Mathematics
Class XII

Time: 3 hours **Total Marks: 100**

1. All questions are compulsory.
2. The question paper consist of 29 questions divided into three sections A, B, C and D. Section A comprises of 4 questions of one mark each, section B comprises of 8 questions of two marks each, section C comprises of 11 questions of four marks each and section D comprises of 6 questions of six marks each.
3. Use of calculators is not permitted.

SECTION – A

1. Show that the relation R, in set of real numbers defined as $R = \{a,\ b:\ a \le b\}$, is transitive.

2. Find the principal values of $\tan^{-1}(-1)$

3. Find the number of all possible matrices of order 3 × 3 with each entry 0 or 1.

4. If $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$, then find the position vector of their mid-point.

SECTION – B

5. Evaluate: $\int_0^p \frac{\sqrt{x}}{\sqrt{x} + \sqrt{p-x}} dx$

6. Find the area of the parallelogram having adjacent sides $\vec{a}$ and $\vec{b}$ given by $2\hat{i} + \hat{j} + k$ and $3\hat{i} + \hat{j} + 4k$ respectively.

7. Find the value of $\tan(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3})$

8. Write the inverse of the matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

9. The contentment obtained after eating x-units of a new dish at a trial function is given by the Function $C(x) = x^3 + 6x^2 + 5x + 3$. If the marginal contentment is defined at the rate of change of (x) with respect to the number of units consumed at an instant, then find the marginal contentment when three units of dish are consumed.

10. If $e^y(x+1)=1$, show that $\frac{dy}{dx}=-e^y$.

11. If $\vec{a}$ and $\vec{b}$ are two vectors of magnitude 3 and $\frac{2}{3}$ respectively such that $\vec{a}\times\vec{b}$ is a unit vector, write the angle between $\vec{a}$ and $\vec{b}$.

12. If A is a square matrix of order 3 such that |Adj A| = 225, find |A'|.

SECTION – C

13. Show that the function

$$f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & x \neq 0 \\ 2 & , x = 0 \end{cases}$$

is continuous at x = 0.

14. If the sum of mean and variance of a binomial distribution for 5 trials is 1.8, find the distribution.

15. Write in the simplest form:

$$y=\cot^{-1}\left(\sqrt{1+x^2}-x\right)$$

OR

Prove that $\sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{16}{65}\right)=\frac{\pi}{2}$

16. Let f: N $\rightarrow$ N be defined by

$$f(n) = \begin{cases} \frac{n+1}{2} & \text{if n is odd} \\ \frac{n}{2} & \text{if n is even} \end{cases}$$

Find whether the function f is bijective or not.

17. If $\hat{a}$ and $\hat{b}$ are two unit vectors and θ is the angle between them, show that

$$\sin\frac{\theta}{2} = \frac{1}{2}\left|\hat{a} - \hat{b}\right|.$$

18. Evaluate: $\int \frac{\sin x}{(1 - \cos x)(2 - \cos x)} dx$

OR

Evaluate: $\int e^x \frac{(x-3)}{(x-1)^3} dx$

19. Find the family of curves passing through the point (x, y) for which the slope of the tangent is equal to the sum of y-coordinate and exponential raise to the power of x-coordinate.

20. From the differential equation of the family of curves $y = A\cos 2x + B\sin 2x$, where A and B are constants.

21. Using properties of determinants, prove the following:

$$\begin{vmatrix} a & b & c \\ a - b & b - c & c - a \\ b + c & c + a & a + b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc.$$

22. Find the equation of the plane passing through the points (1, 2, 3) and (0, −1, 0) and parallel to the line $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z}{-3}$

23. Find the interval in which the function $f(x) = \sin^4 x + \cos^2 x$ is decreasing.

OR

Find the equation of tangent and normal to the curve $y = -3e^{5x}$ where it crosses the y-axis.

SECTION – D

24. If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$. Find A^{-1} and hence, solve the following system of equations:

$$x + 3y + 3z = 2$$
$$x + 4y + 3z = 1$$
$$x + 3y + 4z = 2$$

OR

Find the inverse of the following matrix if exists, using elementary row transformation.

$$\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$$

25. Using integration, find the area of the circle $x^2 + y^2 = 16$ which is exterior to the parabola $y^2 = 6x$.

26. A firm makes items A and B and the total number of items it can make in a day is 24. It takes one hour to make item A and only half an hour to make item B. The maximum time available per day is 16 hours. The profit per item of A is Rs. 300 and Rs. 160 on one item of B. How many items of each type should be produced to maximise the profit? Solve the problem graphically.

27. Find the angle between the lines whose direction cosines are given by the equations: $3l + m + 5n = 0; 6mn - 2nl + 5lm = 0$

28. A window is in the form of a rectangle above which there is a semi-circle. If the perimeter of the window is p cm, show that the window will allow the maximum possible light only when the radius of the semi-circle is $\frac{p}{\pi + 4}$ cm.

OR

Show that the surface area of a closed cuboid with a square base and given volume is the least when it is cube.

29. In a factory which manufactures bulbs, machines X, Y and Z manufacture 1000, 2000, 3000 bulbs respectively. Of their outputs, 1%, 1.5% and 2 % are defective bulbs. A bulb is drawn at random and is found to be defective. What is the probability that the machine X manufactures it?

Mathematics
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Solution

SECTION – A

1. Let $(a, b) \in R$ and $(b, c) \in R$

$\therefore \quad (a \le b)$ and $b \le c \Rightarrow a \le c$

$\therefore \quad (a, c) \in R$

Hence, R is transitive.

2. Let $x = \tan^{-1}(-1)$

$\tan x = -1$

$\tan x = -\tan\frac{\pi}{4}$

$\tan x = \tan\left(\pi - \frac{\pi}{4}\right) \qquad [\because \tan(\pi - \theta)]$

$\tan x = \tan\frac{3\pi}{4}$

$x = \frac{3\pi}{4}$

3. Matrix of order 3 × 3 has 9 elements.
Now the entries have to be either 0 or 1 so that each of the 9 places can be filled with 2 choices 0 or 1.
So $2^9 = 512$ matrices are possible.

4. $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$;

$\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$

$\Rightarrow \quad \vec{a} + \vec{b} = 6\hat{i} + 2\hat{j} - 8\hat{k}$

$\frac{\vec{a} + \vec{b}}{2} = 3\hat{i} + \hat{j} - 4\hat{k}$

SECTION – B

5. Let $I = \int_0^p \frac{\sqrt{x}}{\sqrt{x}+\sqrt{p-x}}dx$ ...(1)

According to property,

$\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$I = \int_0^p \frac{\sqrt{p-x}}{\sqrt{p-x}+\sqrt{x}}dx$ ...(2)

Adding equations (1) and (2), we get

$2I = \int_0^p \frac{\sqrt{x}+\sqrt{p-x}}{\sqrt{x}+\sqrt{p-x}}dx$

$= \int_0^p 1dx = [x]_0^p = p - 0 = p$

Thus, $2I = p \Rightarrow I = \frac{p}{2}$

6. Area of a parallelogram = Cross product of the vectors representing its adjacent sides.

So, required area = $|(2\hat{i} + \hat{j} + k) \times (3\hat{i} + \hat{j} + 4k)|$

Now $(2\hat{i} + \hat{j} + k) \times (3\hat{i} + \hat{j} + 4k)$

$= \begin{vmatrix} \hat{i} & \hat{j} & k \\ 2 & 1 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 3\hat{i} - 5\hat{j} - k$

Area = $|3\hat{i} - 5\hat{j} - k| = \sqrt{9+25+1} = \sqrt{35}$ sq. units.

7. $\tan\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right)$

$= \tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right)$

$= \tan\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{1}{2}}\right)\right)$

$= \tan\left(\tan^{-1}\left(\frac{17}{6}\right)\right)$

$= \frac{17}{6}$

8. Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then

$$|A| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$$

$$= \cos^2\theta + \sin^2\theta = 1 \neq 0$$

Since $|A| \neq 0$, therefore A^{-1} exist.

$$\therefore A^{-1} = \frac{\text{Adj}A}{|A|} = \frac{1}{1}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

9. Contentment function : $C(x) = x^3 + 6x^2 + 5x + 3$ (given)

$\therefore$ Marginal contentment = $\dfrac{dC(x)}{dx}$

$$= 3x^2 + 12x + 5$$

When 3 units of dish are consumed, then

$$\left(\frac{dC(x)}{dx}\right)_{x=3} = 3(3)^2 + 12(3) + 5$$

$$= 27 + 36 + 5$$

$$= 68 \text{ units.}$$

10. On differentiating $e^y(x+1) = 1$ w.r.t x, we get

$$e^y + (x+1)e^y \frac{dy}{dx} = 0$$

$$\Rightarrow e^y + \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -e^y$$

11. We know that $\sin\theta = \dfrac{\left|\vec{a}\times\vec{b}\right|}{\left|\vec{a}\right|\left|\vec{b}\right|}$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

Since $\left|\vec{a}\right| = 3$ (given), $\left|\vec{b}\right| = \dfrac{2}{3}$ (given), $\left|\vec{a}\times\vec{b}\right| = 1$ (given)

$$\Rightarrow \sin\theta = \frac{1}{(3)\left(\frac{2}{3}\right)}$$

$$\Rightarrow \sin\theta = \frac{1}{2}$$

$$\Rightarrow \sin\theta = \sin\frac{\pi}{6}$$

$$\Rightarrow \theta = \frac{\pi}{6}$$

Thus, the angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{6}$.

12. $\left|\text{Adj}\,A\right| = \left|A\right|^{3-1} = 225$ (given)

.....[Since $\left|\text{Adj}\,A\right| = \left|A\right|^{n-1}$, where A is a square matrix of order n]

$$\Rightarrow \left|A\right|^2 = 225$$

$$\Rightarrow \left|A\right| = \pm 15$$

Now, $\left|A'\right| = \left|A\right| = \pm 15$

SECTION – C

13. $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\left(\dfrac{\sin x}{x} + \cos x\right)$

$= 1 + 1 = 2$

Also $f(0) = 2$

$\therefore \quad \lim\limits_{x\to 0} f(x) = f(0)$

Hence, f(x) is continuous at x = 0

14. By hypothesis,

$np + npq = 1.8$ and $n = 5$

$\Rightarrow 5(p + pq) = 1.8$

$\Rightarrow p + p(1 - p) = 0.36$

$\Rightarrow p^2 - 2p + 0.36 = 0$

$\Rightarrow p = \dfrac{2 \pm \sqrt{4 - 1.44}}{2} = \dfrac{2 \pm 1.6}{2}$

$\therefore p = \dfrac{0.4}{2} = \dfrac{1}{5}$ (Reject : $p = 1.8$)

Binomial Distribution $= (q + p)^n$

$= \left(\dfrac{4}{5} + \dfrac{1}{5}\right)^5$

15. Let $y = \cot^{-1}\left(\sqrt{1 + x^2} - x\right)$

Let $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$y = \cot^{-1}\left(\sqrt{1 + \tan^2\theta} - \tan\theta\right)$$

$$y = \cot^{-1}(\sec\theta - \tan\theta)$$

$$y = \cot^{-1}\left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)$$

$$y = \cot^{-1}\left(\frac{1 - \sin\theta}{\cos\theta}\right)$$

$$y = \cot^{-1}\left[\frac{1 - \cos\left(\frac{\pi}{2} - \theta\right)}{\sin\left(\frac{\pi}{2} - \theta\right)}\right]$$

$$y = \cot^{-1}\left[\frac{2\sin^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}{2\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}\right]$$

$$y = \cot^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]$$

$$y = \cot^{-1}\left[\cot\left(\frac{\pi}{2} - \frac{\pi}{4} + \frac{\theta}{2}\right)\right]$$

$$y = \frac{\pi}{4} + \frac{\theta}{2}$$

$$\therefore y = \frac{\pi}{4} + \frac{1}{2}\tan^{-1}x.$$

OR

$$\text{To prove}: \sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{16}{65}\right)=\frac{\pi}{2}$$

$$\text{Let } \sin^{-1}\left(\frac{4}{5}\right)=x$$

$$\Rightarrow \sin x=\frac{4}{5}$$

$$\Rightarrow \cos x=\sqrt{1-\sin^2 x}=\frac{3}{5}$$

$$\sin^{-1}\left(\frac{5}{13}\right)=y$$

$$\Rightarrow \sin y=\frac{5}{13}$$

$$\Rightarrow \cos y=\sqrt{1-\sin^2 y}=\frac{12}{13}$$

$$\sin^{-1}\left(\frac{16}{65}\right)=z$$

$$\Rightarrow \sin z=\frac{16}{65}$$

$$\Rightarrow \cos z=\sqrt{1-\sin^2 z}=\frac{63}{65}$$

$$\tan x=\frac{4}{3},\tan y=\frac{5}{12},\tan z=\frac{16}{63}$$

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x.\tan y}=\frac{\frac{4}{3}+\frac{5}{12}}{1-\frac{20}{36}}=\frac{63}{16}=\cot z$$

$$\tan(x+y)=\tan\left(\frac{\pi}{2}-z\right)\Rightarrow x+y+z=\frac{\pi}{2}$$

$$\therefore \sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{16}{65}\right)=\frac{\pi}{2}$$

16. $f: N \to N$

$$f(n) = \begin{cases} \dfrac{n+1}{2} & \text{if } n \text{ is odd} \\ \dfrac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

Let $f(n_1) = f(n_2)$

Case 1: n_1, n_2 are odd

Let $f(n_1) = f(n_2)$

$\Rightarrow \dfrac{n_1+1}{2} = \dfrac{n_2+1}{2}$

$\Rightarrow n_1 = n_2$

Case 2: n_1, n_2 are even

$f(n_1) = f(n_2) \Rightarrow \dfrac{n_1}{2} = \dfrac{n_2}{2} \Rightarrow n_1 = n_2$

Case3: n_1 is odd and n_2 is even

$f(n_1) = f(n_2) \Rightarrow \dfrac{n_1+1}{2} = \dfrac{n_2}{2}$

$\Rightarrow n_1 + 1 = n_2$

$\Rightarrow n_1 \neq n_2$

Hence,

$f(n_1) = f(n_2)$ does not imply $n_1 = n_2 \ \forall \ n_1, n_2 \in N$

$\therefore f$ is not one $-$ one

Function f is onto and hence, f is surjective.

So f is not bijective.

17. $\hat{a}$ and $\hat{b}$ are two unit vectors

$$\left|\hat{a}-\hat{b}\right|^2 = \left(\hat{a}-\hat{b}\right).\left(\hat{a}-\hat{b}\right)$$

$$\left|\hat{a}-\hat{b}\right|^2 = \left|\hat{a}\right|^2 + \left|\hat{b}\right|^2 - 2\left|\hat{a}\right|\left|\hat{b}\right| \cos\theta$$

$$1^2 + 1^2 - 2(1)(1)\cos\theta$$

$$= 2 - 2\cos\theta$$

$$= 2(1 - \cos\theta)$$

$$= 2.2\sin^2\frac{\theta}{2}$$

$$= 4\sin^2\frac{\theta}{2}$$

$$\sin^2\frac{\theta}{2} = \frac{1}{4}\left|\hat{a} - \hat{b}\right|^2$$

$$\sin\frac{\theta}{2} = \frac{1}{2}\left|\hat{a} - \hat{b}\right|.$$

18. $\int \frac{\sin x}{(1\text{-}\cos x)(2\text{-}\cos x)} dx$

Here substitute $-\cos x = t$, Hence $\sin x\, dx = dt$

$$\frac{\sin x}{(1 - \cos x)(2 - \cos x)} dx = \frac{dt}{(1+t)(2+t)}$$

Let $\frac{1}{(1+t)(2+t)} = \frac{A}{(1+t)} + \frac{B}{(2+t)}$

$1 = A(2 + t) + B(1 + t)$

Solving the equation we get

$B = -1$

$A = 1$

$$\int \frac{dt}{(1+t)(2+t)} = \int \frac{dt}{1+t} - \int \frac{dt}{2+t}$$

$$= \log|1+t| - \log|2+t| + C$$

$$= \log\left|\frac{1+t}{2+t}\right| + C$$

And so

$$\int \frac{\sin x}{(1 - \cos x)(2 - \cos x)} dx = \log\left|\frac{1 - \cos x}{2 - \cos x}\right| + C$$

OR

Let $I = \int e^{x} \frac{(x-3)}{(x-1)^{3}} dx$

$= \int e^{x} \frac{(x-1-2)}{(x-1)^{3}} dx$

$= \int e^{x} \left[\frac{(x-1)}{(x-1)^{3}} + \frac{(-2)}{(x-1)^{3}} \right] dx$

$= \int e^{x} \left[\frac{1}{(x-1)^{2}} + \frac{(-2)}{(x-1)^{3}} \right] dx$

Put $\frac{1}{(x-1)^{2}} = f(x)$, then $f'(x) = \frac{(-2)}{(x-1)^{3}}$

Now $\int e^{x}(f(x)+f'(x))dx = e^{x}f(x) + c$

we get

$I = e^{x} \frac{1}{(x-1)^{2}} + c$

19. We know that the slope of the tangent is given by $\frac{dy}{dx}$

According to question, $\frac{dy}{dx} = y + e^{x}$

Or, $\frac{dy}{dx} - y = e^{x}$ ...... (i)

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$

where, P = -1 and $Q = e^{x}$

Therefore,

$I.F. = e^{\int P dx} = e^{-\int 1 dx} = e^{-x}$

Solution of (i) is given by

$y.e^{-x} = \int e^{x} e^{-x} dx + c$

$\Rightarrow y.e^{-x} = x + c$

This is the required family of curves.

20. $y = A\cos 2x + B\sin 2x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = A(-\sin 2x).2 + B\cos 2x.2$$

$$\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x$$

Again differentiating w. r. t. x, we get

$$\frac{d^2y}{dx^2} = -2A\cos\ 2x \cdot 2 + 2B(-\sin 2x) \cdot 2$$

$$\frac{d^2y}{dx^2} = -4(A\cos\ 2x + B\sin 2x)$$

$$\frac{d^2y}{dx^2} = -4y \Rightarrow \frac{d^2y}{dx^2} + 4y = 0.$$

21. L.H.S $= \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$

Applying: $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix}$$

Applying: $R_3 \to R_3 - 2R_1$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 0 & c+a-2b & a+b-2c \end{vmatrix}$$

Expanding along C_1

$$= (a+b+c).1\begin{vmatrix} b-c & c-a \\ c+a-2b & a+b-2c \end{vmatrix}$$

Applying: $R_2 \to R_2 + 2R_1$

$$= (a+b+c)\begin{vmatrix} b-c & c-a \\ a-c & b-a \end{vmatrix}$$

$$= (a+b+c)\left[(b-c)(b-a)-(a-c)(c-a)\right]$$

$$= (a+b+c)\left[b^2-ab-bc+ac+\left(a^2+c^2-2ca\right)\right]$$

$$= (a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right)$$

$$= a^3+b^3+c^3-3abc = \text{R.H.S.}$$

22. Let the plane through (1, 2, 3) be $a(x-1)+b(y-2)+c(z-3)=0$ ...(1)

This plane is parallel to the line

$$\frac{x-1}{2}=\frac{y+2}{3}=\frac{z}{-3}$$

$\therefore \quad a\times 2+b\times 3+c\times(-3)=0$

$\Rightarrow \quad 2a+3b-3c=0$ ...(2)

Also (1) passes through (0,-1,0) So

a + 3b + 3c = 0…………..(3)

Solving (2) and (3), we get

$$\frac{a}{9+9}=\frac{b}{-3-6}=\frac{c}{6-3}$$

$$\Rightarrow \quad \frac{a}{6}=\frac{b}{-3}=\frac{c}{1}$$

Hence the required plane is given by

$$6(x-1)-3(y-2)+1(z-3)=0$$

$$\Rightarrow \quad 6x-3y+z=3.$$

23. Differentiating f(x) we get

$$f'(x) = 4\sin^3 x\cos x - 2\cos x\sin x$$

$$= 4\sin x\cos x\left(\sin^2 x - \frac{1}{2}\right)$$

$$= 4\sin x\cos x\left(\sin x - \frac{1}{\sqrt{2}}\right)\left(\sin x + \frac{1}{\sqrt{2}}\right)$$

$= f'(x) = 0$

$\Rightarrow x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}$

Checking the sign of f′(x) in each of these intervals

$f'(x) < 0 \text{ in } \left[0, \frac{\pi}{4}\right].$

So f(x) is decreasing in $\left[0, \frac{\pi}{4}\right]$

OR

The given curve crosses the y-axis. When $x = 0$

So y becomes $y = -3e^o = -3.$

So the curve intersects the y-axis at point (0, - 3)

Differentiating $y = -3e^{5x}$ with respect to x

$$\frac{dy}{dx} = -15e^{5x}$$

$$\left.\frac{dy}{dx}\right]_{(0,-3)} = -15e^o = -15$$

Therefore, equation of tangent becomes

y + 3 = –15(x – 0)

Which is 15x + y + 3 = 0

And also slope of the normal at (0, -3) would be $\frac{1}{15}$ so the equation of the normal is

$$y + 3 = \frac{1}{15}(x - 0)$$

$$x - 15y - 45 = 0$$

SECTION – D

24. Given $A=\begin{bmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}$

$|A| = 1(16 - 9) - 3\,(4 - 3) + 3\,(3 - 4) = 7 - 3 - 3 = 1$

$|A| \neq 0$

So A^{-1} exists.

$$\text{So}, A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{1}\begin{bmatrix}7 & -3 & -3\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$$

The system of equation can be written in the form AX = B, where

$$A=\begin{bmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, \text{and } B=\begin{bmatrix}2\\ 1\\ 2\end{bmatrix}$$

Therefore,

$$\text{Thus, } X = A^{-1}B = \frac{1}{1}\begin{bmatrix}7 & -3 & -3\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}\begin{bmatrix}2\\ 1\\ 2\end{bmatrix} = \begin{bmatrix}5\\ -1\\ 0\end{bmatrix}$$

Hence, x = 5, y = − 1 and z = 0

OR

Consider, $A = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$

We write $A = IA$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -11 \\ 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \text{[By performing } R_2 \to R_2 + 3R_1\text{]}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -11 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \qquad \text{[By performing } R_3 \to R_3 - 2R_1\text{]}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & 9 & -11 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \qquad \text{[By performing } R_1 \to R_1 + 3R_3\text{]}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -\frac{11}{9} \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -2 & 0 & 1 \end{bmatrix} A \qquad \text{[By performing } R_2 \to \frac{1}{9} R_2\text{]}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -\frac{11}{9} \\ 0 & 0 & \frac{25}{9} \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -\frac{5}{3} & \frac{1}{9} & 1 \end{bmatrix} A \qquad \text{[By performing } R_3 \to R_3 + R_2\text{]}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -\frac{11}{9} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} A \qquad \text{[By performing } R_3 \to \frac{9}{25} R_3\text{]}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ -\frac{2}{5} & \frac{36}{225} & \frac{11}{25} \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} A \qquad \left[\text{By performing } R_2 \to R_2 + \frac{11}{9} R_3\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} A \qquad [\text{By performing } R_1 \to R_1 - 10R_3]$$

Hence, we obtain

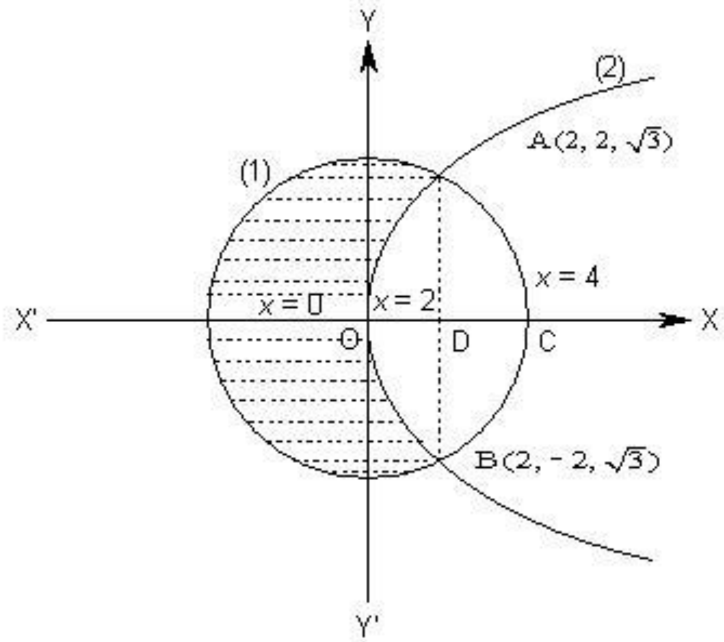
$\Rightarrow$ B is inverse of A by definition.

Hence

$$A^{-1} = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix}$$

25. $x^2+y^2=16$ ......(1)

$y^2=6x$ ......(2)



Points of intersection of curve (i) and (ii) is

$$x^2+6x-16=0$$

$\Rightarrow \quad (x+8)(x-2)=0$

$\Rightarrow \quad x=2 \qquad (\because x\neq -8)$

$y^2 = 12$

$y = \pm\, 2\sqrt{3}$

$\therefore \quad A\left(2,\, 2\sqrt{3}\right)$ and $B\left(2, -2\sqrt{3}\right)$

Also $\quad C(4,0)$.

Area $\quad OBCAO = 2$ (Area ODA + Area DCA)

$$=2\left[\int_0^2 y_2 dx+\int_2^4 y_1 dx\right]$$

$$=2\left[\int_0^2 \sqrt{6x}\, dx+\int_2^4 \sqrt{16-x^2}\, dx\right]$$

$$=2\left[\sqrt{6}.\left\{\frac{2}{3}x^{3/2}\right\}_0^2+\left\{\frac{x\sqrt{16-x^2}}{2}+\frac{16}{2}\sin^{-1}\frac{x}{4}\right\}_2^4\right]$$

$$=2\left[\frac{2\sqrt{6}}{3}.2\sqrt{2}+\left\{0+8\sin^{-1}1\right\}-\left\{\frac{2.2\sqrt{3}}{2}+8\sin^{-1}\frac{1}{2}\right\}\right]$$

$$=\frac{16\sqrt{3}}{3}+16.\frac{\pi}{2}-\left(4\sqrt{3}+16.\frac{\pi}{6}\right)$$

$$=\left(\frac{4\sqrt{3}}{3}+\frac{16}{3}\pi\right)\text{sq. units.}$$

Required, Area = Area of circle $-\left(\frac{4\sqrt{3}}{3}+\frac{16}{3}\pi\right)$

$$=16\pi-\frac{4\sqrt{3}}{3}-\frac{16}{3}\pi$$

$$=-\frac{32}{3}\pi-\frac{4\sqrt{3}}{3}=\frac{4}{3}\left(8\pi-\sqrt{3}\right)\text{sq. units}$$

26. Let the number of items of type A and B produced be x and y respectively.

The L. P.P. is Maximise:

z = 300x + 160y

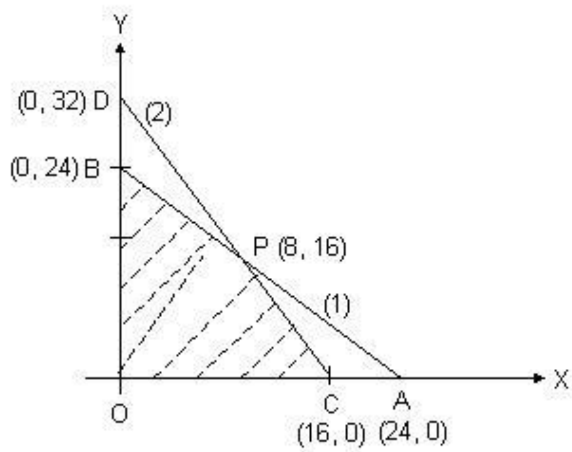
Subject to the constraints

$x + y \leq 24$

$x.1 + y.\frac{1}{2} \leq 16$

$x \geq 0,\ y \geq 0.$

Graphically



These meet at P (8, 16).

The feasible region is OCPB.

The value of z = 300x + 160y

at O(0, 0) is zero

at C(16, 0) is 4800

at B(0 ,24) is 3840

at P(8, 16) is 4960

Clearly value is max. at P(8, 16) ⇒ 8 items of type A and 16 of type B should be produced for maximum profit.

27. We have:

$3l + m + 5n = 0 \quad ...(1)$

$6mn - 2nl + 5lm = 0 \quad ...(2)$

From (1), $m = -(3l + 5n) \quad ...(3)$

$-6(3l + 5n)\ n - 2nl - 5l\ (3l + 5n) = 0$

$-18ln - 30n^2 - 2nl - 15l^2 - 25nl = 0$

$-30n^2 - 45nl - 15l^2 = 0$

$2n^2 + 3ln + l^2 = 0 \quad$ [Divide by (-15)]

$(2n + l)(n + l) = 0$

Either $2n + l = 0$ or $n + l = 0$

(I) when $2n + l = 0 \quad \Rightarrow l = -2n$

from (3), $m = -(6n + 5n) = n$

(II) when $n + l = 0 \quad \Rightarrow l = -n$

from (3), $m = -(-3n + 5n) = -2n$

Direction ratios of two lines are

$-2n, n, n$ and $-n, -2n, n$

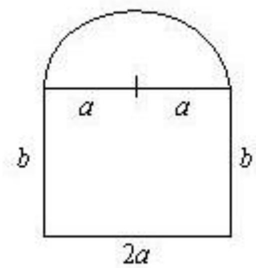
$-2, 1, 1$ and $1, 2, -1$

Angle between two lines

$$\cos\theta = \left| \frac{(-2)1 + (1)(2) + (1)(-1)}{\sqrt{(-2)^2 + 1^2 + 1^2}\ \sqrt{1^2 + 2^2 + (-1)^2}} \right| = \frac{1}{6}$$

Hence $\theta = \cos^{-1}\left(\frac{1}{6}\right)$

28. Let 2a cm be the length and b cm be the breadth of the rectangle. Then a cm is the radius of the semi-circle.



By hypothesis, perimeter

$$p = 2a + b + b + \pi a$$

$$\Rightarrow \quad 2b = p - (\pi + 2)a \qquad .....(1)$$

Also A = Area of the window

$$= \frac{1}{2}\pi a^2 + 2a \times b$$

$$= \frac{1}{2}\pi a^2 + a \times [p - (\pi + 2)a] \qquad [By(1)]$$

$$= pa - \frac{1}{2}(\pi + 4)a^2$$

$$\Rightarrow \quad \frac{dA}{da} = p - (\pi + 4)a$$

For max. or min.,

$$\frac{dA}{da} = 0$$

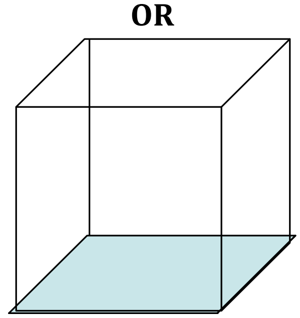
$$\Rightarrow \quad p - (\pi + 4)\, a = 0$$

$$\Rightarrow \quad a = \frac{p}{\pi + 4}.$$

Also $\dfrac{d^2A}{da^2} = -(\pi + 4) < 0.$

$\therefore$ The light will be maximum when the radius of the semi-circle is, $a = \dfrac{p}{\pi + 4}$.

OR



Let x be side of the square base and y be the height of the cuboid

Volume (v) = x. x. y = x^2 y

Surface area (S) = 2 (x.x + x.y + x.y) = $2x^2 + 4xy = 2x^2 + 4x\dfrac{V}{x^2}$

$S = 2x^2 + \dfrac{4V}{x} \Rightarrow \dfrac{dS}{dx} = 4x - \dfrac{4V}{x^2}$

For minimum S, $\dfrac{dS}{dx} = 0 \Rightarrow 4x - \dfrac{4V}{x^2} = 0 \Rightarrow x^3 = V \Rightarrow x = \sqrt[3]{V}$

$$\frac{d^2S}{dx^2} = 4 + \frac{8V}{x^3} \Rightarrow \left.\frac{d^2S}{dx^2}\right]_{x=\sqrt[3]{V}} = 4 + \frac{8V}{V} > 0$$

$\therefore$ For $x = \sqrt[3]{V}$, surface area is minimum

$\Rightarrow x^3 = V \Rightarrow x^3 = x^2y\left[\text{From (i)}\right] \Rightarrow x = y \Rightarrow$ cuboid is a cube

29. B_1: the bulb is manufactured by machine X

B_2: the bulb is manufactured by machine Y

B_3: the bulb is manufactured by machine Z

$P(B_1) = 1000/(1000 + 2000 + 3000) = 1/6$

$P(B_2) = 2000/(1000 + 2000 + 3000) = 1/3$

$P(B_3) = 3000/(1000 + 2000 + 3000) = 1/2$

$P(E|B_1)$ = Probability that the bulb drawn is defective given that it is manufactured by machine X = 1% = 1/ 100

Similarly, $P(E|B_2)$ = 1.5% = 15/ 100 = 3/ 200

$P(E|B_3)$ = 2% = 2/100

$$P(B_1|E) = \frac{P(B_1)(PE|B_1)}{P(B_1)(PE|B_1) + P(B_2)(PE|B_2) + P(B_3)(PE|B_3)}$$

$$= \frac{\frac{1}{6} \times \frac{1}{100}}{\frac{1}{6} \times \frac{1}{100} + \frac{1}{3} \times \frac{3}{200} + \frac{1}{2} \times \frac{2}{100}}$$

$$= \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{2} + 1}$$

$$= \frac{1}{1+3+6} = \frac{1}{10}$$