Or $$\sum_{n=2}^{\infty} (n+1) c_{n+1} x^n + c_1 + 2c_2 x - \sum_{n=2}^{\infty} c_{n-2} x^n = 0$$

Or $$c_1 + 2c_2 x + \sum_{n=2}^{\infty} \left[(n+1) C_{n+1} - C_{n-2} \right] x^n = 0$$

Equating coefficients on both sides

$$c_1 = 0,\ c_2 = 0$$

$$(n+1) c_{n+1} - c_{n-2} = 0,\ n = 2, 3, \ldots\ldots\ldots$$

That is $c_{n+1} = \dfrac{c_{n-2}}{n+1}$, $n = 2, 3, \ldots\ldots\ldots$

Put n = 2, $c_3 = \dfrac{c_0}{3} = \dfrac{c_0}{3.1!}$

n = 3, $c_4 = \dfrac{c_1}{4} = 0$ (As $c_1 = 0$)

n = 4, $c_5 = \dfrac{c_2}{5} = 0$ (As $c_2 = 0$)

n = 5, $c_6 = \dfrac{c_3}{6} = \dfrac{c_0}{3.6} = \dfrac{c_0}{3^2.2!}$

n = 6, $c_7 = \dfrac{c_4}{7} = 0$

n = 7, $c_8 = \dfrac{c_5}{8} = 0$

n = 8, $c_9 = \dfrac{c_6}{9} = \dfrac{c_0}{3.6.9} = \dfrac{c_0}{3^3.3!}$

n = 9, $c_{10} = \dfrac{c_7}{10} = 0$

n = 10, $c_{11} = \dfrac{c_8}{11} = 0$

---------------, and so on

Then the series solution is

$$y = c_0 + c_3 x^3 + c_6 x^6 + c_9 x^9 + \ldots\ldots$$

$$= c_0 + \frac{c_0}{3.1!} x^3 + \frac{c_0}{3^2.2!} x^6 = \frac{c_0}{3^3.3!} x^9 + \ldots\ldots$$

$$= c_0 \left[1 + \frac{x^3}{3.1!} + \frac{x^6}{3^2.2!} + \frac{x^9}{3^3.3!} + \ldots\ldots \right]$$

$$= c_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n.n!}$$

i.e. $\boxed{y = c_0 e^{\frac{x^3}{3}}}$

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The given equation is

$$(x-3) y' + 2y = 0 \qquad \text{------------- (1)}$$

Let $y = \sum_{n=0}^{\infty} c_n x^n$ be the series solution of (1)

Then on differentiating $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$

Substituting in equation (1)

$$(x-3)\sum_{n=1}^{\infty} n c_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

i.e. $$\sum_{n=1}^{\infty} n c_n x^n - 3\sum_{n=1}^{\infty} n c_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

i.e. $$\sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} 3(n+1) c_{n+1} x^n + \sum_{n=0}^{\infty} 2 c_n x^n = 0$$

i.e. $$\sum_{n=0}^{\infty} \left[n c_n - 3(n+1) c_{n+1} + 2c_n \right] x^n = 0 \quad \left(\text{As} \sum_{n=1}^{\infty} n c_n x^n = \sum_{n=0}^{\infty} n c_n x^n \right)$$

Equating coefficients on both sides,

$$(n+2)c_n - 3(n+1)c_{n+1} = 0, \; n = 0,1,2,\ldots\ldots$$

That is $$c_{n+1} = \frac{(n+2)c_n}{3(n+1)}, \; n = 0,1,2,\ldots\ldots$$

Then for n = 0, $c_1 = \frac{2c_0}{3}$

n = 1, $c_2 = \frac{3c_1}{3(2)} = \frac{3c_0}{3^2}$

n = 2, $c_3 = \frac{4c_2}{3(3)} = \frac{4c_0}{3^3}$

n = 3, $c_4 = \frac{5c_3}{3(4)} = \frac{5c_0}{3^4}$ ---------------, and so on

In general $c_n = \frac{(n+1)c_0}{3^n}$

Thus the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

i.e. $$\boxed{y(x) = c_0 \sum_{n=0}^{\infty} \frac{(n+1)}{3^n} x^n}$$

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The given equation is $y'' + xy' + y = 0$ --------------- (1)

Let $y = \sum_{n=0}^{\infty} c_n x^n$ be the series solution

On differentiating $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$

And $$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

Substituting in (1)

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + x\sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

i.e. $$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

i.e. $$2c_2 + \sum_{n=1}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=1}^{\infty} n c_n x^n + c_0 + \sum_{n=1}^{\infty} c_n x^n = 0$$

i.e. $$c_0 + 2c_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) c_{n+2} + n c_n + c_n \right] x^n = 0$$

i.e. $$c_0 + 2c_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) c_{n+2} + c_n (n+1) \right] x^n = 0$$

Equating coefficients on both sides

$$c_0 + 2c_2 = 0$$

$$\therefore\ c_2 = -\frac{c_0}{2}$$

And $(n+2)(n+1)c_{n+2} + (n+1)c_n = 0,\ n = 1, 2, \ldots\ldots\ldots$

That is $c_{n+2} = \frac{-c_n}{n+2},\ n = 1, 2, \ldots\ldots$

Put $n = 1,\ c_3 = \frac{-c_1}{3}$

$n = 2,\ c_4 = \frac{-c_2}{4} = \frac{c_0}{2.4}$

$n = 3,\ c_5 = \frac{-c_3}{5} = \frac{c_1}{3.5}$

$n = 4,\ c_6 = \frac{-c_4}{6} = \frac{c_2}{4.6} = \frac{-c_0}{2.4.6}$

$n = 5,\ c_7 = \frac{-c_5}{7} = \frac{-c_1}{3.5.7}$

$n = 6,\ c_8 = \frac{-c_6}{8} = -\frac{c_2}{4.6.8} = \frac{c_0}{2.4.6.8}$

$n = 7,\ c_9 = \frac{-c_7}{9} = \frac{c_1}{3.5.7.9}$

$n = 8,\ c_{10} = \frac{-c_8}{10} = \frac{-c_0}{2.4.6.8.10}$

--------------, and so on

Then the series solution is

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \ldots\ldots$$

$$= c_0 + c_1x - \frac{c_0x^2}{2} - \frac{c_1x^3}{3} + \frac{c_0x^4}{2.4} + \frac{c_1x^5}{3.5} - \frac{c_0x^6}{2.4.6} + \ldots\ldots\ldots$$

$$= c_0\left(1 - \frac{x^2}{2.1!} + \frac{x^4}{2^2.2!} - \frac{x^6}{2^3.3!} + \ldots\ldots\right)$$

$$+ c_1\left(x - \frac{x^3}{3} + \frac{x^5}{3.5} - \frac{x^7}{3.5.7} + \ldots\ldots\ldots\right)$$

i.e. $$\boxed{y = c_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}.n!}x^{2n} + c_1\sum_{n=0}^{\infty}\frac{(-2)^n n!}{(2n+1)!}x^{2n+1}}$$

[As $x - \frac{x^3}{3} + \frac{x^5}{3.5} - \frac{x^7}{3.5.7} + \ldots\ldots$

$$= x - \frac{x^3.2}{2.3} + \frac{2.4x^5}{2.3.4.5} - \frac{2.4.6x^7}{3.5.7} + \ldots\ldots\ldots$$

$$= x - \frac{2x^3}{3!} + \frac{2^2 2!}{5!}x^5 - \frac{2^3 3!}{7!}x^7 + \ldots\ldots$$

$$= \sum_{n=0}^{\infty}\frac{(-2)^n n!}{(2n+1)!}x^{2n+1}\ ]$$

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Assume there is a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$

And $y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$

$$= \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$$

Substitute in differential equation $y''-y=0$

We get $\sum_{n=0}^{\infty}\left[(n+2)(n+1)c_{n+2}-c_n\right]x^n=0$

That is $c_{n+2}=\dfrac{c_n}{(n+1)(n+2)}$

Put $\quad n=0,\quad c_2=\dfrac{c_0}{1.2}=\dfrac{c_0}{2!}$

$n=1,\quad c_3=\dfrac{c_1}{2.3}=\dfrac{c_1}{3!}$

$n=2,\quad c_4=\dfrac{c_2}{3.4}=\dfrac{c_0}{1.2.3.4}=\dfrac{c_0}{4!}$

$n=3,\quad c_5=\dfrac{c_3}{4.5}=\dfrac{c_1}{2.3.4.5}=\dfrac{c_1}{5!}$

Then solution is

$$y=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+\dots\dots$$

$$=\frac{c_0}{0!}+\frac{c_1x}{1!}+\frac{c_0x^2}{2!}+\frac{c_1x^3}{3!}+\frac{c_0x^4}{4!}+\frac{c_1x^5}{5!}+\dots\dots$$

i.e. $\boxed{y=c_0\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}+c_1\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}}$

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The given equation is $y''=xy$ ---------- (1)

Let $y=\sum_{n=0}^{\infty}c_nx^n$ be the series solution of (1)

On differentiating

$$y'=\sum_{n=1}^{\infty}nc_nx^{n-1}$$

And $\quad y''=\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}$

$$=\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$$

Substituting in (1)

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n=x\sum_{n=0}^{\infty}c_nx^n$$

i.e. $\quad \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n=\sum_{n=1}^{\infty}c_{n-1}x^n$

i.e. $\quad \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n=\sum_{n=1}^{\infty}c_{n-1}x^n$

Or $\quad \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n-\sum_{n=1}^{\infty}c_{n-1}x^n=0$

Or $\quad 2c_2+\sum_{n=1}^{\infty}(n+2)(n+1)c_{n+2}x^n-\sum_{n=1}^{\infty}c_{n-1}x^n=0$

Or $\quad 2c_2+\sum_{n=1}^{\infty}\left[(n+2)(n+1)c_{n+2}-c_{n-1}\right]x^n=0$

Equating coefficients on both sides

$2c_2=0$

$\therefore\ c_2=0$

And $(n+2)(n+1)c_{n+2}-c_{n-1}=0$, $n=1,2,\dots\dots\dots$

That is $c_{n+2}=\dfrac{c_{n-1}}{(n+1)(n+2)}$, $n=1,2,\dots\dots$

Put $\quad n=1,\ c_3=\dfrac{c_0}{2.3}$

$n=2,\ c_4=\dfrac{c_1}{3.4}$

$n=3,\ c_5=\dfrac{c_2}{4.5}=0 \quad (\text{As } c_2=0)$

$n=4,\ c_6=\dfrac{c_3}{5.6}=\dfrac{c_0}{2.3.4.6}$

Similarly,

$n=5,\ c_7=\dfrac{c_4}{6.7}=\dfrac{c_1}{3.4.6.7}$

$n=6,\ c_8=\dfrac{c_5}{7.8}=0$

$n=7,\ c_9=\dfrac{c_6}{8.9}=\dfrac{c_0}{2.3.5.6.8.9.}$

$n=8,\ c_{10}=\dfrac{c_7}{9.10}=\dfrac{c_1}{3.4.6.7.9.10}$ ---------------, and so on

In general, we have

$$c_{3n+1}=\frac{c_1}{3.4.6.7..........(3n)(3n+1)},\ n=4,3,.......$$

And $\quad c_{3n}=\dfrac{c_0}{2.3.5.6.........(3n-1)(3n)},\ n=4,3,.......$

Thus the general solution of equation (1) is

$$y=c_0\left[1+\frac{x_3}{2.3}+\frac{x^6}{2.3.5.6}+..........+\frac{x^{3n}}{2.3.............(3n-1)(3n)}+..........\right]$$
$$+c_1\left[x+\frac{x^4}{3.4}+\frac{x^7}{3.4.6.7}+........+\frac{x^{3n+1}}{3.4......(3n)(3n+1)}+..........\right]$$

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The given equation is

$y''-xy'-y=0 \qquad$ -------------- (1)

Let $y=\sum_{n=0}^{\infty}c_n x^n$ be the series solution of (1)

On differentiating $y'=\sum_{n=1}^{\infty}n c_n x^{n-1}$

And $\quad y''=\sum_{n=2}^{\infty}n(n-1)c_n x^{n-2}$

$$=\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$$

Substituting in (1)

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n-x\sum_{n=1}^{\infty}n c_n x^{n-1}-\sum_{n=0}^{\infty}c_n x^n=0$$

Or $\quad \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n-\sum_{n=1}^{\infty}n c_n x^n-\sum_{n=0}^{\infty}c_n x^n=0$

Or $\quad \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n-\sum_{n=0}^{\infty}n c_n x^n-\sum_{n=0}^{\infty}c_n x^n=0$

Or $\quad \sum_{n=0}^{\infty}\left[(n+2)(n+1)c_{n+2}-n c_n-c_n\right]x^n=0$

Equating coefficients on both sides

$$(n+2)(n+1)c_{n+2}-(n+1)c_n=0,\ n=0,1,2,.......$$

That is $\ c_{n+2}=\dfrac{(n+1)c_n}{(n+2)(n+1)},\ n=0,1,2,.......$

That is $c_{n+2}=\dfrac{c_n}{n+2},\ n=0,1,2,......$

Put $\quad n=0,\ c_2=\frac{c_0}{2}$

$n=1,\ c_3=\frac{c_1}{3}$

$n=2,\ c_4=\frac{c_2}{4}=\frac{c_0}{2.4}$

$n=3,\ c_5=\frac{c_3}{5}=\frac{c_1}{3.5}$

$n=4,\ c_6=\frac{c_4}{6}=\frac{c_0}{2.4.6}$

$n=5,\ c_7=\frac{c_5}{7}=\frac{c_1}{3.5.7}$ ----------, and so on

In general solution we have

$$c_{2n}=\frac{c_0}{2.4.6..........(2n)}$$

And $c_{2n+1}=\frac{c_1}{3.5.7......(2n+1)},\ n=1,2,3,........$

Then the series solution is

$$y=c_0+c_1x+c_2x^2+c_3x^3+.......$$

$$=c_0\left[1+\frac{x^2}{2}+\frac{x^4}{2.4}+........+\frac{x^{2n}}{2.4.6........(2n)}+........\right]$$

$$+c_1\left[x+\frac{x^3}{3}+\frac{x^5}{3.5}+.........+\frac{x^{2n+1}}{3.5.7.........(2n+1)}+..........\right]$$

$$=c_0\left[1+\frac{x^2}{2}+\frac{x^4}{2^2.2!}+..........+\frac{x^{2n}}{2^n.n!}+.........\right]$$

$$+c_1\left[x+\frac{2x^3}{2.3}+\frac{2^2.2x^5}{5.4.3.2.1}+........+\frac{2^n n!x^{2n+1}}{(2n+1)!}+.......\right]$$

$$=c_0\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n(n)!}+c_1\sum_{n=0}^{\infty}\frac{2^n n!x^{2n+1}}{(2n+1)!}$$

Now $\quad y'=c_0\sum_{n=1}^{\infty}\frac{2_n x^{2n-1}}{2.4.6..........(2n)}+c_1\sum_{n=0}^{\infty}\frac{(2n+1)x^{2n}}{1.3.5.........(2n+1)}$

From the given initial conditions

$$y(0)=1,\ \text{and}\ y'(0)=0$$

We have $c_0=1$, and $c_1=0$

Then the required solution of given initial value problem is

$$y=\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n.n!}$$

Or $\quad \boxed{y=e^{\frac{x^2}{2}}}$

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The given equation is

$$y''+x^2y=0 \quad \text{--------- (1)}$$

Let $y=\sum_{n=0}^{\infty}c_nx^n$ be the series solution of (1)

On differentiating $y'=\sum_{n=1}^{\infty}nc_nx^{n-1}$

And $\quad y''=\sum_{n=0}^{\infty}n(n-1)c_nx^{n-2}$

$$=\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$$

Substituting in (1)

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + x^2\sum_{n=0}^{\infty}c_n x^n = 0$$

Or $$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + \sum_{n=0}^{\infty}c_n x^{n+2} = 0$$

Or $$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + \sum_{n=2}^{\infty}c_{n-2} x^{n} = 0$$

Or $$2c_2 + 6c_3x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)c_{n+2} + c_{n-2}\right]x^n = 0$$

Equating coefficients on both sides

$$c_2 = c_3 = 0$$

And $(n+2)(n+1)c_{n+2} + c_{n-2} = 0$, $n = 2, 3, ...$

That is $c_{n+2} = -\dfrac{c_{n-2}}{(n+2)(n+1)}$, $n = 2, 3, ...$

Or $c_{n+4} = \dfrac{-c_n}{(n+4)(n+3)}$, $n = 0, 1, 2, ...$

Put n = 0, $c_4 = \dfrac{-c_0}{4.3}$

n = 1, $c_5 = \dfrac{-c_1}{5.4}$

n = 2, $c_6 = \dfrac{-c_2}{6.5} = 0$ (As $c_2 = 0$)

n = 3, $c_7 = \dfrac{-c_3}{7.6} = 0$ (As $c_3 = 0$)

n = 4, $c_8 = -\dfrac{c_4}{8.7} = \dfrac{c_0}{8.7.4.3}$

Similarly, n = 5, $c_9 = \dfrac{-c_5}{9.8} = \dfrac{c_1}{9.8.5.4}$

n = 6, $c_{10} = \dfrac{-c_6}{10.9} = 0$

n = 7, $c_{11} = \dfrac{-c_7}{11.10} = 0$

n = 8, $c_{12} = \dfrac{-c_8}{12.11} = \dfrac{-c_0}{12.11.8.7.4.3}$

n = 9, $c_{13} = -\dfrac{c_9}{13.12} = \dfrac{-c_1}{13.12.9.8.5.4}$

---------------, and so on

Hence the solution of equation (1) is

$$y = c_0 + c_1x + c_2x^3 + c_3x^3 + c_4x^4 + c_5x^5 + ...$$

$$= c_0\left[1 - \frac{x^4}{3.4} + \frac{x^6}{3.4.7.8} - \frac{x^{12}}{3.4.7.8.11.12} + ...\right]$$

$$+ c_1\left[x - \frac{x^5}{4.5} + \frac{x^9}{4.5.8.9} - \frac{x^{13}}{4.5.8.9.12.13} + ...\right]$$

$$= c_0\left[1 + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{4(n+1)}}{3.4.7.8........(4n+5)(4n+4)}\right]$$

$$+ c_1x\left[1 + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{4n+4}}{3.4.7.8........(4n+4)(4n+5)}\right]$$

Now $$y' = c_0\left[-\frac{4x^3}{3.4} + \frac{8x^7}{3.4.7.8} - \frac{12x^{11}}{3.4.7.8.11.12} + ...\right]$$

$$+ c_1\left[1 - \frac{5x^4}{4.5} + \frac{9x^8}{4.5.8.9} - \frac{13x^{12}}{4.5.8.9.12.13} + ...\right]$$

By given initial conditions

$y(0) = 1$, and $y'(0) = 0$

We have $c_0 = 1$

And $c_1 = 0$

Therefore the required solution of given initial value problem is

$$\boxed{y = 1 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{4n+4}}{3.4.7.8.......(4n+3)(4n+4)}}$$

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The given equation is

$$y'' + x^2 y' + xy = 0 \qquad \text{---------} \ (1)$$

Let $y = \sum_{n=0}^{\infty} c_n x^n$ be the series solution of equation (1)

On differentiating

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

And $$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

Substituting in (1)

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + x^2 \sum_{n=1}^{\infty} n c_n x^{n-1} + x \sum_{n=0}^{\infty} c_n x^n = 0$$

Or $$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=1}^{\infty} n c_n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

Or $$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=2}^{\infty} (n-1) c_{n-1} x^n + \sum_{n=1}^{\infty} c_{n-1} x^n = 0$$

Or $$2c_2 + 3.2c_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=2}^{\infty} (n-1) c_{n-1} x^n + c_0 x + \sum_{n=2}^{\infty} c_{n-1} x^n = 0$$

Or $$2c_2 + (6c_3 + c_0) x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) c_{n+2} + (n-1) c_{n-1} + c_{n-1} \right] x^n = 0$$

Or $$2c_2 + (6c_3 + c_1) x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) c_{n+2} + n c_{n-1} \right] x^n = 0$$

Equating coefficients on both sides

$$c_2 = 0$$

$$6c_3 + c_0 = 0$$

$$\therefore \ c_3 = -\frac{c_0}{6}$$

And $(n+2)(n+1) c_{n+2} + n c_{n-1} = 0$, $n = 2, 3, ...$

That is $c_{n+2} = -\frac{n c_{n-1}}{(n+1)(n+2)}$, $n = 2, 3, ...$

Put n = 2, $c_4 = \frac{-2c_1}{3.4} = -\frac{2^2 c_1}{4!}$

n = 3, $c_5 = \frac{-3c_2}{4.5} = 0$

n = 4, $c_6 = \frac{-4c_3}{5.6} = \frac{4c_0}{5.6.6} = \frac{4^2 c_0}{6!}$

n = 5, $c_7 = -\frac{5c_4}{6.7} = \frac{2.5c_1}{3.4.6.7} = \frac{2^2 5^2 c_1}{7!}$

n = 6, $c_8 = -\frac{6c_5}{7.8} = 0$

n = 7, $c_8 = \frac{-7c_6}{8.9} = \frac{-4.7c_0}{5.6.6.8.9} = \frac{-4^2.7^2 c_0}{9!}$

n = 8, $c_{10} = \frac{-8c_7}{9.10} = \frac{-2^2.5^2.8c_1}{7!9.10} = \frac{-2^2.5^2.8^2 c_1}{10!}$

n = 9, $c_{11} = -\frac{9c_8}{10.11} = 0$

n = 10, $c_{12} = \frac{10c_9}{11.12} = \frac{4.7.10c_0}{5.6.6.8.9.11.12} = \frac{4^2 7^2 10^2 c_0}{12!}$

Then the series solution is

$$y = c_0 + c_1 x + c_1 x^2 + c_3 x^3 + c_4 x^4 + \dots\dots$$

$$= c_0 + c_1 x - \frac{c_0 x^3}{6} - \frac{2^2 c_1}{4!} x^4 + \frac{4^2 c_0}{6!} x^6 + \frac{2^2.5^2 c_1}{7!} x^7$$

$$- \frac{4^2.7^2 c_0}{9!} x^9 - \frac{2^2.5^2.8^2 c_1}{10!} x^{10} + \dots\dots$$

$$= c_0 \left[-\frac{1}{6} x^3 + \frac{4^2}{4!} x^6 - \frac{4^2.7^2}{9!} x^9 + \dots\dots \right]$$

$$+ c_1 \left[x - \frac{2^2}{4!} x^4 + \frac{2^2.5^2}{7!} x^7 - \frac{2^2.5^2.8^2}{10!} x^{10} + \dots\dots \right]$$

$$= -\frac{c_0 x^3}{6} + c_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4^2.7^2 \dots\dots (3n+1)^2 x^{3n+3}}{(3n+3)!}$$

$$+ c_1 x + c_1 \sum_{n=1}^{\infty} \frac{(-1)^n 2^2.5^2 \dots\dots (3n-1)^2 x^{3n+1}}{(3n+1)!}$$

Now $\quad y'(x) = c_0 \left[\frac{-3x^2}{6} + \frac{4^2.6x^5}{4!} - \frac{4^2.7^2.9x^8}{9!} + \dots\dots \right]$

$$+ c_1 \left[1 - \frac{2^2.4x^3}{4!} + \frac{2^2.5^2.7x^6}{7!} - \frac{2^2.5^2.8^2.10x^9}{10!} + \dots\dots \right]$$

By given initial conditions

$$y(0) = 0, \text{ and } y'(0) = 1$$

We have $c_0 = 0$, and $c_1 = 1$

Then the required solution of given initial value problem is

$$\boxed{y = x + \sum_{n=1}^{\infty} \frac{(-1)^n 2^2.5^2 \dots\dots (3n-1)^2 x^{3n+1}}{(3n+1)!}}$$

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(A)

The given equation is

$$x^2 y'' + xy' + x^2 y = 0 \quad \text{--------------- (1)}$$

Let $y = \sum_{n=0}^{\infty} c_n x^n$ be the series solution of (1)

On differentiating $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$

And $\quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

Substituting in (1)

$$x^2 \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + x \sum_{n=1}^{\infty} n c_n x^{n-1} + x^2 \sum_{n=0}^{\infty} c_n x^n = 0$$

i.e. $\quad \sum_{n=2}^{\infty} n(n-1) c_n x^n + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2} = 0$

i.e. $\quad \sum_{n=2}^{\infty} n(n-1) c_n x^n + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=2}^{\infty} c_{n-2} x^n = 0$

i.e. $\quad \sum_{n=2}^{\infty} n(n-1) c_n x^n + c_1 x + \sum_{n=2}^{\infty} n c_n x^n + \sum_{n=2}^{\infty} c_{n-2} x^n = 0$

i.e. $\quad c_1 x + \sum_{n=2}^{\infty} \left[n(n-1) c_n + n c_n + c_{n-2} \right] x^n = 0$

i.e. $\quad c_1 x + \sum_{n=2}^{\infty} \left[n^2 c_n + c_{n-2} \right] x^n = 0$

Equating coefficients on both sides

$$c_1 = 0$$

And $\quad n^2 c_n + c_{n-2} = 0, \; n = 2, 3, \dots\dots$

That is $c_n = -\frac{c_{n-2}}{n^2}, \; n = 2, 3, \dots\dots$

Put $\quad n=2, \quad c_2 = -\frac{c_0}{2^2} = -\frac{c_0}{2^2(1!)^2}$

$n=3, \quad c_3 = \frac{-c_1}{3^2} = 0 \quad (\text{As } c_1 = 0)$

$n=4, \quad c_4 = \frac{-c_2}{4^2} = \frac{c_0}{2^2.4^2} = \frac{c_0}{2^4.(2!)^2}$

$n=5, \quad c_5 = \frac{-c_3}{5^2} = 0$

$n=6, \quad c_6 = \frac{-c_4}{6^2} = -\frac{c_0}{6^2.2^2(2!)^2} = \frac{-c_0}{2^6(3!)^2}$

--------------, and so on

Then the series solutions of equation (1) is

$$y = c_0\left[1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \dots\dots\right]$$

$$= c_0\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right]$$

Or $\quad y = c_0 J_0(x)$

Where $J_0(x) = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ is called Bessel function of zero order

By given initial conditions

$y(0) = 1$, and $y'(0) = 0$

Now $\quad y'(x) = c_0\left[\frac{-2x}{2^2} + \frac{4x^3}{2^4(2!)^2} - \frac{6x^3}{2^3(3!)^2} + \dots\dots\right]$

Then the initial conditions give $c_0 = 1$

Hence the required solution of

$$\boxed{y = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = J_0(x)}$$