Chapter 8

IMPROPER INTEGRALS

§ 8.1. Improper Integrals with Infinite Limits

Let the function $f(x)$ be defined for all $x \geqslant a$ and integrable on any interval $[a, A]$. Then $\lim\limits_{A \to +\infty} \int\limits_a^A f(x)\,dx$ is called the *improper integral* of the function $f(x)$ in the interval $[a, +\infty]$ and is denoted by the symbol $\int\limits_a^{+\infty} f(x)\,dx$. We similarly define the integrals $\int\limits_{-\infty}^{B} f(x)\,dx$ and $\int\limits_{-\infty}^{+\infty} f(x)\,dx$.

Thus,

$$\int\limits_a^{+\infty} f(x)\,dx = \lim_{A \to +\infty} \int\limits_a^A f(x)\,dx;$$

$$\int\limits_{-\infty}^{B} f(x)\,dx = \lim_{A \to -\infty} \int\limits_A^B f(x)\,dx;$$

$$\int\limits_{-\infty}^{+\infty} f(x)\,dx = \lim_{A \to -\infty} \int\limits_A^C f(x)\,dx + \lim_{B \to +\infty} \int\limits_C^B f(x)\,dx.$$

If the above limits exist and are finite, the appropriate integrals are called *convergent;* otherwise, they are called *divergent*.

Comparison test. Let $f(x)$ and $g(x)$ be defined for all $x \geqslant a$ and integrable on each interval $[a, A]$, $A \geqslant a$. If $0 \leqslant f(x) \leqslant g(x)$ for all $x \geqslant a$, then from convergence of the integral $\int\limits_a^{\infty} g(x)\,dx$ it follows that the integral $\int\limits_a^{\infty} f(x)\,dx$ is also convergent, and $\int\limits_a^{\infty} f(x)\,dx \leqslant$

$\leqslant \int_a^\infty g(x)\,dx$; from divergence of the integral $\int_a^\infty f(x)\,dx$ it follows that the integral $\int_a^\infty g(x)\,dx$ is also divergent.

Special comparison test. If as $x \to \infty$ the function $f(x) \geqslant 0$ is an infinitesimal of order $\lambda > 0$ as compared with $\frac{1}{x}$, then the integral $\int_a^{+\infty} f(x)\,dx$ converges for $\lambda > 1$ and diverges for $\lambda \leqslant 1$.

Absolute and conditional convergence. Let the function $f(x)$ be defined for all $x \geqslant a$. If the integral $\int_a^\infty |f(x)|\,dx$ converges, then the integral $\int_a^\infty f(x)\,dx$ also converges and is called *absolutely convergent.* In this case

$$\left| \int_a^\infty f(x)\,dx \right| \leqslant \int_a^\infty |f(x)|\,dx.$$

If the integral $\int_a^\infty f(x)\,dx$ converges, and $\int_a^\infty |f(x)|\,dx$ diverges, then the integral $\int_a^\infty f(x)\,dx$ is called *conditionally convergent.*

The change of the variable in an improper integral is based on the following theorem.

Theorem. *Let the function $f(x)$ be defined and continuous for $x \geqslant a$. If the function $x = \varphi(t)$, defined on the interval $\alpha < t < \beta$ (α and β may also be improper numbers $-\infty$ and ∞), is monotonic, has a continuous derivative $\varphi'(t) \neq 0$ and $\lim\limits_{t \to \alpha+0} \varphi(t) = a$, $\lim\limits_{t \to \beta-0} \varphi(t) = +\infty$, then*

$$\int_a^\infty f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt.$$

Integration by parts involves no difficulties.

8.1.1. Evaluate the following improper integrals with infinite limits or prove their divergence taking advantage of their definition.

(a) $\int\limits_{e^2}^{\infty} \frac{dx}{x \ln^3 x}$; (b) $\int\limits_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 5}$; (c) $\int\limits_0^\infty x \sin x\,dx$.

Solution. (a) By definition,

$$\int_{e^2}^{\infty} \frac{dx}{x \ln^3 x} = \lim_{A \to +\infty} \int_{e^2}^{A} \frac{dx}{x \ln^3 x} = \lim_{A \to +\infty} \left(-\frac{1}{2 \ln^2 x} \bigg|_{e^2}^{A} \right) =$$

$$= \lim_{A \to +\infty} \left(\frac{1}{8} - \frac{1}{2 \ln^2 A} \right) = \frac{1}{8}.$$

(b) By definition,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5} = \lim_{B \to -\infty} \int_{B}^{0} \frac{dx}{x^2+2x+5} + \lim_{A \to +\infty} \int_{0}^{A} \frac{dx}{x^2+2x+5}$$

(instead of the point $x=0$ any other finite point of the x-axis may be taken as an intermediate limit of integration).

Compute each of the limits standing in the right side of the above equality:

$$\lim_{B \to -\infty} \int_{B}^{0} \frac{dx}{x^2+2x+5} = \lim_{B \to -\infty} \frac{1}{2} \arctan \frac{x+1}{2} \bigg|_{B}^{0} = \frac{1}{2} \arctan \frac{1}{2} + \frac{\pi}{4},$$

$$\lim_{A \to +\infty} \int_{0}^{A} \frac{dx}{x^2+2x+5} = \lim_{A \to +\infty} \frac{1}{2} \arctan \frac{x+1}{2} \bigg|_{0}^{A} = \frac{\pi}{4} - \frac{1}{2} \arctan \frac{1}{2}.$$

Hence,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5} = \frac{\pi}{2}.$$

(c) By definition,

$$\int_{0}^{\infty} x \sin x \, dx = \lim_{A \to +\infty} \int_{0}^{A} x \sin x \, dx.$$

Putting $u=x$, $dv = \sin x \, dx$ and integrating by parts, we get

$$\lim_{A \to +\infty} \int_{0}^{A} x \sin x \, dx = \lim_{A \to +\infty} \left(-x \cos x \Big|_{0}^{A} + \int_{0}^{A} \cos x \, dx \right) =$$

$$= \lim_{A \to +\infty} (-A \cos A + \sin A).$$

But the last limit does not exist. Consequently, the integral $\int_{0}^{\infty} x \sin x \, dx$ diverges.

8.1.2. Evaluate the following improper integrals with infinite limits on the basis of their definition:

(a) $\int\limits_2^{\infty} \frac{x\,dx}{\sqrt{(x^2-3)^3}}$; (b) $\int\limits_1^{\infty} \frac{dx}{x+x^3}$; (c) $\int\limits_0^{\infty} \frac{x\,dx}{\sqrt{(4x^2+1)^3}}$;

(d) $\int\limits_1^{\infty} \frac{dx}{x^2(1+x)}$; (e) $\int\limits_{-\infty}^{\infty} \frac{dx}{x^2-6x+10}$; (f) $\int\limits_0^{\infty} e^{-x}\sin x\,dx$.

Solution. (a) By definition

$$\int\limits_2^{\infty} \frac{x\,dx}{\sqrt{(x^2-3)^3}} = \lim_{A\to+\infty} \int\limits_2^{A} \frac{x\,dx}{\sqrt{(x^2-3)^3}} = \lim_{A\to+\infty} \left[\frac{1}{2}\,\frac{(x^2-3)^{-1/2}}{-1/2}\Bigg|_2^A\right] =$$

$$= -\lim_{A\to+\infty} \left[\frac{1}{\sqrt{A^2-3}} - 1\right] = 1.$$

8.1.3. Prove that the integrals of the form

$$\int\limits_a^{+\infty} e^{-px}\,dx \quad \text{and} \quad \int\limits_{-\infty}^{b} e^{px}\,dx$$

converge for any constant $p>0$ and diverge for $p<0$.

8.1.4. Test the integral

$$\int\limits_0^{\infty} \frac{dx}{1+2x^2+3x^4}$$

for convergence.

Solution. The integrand

$$f(x) = \frac{1}{1+2x^2+3x^4}$$

is positive and is an infinitesimal of order $\lambda=4$ as compared with $\frac{1}{x}$ as $x\to\infty$. Since $4>1$, the integral converges according to the special comparison test.

8.1.5. Test the integral

$$\int\limits_1^{\infty} \frac{dx}{x+\sin^2 x}$$

for convergence.

Solution. The integrand $f(x)=\frac{1}{x+\sin^2 x}$ is continuous and positive for $x \geqslant 1$.

As $x\to\infty$ the function $f(x)$ is an infinitesimal of order $\lambda=1$ as compared with $\frac{1}{x}$; according to the special comparison test the integral diverges.

8.1.6. Test the following integrals for convergence:

(a) $\int_1^{\infty} \frac{\ln(x^2+1)}{x}\,dx$; (b) $\int_1^{\infty} \frac{\tan\frac{1}{x}}{1+x\sqrt{x}}\,dx$;

(c) $\int_1^{\infty} \frac{2+\cos x}{\sqrt{x}}\,dx$; (d) $\int_2^{\infty} \frac{3+\arcsin\frac{1}{x}}{1+x\sqrt{x}}\,dx$; (e) $\int_1^{\infty} \frac{\arctan x}{x}\,dx$.

8.1.7. Test the integral

$$\int_1^{\infty} \frac{(x+\sqrt{x+1})\,dx}{x^2+2\sqrt[5]{x^4+1}}$$

for convergence.

Solution. The integrand is continuous and positive for $x \geqslant 1$. Determine its order of smallness λ with respect to $\frac{1}{x}$ as $x \to \infty$; since

$$\frac{x+\sqrt{x+1}}{x^2+2\sqrt[5]{x^4+1}} = \frac{1}{x} \times \frac{1+\sqrt{\frac{1}{x}+\frac{1}{x^2}}}{1+2\sqrt[5]{\frac{1}{x^6}+\frac{1}{x^{10}}}},$$

the order of smallness $\lambda = 1$. According to the special comparison test the integral $\int_1^{\infty} \frac{x+\sqrt{x+1}}{x^2+2\sqrt[5]{x^4+1}}\,dx$ diverges.

8.1.8. Test the integral

$$\int_3^{\infty} \frac{dx}{\sqrt{x(x-1)(x-2)}}$$

for convergence.

Solution. Since the function

$$f(x) = \frac{1}{\sqrt{x^3\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)}} = \frac{1}{x^{\frac{3}{2}}} \times \frac{1}{\sqrt{\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)}}$$

is an infinitesimal of order $\lambda = \frac{3}{2}$ with respect to $\frac{1}{x}$ as $x \to +\infty$, according to the special comparison test the integral converges.

8.1.9. Test the integral

$$\int_2^{\infty} \frac{\sqrt[7]{3+2x^2}}{\sqrt[5]{x^3-1}}\,dx$$

for convergence.

Solution. The integrand is continuous and positive for $x \geqslant 2$. Determine its order of smallness with respect to $\frac{1}{x}$ as $x \to +\infty$:

$$\frac{\sqrt[7]{3+2x^2}}{\sqrt[5]{x^3-1}} = \frac{1}{x^{\frac{11}{35}}} \times \frac{\sqrt[7]{2+\frac{3}{x^2}}}{\sqrt[5]{1-\frac{1}{x^3}}}.$$

Since the second multiplier has the limit $\sqrt[7]{2}$ as $x \to +\infty$, we have $\lambda = \frac{11}{35} < 1$. Consequently, the given integral diverges.

8.1.10. Test the integral

$$\int_1^\infty \left(1 - \cos\frac{2}{x}\right) dx$$

for convergence.

Solution. The integrand

$$f(x) = 1 - \cos\frac{2}{x} = 2\sin^2\frac{1}{x}$$

is positive and continuous for $x \geqslant 1$. Since $2\sin^2\frac{1}{x} \sim 2\left(\frac{1}{x}\right)^2 = \frac{2}{x^2}$, the given integral converges (by the special comparison test).

8.1.11. Test the integral

$$\int_1^\infty \ln\frac{e^{\frac{1}{x}}+(n-1)}{n}\, dx, \quad n > 0$$

for convergence.

Solution. Transform the integrand:

$$f(x) = \ln\frac{e^{\frac{1}{x}}+(n-1)}{n} = \ln\left[1 + \frac{e^{\frac{1}{x}}-1}{n}\right].$$

Since the function $\frac{e^{\frac{1}{x}}-1}{n}$ is an infinitesimal as $x \to \infty$, then $f(x) \sim \frac{e^{\frac{1}{x}}-1}{n} \sim \frac{1}{nx}$. In other words, $\lim\limits_{x\to\infty}\frac{f(x)}{1/x} = \frac{1}{n}$. According to the special comparison test the given integral diverges.

8.1.12. Test the integral

$$\int_1^\infty \frac{1-4\sin 2x}{x^3+\sqrt[3]{x}}\, dx$$

for convergence.

Solution. The function $f(x)=\frac{1-4\sin 2x}{x^3+\sqrt[3]{x}}$ changes its sign together with the change in sign of the numerator. Test the integral

$$\int_1^\infty \frac{|1-4\sin 2x|}{x^3+\sqrt[3]{x}}\,dx$$

for convergence. Since $\frac{|1-4\sin 2x|}{x^3+\sqrt[3]{x}}<\frac{5}{x^3}$, and the integral $\int_1^\infty \frac{5dx}{x^3}$ converges, the integral $\int_1^\infty \frac{|1-4\sin 2x|}{x^3+\sqrt[3]{x}}\,dx$ converges as well (according to the comparison test). Thus, the given integral converges absolutely.

8.1.13. Prove that the Dirichlet integral

$$I=\int_0^\infty \frac{\sin x}{x}\,dx$$

converges conditionally.

Solution. Let us represent the given integral as the sum of two integrals:

$$I=\int_0^\infty \frac{\sin x}{x}\,dx=\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx+\int_{\frac{\pi}{2}}^\infty \frac{\sin x}{x}\,dx.$$

The first is a proper integral $\left(\text{since } \lim\limits_{x\to 0}\frac{\sin x}{x}=1\right)$. Applying the method of integration by parts to the second integral, we have

$$\int_{\frac{\pi}{2}}^\infty \frac{\sin x}{x}\,dx=\lim_{A\to\infty}\int_{\frac{\pi}{2}}^A \frac{\sin x}{x}\,dx=$$

$$=\lim_{A\to\infty}\left[-\frac{\cos x}{x}\Big|_{\frac{\pi}{2}}^A-\int_{\frac{\pi}{2}}^A \frac{\cos x\,dx}{x^2}\right]=-\int_{\frac{\pi}{2}}^\infty \frac{\cos x}{x^2}\,dx.$$

But the improper integral $\int_{\frac{\pi}{2}}^\infty \frac{\cos x}{x^2}\,dx$ converges absolutely, since

$\frac{|\cos x|}{x^2}\leqslant\frac{1}{x^2}$, and the integral $\int_{\frac{\pi}{2}}^\infty \frac{dx}{x^2}$ converges.

Therefore, the integral $\int\limits_0^\infty \frac{\sin x}{x}\,dx$ converges.

Reasoning in a similar way it is easy to prove that the integral $\int\limits_{\frac{\pi}{2}}^\infty \frac{\cos x}{x}\,dx$ also converges. Now let us prove that the integral $\int\limits_{\frac{\pi}{2}}^\infty \frac{|\sin x|}{x}\,dx$ diverges. Indeed,

$$\frac{|\sin x|}{x} \geqslant \frac{\sin^2 x}{x} = \frac{1-\cos 2x}{2x},$$

but the integral

$$\int\limits_{\frac{\pi}{2}}^\infty \frac{1-\cos 2x}{2x}\,dx = \lim_{A\to\infty} \frac{1}{2}\int\limits_{\frac{\pi}{2}}^{A} \frac{dx}{x} - \frac{1}{2}\int\limits_{\frac{\pi}{2}}^\infty \frac{\cos 2x}{x}\,dx =$$

$$= \frac{1}{2}\lim_{A\to\infty} \ln A - \frac{1}{2}\ln\frac{\pi}{2} - \frac{1}{2}\int\limits_{\frac{\pi}{2}}^\infty \frac{\cos 2x}{x}\,dx$$

diverges, since $\lim\limits_{A\to\infty} \ln A = \infty$, and the integral $\int\limits_{\frac{\pi}{2}}^\infty \frac{\cos 2x}{x}\,dx$ converges.

8.1.14. Prove that the following integrals converge

(a) $\int\limits_0^\infty \sin(x^2)\,dx$; $\int\limits_0^\infty \cos(x^2)\,dx$; (b) $\int\limits_0^\infty 2x\cos(x^4)\,dx$.

Solution. (a) Putting $x=\sqrt{t}$, we find

$$\int\limits_0^\infty \sin(x^2)\,dx = \frac{1}{2}\int\limits_0^\infty \frac{\sin t}{\sqrt{t}}\,dt.$$

Let us represent the integral on the right side as the sum of two integrals:

$$\int\limits_0^\infty \frac{\sin t}{\sqrt{t}}\,dt = \int\limits_0^{\frac{\pi}{2}} \frac{\sin t}{\sqrt{t}}\,dt + \int\limits_{\frac{\pi}{2}}^\infty \frac{\sin t}{\sqrt{t}}\,dt.$$

The first summand is a proper integral, since $\lim\limits_{t\to+0} \frac{\sin t}{\sqrt{t}} = 0$. Let us apply to the second summand the method of integration by parts, putting

$$u=1/\sqrt{t}, \quad \sin t\, dt = dv,$$

$$\int_{\pi/2}^{\infty} \frac{\sin t}{\sqrt{t}}\, dt = -\frac{\cos t}{\sqrt{t}}\Big|_{\pi/2}^{\infty} - \frac{1}{2}\int_{\pi/2}^{\infty} \frac{\cos t\, dt}{t^{3/2}} = -\frac{1}{2}\int_{\pi/2}^{\infty} \frac{\cos t}{t^{3/2}}\, dt.$$

The last integral converges absolutely, since $\frac{|\cos t|}{t^{3/2}} \leqslant \frac{1}{t^{3/2}}$, and the integral $\int\limits_{\frac{\pi}{2}}^{\infty} \frac{dt}{t^{3/2}}$ converges. We can prove analogously that the integral $\int\limits_{0}^{\infty} \cos(x^2)\, dx$ is convergent. The integrals considered are called *Fresnel's integrals*. They are used in explaining the phenomenon of light diffraction.

(b) By the substitution $x^2=t$ this integral is reduced to the integral $\int\limits_{0}^{\infty} \cos(t^2)\, dt$. The latter integral converges as has just been proved.

Note. Fresnel's integrals show that an improper integral can converge even when the integrand does not vanish as $x \to \infty$. The last convergent integral considered in item (b) shows that an improper integral can converge even if the integrand is not bounded. Indeed, at $x=\sqrt[4]{n\pi}$ $(n=0, 1, 2, \ldots)$ the integrand attains the values $\pm\sqrt[4]{n\pi}$, i.e. it is unbounded.

8.1.15. Evaluate the improper integral

$$\int_{0}^{\infty} \frac{dx}{(1+x^2)^n}, \quad n \text{ natural number}.$$

Solution. Make the substitution $x=\tan t$, where $0 \leqslant t < \frac{\pi}{2}$. Then $x=0$ at $t=0$, $x \to +\infty$ as $t \to \frac{\pi}{2}-0$ and $x'_t = \frac{1}{\cos^2 t} \neq 0$. Consequently, by the theorem on changing a variable in an improper integral

$$\int_{0}^{\infty} \frac{dx}{(1+x^2)^n} \int_{0}^{\frac{\pi}{2}} \frac{1}{\sec^{2n} t} \times \sec^2 t\, dt = \int_{0}^{\frac{\pi}{2}} \cos^{2n-2} t\, dt.$$

On changing the variable we obtain the proper integral which was computed in Problem **6.6.9.**

Therefore,

$$\int_0^\infty \frac{dx}{(1+x^2)^n} = \begin{cases} \pi/2, \ n=1, \\ \dfrac{1\cdot 3\cdot 5 \ldots (2n-3)}{2\cdot 4\cdot 6 \ldots (2n-2)} \cdot \dfrac{\pi}{2}, \ n>1. \end{cases}$$

8.1.16. Compute the integral $I = \int_0^\infty \frac{x^2}{1+x^4}\,dx$.

Solution. Apply the substitution

$$x = 1/t; \quad dx = -(1/t^2)\,dt; \quad t_1 = \infty, \quad t_2 = 0;$$

$$I = \int_0^\infty \frac{x^2}{1+x^4}\,dx = \int_0^\infty \frac{(1/t^4)\,dt}{1+1/t^4} = \int_0^\infty \frac{dt}{t^4+1}.$$

If another integral I is added to the right and left sides then we get

$$2I = \int_0^\infty \frac{1+t^2}{1+t^4}\,dt = \int_0^\infty \frac{1/t^2+1}{t^2+1/t^2}\,dt.$$

Make the substitution $z = t - 1/t$, $(1+1/t^2)\,dt = dz$. Then, as $t \to +0$, $z \to -\infty$ and as $t \to +\infty$, $z \to +\infty$. Hence,

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dz}{z^2+2} = \frac{1}{2}\left[\lim_{B\to-\infty}\int_B^0 \frac{dz}{z^2+2} + \lim_{A\to+\infty}\int_0^A \frac{dz}{z^2+2}\right] =$$

$$= -\frac{1}{2\sqrt{2}}\lim_{B\to-\infty}\arctan\frac{B}{\sqrt{2}} + \frac{1}{2\sqrt{2}}\lim_{A\to+\infty}\arctan\frac{A}{\sqrt{2}} =$$

$$= \frac{1}{2\sqrt{2}}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{\pi}{2\sqrt{2}}.$$

8.1.17. Evaluate the following improper integrals:

(a) $\int_0^\infty \frac{\ln x}{1+x^2}\,dx$; (b) $\int_0^\infty e^{-x^2} x^{2m+1}\,dx$.

8.1.18. Compute the integral

$$I = \int_1^\infty \frac{\sqrt{x^3-x^2+1}}{x^5+x^2+1}\,dx$$

accurate to two decimal places.

Solution. Represent the given integral in the form of a sum of two integrals

$$I_1 = \int_1^N \frac{\sqrt{x^3-x^2+1}}{x^5+x^2+1}\,dx, \quad I_2 = \int_N^\infty \frac{\sqrt{x^3-x^2+1}}{x^5+x^2+1}\,dx.$$

Compute the former with the required accuracy, using Simpson's formula, and estimate the latter. Since for $x \geqslant 1$ we have

$$0 < \frac{\sqrt{x^3 - x^2 + 1}}{x^5 + x^2 + 1} < \frac{x^{3/2}}{x^5} = x^{-7/2},$$

then

$$0 < I_2 = \int_N^\infty x^{-7/2}\, dx = \frac{2}{5} N^{-5/2}.$$

At $N = 7$ we get the estimate $I_2 < \frac{2}{5} \times \frac{1}{49\sqrt{7}} < 0.0031$.

Computation of the integral

$$I_1 = \int_1^7 \frac{\sqrt{x^3 - x^2 + 1}}{x^5 + x^2 + 1}\, dx$$

by Simpson's formula for a step $h = 1$ gives

$$S_1 = 0.2155,$$

and for a step $\frac{h}{2} = 0.5$

$$S_{0.5} = 0.2079.$$

Since the difference between the values is 0.0076, the integral I_1 gives a more accurate value $S_{0.5} = 0.2079$ with an error of the order $\frac{0.0076}{15} \cong 0.0005$.

Consequently, the sought-for integral is approximately equal to

$$I \approx 0.208$$

with an error not exceeding 0.004, or $I = 0.21$ with all true decimal places.

§ 8.2. Improper Integrals of Unbounded Functions

If the function $f(x)$ is defined for $a \leqslant x < b$, integrable on any interval $[a, b - \varepsilon]$, $0 < \varepsilon < b - a$ and unbounded to the left of the point b, then, by definition, we put

$$\int_a^b f(x)\, dx = \lim_{\varepsilon \to +0} \int_a^{b-\varepsilon} f(x)\, dx.$$

If this limit is existent and finite, then the improper integral is said to be *convergent*. Otherwise it is called *divergent*.

Analogously, if the function $f(x)$ is unbounded to the right from the point a, then

$$\int_a^b f(x)\, dx = \lim_{\varepsilon \to +0} \int_{a+\varepsilon}^{b} f(x)\, dx.$$

Finally, if the function is unbounded in the neighbourhood of an interior point c of the interval $[a, b]$, then, by definition,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Let the function $f(x)$ be continuous on the interval $[a, b]$ except at a finite number of points. If there exists a function $F(x)$ continuous on $[a, b]$ for which $F'(x) = f(x)$ except at a finite number of points, then the Newton-Leibniz formula

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

holds good.

Sometimes the function $F(x)$ is called a *generalized antiderivative* for the function $f(x)$ on the interval $[a, b]$.

For the functions defined and positive on the interval $a \leqslant x < b$ convergence tests (comparison tests) analogous to the comparison tests for improper integrals with infinite limits are valid.

Comparison test. Let the functions $f(x)$ and $g(x)$ be defined on the interval $a \leqslant x < b$ and integrable on each interval $[a, b-\varepsilon]$, $0 < \varepsilon < b - a$. If $0 \leqslant f(x) \leqslant g(x)$, then from the convergence of the integral $\int_a^b g(x)\,dx$ follows the convergence of the integral $\int_a^b f(x)\,dx$, and $\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx$; from the divergence of the integral $\int_a^b f(x)\,dx$ follows the divergence of the integral $\int_a^b g(x)\,dx$.

Special comparison test. If the function $f(x) \geqslant 0$ is defined and continuous on the interval $a \leqslant x < b$ and is an infinitely large quantity of the order λ as compared with $\frac{1}{b-x}$ as $x \to b-0$, then the integral $\int_a^b f(x)\,dx$ converges for $\lambda < 1$ and diverges for $\lambda \geqslant 1$.

In particular, the integral

$$\int_a^b \frac{dx}{(b-x)^\lambda}$$

converges for $\lambda < 1$ and diverges for $\lambda \geqslant 1$.

Absolute and conditional convergence. Let the function $f(x)$ be defined on the interval $a \leqslant x < b$ and integrable on each interval

$[a, b-\varepsilon]$; then from the convergence of the integral $\int_a^b |f(x)|\,dx$ follows the convergence of the integral $\int_a^b f(x)\,dx$.

In this case the integral $\int_a^b f(x)\,dx$ is called *absolutely convergent*. But if the integral $\int_a^b f(x)\,dx$ converges, and the integral $\int_a^b |f(x)|\,dx$ diverges, then the integral $\int_a^b f(x)dx$ is called *conditionally convergent*.

Analogous tests are also valid for improper integrals $\int_a^b f(x)\,dx$, where $f(x)$ is unbounded to the right from the point a.

8.2.1. Proceeding from the definition, evaluate the following improper integrals (or prove their divergence):

(a) $\int_1^e \frac{dx}{x\sqrt[3]{\ln x}}$; (b) $\int_0^{\frac{\pi}{2}} \frac{dx}{\cos x}$;

(c) $\int_1^3 \frac{dx}{\sqrt{4x-x^2-3}}$; (d) $\int_0^2 \frac{dx}{\sqrt{|1-x^2|}}$;

(e) $\int_0^1 \frac{x^3+\sqrt[3]{x}-2}{\sqrt[5]{x^3}}\,dx$; (f) $\int_0^1 \frac{dx}{1-x^3}$.

Solution. (a) The integrand $f(x)=\frac{1}{x\sqrt[3]{\ln x}}$ is unbounded in the neighbourhood of the point $x=1$. It is integrable on any interval $[1+\varepsilon, e]$, since it is a continuous function.

Therefore

$$\int_1^e \frac{dx}{x\sqrt[3]{\ln x}} = \lim_{\varepsilon\to+0}\int_{1+\varepsilon}^e \frac{dx}{x\sqrt[3]{\ln x}} = \lim_{\varepsilon\to+0}\left[\frac{3}{2}\sqrt[3]{\ln^2 x}\,\Big|_{1+\varepsilon}^{e}\right] =$$

$$= \lim_{\varepsilon\to+0}\left[\frac{3}{2}-\frac{3}{2}\sqrt[3]{\ln^2(1+\varepsilon)}\right] = \frac{3}{2}.$$

(b) The integrand $f(x)=\frac{1}{\cos x}$ is unbounded in the neighbourhood of the point $x=\frac{\pi}{2}$ and integrable on any interval $\left[0, \frac{\pi}{2}-\varepsilon\right]$ as a continuous function. Therefore

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\cos x} = \lim_{\varepsilon \to +0} \int_0^{\frac{\pi}{2}-\varepsilon} \frac{dx}{\cos x} =$$

$$= \lim_{\varepsilon \to +0} \ln \tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\Big|_0^{\frac{\pi}{2}-\varepsilon} = \lim_{\varepsilon \to +0} \ln \tan\left(\frac{\pi}{2}-\frac{\varepsilon}{2}\right) = \infty.$$

Hence, the given integral diverges.

(c) The integrand is unbounded in the neighbourhood of the points $x=1$ and $x=3$. Therefore, by definition,

$$\int_1^3 \frac{dx}{\sqrt{4x-x^2-3}} = \int_1^2 \frac{dx}{\sqrt{4x-x^2-3}} + \int_2^3 \frac{dx}{\sqrt{4x-x^2-3}}$$

(instead of the point $x=2$ we can take any other interior point of the interval [1, 3]). Let us now compute each summand separately:

$$\int_1^2 \frac{dx}{\sqrt{4x-x^2-3}} = \lim_{\varepsilon \to +0} \int_{1+\varepsilon}^2 \frac{dx}{\sqrt{1-(x-2)^2}} = \lim_{\varepsilon \to +0} \arcsin(x-2)\Big|_{1+\varepsilon}^2 =$$

$$= \lim_{\varepsilon \to +0} [0-\arcsin(\varepsilon-1)] = \frac{\pi}{2};$$

$$\int_2^3 \frac{dx}{\sqrt{4x-x^2-3}} = \lim_{\varepsilon \to +0} \int_2^{3-\varepsilon} \frac{dx}{\sqrt{1-(x-2)^2}} = \lim_{\varepsilon \to +0} \arcsin(x-2)\Big|_2^{3-\varepsilon} =$$

$$= \lim_{\varepsilon \to +0} [\arcsin(1-\varepsilon)-0] = \frac{\pi}{2}.$$

Hence,

$$\int_1^3 \frac{dx}{\sqrt{4x-x^2-3}} = \frac{\pi}{2}+\frac{\pi}{2} = \pi.$$

(d) The integrand $f(x) = \frac{1}{\sqrt{|1-x^2|}}$ is unbounded in the neighbourhood of the point $x=1$, which is an interior point of the interval of integration. Therefore, by definition,

$$\int_0^2 \frac{dx}{\sqrt{|1-x^2|}} = \int_0^1 \frac{dx}{\sqrt{|1-x^2|}} + \int_1^2 \frac{dx}{\sqrt{|1-x^2|}}.$$

Evaluate each summand separately. If $0 \leqslant x < 1$, then

$$\int_0^1 \frac{dx}{\sqrt{|1-x^2|}} = \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{\varepsilon \to +0} \int_0^{1-\varepsilon} \frac{dx}{\sqrt{1-x^2}} =$$

$$= \lim_{\varepsilon \to +0} \arcsin x \Big|_0^{1-\varepsilon} = \lim_{\varepsilon \to +0} [\arcsin(1-\varepsilon) - 0] = \frac{\pi}{2}.$$

If $1 < x \leqslant 2$, then

$$\int_1^2 \frac{dx}{\sqrt{|1-x^2|}} = \int_1^2 \frac{dx}{\sqrt{x^2-1}} = \lim_{\varepsilon \to +0} \int_{1+\varepsilon}^2 \frac{dx}{\sqrt{x^2-1}} =$$

$$= \lim_{\varepsilon \to +0} \ln(x+\sqrt{x^2-1}) \Big|_{1+\varepsilon}^2 =$$

$$= \lim_{\varepsilon \to +0} [\ln(2+\sqrt{3}) - \ln(1+\varepsilon+\sqrt{(1+\varepsilon)^2-1})] = \ln(2+\sqrt{3}).$$

Hence,

$$\int_0^2 \frac{dx}{\sqrt{|1-x^2|}} = \frac{\pi}{2} + \ln(2+\sqrt{3}).$$

(e) Represent the given integral as a sum of three items, dividing each term of the numerator by $\sqrt[5]{x^3}$,

$$\int_0^1 \frac{x^3+\sqrt[3]{x}-2}{\sqrt[5]{x^3}}\, dx = \int_0^1 x^{12/5}\, dx + \int_0^1 \frac{dx}{x^{4/15}} - 2\int_0^1 \frac{dx}{x^{3/5}}.$$

The first summand is a proper integral evaluated by the Newton-Leibniz formula:

$$\int_0^1 x^{12/5}\, dx = \frac{5}{17} x^{17/5} \Big|_0^1 = \frac{5}{17}.$$

The second and third summands are unbounded to the right of the point $x = 0$. Therefore,

$$\int_0^1 \frac{dx}{x^{4/15}} = \lim_{\varepsilon \to +0} \int_\varepsilon^1 \frac{dx}{x^{4/15}} = \lim_{\varepsilon \to +0} \frac{15}{11} x^{11/15} \Big|_\varepsilon^1 = \frac{15}{11};$$

analogously,

$$\int_0^1 \frac{dx}{x^{3/5}} = \lim_{\varepsilon \to +0} \int_\varepsilon^1 \frac{dx}{x^{3/5}} = \lim_{\varepsilon \to +0} \frac{5}{2} x^{2/5} \Big|_\varepsilon^1 = \frac{5}{2}.$$

Hence,

$$\int_0^1 \frac{x^3+\sqrt[3]{x}-2}{\sqrt[5]{x^3}}\,dx=\frac{5}{17}+\frac{15}{11}-2\cdot\frac{5}{2}=-\frac{625}{187}.$$

(f) Represent the integrand $f(x)=\frac{1}{1-x^3}$ in the form of a sum of partial fractions:

$$f(x)=\frac{1}{1-x^3}=\frac{1}{(1-x)(1+x+x^2)}=\frac{1}{3}\left[\frac{1}{1-x}+\frac{x+2}{1+x+x^2}\right].$$

Then $\int_0^1 \frac{dx}{1-x^3}=\frac{1}{3}\int_0^1 \frac{dx}{1-x}+\frac{1}{3}\int_0^1 \frac{x+2}{1+x+x^2}\,dx$. Since

$$\int_0^1 \frac{dx}{1-x}=\lim_{\varepsilon\to+0}\int_0^{1-\varepsilon}\frac{dx}{1-x}=-\lim_{\varepsilon\to+0}\ln(1-x)\Big|_0^{1-\varepsilon}=\infty,$$

the given integral diverges. There is no need to compute the second summand representing a proper integral.

Note. Evaluation of the improper integrals from Problem **8.2.1** (a to f) can be considerably simplified by using a generalized antiderivative and applying the Newton-Leibniz formula. For instance, in Problem **8.2.1** (a) the function $F(x)=\frac{3}{2}\sqrt[3]{\ln^2 x}$ is continuous on the interval $[1, e]$ and differentiable at each point of the interval $1<x\leqslant e$, and $F'(x)=f(x)$ on this interval. Therefore

$$\int_1^e \frac{dx}{x\sqrt[3]{\ln x}}=\frac{3}{2}\sqrt[3]{\ln^2 x}\Big|_1^e=\frac{3}{2}.$$

8.2.2. Proceeding from the definition, compute the following improper integrals (or prove their divergence):

(a) $\int_0^{3a} \frac{2x\,dx}{(x^2-a^2)^{2/3}}$; (b) $\int_0^{2/\pi} \sin\frac{1}{x}\cdot\frac{dx}{x^2}$;

(c) $\int_0^1 \cos\frac{\pi}{1-x}\cdot\frac{dx}{(1-x)^2}$; (d) $\int_2^6 \frac{dx}{\sqrt[3]{(4-x)^2}}$;

(e) $\int_{-1}^{-2} \frac{dx}{x\sqrt{x^2-1}}$; (f) $\int_1^2 \frac{dx}{x\ln^p x}$.

8.2.3. Evaluate the following improper integrals:

(a) $\int\limits_{-3}^{3} \frac{x^2\,dx}{\sqrt{9-x^2}}$; (b) $\int\limits_{0}^{2} \sqrt{\frac{2+x}{2-x}}\,dx$.

Solution. (a) Find the indefinite integral

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \frac{1}{2}\left(9 \arcsin \frac{x}{3} - x\sqrt{9-x^2}\right) + C.$$

The function $F(x) = \frac{1}{2}\left(9 \arcsin \frac{x}{3} - x\sqrt{9-x^2}\right)$ is a generalized antiderivative for $f(x) = \frac{x^2}{\sqrt{9-x^2}}$ on the interval [—3, 3], since it is continuous on this interval and $F'(x) = f(x)$ at each point of the interval (—3, 3). Therefore, applying the Newton-Leibniz formula, we get

$$\int\limits_{-3}^{3} \frac{x^2\,dx}{\sqrt{9-x^2}} = \frac{1}{2}\left(9 \arcsin \frac{x}{3} - x\sqrt{9-x^2}\right)\Bigg|_{-3}^{3} = \frac{9}{2}\pi.$$

(b) Transform the integrand

$$f(x) = \sqrt{\frac{2+x}{2-x}} = \frac{2+x}{\sqrt{4-x^2}} = \frac{2}{\sqrt{4-x^2}} + \frac{x}{\sqrt{4-x^2}}.$$

The indefinite integral is equal to

$$\int \sqrt{\frac{2+x}{2-x}}\,dx = 2 \arcsin \frac{x}{2} - \sqrt{4-x^2} + C.$$

The function $F(x) = 2 \arcsin \frac{x}{2} - \sqrt{4-x^2}$ is a generalized antiderivative for $f(x)$ on the interval [0, 2], since it is continuous on this interval and $F'(x) = f(x)$ on the interval [0, 2).

Therefore, applying the Newton-Leibniz formula, we get

$$\int\limits_{0}^{2} \sqrt{\frac{2+x}{2-x}}\,dx = \left(2 \arcsin \frac{x}{2} - \sqrt{4-x^2}\right)\Bigg|_{0}^{2} = \pi + 2.$$

8.2.4. Test the integral

$$\int\limits_{-1}^{1} \frac{dx}{x\sqrt[3]{x}}$$

for convergence.

Solution. At the point $x=0$ the integrand goes to infinity. Both integrals $\int\limits_{-1}^{0} \frac{dx}{x\sqrt[3]{x}}$ and $\int\limits_{0}^{1} \frac{dx}{x\sqrt[3]{x}}$ diverge, since $\lambda = \frac{4}{3} > 1$. Consequently, the given integral diverges. If this were ignored, and the Newton-Leibniz formula formally applied to this integral, we would obtain the wrong result:

$$\int_{-1}^{1} \frac{dx}{x\sqrt[3]{x}} = \left(-\frac{3}{\sqrt[3]{x}}\right)\Bigg|_{-1}^{1} = -6.$$

And this is because the integrand is positive.

8.2.5. Test the following improper integrals for convergence:

(a) $\int_0^1 \frac{e^x}{\sqrt{1-\cos x}}\,dx$; b) $\int_0^1 \frac{\sin x+\cos x}{\sqrt[5]{1-x^3}}\,dx$.

Solution. (a) The integrand is infinitely large as $x \to +0$. Since

$$\sqrt{1-\cos x} = \sqrt{2}\sin\frac{x}{2} \sim \frac{\sqrt{2}}{2}x \text{ as } x \to +0,$$

the integrand has the order $\lambda = 1$ as compared with $\frac{1}{x}$. According to the special comparison test the given integral diverges.

(b) Rewrite the integrand in the following way:

$$f(x) = \frac{\sin x+\cos x}{\sqrt[5]{1+x+x^2}} \cdot \frac{1}{\sqrt[5]{1-x}}.$$

This function is infinitely large as $x \to 1$, its order is equal to $\lambda = \frac{1}{5}$ as compared with $\frac{1}{1-x}$, since the first multiplier tends to 1 as $x \to 0$. Therefore, by the special comparison test, the given integral converges.

8.2.6. Test the following improper integrals for convergence:

(a) $\int_0^2 \frac{\ln\left(1+\sqrt[5]{x^3}\right)}{e^{\sin x}-1}\,dx$; (b) $\int_1^2 \frac{\sqrt{x^2+1}}{\sqrt[3]{16-x^4}}\,dx$;

(c) $\int_0^1 \frac{\cos x\,dx}{\sqrt[4]{x}-\sin x}$.

Solution. (a) The integrand $f(x) = \frac{\ln\left(1+\sqrt[5]{x^3}\right)}{e^{\sin x}-1}$ is positive in the interval (0, 2) and is not defined at $x=0$. Let us show that $\lim_{x\to+0} f(x) = \infty$. Indeed, since

$$e^{\sin x}-1 \sim \sin x \sim x, \quad \ln\left(1+\sqrt[5]{x^3}\right) \sim \sqrt[5]{x^3} \text{ as } x \to 0,$$

we have

$$\lim_{x \to 0} \frac{\ln\left(1+\sqrt[5]{x^3}\right)}{e^{\sin x}-1} = \lim_{x \to 0} \frac{\sqrt[5]{x^3}}{x} = \lim_{x \to 0} \frac{1}{\sqrt[5]{x^2}} = \infty.$$

At the same time we have shown that $f(x) \sim \frac{1}{\sqrt[5]{x^2}}$ as $x \to 0$, i.e. that $f(x)$ is an infinitely large quantity of order $\lambda = \frac{2}{5} < 1$ as compared with $\frac{1}{x}$. Consequently, by the special comparison test, the given integral converges.

(b) Determine the order of the infinitely large function $f(x) = \frac{\sqrt{x^2+1}}{\sqrt[3]{16-x^4}}$ in the neighbourhood of the point $x=2$ with respect to $\frac{1}{2-x}$. To this end transform the expression for $f(x)$:

$$f(x) = \frac{\sqrt{x^2+1}}{\sqrt[3]{16-x^4}} = \frac{\sqrt{x^2+1}}{\sqrt[3]{4+x^2}\,\sqrt[3]{2+x}} \cdot \frac{1}{\sqrt[3]{2-x}}.$$

Hence it is obvious that the function $f(x)$ is an infinitely large quantity of order $\lambda = \frac{1}{3} < 1$ as $x \to 2$. According to the special comparison test the given integral converges.

(c) The integrand $f(x) = \frac{\cos x}{\sqrt[4]{x} - \sin x}$ is unbounded in the neighbourhood of the point $x=0$. Since

$$f(x) = \frac{\cos x}{\sqrt[4]{x} - \sin x} = \frac{\cos x}{\sqrt[4]{x}\left(1 - \frac{\sin x}{\sqrt[4]{x}}\right)} \sim \frac{1}{\sqrt[4]{x}} \qquad (x \to +0),$$

as $x \to +0$ the function $f(x)$ is an infinitely large quantity of order $\lambda = \frac{1}{4} < 1$ as compared with $\frac{1}{x}$ and, by the special comparison test, the integral converges.

8.2.7. Investigate the following improper integrals for convergence:

(a) $\int_0^1 \frac{e^x\,dx}{\sqrt{1-x^3}}$; (b) $\int_0^1 \frac{x^2\,dx}{\sqrt[3]{(1-x^2)^5}}$;

(c) $\int_0^1 \sqrt{\frac{x}{1-x^4}}\,dx$; (d) $\int_0^1 \frac{dx}{1-x^3+x^5}$;

(e) $\int_0^1 \frac{dx}{x-\sin x}$; (f) $\int_0^2 \frac{\ln\left(\sqrt[4]{x}+1\right)}{e^{\tan x}-1}\,dx$.

8.2.8. Prove that the integral

$$\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}\,dx$$

converges.

Solution. For $0 < x \leqslant 1$

$$0 \leqslant \left|\frac{\sin\frac{1}{x}}{\sqrt{x}}\right| \leqslant \frac{1}{\sqrt{x}}.$$

But the integral $\int_0^1 \frac{dx}{\sqrt{x}}$ converges, therefore, by the comparison test, the integral $\int_0^1 \left|\frac{\sin(1/x)}{\sqrt{x}}\right| dx$ also converges, and consequently the given integral converges absolutely.

8.2.9. Prove the convergence of the integral

$$I = \int_0^{\frac{\pi}{2}} \ln \sin x\,dx$$

and evaluate it.

Solution. Integrate by parts, putting $u = \ln(\sin x)$, $dx = dv$:

$$\int_0^{\frac{\pi}{2}} \ln \sin x\,dx = x \ln \sin x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} x\,\frac{\cos x}{\sin x}\,dx = -\int_0^{\frac{\pi}{2}} \frac{x}{\tan x}\,dx.$$

Since $\lim\limits_{x \to +0} \frac{x}{\tan x} = 1$, $\lim\limits_{x \to \frac{\pi}{2} - 0} \frac{x}{\tan x} = 0$, the last integral is a proper one. Consequently, the initial integral converges.

Now make the substitution $x = 2t$ in integral I. Then $dx = 2dt$; $x = 0$ at $t_1 = 0$; $x = \frac{\pi}{2}$ at $t_2 = \frac{\pi}{4}$. On substituting we get:

$$\int_0^{\pi/2} \ln \sin x\,dx = 2\int_0^{\pi/4} \ln \sin 2t\,dt = 2\int_0^{\pi/4} (\ln 2 + \ln \sin t + \ln \cos t)\,dt =$$

$$= 2t \ln 2 \Big|_0^{\pi/4} + 2\int_0^{\pi/4} \ln \sin t\,dt + 2\int_0^{\pi/4} \ln \cos t\,dt =$$

$$= \frac{\pi}{2} \ln 2 + 2\int_0^{\pi/4} \ln \sin t\,dt + 2\int_0^{\pi/4} \ln \cos t\,dt.$$

In the last integral make the substitution $t=\pi/2-z$. Then $dt=-dz$; $t=0$ at $z_1=\pi/2$; $t=\pi/4$ at $z_2=\pi/4$. Hence,

$$2\int_0^{\pi/4} \ln\cos t\,dt=-2\int_{\pi/2}^{\pi/4} \ln\cos\left(\frac{\pi}{2}-z\right)dz=2\int_{\pi/4}^{\pi/2} \ln\sin z\,dz.$$

Thus,

$$I=\int_0^{\pi/2} \ln\sin x\,dx=\frac{\pi}{2}\ln 2+2\int_0^{\pi/4} \ln\sin t\,dt+2\int_{\pi/4}^{\pi/2} \ln\sin z\,dz=$$

$$=\frac{\pi}{2}\ln 2+2\int_0^{\pi/2} \ln\sin t\,dt=\frac{\pi}{2}\ln 2+2I.$$

Whence

$$I=\int_0^{\pi/2} \ln\sin x\,dx=-\frac{\pi}{2}\ln 2.$$

8.2.10. Compute the integral

$$\int_0^1 \frac{x^n\,dx}{\sqrt{1-x^2}}\ (n \text{ a natural number}).$$

Solution. The integrand is an infinitely large quantity of order $\lambda=\frac{1}{2}$ with respect to $\frac{1}{1-x}$ as $x\to 1-0$. Therefore, the integral converges.

Make the substitution $x=\sin t$ in the integral. Then $dx=\cos t\,dt$, $x=0$ at $t=0$, $x=1$ at $t=\pi/2$. On substituting we get

$$\int_0^1 \frac{x^n\,dx}{\sqrt{1-x^2}}=\int_0^{\pi/2} \frac{\sin^n t\cdot\cos t\,dt}{\cos t}=\int_0^{\pi/2} \sin^n t\,dt.$$

The last integral is evaluated in Problem **6.6.9**:

$$\int_0^{\pi/2} \sin^n t\,dt=\begin{cases}\dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{1}{2}\dfrac{\pi}{2}, & n \text{ even},\\[2mm] \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{2}{3}, & n \text{ odd}.\end{cases}$$

Consequently, the given integral is also computed by the same formula.

8.2.11. Evaluate the following improper integrals (or prove their divergence):

$$\text{(a)} \int_0^{\frac{1}{2}} \frac{dx}{x\ln x};\quad \text{(b)} \int_1^2 \frac{dx}{x\sqrt{\ln x}};\quad \text{(c)} \int_0^1 \frac{3x^2+2}{\sqrt[3]{x^2}}\,dx.$$

8.2.12. Compute the improper integral

$$I_n=\int_0^1 x^m \ln^n x\,dx \quad (n \text{ natural}, \ m>-1).$$

Solution. At $n=0$ the integral is evaluated directly:

$$I_0=\int_0^1 x^m\,dx=\left.\frac{x^{m+1}}{m+1}\right|_0^1=\frac{1}{m+1}.$$

For $n>0$ integrate I_n by parts, putting

$$u=\ln^n x; \qquad dv=x^m\,dx;$$

$$du=n\ln^{n-1}x\frac{dx}{x}; \quad v=\frac{x^{m+1}}{m+1}.$$

We get

$$I_n=\left.\frac{x^{m+1}}{m+1}\ln^n x\right|_0^1-\frac{n}{m+1}\int_0^1 x^m\ln^{n-1}x\,dx=-\frac{n}{m+1}I_{n-1}.$$

This gives a formula by means of which one can reduce I_n to I_0 for any natural n:

$$I_n=-\frac{n}{m+1}I_{n-1}=+\frac{n(n-1)}{(m+1)^2}I_{n-2}=\ldots=\frac{(-1)^n n!}{(m+1)^n}I_0$$

And finally,

$$I_n=\frac{(-1)^n n!}{(m+1)^{n+1}}.$$

8.2.13. Compute the integral

$$I=\int_{0.3}^{2.0}\frac{e^{-x}\,dx}{\sqrt[4]{2+x-x^2}}$$

accurate to 0.03.

Solution. The integral has a singularity at the point $x=2$, since $2+x-x^2=(2-x)(1+x)$. Let us represent it as the sum of two integrals:

$$I_1=\int_{0.3}^{2-\varepsilon}\frac{e^{-x}\,dx}{\sqrt[4]{2+x-x^2}}, \quad I_2=\int_{2-\varepsilon}^{2}\frac{e^{-x}\,dx}{\sqrt[4]{2+x-x^2}}.$$

Now compute the first integral to the required accuracy, and estimate the second one. For $\varepsilon\leqslant 0.1$ we have

$$0<I_2<\frac{e^{-1.9}}{\sqrt[4]{2.9}}\int_{2-\varepsilon}^{2}\frac{dx}{\sqrt[4]{2-x}}=0.115\times\frac{4}{3}\varepsilon^{\frac{3}{4}}=0.153\,\varepsilon^{\frac{3}{4}}.$$

Putting $\varepsilon = 0.1$, we get the estimate $I_2 < 0.028$. Evaluation of the integral

$$I_1 = \int_{0.3}^{1.9} \frac{e^{-x}\,dx}{\sqrt[4]{2+x-x^2}}$$

by Simpson's formula with a step $h = 0.8$ gives

$$S_{0.8} = 0.519,$$

and with a step $h/2 = 0.4$,

$$S_{0.4} = 0.513.$$

And so, integral I_1 gives the more accurate value, 0.513, with an error not exceeding 0.001. Taking into consideration that integral I_2 is positive, we round off the obtained value to

$$I \approx 0.52$$

with an error not exceeding 0.03.

Note. By putting $\varepsilon = 0.01$, we get the estimate $I_2 < 0.005$, but the computation of the integral

$$I_1 = \int_{0.3}^{1.99} \frac{e^{-x}\,dx}{\sqrt[4]{2+x-x^2}}$$

would involve much more cumbersome calculations.

8.2.14. Investigate the following integrals for convergence:

(a) $\int_0^1 \frac{dx}{\sqrt{\sin x}}$; (b) $\int_0^1 \frac{dx}{e^x - \cos x}$;

(c) $\int_0^1 \frac{\cos^2 x\,dx}{(1-x)^2}$; (d) $\int_0^1 \frac{\tan x\,dx}{\sqrt{1-x^2}}$; (e) $\int_{1/2}^{6/5} \frac{\sin x\,dx}{\sqrt{|1-x^2|}}$.

§ 8.3. Geometric and Physical Applications of Improper Integrals

8.3.1. Find the area of the figure bounded by the curve $y = \frac{1}{1+x^2}$ (the witch of Agnesi) and its asymptote.

Solution. The function $y = \frac{1}{1+x^2}$ is continuous throughout the entire number scale, and $\lim_{x\to\infty} y = 0$. Consequently, the x-axis is the asymptote of the given curve which is shown in Fig. 118. It is required to find the area S of the figure that extends without bound along the

x-axis. In other words, it is required to evaluate the improper integral $S=\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$. By virtue of the symmetry of the figure about the y-axis we have

$$S=\int_{-\infty}^{\infty}\frac{dx}{1+x^2}=2\int_{0}^{\infty}\frac{dx}{1+x^2}=2\lim_{A\to\infty}\arctan x\Big|_0^A=2\cdot\frac{\pi}{2}=\pi.$$

8.3.2. Find the surface area generated by revolving about the x-axis the arc of the curve $y=e^{-x}$ between $x=0$ and $x=+\infty$.

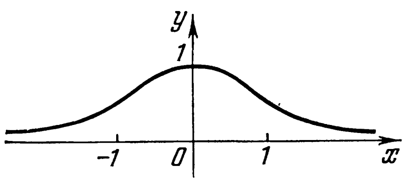

Fig. 118

Solution. The area of the surface is equal to the improper integral

$$S=2\pi\int_0^{+\infty}e^{-x}\sqrt{1+e^{-2x}}\,dx.$$

Making the substitution $e^{-x}=t$, $dt=-e^{-x}\,dx$, we get $x=0$ at $t=1$, $x=\infty$ at $t=0$; hence

$$S=2\pi\int_0^1\sqrt{1+t^2}\,dt=2\pi\cdot\frac{1}{2}\left[t\sqrt{1+t^2}+\ln(t+\sqrt{1+t^2})\right]_0^1=$$

$$=\pi\left[\sqrt{2}+\ln(1+\sqrt{2})\right].$$

8.3.3. Compute the area enclosed by the loop of the folium of Descartes

$$x^3+y^3-3axy=0.$$

Solution. The folium of Descartes is shown in Fig. 86. Let us represent the curve in polar coordinates:

$$x=\rho\cos\varphi;\quad y=\rho\sin\varphi.$$

Then $\rho^3\cos^3\varphi+\rho^3\sin^3\varphi-3a\rho^2\cos\varphi\sin\varphi=0$, whence, cancelling ρ^2, we get

$$\rho=\frac{3a\cos\varphi\sin\varphi}{\cos^3\varphi+\sin^3\varphi}.$$

Since the loop of the curve corresponds to the variation of φ between 0 and $\frac{\pi}{2}$ the sought-for area is equal to

$$S=\frac{1}{2}\int_0^{\frac{\pi}{2}}\rho^2\,d\varphi=\frac{9a^2}{2}\int_0^{\frac{\pi}{2}}\frac{\sin^2\varphi\cos^2\varphi}{(\sin^3\varphi+\cos^3\varphi)^2}\,d\varphi.$$

To evaluate the obtained proper integral make the substitution $\tan\varphi = t$; $\frac{d\varphi}{\cos^2\varphi} = dt$; $\varphi = 0$ at $t = 0$, $\varphi = \frac{\pi}{2}$ at $t = \infty$. Thus we get

$$S = \frac{9a^2}{2}\int_0^\infty \frac{t^2\,dt}{(1+t^3)^2} = \frac{9a^2}{2}\lim_{A\to\infty}\int_0^A \frac{t^2\,dt}{(1+t^3)^2} = -\frac{3a^2}{2}\lim_{A\to\infty}\left[\frac{1}{1+t^3}\right]_0^A = \frac{3}{2}a^2.$$

8.3.4. Find the volume of the solid generated by revolving the cissoid $y^2 = \frac{x^3}{2a-x}$ about its asymptote $x = 2a$.

Solution. The cissoid is shown in Fig. 119. Transfer the origin of coordinates to the point O' $(2a, 0)$ without changing the direction of the axes. In the new system of coordinates $X = x - 2a$, $Y = y$ the equation of the cissoid has the following form:

$$Y^2 = \frac{(X+2a)^3}{-X}.$$

The volume of the solid of revolution about the axis $X = 0$, i. e. about the asymptote, is expressed by the integral

$$V = \pi\int_{-\infty}^{\infty} X^2\,dY = 2\pi\int_0^\infty X^2\,dY.$$

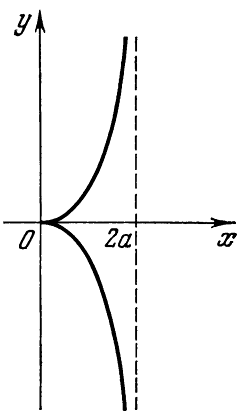

Fig. 119

Let us pass over to the variable X. For this purpose we find $dY = Y'\,dX$. Differentiating the equation of the cissoid in the new coordinates as an identity with respect to X, we get

$$2YY' = -\frac{3(X+2a)^2X - (X+2a)^3}{X^2} = -\frac{2(X+2a)^2(X-a)}{X^2}$$

whence for $Y > 0$ we have

$$Y' = -\frac{(X+2a)^2(X-a)}{X^2Y} = -\frac{(X+2a)(X-a)}{X^2\sqrt{-(X+2a)/X}}.$$

Hence,

$$V = -2\pi\int_{-2a}^{0} \frac{(X+2a)(X-a)}{\sqrt{-(X+2a)/X}}\,dX.$$

Make the substitution $(X+2a)/X = -t^2$; $X = -2a$ at $t = 0$, $X = 0$ at $t = \infty$. Then:

$$X = -\frac{2a}{1+t^2};\quad dX = \frac{4at}{(1+t^2)^2}dt;\quad X + 2a = \frac{2at^2}{1+t^2};$$

$$X - a = -\frac{3a+at^2}{1+t^2};$$

whence

$$V = 2\pi \int_0^\infty \frac{2at^2(3a+at^2)\,4at\,dt}{t(1+t^2)(1+t^2)(1+t^2)^2} =$$

$$= 48\pi a^3 \int_0^\infty \frac{t^2}{(1+t^2)^4}\,dt + 16\pi a^3 \int_0^\infty \frac{t^4}{(1+t^2)^4}\,dt.$$

Putting $t = \tan z$, $dt = \sec^2 z\,dz$, we get $t = 0$ at $z = 0$, $t = \infty$ at $z = \pi/2$. Hence,

$$V = 48\pi a^3 \int_0^{\pi/2} \sin^2 z \cos^4 z\,dz + 16\pi a^3 \int_0^{\pi/2} \cos^2 z \sin^4 z\,dz =$$

$$= 48\pi a^3 \int_0^{\pi/2} \cos^4 z\,dz - 48\pi a^3 \int_0^{\pi/2} \cos^6 z\,dz +$$

$$+ 16\pi a^3 \int_0^{\pi/2} \sin^4 z\,dz - 16\pi a^3 \int_0^{\pi/2} \sin^6 z\,dz.$$

Using the known formulas for the integrals $\int_0^{\pi/2} \sin^n x\,dx$, $\int_0^{\pi/2} \cos^n x\,dx$ (see Problem **6.6.9**), we get

$$V = 64\pi a^3 \frac{\pi}{2} \cdot \frac{1\times 3}{2\times 4} - 64\pi a^3 \frac{\pi}{2} \cdot \frac{1\times 3\times 5}{2\times 4\times 6} = 2\pi^2 a^3.$$

8.3.5. Prove that the area of the region bounded by the curve $y = \frac{1}{\sqrt{1-x^2}}$, the axis of abscissas, the axis of ordinates and the asymptote $x = 1$ is finite and equals $\frac{\pi}{2}$.

8.3.6. Prove that the area of the region bounded by the curve $y = \frac{1}{\sqrt[3]{x^2}}$, the axis of abscissas and the straight lines $x = \pm 1$ is finite and equals 6, and the area of the region contained between the curve $y = \frac{1}{x^2}$, the axis of abscissas and the straight lines $x = \pm 1$ is infinite.

8.3.7. Find the volumes of the solids enclosed by the surfaces generated by revolving the lines $y = e^{-x}$, $x = 0$, $y = 0$ $(0 \leqslant x < +\infty)$:

(a) about the x-axis,

(b) about the y-axis.

8.3.8. Compute the area contained between the cissoid $y^2 = \frac{x^3}{2a-x}$ and its asymptote.

8.3.9. Compute the area bounded by the curve $y = e^{-2x}$ (at $x > 0$) and the axes of coordinates.

8.3.10. Find the volume of the solid generated by revolving, about the x-axis, the infinite branch of the curve $y = 2\left(\frac{1}{x} - \frac{1}{x^2}\right)$ for $x \geqslant 1$.

8.3.11. Let a mass m be located at the origin O and attract a material point M found on the x-axis at a distance x from O and having a mass of 1, with a force $F = \frac{m}{x^2}$ (according to Newton's law). Find the work performed by the force F as the point M moves along the x-axis from $x = r$ to infinity.

Solution. The work will be negative, since the direction of the force is opposite to the direction of motion, hence

$$A = \int_r^\infty -\frac{m}{x^2}\,dx = \lim_{N \to \infty} \int_r^N -\frac{m}{x^2}\,dx = -\frac{m}{r}.$$

During the reverse displacement of the point M from infinity to the point $x = r$ the force of Newtonian attraction will perform positive work $\frac{m}{r}$. This quantity is called the *potential* of the force under consideration at the point $x = r$ and serves as the measure of potential energy accumulated at a point.

8.3.12. In studying a decaying current resulting from a discharge "ballistic" instruments are sometimes used whose readings are proportional to the "integral current intensity" $g = \int_0^\infty I\,dt$ or the "integral square of current intensity" $S = \int_0^\infty I^2\,dt$ and not to the instantaneous value of the current intensity I or to its square I^2. Here t is time measured from the beginning of the discharge; I is alternating-current intensity depending on time. Theoretically, the process continues indefinitely, though, practically, the current intensity becomes imperceptible already after a finite time interval. To simplify the formulas we usually assume the time interval to be infinite in all calculations involved.

Compute g and S for the following processes:

(a) $I = I_0 e^{-kt}$ (a simple aperiodic process); k is a constant coefficient, which is greater than zero.

(b) $I = I_0 e^{-kt} \sin \omega t$ (simple oscillating process); coefficients k and ω are constant.

Solution.

$$(a)\ g = \int_0^\infty I_0 e^{-kt}\,dt = \lim_{A\to\infty}\int_0^A I_0 e^{-kt}\,dt = I_0 \lim_{A\to\infty}\left[\frac{-e^{-kt}}{k}\right]_0^A = I_0/k;$$

$$S = \int_0^\infty I_0^2 e^{-2kt}\,dt = \frac{I_0^2}{2k};$$

$$(b)\quad g = \int_0^\infty I_0 e^{-kt}\sin\omega t\,dt = \lim_{A\to\infty}\int_0^A I_0 e^{-kt}\sin\omega t\,dt =$$

$$= \frac{I_0}{\omega^2+k^2}\lim_{A\to\infty}\left[(\omega\cos\omega t + k\sin\omega t)\,e^{-kt}\right]_0^A = \frac{I_0\omega}{\omega^2+k^2};$$

$$S = \int_0^\infty I_0^2 e^{-2kt}\sin^2\omega t\,dt = \lim_{A\to\infty}\int_0^A I_0^2 e^{-2kt}\,\frac{1-\cos 2\omega t}{2}\,dt =$$

$$= -\frac{I_0^2}{4k}\lim_{A\to\infty}\left[1-\frac{1}{\omega^2+k^2}(k^2\cos 2\omega t + \omega k\sin 2\omega t)\right]e^{-2kt}\Big|_0^A =$$

$$= \frac{I_0^2\omega^2}{4k(k^2+\omega^2)}.$$

8.3.13. Let an infinitely extended (in both directions) beam lying on an elastic foundation be bent by a concentrated force P. If the x-axis is brought to coincidence with the initial position of the axis of the beam (before the latter is bent) and the y-axis is drawn through the point O (at which the force is applied) and directed downwards, then, on bending, the beam axis will have the following equation

$$y = \frac{P\alpha}{2k} e^{-\alpha|x|}(\cos\alpha x + \sin\alpha|x|),$$

where α and k are certain constants. Compute the potential energy of elastic deformation by the formula

$$W = Ee\int_0^\infty (y'')^2\,dx \qquad (E,\ e\ \text{const}).$$

Solution. Find y'':

$$y'' = \frac{P\alpha^3}{k}e^{-\alpha x}\left[(\cos\alpha x + \sin\alpha x) - 2(-\sin\alpha x + \cos\alpha x) +\right.$$

$$\left. + (-\sin\alpha x - \cos\alpha x)\right] = \frac{P\alpha^3}{k}e^{-\alpha x}(\sin\alpha x - \cos\alpha x).$$

Hence,

$$W=\frac{P^2\alpha^6Ee}{k^2}\int_0^\infty e^{-2\alpha x}(1-2\sin\alpha x\cos\alpha x)\,dx=$$

$$=\frac{P^2\alpha^6Ee}{k^2}\left[\frac{1}{2\alpha}-\frac{2\alpha}{4\alpha^2+4\alpha^2}\right]=\frac{P^2\alpha^5Ee}{4k^2}.$$

8.3.14. What work has to be performed to move a body of mass m from the Earth's surface to infinity?

8.3.15. Determine the work which has to be done to bring an electric charge $e_2=1$ from infinity to a unit distance from a charge e_1.

§ 8.4. *Additional Problems*

8.4.1. Prove that the integral

$$\int_1^\infty \frac{dx}{x^p \ln^q x}$$

converges for $p>1$ and $q<1$.

8.4.2. Prove that the integral

$$\int_0^\infty x^p \sin x^q\,dx,\qquad q\neq 0$$

converges absolutely for $-1<(p+1)/q<0$ and converges conditionally for $0\leqslant(p+1)/q<1$.

8.4.3. Prove that the Euler integral of the first kind (beta function)

$$B(p,\,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx$$

converges for $p>0$ and $q>0$.

8.4.4. Prove that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \sin\alpha x\cdot\sin\beta x\,dx=0,$$

if $|\alpha|\neq|\beta|$.

8.4.5. Prove that

$$I=\int_0^\infty e^{-x^2}\cdot x^{2n+1}\,dx=\frac{n!}{2}\quad (n\text{ natural}).$$

8.4.6. Prove that if the integral $\int\limits_{a}^{\infty} \frac{f(x)}{x} dx$ converges for any positive a and if $f(x)$ tends to A as $x \to 0$, then the integral

$$\int\limits_{0}^{\infty} \frac{f(\alpha x) - f(\beta x)}{x} dx \quad (\alpha > 0, \quad \beta > 0)$$

converges and equals $A \ln(\beta/\alpha)$.

8.4.7. Prove that

$$\int\limits_{0}^{\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x} dx = \int\limits_{0}^{\infty} \frac{\cos \alpha x - \cos \beta x}{x} dx = \ln \frac{\beta}{\alpha}.$$

8.4.8. At what values of m does the integral $\int\limits_{0}^{\pi/2} \frac{1 - \cos x}{x^m} dx$ converge?

8.4.9. Prove that the integral $\int\limits_{0}^{\pi} \frac{dx}{(\sin x)^k}$ converges if $k < 1$, and diverges if $k \geqslant 1$.

8.4.10. Prove that the integral $\int\limits_{0}^{\infty} \frac{\sin x (1 - \cos x)}{x^s} dx$ converges if $0 < s < 4$, and converges absolutely if $1 < s < 4$.

8.4.11. Suppose the integral

$$\int\limits_{a}^{+\infty} f(x)\, dx \tag{1}$$

converges and the function $\varphi(x)$ is bounded.

Does the integral

$$\int\limits_{a}^{+\infty} f(x)\, \varphi(x)\, dx \tag{2}$$

necessarily converge?

What can be said about the convergence of integral (2), if integral (1) converges absolutely?

8.4.12. Prove the validity of the relation

$$f(x) = 2f(\pi/4 + x/2) - 2f(\pi/4 - x/2) - x \ln 2,$$

where $f(x) = -\int\limits_{0}^{x} \ln \cos y\, dy$.

Compute with the aid of the relation obtained

$$f\left(\frac{\pi}{2}\right)=-\int_{0}^{\frac{\pi}{2}}\ln\cos y\,dy.$$

8.4.13. Deduce the reduction formula for the integral

$$I_n=\int_{0}^{\frac{\pi}{2}}\ln\cos x\cdot\cos 2nx\,dx \qquad (n \text{ natural})$$

and evaluate this integral.

ANSWERS AND HINTS

Chapter I

1.1.5. (b) *Hint.* Prove by the rule of contraries, putting $2=\frac{p^2}{q^2}$, where p and q are positive integers without common multipliers.

1.1.8. *Hint.* You may take $k=\frac{s^2-2}{2s}$.

1.1.9. (b) $x \geqslant 4$, $x \leqslant 0$; (c) $-4 \leqslant x \leqslant 2$.

1.1.11. (a) $x<-1$ or $x \geqslant 1$. *Hint.* The equality is valid for those values of x for which $\frac{x-1}{x+1} \geqslant 0$; (b) $2 \leqslant x \leqslant 3$. *Hint.* The equality holds true for those values of x for which $x^2-5x+6 \leqslant 0$.

1.1.13. (a) $x<\frac{2}{5}$ or $x>8$; (b) $x<0$ or $0<x<5$. *Hint.* The inequality $|a-b|>|a|-|b|$ holds good when a and b are opposite in sign or when $|a|<|b|$.

1.2.3. 0; $\frac{a+2}{[a(a^2+3a+3)]}$; $(a^3+a)(a^3-1)$.

1.2.4. b^2+ab+a^2; $\frac{(a+h)^3}{8}-1$. **1.2.6.** $4\sqrt{2}+1$; $\frac{\sqrt{2}+1}{2}$; $2\sqrt{10}-5$.

1.2.11. $f(x)=10+5\times 2^x$.

1.2.13. $f(3x)=\frac{45x^2+1}{2-3x}$; $f(x^3)=\frac{5x^6+1}{2-x^3}$;

$$3f(x)=\frac{15x^2+3}{2-x}; \quad [f(x)]^3=\frac{125x^6+75x^4+15x^2+1}{8-12x+6x^2-x^3}.$$

1.2.14. $f(2)=5$; $f(0)=4$; $f(0.5)=4$; $f(-0.5)=\frac{\sqrt{3}}{3}$; $f(3)=8$.

1.2.15. *Hint.* From $x_{n+1}=x_n+d$ it follows that $y_{n+1}=a^{x_{n+1}}=a^{x_n+d}=a^{x_n}a^d$.

1.2.16. $x=\pm 2$; ± 3. **1.2.17.** $f(x)=x^2-5x+6$. **1.2.18.** $f(x)=23$; $\varphi(x)=527$.

1.2.19. $x \leqslant -1$ or $x \geqslant 2$. **1.2.20.** $P=2b+2\left(1-\frac{b}{h}\right)x$; $S=b\left(1-\frac{x}{h}\right)x$.

1.2.21. (b) (2, 3); (c) $(-\infty, -1)$ and $(2, \infty)$; (d) $x=\frac{\pi}{2}+2k\pi$ $(k=0, \pm 1, \pm 2, \ldots)$. *Hint.* Since $\sin x \leqslant 1$, the function is defined only when $\sin x=1$; (g) $(-\infty, 2)$ and $(3, \infty)$; (h) $[1, 4)$; (i) $(-2, 0)$ and $(0, 1)$; (j) $-\frac{\pi}{2}+2k\pi<x<\frac{\pi}{2}+2k\pi$ $(k=0, \pm 1, \pm 2, \ldots)$.

1.2.22. (d) The function is defined over the entire number scale, except the points $x = \pm 2$.

1.2.24. (a) $(-\infty, \infty)$; (b) $(3-2\pi, 3-\pi)$ and $(3, 4)$; (c) $[-1, 3]$; (d) $(-1, 0)$ and $(0, \infty)$. **1.2.25.** (b) $5 \leqslant x \leqslant 6$.

1.2.26. (a) $2k\pi \leqslant x \leqslant (2k+1)\pi$ $(k=0, \pm 1, \pm 2, \ldots)$; (b) $\left[-\frac{3}{2}, -1\right]$.

1.3.3. (b) *Hint*. Consider the difference $\frac{x_2}{1+x_2^2} - \frac{x_1}{1+x_1^2}$.

1.3.4. (b) It increases for $-\frac{5\pi}{6} + k\pi < x < \frac{\pi}{6} + k\pi$ $(k=0, \pm 1, \pm 2, \ldots)$ and decreases on the other intervals.

1.3.7. The function decreases on the interval $0 < x \leqslant \frac{\pi}{4}$ from $+\infty$ to 2 and increases on the interval $\frac{\pi}{4} \leqslant x < \frac{\pi}{2}$ from 2 to $+\infty$.

1.3.9. (c) The function is neither even, nor odd, (d) even.

1.3.10. (a) Even; (b) odd; (c) odd; (d) neither even, nor odd; (e) even.

1.3.12. (a) $|A| = 5$, $\omega = 4$, $\varphi = 0$, $T = \frac{\pi}{2}$; (b) $|A| = 4$, $\omega = 3$, $\varphi = \frac{\pi}{4}$, $T = \frac{2\pi}{3}$; (c) $|A| = 5$, $\omega = \frac{1}{2}$, $\varphi = \arctan \frac{4}{3}$, $T = 4\pi$. *Hint*. $3 \sin \frac{x}{2} + 4 \cos \frac{x}{2} = 5 \sin\left(\frac{x}{2} + \varphi\right)$, where $\cos\varphi = \frac{3}{5}$, $\sin\varphi = \frac{4}{5}$. **1.3.13.** (b) $T = 2\pi$; (c) $T = 1$.

1.3.16. The greatest value $f(1) = 2$. *Hint*. The function reaches the greatest value at the point where the quadratic trinomial $2x^2 - 4x + 3$ reaches the least value.

1.3.17. (a) Even; (b) even; (c) odd; (d) even.

1.3.18. (a) $T = \pi$; (b) $T = 6\pi$.

1.3.19. *Hint*. (a) Assume the contrary. Then

$$x + T + \sin(x+T) = x + \sin x,$$

whence $\cos\left(x + \frac{T}{2}\right) = -\frac{T}{2 \sin \frac{T}{2}}$, which is impossible for any constant T, since the left side is not constant; (b) suppose the contrary. Then $\cos\sqrt{x+T} = \cos\sqrt{x}$, whence either $\sqrt{x+T} + \sqrt{x} = 2\pi k$, or $\frac{T}{\sqrt{x+T}+\sqrt{x}} = 2\pi k$ $(k=0, \pm 1, \pm 2, \ldots)$, which is impossible, since the left-hand members of these equalities are functions of a continuous argument x.

1.4.6. (a) $x = \frac{1 + \arcsin y}{3}$; (b) $x = 3 \sin y$; (c) $x = y^{\frac{1}{\log 5}}$ $(y > 0)$; (d) $x = \frac{\log_2 y}{\log_2 y - 1} = \frac{\log y}{\log \frac{y}{2}}$ $(0 < y < 2$ or $2 < y < \infty)$.

1.6.3. (a) $\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, \ldots$; (b) $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$; (c) 2; 2.25; $2\frac{10}{27}$; $2\frac{113}{256}$, ...

1.6.9. *Hint*. The inequality $\left|\frac{2n+3}{n+1} - 2\right| < \varepsilon$ is satisfied for $n > N =$

$=E\left(\frac{1}{\varepsilon}-1\right)$. At $\varepsilon=0.1$ the inequality is fulfilled beginning with $n=10$, at $\varepsilon=0.01$ beginning with $n=100$, at $\varepsilon=0.001$ beginning with $n=1000$.

1.6.10. *Hint*. Verify that the sequence $\{x_{2n-1}\}$ tends to 1 as $n\to\infty$, and the sequence $\{x_{2n}\}$ tends to 0 as $n\to\infty$.

1.6.12. (a) It has; (b) it does not have; (c) it has; (d) it does not have.

1.6.14. *Hint*. (a) $|x_n|\leqslant\frac{2}{n}$; (b) $|x_n|\leqslant\frac{1}{n}$.

1.6.19. *Hint*. For $a>1$ put $\sqrt[n]{a}=1+\alpha_n\ (\alpha_n>0)$ and, with the aid of the inequality $a=(1+\alpha_n)^n>n\alpha_n$, prove that α_n is an infinitesimal. For $a<1$ put $\sqrt[n]{a}=\frac{1}{1+\alpha_n}\ (\alpha_n>0)$ and make use of the inequality $\frac{1}{a}=(1+\alpha_n)^n>n\alpha_n$.

1.7.1. (b) $\frac{5}{4}$; (c) 0; (e) $\frac{1}{2}$. **1.7.2.** (b) $\frac{1}{16}$; (e) 1; (f) 1.

1.7.4. (b) 1; (f) 0. *Hint*. Multiply and divide by imperfect of a sum, square and then divide by $n^{\frac{4}{3}}$; (g) $-\frac{1}{3}$; (h) 1. *Hint*. Represent each summand of x_n in the form of the difference

$$\frac{1}{1\times 2}=1-\frac{1}{2},\quad \frac{1}{2\times 3}=\frac{1}{2}-\frac{1}{3};\ \ldots;\ \frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1},$$

which will bring x_n to the form $x_n=1-\frac{1}{n+1}$.

1.7.5. (a) $\frac{1}{2}$; (b) 1; (c) 0, (d) $-\frac{1}{2}$. *Hint*. The quantity $\frac{1}{2n}$ is an infinitesimal, and $\cos n^3$ is a bounded quantity; (e) 0; (f) $\frac{4}{3}$.

1.8.6. (b) *Hint*. The sequence is bounded due to the fact that $n!=1\times 2\times\times 3\times\ldots\times n\geqslant 2^{n-1}$ and therefore

$$x_n\leqslant 2+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\ldots+\left(\frac{1}{2}\right)^{n-1}=3-\left(\frac{1}{2}\right)^{n-1}<3.$$

1.8.7. (b) 0. *Hint*. Take advantage of the fact that $\frac{x_{n+1}}{x_n}=\frac{2}{n+3}<1$.

1.8.9. *Hint*. For all n, beginning with a certain value, the inequalities $\frac{1}{n}<a<n$ are fulfilled; therefore $\frac{1}{\sqrt[n]{n}}<\sqrt[n]{a}<\sqrt[n]{n}$, and $\lim\sqrt[n]{n}=$ $=\lim\frac{1}{\sqrt[n]{n}}=1$.

1.8.10. *Hint*. The sequence $\{y_n\}$ decreases, since $y_{n+1}=a^{\frac{1}{2^{n+1}}}=a^{\frac{1}{2^n\times 2}}=$ $=\sqrt{y_n}\ (y_n>1)$.

The boundedness of the sequence from below follows from $a>1$. Denote $\lim\limits_{n\to\infty}y_n$ by b and from the relation $y_{n+1}=\sqrt{y_n}$ find $b=1$.

1.8.11. *Hint*. Ascertain that the sequence increases. Establish the boundedness from the inequalities

$$\frac{1}{n^2}<\frac{1}{n(n-1)}=\frac{1}{n-1}-\frac{1}{n}\quad (n\geqslant 2);$$

$$x_n<1+\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n}\right)=2-\frac{1}{n}.$$

1.8.12. *Hint.* Transform x_n into the form $x_n=\dfrac{2n}{\sqrt{n^2+1}+n}$ and take advantage of the inequalities

$$\frac{2n}{2n+1}<\frac{2n}{\sqrt{n^2+1}+n}<1.$$

1.8.13. *Hint.* See Problem **1.8.7** (a).

1.8.14. *Hint.* Establish the boundedness of the sequence by comparing x_n with the sum of some geometric progression.

1.9.2. (b) *Hint.* Choose the sequences

$$x_n=\frac{1}{n} \quad\text{and}\quad x_n'=-\frac{1}{n} \quad (n=1, 2, \ldots)$$

and ascertain that the sequences of appropriate values of the function have different limits:

$$\lim 2^{\frac{1}{x_n}}=+\infty, \qquad \lim 2^{\frac{1}{x_n'}}=0.$$

1.9.3. (e) *Hint.* Take advantage of the inequality

$$\frac{\pi}{2}-\operatorname{arc\,tan} x<\tan\left(\frac{\pi}{2}-\operatorname{arc\,tan} x\right)=\frac{1}{x} \quad (x>0).$$

(f) *Hint.* Transform the difference

$$\sin x-\frac{1}{2}=\sin x-\sin\frac{\pi}{6}$$

into a product and apply the inequality $|\sin\alpha|\leqslant|\alpha|$.

1.10.1. (d) $\frac{p}{q}$; (e) $\frac{5}{6}$; (f) $-\frac{1}{12}$. *Hint.* Multiply the numerator and denominator by imperfect trinomial square $\left(\sqrt[3]{10-x}+2\right)$; (g) $\frac{34}{23}$; (h) $\log_a 6$. *Hint.* $\lim\limits_{x\to 3}\left[\log_a\dfrac{x-3}{\sqrt{x+6}-3}\right]=\log_a\left[\lim\limits_{x\to 3}\dfrac{(x-3)(\sqrt{x+6}+3)}{x-3}\right]=\log_a 6$; (i) $\frac{2}{3}$; (j) $\frac{7}{12}$.

1.10.2. (e) $\frac{1}{2}$. *Hint.* On removing the irrationality to the denominator divide the numerator and denominator by x.

1.10.3. (b) 32. (c) $\frac{5}{3}$. *Hint.* Put $x=z^{15}$; (f) ∞. *Hint.* Put $\frac{\pi}{2}-x=z$; $x=\frac{\pi}{2}-z$; $z\longrightarrow 0$ as $x\longrightarrow\frac{\pi}{2}$; (g) -3. *Hint.* Put $\sin x=y$.

1.10.5. (b) $e^{\frac{1}{3}}$; (c) e^{-1}; (d) e^{mk}; (f) 4; (g) $\frac{1}{a}$; (h) 2.

1.10.7. (b) $\frac{1}{4}$. **1.10.8.** (b) 1; (c) $\frac{1}{e}$; (d) $e^{\cot a}$.

1.10.11. (a) $\frac{1}{2}$; (b) $-\frac{3}{4}$; (c) $\frac{1}{2}$; (d) $\frac{2}{5}$; (e) 0; (f) -1.

1.10.12. (a) $\frac{1}{20}$; (b) -2; (c) $\frac{\pi}{2}$; (d) $\frac{1}{2}$; (e) -24.

1.10.13. (a) e^4; (b) -1; (c) $2\ln a$; (d) e^3; (e) $e^{-\frac{1}{2}}$; (f) e^{-1}; (g) 1; (h) 1; (i) 9; (j) 1; (k) $\alpha-\beta$. *Hint.*

$$\lim_{x\to 0}\frac{e^{\alpha x}-e^{\beta x}}{x}=\lim_{x\to 0} e^{\beta x}\frac{e^{(\alpha-\beta)x}-1}{x}=\alpha-\beta.$$

1.10.14. (a) $\sqrt{2}$. *Hint.* Replace $\arccos(1-x)$ by $\arcsin\sqrt{2x-x^2}$; (b) 1; (c) a.

1.11.5. (b) It is of the third order of smallness. *Hint.*

$$\lim_{\alpha\to 0}\frac{\tan\alpha-\sin\alpha}{\alpha^3}=\frac{1}{2}.$$

1.11.6. (b) They are of the same order; (c) they are equivalent.

1.11.8. (a) $100x$ is an infinitesimal of the same order as x; (b) x^2 is an infinitesimal of an order higher than x; (c) $6\sin x$ is an infinitesimal of the same order as x; (d) $\sin^3 x$ is an infinitesimal of an order higher than x; (e) $\sqrt[3]{\tan^2 x}$ is an infinitesimal of an order of smallness lower than x.

1.11.9. (a) It is of the fourth order of smallness; (b) of the first order of smallness; (c) of the third order of smallness; (d) of the third order of smallness; (e) of the first order of smallness; (f) of the order of smallness $\frac{1}{2}$; (g) of the first order of smallness; (h) of the first order of smallness; (i) of the second order of smallness. *Hint.* Multiply and divide the difference $\cos x-\sqrt[3]{\cos x}$ by imperfect trinomial square; (j) of the first order of smallness.

1.11.10. The diagonal d is of the first order of smallness; the area S is of the second order of smallness; the volume V is of the third order of smallness.

1.12.3. (b) 4; (f) 3; (g) $\frac{1}{2}$; (i) 2. **1.12.6.** (a) 1; (b) 2.

1.12.7. (a) 1; (b) $\frac{1}{3}$. **1.12.8.** (a) $\frac{3}{5}$; (b) $\frac{4}{5}$; (c) $\frac{3}{2}$; (d) $\frac{3}{2}$; (e) $\frac{2}{9}$; (f) $\frac{3}{4}$; (g) -2; (h) 1. **1.12.9.** 10.14. *Hint.* $1042=10^3\times(1+0.042)$.

1.13.1. (b) $f(1-0)=-2$, $f(1+0)=2$; (f) $f(2-0)=-\infty$; $f(2+0)=+\infty$.

1.13.3. (a) $f(-0)=\frac{1}{2}$, $f(+0)=0$; (b) $f(-0)=0$, $f(+0)=+\infty$; (c) $f(-0)=-1$, $f(+0)=1$.

1.14.2. (b) The function has a discontinuity of the first kind at the point $x=3$. The jump is equal to 27.

1.14.3. (c) The function is continuous everywhere; (e) the function has a discontinuity of the first kind at the point $x=0$; the jump equals π. *Hint.* $\arctan(-\infty)=-\frac{\pi}{2}$, $\arctan(+\infty)=+\frac{\pi}{2}$.

1.14.6. (b) At the point $x_0=5$ there is a discontinuity of the first kind: $f(5-0)=-\frac{\pi}{2}$, $f(5+0)=\frac{\pi}{2}$; (c) at the point $x_0=0$, a discontinuity of the first kind: $f(-0)=1$, $f(+0)=0$; (d) at the point $x_0=\frac{\pi}{2}$, an infinite discontinuity of the second kind:

$$f\left(\frac{\pi}{2}-0\right)=+\infty,\quad f\left(\frac{\pi}{2}+0\right)=-\infty.$$

1.14.7. (a) At the point $x=0$ there is a removable discontinuity. To remove the discontinuity it is sufficient to redefine the function, putting $f(0)=1$; (b) at the point $x=0$ there is a removable discontinuity. To remove the discontinuity it is sufficient to extend the function putting $f(0)=1$; (c) at the point $x=0$ there is a discontinuity of the second kind: $f(-0)=0$, $f(+0)=+\infty$; (d) at the points $x=(2k+1)\frac{\pi}{2}$ ($k=0, \pm 1, \pm 2, \ldots$), removable discontinuities, since

$$f(x)=\lim_{n\to\infty}(\sin x)^{2n}=\begin{cases}0 \text{ if } |\sin x|<1,\\ 1 \text{ if } |\sin x|=1;\end{cases}$$

(e) at the points $x=k\pi$ ($k=0, \pm 1, \pm 2, \ldots$), discontinuities of the 1st kind, since

$$f(x)=\frac{|\sin x|}{\sin x}\begin{cases}1 \text{ if } \sin x>0,\\ -1 \text{ if } \sin x<0;\end{cases}$$

(f) at the points $x=n=0, \pm 1, \pm 2, \ldots$ removable discontinuities, since

$$f(x)=\begin{cases}-1 \text{ if } x=n,\\ 0 \text{ if } x\neq n.\end{cases}$$

1.14.8. (a) At the point $x=1$ there is an infinite discontinuity of the second kind; (b) at the point $x=-2$, a discontinuity of the first kind (the jump being equal to 2); (c) at the point $x=0$, an infinite discontinuity of the second kind, at the point $x=1$, a discontinuity of the first kind (the jump being equal to -4); (d) at the point $x=1$, an infinite discontinuity of the second kind.

1.14.9. (a) $f(0)=1$; (b) $f(0)=-\frac{3}{2}$; (c) $f(0)=\frac{1}{2}$; (d) $f(0)=2$.

1.15.2. (b) The function is continuous on the interval $(0, +\infty)$.

1.15.3. (b) The function is continuous everywhere. At the only possible point of discontinuity $x=0$ we have

$$\lim_{x\to -0} y=\lim_{u\to 1} u^2=1; \qquad \lim_{x\to +0} y=\lim_{u\to -1} u^2=1; \qquad y|_{x=0}=y|_{u=-1}=1;$$

(c) at the points $x=\frac{\pi}{2}+n\pi$ ($n=0, \pm 1, \pm 2, \ldots$) there are removable discontinuities, since $\lim\limits_{x\to\frac{\pi}{2}} y=\lim\limits_{u\to\pm\infty} y=-1$.

1.16.2. Yes. **1.16.12.** 1.53. **1.16.13.** No. For instance, the function $y=x^2$ on the interval $[-1, 1]$.

1.17.1. (a) *Hint*. Multiply the obvious inequalities:

$$\sqrt{1\cdot n}<\frac{n+1}{2};$$

$$\sqrt{2(n-1)}<\frac{n+1}{2};$$

$$\cdots\cdots\cdots\cdots$$

$$\sqrt{(n-1)\cdot 2}<\frac{n+1}{2};$$

$$\sqrt{n\cdot 1}<\frac{n+1}{2}.$$

(b) *Hint*. Let $A=\frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}\times\ldots\times\frac{2n-1}{2n}$,

$$B=\frac{2}{3}\times\frac{4}{5}\times\frac{6}{7}\times\ldots\times\frac{2n}{2n+1}.$$

Then $A<B$ since $\frac{2n-1}{2n}<\frac{2n}{2n+1}$ and $A^2<AB=\frac{1}{2n+1}$.

1.17.2. (a) *Hint.* Extract the 101st root from both sides of the inequality and reduce both sides by 101^2.

(b) Multiply the obvious inequalities:

$$99 \times 101 < 100^2,$$
$$98 \times 102 < 100^2,$$
$$\dots\dots\dots$$
$$2 \times 198 < 100^2,$$
$$1 \times 100 \times 199 \times 200 < 100^4.$$

1.17.3. (a) $-3 < x < -1$ or $1 < x < 3$; (b) $x < -\frac{1}{3}$ or $x > \frac{5}{3}$; (c) the inequality has no solutions, since it is equivalent to the contradictory system $x-2 > 0$, $x(4x^2-x+4) < 0$. **1.17.4.** Yes. **1.17.5.** (a) No; (b) Yes.

1.17.7. *Hint.* Apply the method of mathematical induction. At $n=1$ the relation is obvious. Supposing that the inequality

$$(1+x_1)(1+x_2)\,\dots\,(1+x_{n-1}) \geqslant 1+x_1+x_2+\dots+x_{n-1}$$

holds true, multiply both its sides by $1+x_n$ and take into consideration the conditions $1+x_n > 0$, $x_i \cdot x_n > 0$ $(i=1, 2, \dots, n-1)$.

1.17.8. (a) $[1, +\infty)$; (b) $(2n\pi)^2 \leqslant x \leqslant (2n+1)^2\pi^2$ $(n=0, 1, 2, \dots)$; (c) $x=0, \pm 1, \pm 2, \dots$; (d) $(-\infty, 0)$ for $f(x)$; $g(x)$ is nowhere defined; (e) $[-4, -2]$ or $[2, 4]$; (f) $x=(2n+1)\frac{\pi}{2}$ $(n=0, \pm 1, \pm 2, \dots)$.

1.17.9. (a) No: $\varphi(0)=1$, and $f(0)$ is not defined; (b) No: $f(x)$ is defined for all $x \neq 0$, and $\varphi(x)$ only for $x > 0$; (c) No: $f(x)$ is defined for all x, and $\varphi(x)$ only for $x \geqslant 0$; (d) Yes; (e) No: $f(x)$ is defined only for $x > 2$, and $\varphi(x)$ for $x > 2$ and for $x < 1$.

1.17.10. (a) $(0, \infty)$; (b) $[1, \infty)$. **1.17.11.** $V=8\pi(x-3)(6-x)$, $3 < x < 6$.

1.17.12. (a) $x=5$. *Hint.* The domain of definition is specified by the inequalities $x+2 \geqslant 0$, $x-5 \geqslant 0$, $5-x \geqslant 0$, which are fulfilled only at the point $x=5$. Verify that the number $x=5$ satisfies the given inequality. (b) *Hint.* The domain of definition is specified by the contradictory inequalities $x-3 > 0$; $2-x > 0$.

1.17.17. (a) $f(x)=\frac{2}{1+x^2}+\frac{x}{1+x^2}$; (b) $a^x=\frac{a^x+a^{-x}}{2}+\frac{a^x-a^{-x}}{2}$ (see Problem **1.17.16**).

1.17.18. An even extension defines the function

$$\varphi(x)=\begin{cases} f(x)=x^2+x & \text{for } 0 \leqslant x \leqslant 3, \\ f(-x)=x^2-x & \text{for } -3 \leqslant x < 0. \end{cases}$$

An odd extension defines the function

$$\psi(x)=\begin{cases} f(x)=x^2+x & \text{for } 0 \leqslant x \leqslant 3, \\ -f(-x)=-x^2+x & \text{for } -3 \leqslant x < 0. \end{cases}$$

1.17.21. *Hint.* If the function $f(x)$ has a period T_1, and the function $\varphi(x)$ has a period T_2, and $T_1=n_1 d$, $T_2=n_2 d$ (n_1, n_2 positive integers), then the period of the sum and the product of these functions will be $T=nd$, where n is the least common multiple of the numbers n_1 and n_2.

1.17.22. *Hint.* For any rational number r

$$\lambda(x+r)=\lambda(x)=\begin{cases} 1 & \text{for rational } x, \\ 0 & \text{for irrational } x. \end{cases}$$

But there is no least number in the set of positive rational numbers.

1.17.23. *Hint.* If we denote the period of the function $f(x)$ by T, then from $f(T)=f(0)=f(-T)$ we get

$$\sin T+\cos aT=1=\sin(-T)+\cos(-aT),$$

whence $\sin T=0$, $\cos aT=1$, and hence $T=k\pi$, $aT=2n\pi$, $a=\frac{2n}{k}$ is rational.

1.17.25. The difference of two increasing functions is not necessarily a monotonic function. For example, the functions $f(x)=x$ and $g(x)=x^2$ increase for $x\geqslant 0$, but their difference $f(x)-g(x)=x-x^2$ is not monotonic for $x\geqslant 0$: it increases on $\left[0, \frac{1}{2}\right]$ and decreases on $\left[\frac{1}{2}, \infty\right)$.

1.17.26. Example:

$$y=\begin{cases} x & \text{if } x \text{ is rational,} \\ -x & \text{if } x \text{ is irrational.}\end{cases}$$

1.17.27. (a) $x=\frac{1}{2}\ln\frac{1+y}{1-y}$ $(-1<y<1)$;

(b)

$$x=\begin{cases} y & \text{for } -\infty<y<1, \\ \sqrt{y} & \text{for } 1\leqslant y\leqslant 16, \\ \log_2 y & \text{for } 16<y<\infty. \end{cases}$$

1.17.28. *Hint.* The functions $y=x^2+2x+1$ $(x\geqslant -1)$ and $y=-1+\sqrt{x}$ $(x\geqslant 0)$ are mutually inverse, but the equation $y=x$, i. e. $x^2+2x+1=x$ has no real roots (see Problem **1.4.4**).

1.17.30. (c) *Hint.* If E is the domain of definition of the function $f(x)$, then the function $y=f[f(x)]$ is defined only for those $x\in E$ for which $f(x)\in E$. How the points of the desired graph are plotted is shown in Fig. 120.

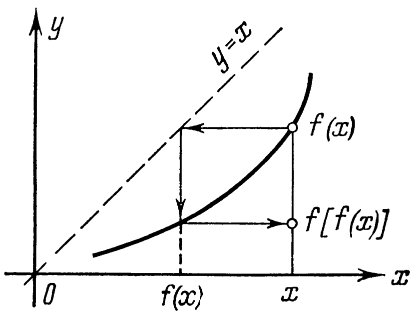

Fig. 120

1.17.32. *Hint.* The quantity $T=2(b-a)$ is a period: from the conditions of symmetry $f(a+x)=f(a-x)$ and $f(b+x)=f(b-x)$ it follows that

$$f[x+2(b-a)]=f[b+(b+x-2a)]=f(2a-x)=f[a+(a-x)]=f(x).$$

1.17.33. (a) It diverges; (b) it may either converge or diverge. Examples:

$$x_n=\frac{1}{n};\quad y_n=\frac{[1+(-1)^n]}{2};\qquad \lim_{n\to\infty}(x_ny_n)=0,$$

$$x_n=\frac{1}{n}\quad y_n=n^2;\qquad \lim_{n\to\infty}(x_ny_n)=\infty.$$

1.17.34. (a) No. Example: $x_n=n$; $y_n=-n+1$; (b) No.

1.17.35. $\alpha_n=\frac{\pi(n-2)}{n}$ $(n=3, 4, \ldots)$. **1.17.36.** *Hint.* Take into account that $\big||x_n|-|a|\big|\leqslant|x_n-a|$. The converse is incorrect. Example: $x_n=(-1)^{n+1}$.

1.17.38. *Hint.* The sequence α_n may attain only the following values: 0, 1, ..., 9. If this sequence turned out to be monotonic, then the irrational number would be represented by a periodic decimal fraction.

1.17.39. *Hint.* If the sequence $\frac{a_n}{b_n}$ increases, then

$$\frac{a_i}{b_i}<\frac{a_{n+1}}{b_{n+1}},\text{ i. e } b_{n+1}a_i<a_{n+1}b_i\ (i=1, 2, \ldots, n),$$

whence it follows that

$$b_{n+1}(a_1+a_2+\ldots+a_n) < a_{n+1}(b_1+b_2+\ldots+b_n),$$

and hence

$$\frac{a_1+a_2+\ldots+a_{n+1}}{b_1+b_2+\ldots+b_{n+1}}-\frac{a_1+a_2+\ldots+a_n}{b_1+b_2+\ldots+b_n}=$$
$$=\frac{a_{n+1}(b_1+b_2+\ldots+b_n)-b_{n+1}(a_1+a_2+\ldots+a_n)}{(b_1+b_2+\ldots+b_{n+1})(b_1+b_2+\ldots+b_n)}>0.$$

1.17.40. (a) 2; (b) 0; (c) 0. **1.17.41.** *Hint.* From the inequalities $nx-1 < E(nx) \leqslant nx$ it follows that $x-1 < x-\frac{1}{n} < \frac{E(nx)}{n} \leqslant x$.

1.17.42. *Hint.* From the inequalities

$$\sum_{k=1}^{n}(kx-1) \leqslant \sum_{k=1}^{n} E(kx) \leqslant \sum_{k=1}^{n} kx,$$

it follows that

$$x\frac{n+1}{2n}-\frac{1}{n} \leqslant \frac{1}{n^2}\sum_{k=1}^{n} E(kx) \leqslant x\frac{n+1}{2n}.$$

1.17.43. *Hint.* Take advantage of the fact that $\lim\limits_{n\to\infty} a^{\frac{1}{n}} = \lim\limits_{n\to\infty} \sqrt[n]{a} = 1$ (see Problem **1.6.19**), $\lim\limits_{n\to\infty} a^{-\frac{1}{n}} = \frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{a}} = 1$, and for $a > 1$, $|h| < \frac{1}{n}$ the inequalities $a^{-\frac{1}{n}}-1 < a^h-1 < a^{\frac{1}{n}}-1$ take place.

1.17.45. *Hint.* Divide the numerator and denominator by x^m.

1.17.46. (a) $a=1$; $b=-1$; (b) $a=1$; $b=\frac{1}{2}$. *Hint.* To find the coefficient a divide the expression by x and pass over to the limit.

1.17.47. (a)

$$f(x)=\begin{cases} 1 \text{ for } 0 \leqslant x \leqslant 1, \\ x \text{ for } x > 1. \end{cases}$$

(b)

$$f(x)=\begin{cases} 0 \text{ for } x \neq \frac{\pi}{2}+n\pi, \\ 1 \text{ for } x=\frac{\pi}{2}+n\pi \end{cases} \quad (n=0, \pm 1, \pm 2, \ldots).$$

1.17.48. *Hint.* Take advantage of the identity

$$(1-x)(1+x)(1+x^2)\ldots(1+x^{2^n})=1-x^{2^n}.$$

1.17.49. Generally speaking, one can't. For example,

$$\lim_{x\to 0}\frac{\ln(1+x)+\ln(1-x)}{x^2}=\lim_{x\to 0}\frac{\ln(1-x^2)}{x^2}=-1,$$

and if we replace $\ln(1+x)$ by x and $\ln(1-x)$ by $-x$ we will get the wrong result: $\lim\limits_{x\to 0}\frac{x-x}{x^2}=0$.

1.17.50. $\frac{1}{2}$. *Hint.* If α is a central angle subtended by the arc under consideration, then the chord is equal to $2R\sin\frac{\alpha}{2}\sim R\alpha$, and the sagitta to $R(1-\cos\alpha)\sim R\frac{\alpha^2}{2}$.

1.17.51. 2. *Hint.* The difference of the perimeters of a circumscribed and inscribed regular n-gons is equal to

$$2Rn\left(\tan\frac{\pi}{n}-\sin\frac{\pi}{n}\right)=2\pi R\,\frac{\tan\alpha-\sin\alpha}{\alpha}\sim\pi R\alpha^2,$$

where $\alpha=\frac{\pi}{n}$, and the side of an inscribed n-gon is

$$2R\sin\frac{\pi}{n}=2R\sin\alpha\sim 2R\alpha.$$

1.17.52. On the equivalence of $(1+\alpha)^3-1$ and 3α as $\alpha\to 0$.

1.17.53. No, $\log(1+x)=\frac{\ln(1+x)}{\ln 10}\sim\frac{x}{\ln 10}$ as $x\to 0$.

1.17.54. (a) Yes. *Hint.* If the function $\varphi(x)=f(x)+g(x)$ is continuous at the point $x=x_0$, then the function $g(x)=\varphi(x)-f(x)$ is also continuous at this point; (b) No. Example: $f(x)=-g(x)=\operatorname{sign}x$ (see Problem **1.5.11** (p)); both functions are discontinuous at the point $x=0$, and their sum is identically equal to zero, and is, hence, continuous.

1.17.55. (a) No. Example: $f(x)=x$ is continuous everywhere, and $g(x)=\sin\frac{\pi}{x}$ for $x\neq 0$, $g(0)=0$ being discontinuous at the point $x=0$. The product of these functions is a function continuous at $x=0$ since $\lim\limits_{x\to 0}x\sin\frac{\pi}{x}=0$; (b) No. Example: $f(x)=-g(x)=\begin{cases}1 & \text{for } x\geqslant 0,\\ -1 & \text{for } x<0;\end{cases}$ both functions are discontinuous at the point $x=0$, their product $f(x)\,g(x)\equiv -1$ being continuous everywhere.

1.17.56. No. Example: $f(x)=\begin{cases}1 & \text{if } x \text{ is rational},\\ -1 & \text{if } x \text{ is irrational}.\end{cases}$ We may write $f(x)=2\lambda(x)-1$, where $\lambda(x)$ is the Dirichlet function (see Problem **1.14.4** (b)).

1.17.57. (a) $x=0$ is a discontinuity of the second kind, $x=1$ is a discontinuity of the first kind; (b) $x=1$ is a discontinuity of the first kind: $f(1-0)=0$, $f(1+0)=1$; (c) $\varphi(x)$ is discontinuous at all points except $x=0$.

1.17.58. (a) $x=n=0,\ \pm 1,\ \pm 2,\ \ldots$ are discontinuities of the first kind: $\lim\limits_{x\to n-0}y=1$, $\lim\limits_{x\to n+0}y=y\,|_{x=n}=0$. The function has a period of 1; (b) $x=\pm\sqrt{n}$ $(n=\pm 1,\ \pm 2,\ \ldots)$ are points of discontinuity of the first kind:

$$\lim_{x\to\sqrt{n}-0}y=2n-1;\qquad \lim_{x\to\sqrt{n}+0}y=y\,|_{x=\sqrt{n}}=2n.$$

The function is even; (c) $x=\pm\sqrt{n}$ $(n=\pm 1,\ \pm 2,\ \ldots)$ are the points of discontinuity of the first kind; at these points the function passes over from the value 1 to -1 and returns to 1. The function is even;

(d)

$$y=\begin{cases} x & \text{if } |\sin x| < \frac{1}{2}, \text{ i.e. } -\frac{\pi}{6}+\pi n < x < \frac{\pi}{6}+\pi n, \\ \frac{x}{2} & \text{if } |\sin x| = \frac{1}{2}, \text{ i.e. } x=\pm\frac{\pi}{6}+\pi n, \\ 0 & \text{if } |\sin x| > \frac{1}{2}, \text{ i.e. } \frac{\pi}{6}+\pi n < x < \frac{5\pi}{6}+\pi n. \end{cases}$$

$x=\pm\frac{\pi}{6}+\pi n$ are discontinuities of the first kind.

1.17.59. The function $f[g(x)]$ has discontinuities of the first kind at the points $x=-1;\ 0;\ +1$. The function $g[f(x)]$ is continuous everywhere. *Hint.* The function $f(u)$ is discontinuous at $u=0$, and the function $g(x)$ changes sign at the points $x=0,\ \pm 1$. The function $g[f(x)]\equiv 0$, since $f(x)$ attains only the values $0,\ \pm 1$.

1.17.61. *Hint.* Write the function in the form

$$f(x)=\begin{cases} x+1 & \text{for } -2\leqslant x<0, \\ 0 & \text{for } x=0, \\ (x+1)\,2^{-\frac{2}{x}} & \text{for } 0<x\leqslant 2. \end{cases}$$

Make sure that the function increases from -1 to 1 on the interval $[-2, 0)$ and from 0 to $\frac{3}{2}$ on the interval $[0, 2]$. Apply the intermediate value theorem to the intervals $[-2, -1]$ and $[0, 2]$. The function is discontinuous at the point $x=0$: $f(-0)=1,\ f(+0)=0$.

1.17.62. *Hint.* Suppose $\varepsilon>0$ is given and the point $x_0\in[a, b]$ is chosen. We may consider that

$$\varepsilon\leqslant\min\,[f(x_0)-f(a),\quad f(b)-f(x_0)].$$

Choose the points x_1 and x_2, $x_1<x_0<x_2$ so that

$$f(x_1)=f(x_0)-\varepsilon,\quad f(x_2)=f(x_0)+\varepsilon,$$

and put $\delta=\min\,(x_0-x_1,\ x_2-x_0)$.

1.17.63. *Hint.* Apply the intermediate value theorem to the function $g(x)=f(x)-x$.

1.17.64. *Hint.* Apply the intermediate value theorem to the function $f(x)$ on the interval $[x_1, x_n]$, noting that

$$\min\,[f(x_1), \ldots, f(x_n)]\leqslant\frac{1}{n}\,[f(x_1)+f(x_2)+\ldots+f(x_n)]\leqslant\max\,[f(x_1), \ldots, f(x_n)].$$

1.17.65. *Hint.* Apply the intermediate value theorem to the function $g(x)=2^x-\frac{1}{x}$ on the interval $\left[\frac{1}{4}, 1\right]$.

1.17.66. *Hint.* At sufficiently large values of the independent variable the values of the polynomial of an even degree have the same sign as the coefficient at the superior power of x; therefore the polynomial changes sign at least twice.

1.17.67. *Hint.* The inverse function

$$x=\begin{cases} -\sqrt{-y-1} & \text{for } y<-1, \\ 0 & \text{for } y=0, \\ \sqrt{y-1} & \text{for } y>1 \end{cases}$$

is continuous in the intervals $(-\infty, -1)$ and $(1, \infty)$ and has one isolated point $y=0$.

Chapter II

2.1.1. (b) $-\frac{20}{21}$. **2.1.2.** (b) $y'=10x-2$. **2.1.5.** $v_{av}=25$ m/sec. **2.1.6.** (a) $y'=3x^2$; (b) $y'=-\frac{2}{x^3}$. **2.1.7.** The function is non-differentiable at the indicated points. **2.2.1.** (b) $y'=-\frac{2}{3}ax^{-\frac{5}{3}}+\frac{4}{3}bx^{-\frac{7}{3}}$. **2.2.2.** (c) $y'=2x\arctan x+1$. **2.2.3.** (b) -9000. **2.2.4.** (a) $y'=6x^2+3$; (b) $y'=\frac{1}{2\sqrt{x}}-\frac{1}{2x\sqrt{x}}+x^9$; (c) $y'=\frac{-3x^2+2x+2}{(x^2-x+1)^2}$; (d) $y'=-\frac{3\sqrt{x}+8\sqrt{x}+2\sqrt{1/x}}{6\left(x-2\sqrt[3]{x}\right)}$; (e) $y'=\frac{\cos\varphi-\sin\varphi-1}{(1-\cos\varphi)^2}$; (f) $y'=2e^x+\frac{1}{x}$; (g) $y'=2e^x\cos x$; (h) $y'=\frac{x(\cos x-\sin x)-\sin x-e^x}{x^2e^x}$.

2.2.5. (f) $30\ln^4(\tan^3 x)\frac{1}{\sin 6x}$; (g) $\sin\frac{2}{\sqrt{1-x}}\cdot\frac{1}{2(1-x)^{\frac{3}{2}}}$.

2.2.6. (b) $y'=-3(3-\sin x)^2\cos x$; (c) $y'=\frac{2\cos x}{3\sin x\sqrt[3]{\sin^2 x}}+\frac{2\sin x}{\cos^3 x}$;

(d) $y'=\frac{2e^x+2^x\ln 2}{3\sqrt[3]{(2e^x-2^x+1)^2}}+\frac{5\ln^4 x}{x}$; (e) $y'=3\cos 3x-\frac{1}{5}\sin\frac{x}{5}+\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}$; (f) $y'=(2x-5)\cos(x^2-5x+1)-\frac{a}{x^2}\sec^2\frac{a}{x}$; (h) $y'=\frac{1}{x\sqrt{1+\ln^2 x}}+\frac{1}{\arctan x}+\frac{1}{1+x^2}$; (i) $y'=2\ln\arctan\frac{x}{3}\cdot\frac{1}{\arctan\frac{x}{3}}\cdot\frac{3}{9+x^2}$.

2.2.8. (b) $y'=-\frac{1}{\sinh^2(\tan x)}\sec^2 x+\frac{1}{\cosh^2(\cot x)}\operatorname{cosec}^2 x$; (d) $y'=3x\times(x\sinh 2x^3+\cosh x^2\cdot\sinh 2x^2)$; (e) $y'=e^{\sinh ax}e^{bx}(a\cosh ax+b)$.

2.2.9. (c) $y'=\sqrt[3]{x^2}\frac{1-x}{1+x^2}\sin^3 x\cos^2 x\left(\frac{2}{3x}-\frac{1}{1-x}-\frac{2x}{1+x^2}+3\cot x-2\tan x\right)$;

(d) $y'=(\tan x)^{\frac{(x+1)}{2}}\left(\frac{1}{2}\ln\tan x+\frac{x+1}{\sin 2x}\right)$.

2.2.13. (a) $f'(x)=\frac{1}{2}\left(\cosh\frac{x}{2}+\sinh\frac{x}{2}\right)$; (b) $f'(x)=\tanh x$; (c) $f'(x)=\sqrt{\cosh x+1}$; (d) $f'(x)=\frac{1}{\cosh x}$; (e) $f'(x)=4\sinh 4x$; (f) $f'(x)=(a+b)e^{ax}\times(\cosh bx+\sinh bx)=(a+b)e^{(a+b)x}$.

2.2.14. (a) $y'=(\cos x)^{\sin x}(\cos x\ln\cos x-\tan x\sin x)$;

(b) $y'=\frac{\cos 3x}{\sqrt[3]{\sin^2 3x(1-\sin 3x)^4}}$;

(c) $y'=\frac{5x^2+x-24}{3(x-1)^{\frac{1}{2}}(x+2)^{\frac{5}{3}}(x+3)^{\frac{5}{2}}}$.

2.2.17. (a) $y'=\frac{\ln 3}{\sqrt{81^x-1}}\cdot\frac{\tan\sqrt{\arcsin 3^{-2x}}}{\sqrt{\arcsin 3^{-2x}}}$;

(b) $y'=-\frac{\sin \ln^3 x\cdot\ln^2 x}{5x\sqrt[5]{\cos^4\ln^3 x}\left(1+\sqrt[5]{\cos^2\ln^3 x}\right)\sqrt[3]{\left(\arctan\sqrt[5]{\cos\ln^3 x}\right)^2}}$.

2.3.1. (b) $k^n e^{kx}$; (e) $2^{n-1}\sin\left(2x+n\frac{\pi}{2}\right)$; (f) $\frac{1}{4}\sin\left(x+n\frac{\pi}{2}\right)+$ $+\frac{3^n}{2}\sin\left(3x+n\frac{\pi}{2}\right)+\frac{5^n}{4}\sin\left(5x+n\frac{\pi}{2}\right)$.

2.3.4. (b) $e^x(x^2+48x+551)$; (c) $e^{\alpha x}\left\{\sin\beta x\left[\alpha^n-\frac{n(n-1)}{1\cdot 2}\alpha^{n-2}\beta^2+\ldots\right]+\right.$ $\left.+\cos\beta x\left[n\alpha^{n-1}\beta-\frac{n(n-1)(n-2)}{1\times 2\times 3}\alpha^{n-3}\beta^3+\ldots\right]\right\}$.

2.3.6. (a) $\frac{2x^2+3x}{(1+x^2)\sqrt{1+x^2}}$; (b) $\frac{(1+2x^2)\arcsin x}{(1-x^2)^{\frac{3}{2}}}+\frac{3x}{(1-x^2)^2}$; (c) $2e^{-x^2}\times$ $\times(2x^2-1)$.

2.3.8. (a) $x^3\sin x-60x^2\cos x-1140x\sin x+8640\cos x$; (b) $2e^{-x}\times$ $\times(\sin x+\cos x)$; (c) $e^x[3x^2+6nx+3n(n-1)-4]$; (d) $(-1)^n[(4n^2+2n+$ $+1-x^2)\cos x-4nx\sin x]$.

2.3.9. (a) $100!\left[\frac{1}{(x-2)^{101}}-\frac{1}{(x-1)^{101}}\right]$; (b) $\frac{1\times 3\times 5\times\ldots\times 197\times(399-x)}{2^{100}(1-x)^{\frac{201}{2}}}$.

Hint. $y=2(1-x)^{-\frac{1}{2}}-(1-x)^{\frac{1}{2}}$.

2.4.1. (b) $x''_{yy}=-\frac{4\cos x}{(6+\sin x)^3}$.

2.4.3. (b) $y'_x=-\cot\frac{k-1}{2}t$; (d) $y'_x=-2e^{-2ct}$.

2.4.4. (b) $y''_{xx}=\frac{4t}{3(t^2+1)^3}$; (c) $y''_{xx}=\frac{1}{at\cos^3 t}$.

2.4.5. (b) $y'''_{xxx}=-3\sin t\sec^2 t$.

2.4.6. (b) $y'_x=\frac{y}{x}+e^{-\frac{y}{x}}$; (c) $y'_x=\frac{2-x}{y-5}$; (d) $y'_x=-\sqrt[3]{\frac{y}{x}}$.

2.4.7. (b) $y''_{xx}=\frac{(e^x-e^y)(1-e^{x+y})}{(1+e^y)^3}$; (c) $y''_{xx}=\frac{4e^{x-y}}{(e^{x-y}+1)^3}=\frac{4(x+y)}{(x+y+1)^3}$.

2.4.9. (a) $\frac{2a-2x-y}{x+2y-2a}$; (b) $\frac{x+y}{x-y}$; (c) $-\frac{e^x\sin y+e^{-y}\sin x}{e^x\cos y+e^{-y}\cos x}$; (d) $-\frac{1}{e}$.

2.4.10. (a) $-\frac{2y^2+2}{y^5}$; (b) $\frac{111}{256}$. **2.4.11.** (a) $-\frac{c\sin t}{a(b+\cos t)}$; (b) $\frac{t}{2}$;

(c) $\frac{t^2+1}{4t^3}$; (d) $-\frac{e^{t^2}}{2t(2t^2+2t+1)}$; (e) $\frac{(a\cos t-b\sin t)\cos^3\frac{t}{2}}{4\sin\frac{t}{2}}$;

(f) $-\sqrt{\frac{1-4t^2}{2-t^2}}$; (g) $-\sqrt{1-t^2}$.

2.5.1. (b) $6x+2y-9=0$; $2x-6y+37=0$.

2.5.2. (c) $M_1\left(-\frac{2}{\sqrt{3}},\ 5+\frac{10}{3\sqrt{3}}\right)$, $M_2\left(\frac{2}{\sqrt{3}},\ 5-\frac{10}{3\sqrt{3}}\right)$.

2.5.3. (b) $\varphi=\arctan 2\sqrt{2}$. **2.5.8.** (b) $x+y-2=0$; $y=x$.

2.5.15. (a) $\frac{\pi}{4}$; (b) $y=1$, $x+2y-2=0$; (c) $y+\frac{39}{16}=-\frac{2}{3}\left(x+\frac{5}{4}\right)$; (d) $\frac{\pi}{4}$. **2.5.16.** 11.

2.5.17. 26,450. **2.5.19.** $s=at-\frac{gt^2}{2}$; $v=a-gt$; $s_{\max}=s\Big|_{t=\frac{a}{g}}=\frac{a^2}{2g}$.

2.5.20. $v=r'_t=\frac{2\pi a\varepsilon}{P}\sin M\,(1+2\varepsilon\cos M)$. **2.6.3.** $\Delta y\approx dy=0.05$.

2.6.5. (b) $\log 10.21\approx 1.009$; (d) $\cot 45°10'\approx 0.9942$.

2.6.7. (c) $\Delta_y=|\cos x|\,\Delta_x$; (d) $\Delta_y=(1+\tan^2 x)\,\Delta_x$.

2.6.9. (a) $d^2y=4^{-x^2}\,2\ln 4\,(2x^2\ln 4-1)\,dx^2$; (b) $d^2y=\frac{4\ln x-4-\ln^3 x}{x^2\sqrt{(\ln^2 x-4)^3}}\,dx^2$; (c) $d^3y=-4\sin 2x\,dx^3$.

2.6.10. (a) $d^2y=-\frac{4(1+3x^4)}{(1-x^4)^2}\,dx^2$; (b) $d^2y=-\frac{4(1+3x^4)}{(1-x^4)^2}\,dx^2-\frac{4x}{1-x^4}\,dx^2$; in particular at $x=\tan t$, $d^2y=-\frac{4}{\cos^2 2t}\,dt^2$.

2.6.11. $\Delta V=4\pi r^2\,\Delta r+4\pi r\,\Delta r^2+\frac{4}{3}\pi\Delta r^3$ is the volume contained between two spheres of radii r and $r+\Delta r$; $dV=4\pi r^2\,\Delta r$ is the volume of a thin layer with a base area equal to the sphere's surface area $4\pi r^2$ and a height Δr.

2.6.12. $\Delta s=gt\,\Delta t+\frac{1}{2}g\,\Delta t^2$ is the distance covered by a body within the time Δt; $ds=gt\,\Delta t=v\,dt$ is the distance covered by a body which would move at a velocity $v=gt$ during the entire interval of time.

2.7.1. (a) It does not exist; (b) it exists and equals zero.

2.7.2. 90°. *Hint.* Since

$$y=\begin{cases} e^x, & x\geqslant 0\\ e^{-x}, & x<0,\end{cases}$$

$f'_-(0)=-1$, $f'_+(0)=1$.

2.7.3. $f'_-(a)=-\varphi(a)$; $f'_+(a)=\varphi(a)$.

2.7.4. *Hint.* For $x\neq 0$ the derivative

$$f'(x)=-\cos\left(\frac{1}{x}\right)+2x\sin\left(\frac{1}{x}\right).$$

At $x=0$ the derivative equals zero:

$$f'(0)=\lim_{\Delta x\to 0}\frac{\Delta x^2\sin\frac{1}{\Delta x}}{\Delta x}=0.$$

Thus, the derivative $f'(x)$ exists for all x, but has a discontinuity of the second kind at the point $x=0$.

2.7.5. $a=2x_0$, $b=-x_0^2$. **2.7.7.** *Hint.* The formula for the sum of a geometric progression represents an identity with respect to x. Equating the derivatives of both sides of the identity, we get

$$1+2x+3x^2+\ldots+nx^{n-1}=\frac{nx^{n+1}-(n+1)\,x^n+1}{(1-x)^2};$$

multiplying both sides of this equality by x and differentiating again, we get

$$1^2+2^2x+\ldots+n^2x^{n-1}=\frac{1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-nx^{n+2}}{(1-x)^3}.$$

2.7.8. $\sin x+3\sin 3x+\ldots+(2n-1)\sin(2n-1)x=$

$$=\frac{(2n+1)\sin(2n-1)x-(2n-1)\sin(2n+1)x}{4\sin^2 x}.$$

Hint. To prove the identity multiply its left side by $2\sin x$ and apply the formula $2\sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta)$. To deduce the desired formula differentiate both sides of the identity and equate the derivatives.

2.7.9. (a) $\sin 2x\,[f'(\sin^2 x)-f'(\cos^2 x)]$; (b) $e^{f(x)}[e^x f'(e^x)+f'(x)f(e^x)]$;

(c) $\dfrac{\psi'(x)}{\psi(x)}\cdot\dfrac{1}{\ln\varphi(x)}-\dfrac{\varphi'(x)}{\varphi(x)}\cdot\dfrac{\ln\psi(x)}{\ln^2\varphi(x)}$.

2.7.10. (a) No; (b) No; (c) Yes; (d) No.

2.7.11. *Hint*. Differentiate the identity $f(-x)=f(x)$ or $f(-x)=-f(x)$. This fact is easily illustrated geometrically if we take into consideration that the graph of the even function is symmetrical about the y-axis, and the graph of the odd function about the origin.

2.7.12. *Hint*. Differentiate the identity $f(x+T)=f(x)$.

2.7.13. $F'(x)=6x^2$. **2.7.14.** $y'=2|x|$. **2.7.15.** The composite function $f[\varphi(x)]$ may be non-differentiable only at points where $\varphi'(x)$ does not exist and where $\varphi(x)$ attains such values of $\varphi(x)=u$ at which $f'(u)$ does not exist. But the function $y=u^2=|x|^2$ has a derivative $y'=0$ at the point $x=0$, though at this point the function $u=|x|$ has no derivative.

2.7.16. (a) $y''=6|x|$; (b) $y''=2\sin\dfrac{1}{x}-\dfrac{2}{x}\cos\dfrac{1}{x}-\dfrac{1}{x^2}\sin\dfrac{1}{x}$ at $x\neq 0$, $y''(0)$ does not exist, since $y'(x)$ is discontinuous at $x=0$.

2.7.17. *Hint*. (a) Verify that $f^{(k)}\dfrac{1}{k!}=C_n^k$ $(k=0, 1, \ldots, n)$ and take advantage of the property of the binomial coefficients. (b) Designate: $f(x)=u_n$; show that $u'_n=(n-1)u_{n-1}-u_{n-2}$ and use the method of mathematical induction.

2.7.18. *Hint*. Apply the Leibniz formula for the nth derivative of the product of the functions $u=e^{-\frac{x}{a}}$ and $v=x^2$.

2.7.19. $y^{(n)}(0)=\begin{cases}0 & \text{at } n=2k\\ [1\times 3\times\ldots\times(2k-1)]^2 & \text{at } n=2k+1\\ & (k=1, 2, \ldots).\end{cases}$

Hint. Differentiate the identity $n-2$ times and, putting $x=0$, obtain

$$y^{(n)}(0)=(n-2)^2y^{(n-2)}(0)\quad(n\geqslant 2).$$

2.7.21. *Hint*. Take advantage of the definition

$$e^{-x^2}H_{n+1}(x)=(e^{-x^2})^{(n+1)}=(-2xe^{-x^2})^{(n)}$$

and the Leibniz formula for the nth derivative of the product $u=e^{-x^2}$ and $v=-2x$. **2.7.22.** $y'_x=\dfrac{1}{3(y^2+1)}$.

2.7.23. $x_{1,2}=\pm\sqrt{1+\sqrt{1-y}}\quad(-\infty<y\leqslant 1)$,

$x_{3,4}=\pm\sqrt{1-\sqrt{1-y}}\quad(0\leqslant y\leqslant 1)$,

$x'_i=\dfrac{1}{4x_i(1-x_i^2)}\quad(i=1, 2, 3, 4)$ for $x_i\neq 0, \pm 1$.

Hint. Solve the biquadratic equation $x^4-2x^2+y=0$ and find the domains of definition of the obtained functions $x_i(y)$.

2.7.25. (a) $x_1=-3$; $x_2=1$; (b) $x=\pm 1$.

2.7.26. *Hint*. Note that the function $x=2t-|t|=\begin{cases} t, & t\geqslant 0, \\ 3t, & t<0 \end{cases}$ has no derivative at $t=0$. But $t=\begin{cases} x, & x\geqslant 0, \\ \frac{x}{3}, & x<0, \end{cases}$ therefore we can express $y=t^2+t|t|=$ $=\begin{cases} 2t^2, & t\geqslant 0, \\ 0, & t<0 \end{cases}$ through x: $y=\begin{cases} 2x^2, & x\geqslant 0, \\ 0, & x<0. \end{cases}$ This function is differentiable everywhere. **2.7.27.** $a=c=\frac{1}{4}$; $b=\frac{1}{2}$. **2.7.28.** *Hint*. The curves intersect at the points where $\sin ax=1$. Since at these points $\cos ax=0$,

$$y_2'=f'(x)\sin ax+f(x)a\cos ax=f'(x)=y_1',$$

i.e. the curves are tangent.

2.7.30. *Hint*. For $t\neq \pi n$ the equations of the tangent and the normal are reduced to the form:

$$y=\cot\frac{t}{2}(x-at)+2a;\quad y=-\tan\frac{t}{2}(x-at),$$

respectively. For $t=\pi(2k-1)$ $(k=1, 2, \ldots)$ the tangent line $(y=2a)$ touches the circle at the highest point, and the normal $(x=at)$ passes through the highest and lowest points; for $t=2k\pi$ $(k=0, 1, \ldots)$ the tangent line $(x=at)$ passes through both points, and the normal $(y=0)$ touches the circle at the lowest point. **2.7.34.** $\frac{d^2y}{dt^2}+y$. **2.7.35.** The relative error $\delta=\frac{\Delta l}{l}\approx\frac{2d\varphi}{\sin 2\varphi}$. The most reliable result, i.e. the result with the least relative error, corresponds to the value $\varphi=45°$.

Chapter III

3.1.2. (b) Yes; (c) No, since the derivative is non-existent at the point 0. **3.1.5.** $\xi=e-1$. **3.1.7.** No, since $g(-3)=g(3)$. **3.1.9.** (d) *Hint*. Consider the functions

$$f(x)=\arcsin\frac{2x}{1+x^2}+2\arctan x \quad \text{for } |x|>1,$$

$$g(x)=\arcsin\frac{2x}{1+x^2}-2\arctan x \quad \text{for } |x|<1.$$

3.1.15. (a) $\xi=\frac{7}{2}$; (b) $\xi=\frac{2}{\ln 3}$; (c) $\xi=\frac{10\pm\sqrt{52}}{24}$; (d) it is not applicable, since the function has no derivative at the point $x=0$.

3.1.16. $1.26<\ln(1+e)<1.37$. *Hint*. Write the Lagrange formula for the function $f(x)=\ln x$ on the interval $[e, e+1]$ and estimate the right-hand side in the obtained relation: $\ln(1+e)=1+\frac{1}{\xi}$ $(e<\xi<e+1)$.

3.1.17. *Hint*. Apply the Lagrange formula to the function $f(x)=\ln x$ on the interval $[1, 1+x]$, $x>0$, and estimate the right-hand side in the obtained relation $\ln(1+x)=\frac{x}{\xi}$ $(1<\xi<1+x)$. **3.2.1.** (c) 2; (d) 0; (f) $-\frac{1}{2}$.

3.2.3. (b) 0. *Hint*. Represent $\cot x-\frac{1}{x}=\frac{x-\tan x}{x\tan x}$; (c) $\frac{1}{2}$. **3.2.5.** (b) $e^1=e$.

3.2.6. (a) 1; (b) 1. **3.2.9.** (a) $\frac{4}{7}$; (b) $\ln a-1$; (c) 2; (d) $\frac{\pi\sqrt{3}}{6}$; (e) $\frac{1}{a}$; (f) 0; (g) 1; (h) $\ln a$; (i) $e^{-m^2\frac{n}{2}}$; (j) $\frac{2}{\pi}$; (k) -1; (l) e; (m) $\frac{2}{3}$; (n) $\frac{1}{2}$; (o) $\frac{a^2}{2}$; (p) $e^{-\frac{1}{30}}$; (q) 1; (r) $-\frac{1}{2}$. **3.3.5.** (b) 0.34201.

3.3.6. $\sqrt[4]{83}\approx 3.018350$. *Hint.* $\sqrt[4]{83}=\sqrt[4]{81+2}=3\left(1+\frac{2}{81}\right)^{\frac{1}{4}}$. Apply the binomial formula and retain four terms.

3.3.7. *Hints.* (b) Write the Maclaurin formula for the function $f(x)=\tan x$ with the remainder $R_4(x)$; (c) write the Maclaurin formula for the function $f(x)=(1+x)^{\frac{1}{2}}$ with remainders $R_2(x)$ and $R_3(x)$.

3.4.2. (a) $f(x)=\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{5}x^5+o(x^5)$; (b) $f(x)=x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\frac{x^5}{24}+o(x^5)$.

3.4.3. (b) $-\frac{1}{2}$; (c) $-\frac{1}{12}$; (d) $\frac{1}{3}$; (e) 1.

3.4.4. (a) $1+2x+x^2-\frac{2}{3}x^3-\frac{5}{6}x^4-\frac{1}{15}x^5$; (b) $-\frac{x^2}{2}-\frac{x^4}{12}+\frac{x^6}{45}$; (c) $1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^4}{720}$.

3.5.1. (d) The function decreases on the interval $(-\infty, 0)$ and increases on $(0, \infty)$; (e) the function increases on the intervals $\left(-\infty, \frac{1}{2}\right)$ and $(3, +\infty)$ and decreases on $\left(\frac{1}{2}, 3\right)$; (f) the function increases over the entire number scale.

3.5.2. (b) The function increases on the intervals $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right)$ and decreases on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

3.5.8. (a) The function increases throughout the number scale; (b) the function increases on the interval $(-1, 0)$ and decreases on $(0, 1)$; (c) the function decreases throughout the number scale; (d) the function increases on both intervals $(-\infty, 0)$ and $(0, \infty)$ where it is defined; (e) the function decreases on the intervals $(0, 1)$ and $(1, e)$ and increases on $(e, +\infty)$; (f) the function decreases on the intervals $(-\infty, 1)$ and $(1, \infty)$, increases on $(-1, 1)$. **3.5.10.** $a\leqslant 0$. **3.5.11.** $b\geqslant 1$. **3.6.1.** (b) The minimum is $f(1)=f(3)=3$, the maximum $f(2)=4$; (d) the minimum $f\left(\frac{7}{5}\right)=-\frac{1}{24}$. **3.6.2.** (b) The minima are $f(\pm 1)=\sqrt{3}$; the maximum $f(0)=2$.

3.6.3. (b) The maximum is $f(-2)=160$; the minimum $f(0)=2$.

3.6.7. (b) The minimum is $f(0)=0$.

3.6.8. (b) On the interval $[0, 2\pi]$: the minimum is $f\left(\frac{\pi}{2}\right)=-4$; the maximum $f\left(\frac{3\pi}{2}\right)=4$. **3.6.10.** (a) The minimum is $f(0)=0$, the maximum $f(2)=4e^{-2}$; (b) the minimum is $f(-2)=-1$, the maximum $f(2)=1$; (c) the maximum is

$f(0)=0$, the minimum $f\left(\frac{5}{3}\right)=-\frac{25}{9}\sqrt[5]{\frac{1}{9}}$; (d) the maximum is $f(\pm 2)=-1$, the minimum $f(0)=7$; (e) the maximum is $f(-3)=3\sqrt[3]{3}$, the minimum $f(2)=-\sqrt[3]{44}$.

3.6.11. (a) There is no extremum; (b) there is no extremum; (c) the maximum is $f(0)=0$; (d) the minimum is $f(0)=0$.

3.7.1. (c) The greatest value is $f(1)=\frac{1}{e}$, the least value $f(0)=0$; (d) the greatest value is $f\left(\pm\frac{1}{2}\right)=\frac{3}{\sqrt{8}}$, the least value $f(\pm 1)=0$.

3.7.2. (b) The greatest value is $y(0)=\frac{\pi}{2}$, the least value $y\left(\pm\frac{\sqrt{2}}{2}\right)=\frac{\pi}{3}$; (c) the greatest value is $y(4)=6$, the least value $y(0)=0$.

3.7.6. (a) The greatest value is $f(-2)=\frac{16}{3}$, the least value $f(3)=-\frac{37}{4}$; (b) the greatest value is $f(0)=2$, the least value $f(\pm 2)=0$; (c) the greatest value is $f\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}+0.25\ln 3$, the least value $f(\sqrt{3})=\frac{\pi}{3}-0.25\ln 3$; (d) the greatest value is $f\left(\frac{\pi}{3}\right)=\frac{3\sqrt{3}}{2}$, the least value $f\left(\frac{3\pi}{2}\right)=-2$; (e) the greatest value is $f(1)=1$, the least value $f(2)=2(1-\ln 2)$; (f) there is no greatest value, the least value is $f(0)=1$.

3.8.3. $H=R\sqrt{2}$, where H is the height of the cylinder, R is the radius of the sphere. **3.8.7.** $x=a\sin\alpha$, $y=a\cos\alpha$, where $\alpha=0.5\ \text{arc}\tan 2$.

Hint. The problem is reduced to finding the greatest value of the function

$$S=4xy+4x(y-x)=4a^2(\sin 2\alpha-\sin^2\alpha)$$

in the interval $0<\alpha<\frac{\pi}{4}$. **3.8.8.** $P_{\max}=\frac{E^2}{4W_i}$ at $W=W_i$. **3.8.9.** $h=2R=2\sqrt[3]{\frac{3v}{2\pi}}$. **3.8.10.** The radius of the cylinder base is $r=\frac{R}{2}$, where R is the radius of the cone base. **3.8.11.** The equation of the desired straight line is $\frac{x}{2}+\frac{y}{4}=1$.

3.8.12. $x=a-p$ for $a>p$ and $x=0$ for $a\leqslant p$.

3.8.13. $v=\sqrt[3]{\frac{a}{2b}}$. *Hint*. It will take $\frac{1}{v}$ hours to cover one knot. The appropriate expenses are expressed by the formula $T=\frac{a+bv^3}{v}=\frac{a}{v}+bv^2$.

3.8.14. $\varphi=\frac{\pi}{3}$. *Hint*. At the board width a the cross-sectional area of the trough is equal to $a^2(1+\cos\varphi)\sin\varphi$, where φ is the angle of inclination of the walls to the bottom.

3.8.15. $\frac{h}{2}$. *Hint*. The point of fall of the jet is at a distance of $\frac{v\sqrt{2H}}{g}$ from the tank base, where $H=h-x$ is the height at which the orifice should be located, v is the rate of flow; therefore the length of the jet is determined by the expression

$$\sqrt{2gx}\sqrt{\frac{2(h-x)}{g}}=2\sqrt{x(h-x)}.$$

3.8.16. After $\frac{a}{2v}$ hours the least distance will be equal to $\frac{a}{2}$ km.

3.9.1. (b) The intervals of concavity are $\left(-\infty, \frac{1}{3}\right)$ and $(1, \infty)$, of convexity $\left(\frac{1}{3}, 1\right)$; the points of inflection are $\left(\frac{1}{3}, 12\frac{11}{27}\right)$, $(1, 13)$; (c) the intervals of concavity are $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$, of convexity $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$; the points of inflection are $\left(-\sqrt{3}, -\frac{\sqrt{3}}{10}\right)$, $(0, 0)$, $\left(\sqrt{3}, \frac{\sqrt{3}}{10}\right)$; (e) the curve is concave everywhere; (f) the intervals of concavity are $(0, x_1)$ and (x_2, ∞), of convexity (x_1, x_2), where $x_1 = e^{\frac{3-\sqrt{5}}{2}}$, $x_2 = e^{\frac{3+\sqrt{5}}{2}}$; the points of inflection are (x_1, y_1), (x_2, y_2), where

$$y_1 = \left(\frac{3-\sqrt{5}}{2}\right)^2 e^{\frac{\sqrt{5}-3}{2}}, \quad y_2 = \left(\frac{3+\sqrt{5}}{2}\right)^2 e^{-\frac{3+\sqrt{5}}{2}}.$$

3.9.5. (a) The point of inflection is $(3, 3)$; the curve is convex for $x < 3$ and concave for $x > 3$; (b) the abscissa of the point of inflection $x = \arcsin\frac{\sqrt{5}-1}{2}$; the curve is concave in $\left(-\frac{\pi}{2}, \arcsin\frac{\sqrt{5}-1}{2}\right)$, and convex in $\left(\arcsin\frac{\sqrt{5}-1}{2}, \frac{\pi}{2}\right)$.

3.10.1. (c) $y=0$; (d) $x=0$; (i) $y=2x$ as $x \to +\infty$ and $y=-2x$ as $x \to -\infty$. **3.10.3.** (a) $x=3$, $y=x-3$; (b) $y = \pm\frac{\pi x}{2} - 1$; (c) $y=x$; (d) $x = \pm 2$; (e) $y = 2x - \frac{\pi}{2}$.

3.11.2. (a) The function is defined everywhere, it is even. The graph is symmetrical about the y-axis and has no asymptotes. The minimum is $y(0)=1$, maxima $y(1)=y(-1)=\frac{3}{2}$. The points of inflection are $\left(\pm\frac{\sqrt{3}}{3}, \frac{23}{18}\right)$; (b) the function is defined in $(-\infty, -1)$ and $(-1, +\infty)$. The graph has a vertical asymptote $x=-1$ and an inclined asymptote $y=x-3$. The minimum is $y(0)=0$, maximum $y(-4) = -\frac{256}{27}$. The points of inflection are $\left(-6, -\frac{3296}{125}\right)$ and $\left(2, \frac{16}{27}\right)$; (c) the function is defined in $(-\infty, 0)$ and $(0, +\infty)$. The graph has a vertical asymptote $x=0$. The minimum is $y\left(\frac{1}{2}\right)=3$. The point of inflection is $\left(-\frac{\sqrt[3]{2}}{2}, 0\right)$; (d) the function is defined in the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$; it is odd. The graph is symmetrical about the origin, has two vertical asymptotes $x = \pm 1$ and an inclined asymptote $y=x$. The minimum is $y(\sqrt{3}) = +3\frac{\sqrt{3}}{2}$, the maximum $y(-\sqrt{3}) = -\frac{3\sqrt{3}}{2}$. The point of inflection is $(0, 0)$; (e) the function is defined everywhere, it is even. The graph is symmetrical about the y-axis and has a horizontal asymptote $y=0$. The minimum is $y(0)=\sqrt[3]{4}$, the maxima $y(\pm\sqrt{2})=2\sqrt[3]{2}$. The points of inflection are $(\pm 2, \sqrt[3]{4})$; (f) the function is defined in $(-2, +\infty)$. The vertical asymptote is $x=-2$. The minimum is $y(0)=0$, the maximum $y(-0.73)\approx 0.12$. The point of inflection is $(-0.37; 0.075)$; (g) the function is defined everywhere. The horizontal asymptote is $y=0$ as $x\to+\infty$. The maximum is $y\left(\frac{3}{4}\right)=\left(\frac{3}{4e}\right)^3$. The points of inflection are $(0, 0)$,

$$\left(\frac{3-\sqrt{3}}{4}, \left(\frac{3-\sqrt{3}}{4}\right)^3 e^{\sqrt{3}-3}\right), \left(\frac{3+\sqrt{3}}{4}, \left(\frac{3+\sqrt{3}}{4}\right)^3 e^{-3-\sqrt{3}}\right);$$

(h) the function is defined and continuous everywhere. The horizontal asymptote is $y=1$. The minimum is $y(0)=0$, the point $(0, 0)$ being a corner point on the graph: $y'_-(0)=-\frac{\pi}{2}$, $y'_+(0)=+\frac{\pi}{2}$.

3.12.6. 4.4934. **3.12.8.** $x_1=-2.330$; $x_2=0.202$; $x_3=2.128$. **3.12.11.** 0.6705. **3.12.12.** (a) 0.27; 2.25; (b) 0.21. **3.12.13.** (a) 1.17; (b) 3.07. **3.12.14.** 1.325. **3.12.15.** 0.5896 and 2.2805. *Hint.* To approximate the smaller root more precisely write the equation in the form $x=e^{0.8x-1}$, to find a more accurate value of the larger root represent it in the form $x=1.25(1+\ln x)$.

3.13.1. No. *Hint.* Show that at the point $x=1$ the derivative is non-existent: $f'_-(1)=1$; $f'_+(1)=-1$.

3.13.2. *Hint.* Check the equality $f(b)-f(a)=(b-a)f'\left(\frac{a+b}{2}\right)$.

3.13.3. *Hint.* Apply the Rolle theorem to the function $f(x)=a_0x^n+\ldots$ $\ldots+a_{n-1}(x)$ on the interval $[0, x_0]$.

3.13.4. *Hint.* Make sure that the derivative $f'(x)=4(x^3-1)$ has only one real root, $x=1$, and apply the Rolle theorem.

3.13.5. *Hint.* The derivative $f'(x)=nx^{n-1}+p$ has only one real root at an even n and not more than two real roots at an odd n.

3.13.6. *Hint.* The derivative is a polynomial of the third degree and has three roots. Take advantage of the fact that between the roots of the polynomial lies the root of its derivative.

3.13.7. *Hint.* From the correct equality $\lim\limits_{x\to 0}\cos\frac{1}{\xi}=0$ $(0<\xi<x)$, where ξ is determined from the mean value theorem, it does not follow that $\lim\limits_{x\to 0}\cos\frac{1}{x}=0$, since it cannot be asserted that the variable ξ attains all intermediate values in the neighbourhood of zero as $x\to 0$. Moreover, ξ takes on only such a sequence of values E for which $\lim\cos\frac{1}{\xi}=0$ $(\xi\in E)$.

3.13.8. *Hint.* The mistake is that in the Lagrange formula one and the same point ξ is taken for $f(x)$ and $\varphi(x)$.

3.13.9. *Hint.* Apply the Lagrange formula to the function $\ln x$ on the interval $[b, a]$; (b) apply the Lagrange formula to the function z^p on the interval $[y, x]$.

3.13.10. *Hint.* With the aid of the Leibniz formula ascertain that the coefficients of the Chebyshev-Laguerre polynomial alternate in sign, the odd powers of x having negative coefficients. Whence deduce that $L_n(x)>0$ for $x<0$.

3.13.11. *Hint.* Using the Rolle theorem, show that inside the interval $[x_0, x_n]$ there are at least n roots of the first derivative, $n-1$ roots of the second derivative, and so on.

3.13.12. *Hint.* The L'Hospital rule is not applicable here, since the derivatives of both the numerator and denominator vanish at all points where the factor $\sin x$ (which we cancelled in computing the limit of the ratio of derivatives) vanishes.

3.13.13. *Hint.* Write the Taylor formula with the remainder R_2:

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a+\theta_1 h).$$

Comparing it with the expansion given in the problem, get the equality $\frac{f''(a+\theta h)-f''(a)}{h}=\frac{1}{3}f'''(a+\theta_1 h)$ and pass over to the limit as $h\to 0$.

3.13.14. *Hint.* Prove by using the rule of contraries. Suppose that $e=\frac{p}{q}$, where p and q are natural numbers, $p>q>1$, and, using the Taylor formula, get for $n>p$

$$\frac{p}{q}=1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}+\frac{1}{(n+1)!}\left(\frac{p}{q}\right)^\theta \quad (0<\theta<1).$$

Multiply both sides of this equality by $n!$, and noting, that $\frac{p}{q}n!$ and $\left(1+\frac{1}{1!}+\ldots+\frac{1}{n!}\right)n!$ are positive integers and $\frac{1}{n+1}\left(\frac{p}{q}\right)^\theta<\frac{1}{n+1}\cdot\frac{p}{q}<1$, obtain a contradictory result.

3.13.15. *Hint.* Verify that the function

$$f(x)=\begin{cases}\frac{\sin x}{x}, & 0<x\leqslant\frac{\pi}{2},\\ 1, & x=0\end{cases}$$ is continuous on the interval $\left[0,\frac{\pi}{2}\right]$.

Ascertain that the derivative $f'(x)<0$ is inside the interval.

3.13.16. *Hint.* Show that $f'(x)\geqslant 0$. Ascertain that

$$f(0)=1-a\begin{cases}>0 \text{ for } a<1,\\ <0 \text{ for } a>1,\end{cases}$$

and take advantage of the fact that the function increases.

3.13.17. *Hint.* Show that the function $f(x)=xe^x-2$ increases and has opposite signs at the end-points of the interval $(0, 1)$.

3.13.18. *Hint.* Show that the derivative

$$f'(x)=\frac{1}{2}+2x\sin\frac{1}{x}-\cos\frac{1}{x}\quad(x\neq 0)$$

is equal to $\frac{3}{2}$ at the points $x=\frac{1}{(2n+1)\pi}$ $(n=0, \pm 1, \pm 2, \ldots)$, and to $-\frac{1}{2}$ at the points $x=\frac{1}{2n\pi}$, i.e. the derivative changes sign in any vicinity of the origin.

3.13.19. *Hint.* Ascertain that the auxiliary function $\psi(x)=f(x)-\varphi(x)$ increases.

3.13.20. *Hint.* Make sure that at all points of the domain of definition of the function the derivative retains its sign if $ad-bc\neq 0$. But if $ad-bc=0$, i.e. $\frac{a}{c}=\frac{b}{d}$, then the function is constant. **3.13.21.** $p=-6$, $q=14$.

3.13.22. A minimum $f(x_0)=0$ if $\varphi(x_0)>0$ and n is even; a maximum $f(x_0)=0$ if $\varphi(x_0)<0$ and n is even; the point x_0 is not an extremum if n is odd. *Hint.* At an even n, in a certain neighbourhood of the point x_0 the function retains its sign and is either rigorously greater than zero or rigorously less than zero, depending on the sign of $\varphi(x_0)$. At an odd n the function changes sign in a certain neighbourhood of the point x_0.

3.13.23. *Hint.* For $x \neq 0$ $f(x) > 0$, hence $f(0)$ is a minimum. For $x > 0$ the derivative $f'(x) = 2 - \sin\frac{1}{x} + \frac{1}{x}\cos\frac{1}{x}$ is positive at the points $x = \frac{1}{2\pi n}$ and negative at the points $x = \frac{1}{(2n+1)\pi}$. The case $x < 0$ is investigated analogously. **3.13.24.** (a) 1 and 0; (b) 1 and -2.

3.13.25. (a) The least value is non-existent, the greatest value equals 1; (b) the function has neither the greatest, nor the least value.

3.13.30. Yes. *Hint.* Since $f''(x)$ changes sign when passing through the point x_0, the latter is a point of extremum for the function $f'(x)$.

3.13.31. The graph passes through the point $M(-1, 2)$ and has a tangent line $y - 2 = -(x+1)$; M is a point of inflection, the curve being concave downward to the left of the point M, and upward to the right of it. *Hint.* The function $f''(x)$ increases and changes sign when passing through $x = -1$.

3.13.32. $h = \frac{1}{\sigma\sqrt{2}}$.

3.13.33. *Hint.* According to the Rolle theorem, between the roots of the first derivative there is at least one root of the second derivative. When passing through one of these roots the second derivative must change sign.

3.13.35. *Hint.* The polynomial has the form $a_0x^{2n} + a_1x^{2n-2} + \ldots + a_{n-1}x^2 + a_n$. Polynomials of this form with positive coefficients have no real roots.

3.13.36. *Hint.* Take advantage of the fact that a polynomial of an odd degree (and, hence, also its second derivative) has at least one real root and changes sign at least once.

3.13.37. *Hint.* Find $\lim\limits_{x\to\infty}\left(\frac{2x^4 + x^3 + 1}{x^3 - 2x - 1}\right)$.

Chapter IV

4.1.2. $I = x^3 + x^2 + 0.5\ln|2x - 1| + C$.

4.1.7. $I = \frac{2}{3}(x+1)^{\frac{3}{2}} + \frac{2}{3}x^{\frac{3}{2}} + C$. *Hint.* Eliminate the irrationality from the denominator.

4.1.14. $I = \frac{1}{10}\arctan\frac{2x}{5} + C$.

4.1.15. $I = \frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C$.

4.1.18. $I = \ln\left|x + 3 + \sqrt{x^2 + 6x + 1}\right| + C$.

4.1.20. $I = \frac{1}{2\sqrt{70}}\ln\left|\frac{\sqrt{10}x - \sqrt{7}}{\sqrt{10}x + \sqrt{7}}\right| + C$.

4.1.21. (a) $\frac{1}{2}\arctan\frac{x-3}{2} + C$; (b) $\frac{3}{4}(x-4)\sqrt[3]{x} + C$; (c) $3\tan x + 2\cot x + C$; (d) $-\frac{2}{x} + \arctan x + C$.

4.1.22. (a) $\ln(x + \sqrt{1+x^2}) + \arcsin x + C$; (b) $\sin x - \cos x + C$; (c) $-\frac{2}{\ln 5}5^{-x} + \frac{1}{5\ln 2}2^{-x} + C$; (d) $-0.2\cos 5x - x\sin 5\alpha + C$.

4.2.3. $I = \frac{1}{12}\sqrt{(2x-5)^3} + \frac{5}{2}\sqrt{2x-5} - \frac{37}{4\sqrt{2x-5}} + C$.

4.2.8. $I=-2\sqrt{\cos x}+C$. **4.2.10.** $I=\frac{1}{2}(x^3+3x+1)^{\frac{2}{3}}+C$.

4.2.13. (a) $0.75\sqrt[3]{(1+\ln x)^4}+C$; (b) $\ln|\ln x|+C$; (c) $\frac{1}{2}\arcsin\frac{x^2}{\sqrt{3}}+C$; (d) $\frac{1}{na}\arctan\frac{x^n}{a}+C$; (e) $-2\cos\sqrt{x}+C$; (f) $\frac{1}{2}\ln^2 x+\ln|\ln x|+C$.

4.2.14. (a) $-\frac{3}{140}(35-40x+14x^2)(1-x)^{\frac{4}{3}}+C$;

(b) $\frac{2}{3}(\ln x-5)\sqrt{1+\ln x}+C$;

(c) $\left(\frac{2}{3}-\frac{2}{7}\sin^2 x+\frac{2}{11}\sin^4 x\right)\sqrt{\sin^3 x}+C$;

(d) $-\frac{1}{15}(8+4x^2+3x^4)\sqrt{1-x^2}+C$.

4.3.2. $x\arcsin x+\sqrt{1-x^2}+C$.

4.3.14. $-\cos x\ln\tan x+\ln\left|\tan\left(\frac{x}{2}\right)\right|+C$.

4.3.17. $x\ln(x+\sqrt{1+x^2})-\sqrt{1+x^2}+C$.

4.3.18. $\frac{3}{4}x\sqrt[3]{x}\left[(\ln x)^2-\frac{3}{2}\ln x+\frac{9}{8}\right]+C$.

4.3.19. $2\sqrt{1+x}\arcsin x+4\sqrt{1-x}+C$.

4.3.20. $-0.5\left(\frac{x}{\sin^2 x}+\cot x\right)+C$.

4.3.21. $\frac{3^x(\sin x+\cos x\ln 3)}{1+(\ln 3)^2}+C$.

4.3.22. $\left(\frac{1}{3}x^3-x^2+\frac{2}{3}x+\frac{13}{9}\right)e^{3x}+C$.

4.3.23. $(x^4-10x^2+21)\sin x+x(4x^2-20)\cos x+C$.

4.3.24. $\frac{9x^2+18x-11}{27}\cos 3x+\frac{2x+2}{9}\sin 3x+C$.

4.3.25. $\left(\frac{x^3}{3}-x^2+3x\right)\ln x-\frac{x^3}{9}+\frac{x^2}{2}-3x+C$.

4.3.26. $\frac{x^4-1}{4}\arctan x-\frac{x^3}{12}+\frac{x}{4}+C$.

4.3.27. $\frac{x^3}{3}\arccos x-\frac{2+x^2}{9}\sqrt{1-x^2}+C$.

4.3.28. (a) $-\frac{18x^2+6x-13}{72}\sin(6x+2)-\frac{6x+1}{72}\cos(6x+2)+\frac{1}{2}x^3+$ $+\frac{1}{4}x^2-x+C$; (b) $\frac{3}{4}(x^2-7x+1)(2x+1)^{\frac{2}{3}}-\frac{9}{40}(2x-7)(2x+1)^{\frac{5}{3}}+$ $+\frac{27}{320}(2x+1)^{\frac{8}{3}}+C$.

4.4.2. (d) *Hint.* Apply the generalized formula for integration by parts and express I_n from the relation thus obtained

$$I_n=\frac{e^{\alpha x}}{\alpha^2}\sin^{n-1}x(\alpha\sin x-n\cos x)+\frac{n(n-1)}{\alpha^2}I_{n-2}-\frac{n^2}{\alpha^2}I_n.$$

4.4.3. $I_n = -\dfrac{\cos x}{(n-1)\sin^{n-1} x} + \dfrac{n-2}{n-1} I_{n-2} \quad (n \geqslant 2);$

$$I_3 = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2} I_1 = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C.$$

4.4.4. (a) $I_n = \dfrac{1}{n-1}\tan^{n-1} x - I_{n-2}$; $\quad I_1 = -\ln|\cos x| + C$; $\quad I_0 = x + C$; (b) $I_n = \dfrac{1}{n-1}\cot^{n-1} x - I_{n-2}$; $\quad I_1 = \ln|\sin x| + C$; $\quad I_0 = x + C$; (c) $I_n = \dfrac{1}{n} x^{n-1}\sqrt{x^2+a} - \dfrac{n-1}{n}\alpha I_{n-2}$; $I_1 = \sqrt{x^2+a} + C$; $I_0 = \ln|x + \sqrt{x^2+a}| + C$.

Chapter V

5.1.2. $\dfrac{x^2}{2} - 2x + \dfrac{1}{6}\ln\left|\dfrac{x-1}{(x+1)^3}\right| + \dfrac{16}{3}\ln|x+2| + C.$

5.1.5. $2\ln|x-1| - \ln|x| - \dfrac{x}{(x-1)^2} + C.$

5.1.8. $\dfrac{2}{3\sqrt{7}}\arctan\dfrac{2x+1}{\sqrt{7}} - \dfrac{1}{3}\arctan(x+2) + C.$

5.1.10. $5x + \ln x^2 (x+2)^4 |x-2|^3 + C.$

5.1.11. $\dfrac{9x^2+50x+68}{4(x+2)(x+3)^2} + \dfrac{1}{8}\ln\left|\dfrac{(x+1)(x+2)^{16}}{(x+3)^{17}}\right| + C.$

5.1.12. $-\dfrac{1}{x-2} - \arctan(x-2) + C.$

5.1.13. $-\dfrac{1}{6(1+x)} + \dfrac{1}{6}\ln\dfrac{(1+x)^2}{1-x+x^2} + \dfrac{1}{2}\arctan x -$

$$-\frac{1}{3\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C.$$

5.1.14. $\dfrac{x+2}{2(x^2+1)} + 2\arctan x + \ln\dfrac{\sqrt{x+1}}{\sqrt[4]{x^2+1}} + C.$

5.2.2. $4\sqrt[4]{x} + 6\sqrt[6]{x} + 24\sqrt[12]{x} + 24\ln\left|\sqrt[12]{x} - 1\right| + C.$

5.2.4. $-\dfrac{1}{\sqrt{3}}\arctan\dfrac{2t+1}{\sqrt{3}} + \ln\left|\dfrac{\sqrt[3]{(t+2)^4}}{\sqrt[3]{t-1}\cdot\sqrt{t^2+t+1}}\right| +$

$$+C, \quad \text{where } t = \sqrt[3]{\frac{x-1}{x}}.$$

5.2.7. $\sqrt{\dfrac{x+1}{1-x}} + C.$ **5.2.8.** $\dfrac{3}{2}\sqrt[3]{\dfrac{1+x}{1-x}} + C.$

5.2.9. $\left(1 - \dfrac{1}{2}x\right)\sqrt{1-x^2} - \dfrac{3}{2}\arcsin x + C.$

5.3.3. $-2\arctan\left(\dfrac{\sqrt{1+x-x^2}+1}{x} + 1\right) + C.$

5.3.5. $2\ln|\sqrt{x^2+2x+4} - x| - \dfrac{3}{2(\sqrt{x^2+2x+4} - x - 1)} -$

$$-\frac{3}{2}\ln|\sqrt{x^2+2x+4} - x - 1| + C.$$

5.3.6. $\dfrac{1+\sqrt{1-x^2}}{x}+2\arctan\sqrt{\dfrac{1+x}{1-x}}+C.$

5.3.7. $\dfrac{x-1}{\sqrt{2x-x^2}}+C.$ **5.3.8.** $\dfrac{(x+\sqrt{1+x^2})^{15}}{15}+C.$

5.4.2. $5\sqrt{x^2+2x+5}-\ln(x+1+\sqrt{x^2+2x+5})+C.$

5.4.5. $\dfrac{3x^2+x-1}{3}\sqrt{3x^2-2x+1}+C.$

5.4.6. $\dfrac{2x+1}{4}\sqrt{x^2+x+1}+\dfrac{3}{8}\ln|2x+1+2\sqrt{x^2+x+1}|+C.$

5.4.8. $\dfrac{1}{3}(x^2-14x+111)\sqrt{x^2+4x+3}-66\ln|x+2+\sqrt{x^2+4x+3}|+C.$

5.4.9. $\dfrac{1}{64}(32x^2-20x-373)\sqrt{2x^2+5x+7}+\dfrac{3297}{128\sqrt{2}}\ln|4x+5+$
$+2\sqrt{4x^2+10x+14}|+C.$

5.4.10. $\dfrac{3x+5}{8(x+1)^2}\sqrt{x^2+2x}-\dfrac{3}{8}\arcsin\dfrac{1}{(x+1)}+C.$

5.4.11. $-\dfrac{\sqrt{x^2-4x+3}}{x-1}-2\arcsin\dfrac{1}{x-2}+C.$

5.4.12. $-\dfrac{2}{15}\sqrt{\dfrac{x+2}{x+1}}\,\dfrac{8x^2+12x+7}{(x+1)^2}+C.$

5.4.13. $\ln\left|\dfrac{x^2+1+\sqrt{x^4+3x^2+1}}{x}\right|+C.$ *Hint.* First make the substitution $x^2=t$.

5.5.2. $3\arctan\sqrt[3]{x}+C.$ **5.5.4.** $\dfrac{2}{3}\left(2+x^{\frac{2}{3}}\right)^{\frac{9}{4}}-\dfrac{12}{5}\left(2+x^{\frac{2}{3}}\right)^{\frac{5}{4}}+C.$

5.5.5. $\dfrac{3}{22}(1+x^2)^{\frac{11}{3}}-\dfrac{3}{8}(1+x^2)^{\frac{8}{3}}+\dfrac{3}{10}(1+x^2)^{\frac{5}{3}}+C.$

5.5.7. $\dfrac{12}{7}\sqrt[3]{(1+\sqrt[4]{x})^7}-3\sqrt[3]{(1+\sqrt[4]{x})^4}+C.$

5.5.8. $3\ln\dfrac{\sqrt[3]{x}}{1+\sqrt[3]{x}}+\dfrac{3}{1+\sqrt[3]{x}}+C.$

5.5.9. $\dfrac{(1+x^2)^{\frac{3}{2}}(3x^2-2)}{15}+C.$

5.5.10. $\dfrac{\sqrt{1+x^2}\,(2x^2-1)}{3x^3}+C.$

5.5.11. $\dfrac{21}{32}\sqrt[7]{(1+\sqrt[3]{x^4})^8}+C.$

5.5.12. $\dfrac{5}{4}\left(1+\dfrac{1}{x}\right)^{\frac{4}{5}}-\dfrac{5}{9}\left(1+\dfrac{1}{x}\right)^{\frac{9}{5}}+C.$

5.6.2. $\dfrac{1}{3\sin^3 x}-\dfrac{1}{5\sin^5 x}+C.$ **5.6.6.** $\tan x+\dfrac{1}{3}\tan^3 x+C.$

5.6.10. (a) $-\cot x+\dfrac{1}{3}\cot^3 x-\dfrac{1}{5}\cot^5 x-x+C;$

(b) $\dfrac{1}{2}\tan^2 x-\dfrac{1}{2}\ln(1+\tan^2 x)+C=\dfrac{1}{2}\tan^2 x+\ln|\cos x|+C.$

5.6.12. $-\sin x-\frac{1}{3}\sin^3 x+\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|+C.$

5.6.14. $\frac{2}{\sqrt{15}}\arctan\left(\frac{1+2\tan\frac{x}{2}}{\sqrt{15}}\right)+C.$

5.6.22. (a) $-\frac{x}{8}+\frac{\sinh 4x}{32}+C$; (b) $\frac{2}{\sqrt{3}}\arctan\left(\frac{2\tanh\frac{x}{2}+1}{\sqrt{3}}\right)+C.$

5.7.3. $-\frac{1}{8}\ln(x+\sqrt{x^2-1})+\frac{1}{8}x(2x^2-1)\sqrt{x^2-1}+C.$

5.7.4. $\ln(x+\sqrt{x^2+1})-\frac{\sqrt{x^2+1}}{x}+C.$

5.7.7. $I=\arcsin\frac{x+1}{2}+C.$

5.7.8. $I=\frac{x-1}{4\sqrt{x^2-2x+5}}+C.$

5.8.2. $I=4\sqrt{1-x}+2\ln(2-x-2\sqrt{1-x})-2(1+\sqrt{1-x})\ln x+C.$

5.8.5. $I=e^{\alpha t}\frac{\alpha\cos t+\sin t}{\alpha^2+1}+C$, where $t=\arctan x$.

Chapter VI

6.1.9. $I=4\cdot\frac{3+19}{2}=44$ as the area of a trapezoid whose height is $5-1=4$ and bases $4\times1-1=3$ and $4\times5-1=19$.

6.1.12. $s_n=16\frac{1}{4}-\frac{175}{2n}+\frac{125}{4n^2}$; $S_n=16\frac{1}{4}+\frac{175}{2n}+\frac{125}{4n^2}$.

6.2.2. (a) 1; (b) $\frac{3}{2}$; (c) $\frac{\pi}{6}$. **6.2.10.** (a) $\frac{7}{72}$; (b) $\frac{1}{2}\ln\frac{3}{2}$; (c) π; (d) $\frac{\pi}{4}-\arctan\frac{\pi}{4}$; (e) $\ln 2$; (f) 1; (g) $\arctan e-\frac{\pi}{4}$; (h) $\frac{\pi}{16}$; (i) $\frac{14}{15}$; (j) $\frac{4}{3}$; (k) $\frac{\sqrt{3}-\sqrt{2}}{2}$. **6.3.1.** (c) $3 < I < 5$. *Hint*. $M=f(0)=\frac{5}{2}$, $m=f(2)=\frac{3}{2}$. **6.3.11.** (a) $\frac{\sin 2x}{x}$; (b) $-\sqrt{1+x^4}$. **6.3.14.** (b) $\frac{\pi^2}{4}$. **6.3.15.** (b) $\frac{dy}{dx}=-e^{-y}\sin x$. **6.3.23.** (a) $\ln x$; (b) $\frac{3}{x}$. **6.3.24.** (a) $y'_x=\frac{t}{\ln t}$; (b) $y'_x=\frac{\tan t}{t^2}$.

6.3.25. (a) The maximum is at $x=1$, the minimum at $x=-1$; (b) the minima are at $x=-2$; 0; 2, the maxima at $x=\pm 1$.

6.4.3. (a) $\frac{\pi a^4}{16}$ (substitution $x=a\sin t$); (b) $\frac{\sqrt{3}-\sqrt{2}}{2}$ (substitution $x=\tan t$).

6.4.6. (a) $\sqrt{2}-\frac{2}{\sqrt{3}}+\ln\frac{2+\sqrt{3}}{1+\sqrt{2}}$; (b) $2(\sqrt{3}-1)$; (c) $8+\frac{3\sqrt{3}}{2}\pi$.

6.4.15. (a) $2-2\ln 2$; (b) $0.2\ \ln 112$; (c) $\frac{\sin\frac{\pi}{24}}{\sin\frac{\pi}{8}\sin\frac{\pi}{12}}$; (d) $\sqrt{3}-0.5\ \ln(2+$

$+\sqrt{3}$); (e) $0.25 \ln 3$ (substitution $\sin x - \cos x = t$); (f) $a^3\left(\frac{\pi}{4}-\frac{2}{3}\right)$ (substitution $x = a\cos t$); (g) $\frac{\pi a^2}{2}$ (substitution $x = 2a\sin^2 t$); (h) $\frac{\pi}{4}+\frac{1}{2}$.

6.4.16. (a) $\frac{\pi}{6}$; (b) $\frac{\pi}{4}$; (c) $\frac{1}{4}\ln\frac{32}{17}$ (substitution $x^4 = t$); (d) $\frac{\pi}{12}$ (substitution $x^2 = a^2\cos^2 t + b^2 \sin^2 t$).

6.4.17. The substitution $x = \frac{1}{t}$ will not do, since this function is discontinuous at $t = 0$.

6.4.18. The substitution $t = \tan\frac{x}{2}$ will not do, since this function is discontinuous at $x = \pi$.

6.4.19. *Hint.* The inverse function $x = \pm\sqrt{t^5}$ is double-valued. To obtain the correct result it is necessary to divide the initial interval of integration into two parts:

$$\int_{-2}^{2} \sqrt[5]{x^2}\,dx = \int_{-2}^{0} \sqrt[5]{x^2}\,dx + \int_{0}^{2} \sqrt[5]{x^2}\,dx$$

and apply the substitutions $x = -\sqrt{t^5}$ in $-2 < x < 0$ and $x = +\sqrt{t^5}$ in $0 < x < 2$.

6.4.20. It is impossible, since $\sec t \geqslant 1$ and the interval of integration is $[0, 1]$.

6.4.21. It is possible; see Problem **6.4.12.**

6.4.22. *Hint.* On writing $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$, make the substitution $x = -t$ in the first integral.

6.4.23. $\int_{0}^{1} f(\arcsin t)\,dt + \int_{1}^{-1} f(\pi - \arcsin t)\,dt + \int_{-1}^{0} f(2\pi + \arcsin t)\,dt.$

Hint. Represent the given integral as the sum of three integrals for the intervals: $\left(0, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $\left(\frac{3\pi}{2}, 2\pi\right)$ and substitute the variable: $x = \arcsin t$, $x = \pi - \arcsin t$, $x = 2\pi + \arcsin t$ respectively.

6.5.3. (1) If $f(x)$ is an even function, then

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 2\int_{0}^{\pi} f(x)\cos nx\,dx, \text{ and } \int_{-\pi}^{\pi} f(x)\sin x\,dx = 0.$$

(2) If $f(x)$ is an odd function, then $\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$, and $\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 2\int_{0}^{\pi} f(x)\sin n\,dx.$

6.5.4. 0. **6.6.3.** $6 - 2e$. **6.6.5.** $\pi\sqrt{2} - 4$. **6.6.6.** $\pi - 2$. **6.6.13.** (a) $\frac{\pi}{2} - 1$; (b) $-\frac{1}{e}$; (c) $\frac{\pi}{4} - \frac{\sqrt{3}}{9}\pi + \frac{1}{2}\ln\frac{3}{2}$; (d) $\frac{\pi}{4} - \frac{1}{2}$; (e) $\ln 2 - \frac{1}{2}$; (f) $\ln\frac{2}{8}$; (g) $\frac{\pi}{2} - 1$; (h) $\frac{16\pi}{3} - 2\sqrt{3}$.

6.6.14. *Hint.* Integrate by parts twice, putting $u=(\arccos x)^n$ the first time and $u=(\arccos x)^{n-1}$ the second time.

6.6.15. *Hint.* Integrate by parts, putting $u=x$.

6.7.4. (a) 0.601. *Hint.* Estimate $|f^{IV}(x)|$ on the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ and put $2n=6$; (b) 0.7462. **6.7.5.** 0.96

6.8.1.

$$F(x)=\begin{cases} \dfrac{x-x^2}{2} & \text{for } 0\leqslant x\leqslant 1, \\ \dfrac{1}{2} & \text{for } 1<x\leqslant 2, \\ \dfrac{(x-2)^3}{3}+\dfrac{1}{2} & \text{for } 2<x\leqslant 3. \end{cases}$$

Continuity is checked directly. The assertion concerning the derivative requires checking only at the points $x=1$, $x=2$.

6.8.2. *Hint.* Make sure that the function $f(x)$ is continuous both inside the interval (0, 1) and at the end-points ($\lim\limits_{x\to+0} f(x)=f(0)$ and $\lim\limits_{x\to 1-0} f(x)=f(1)$).

6.8.3. No. *Hint.* Consider the function

$$\varphi(x)=\begin{cases} 1 & \text{if } x \text{ is rational,} \\ -1 & \text{if } x \text{ is irrational on the interval } [0, 1]. \end{cases}$$

6.8.4. $1-\sqrt{3}$. *Hint.* $\int\limits_a^b f''(x)\,dx=f'(b)-f'(a)$.

6.8.5. *Hint.* Putting for definiteness $x>0$ and

$$E(x)=n\leqslant x<n+1,$$

take advantage of the additivity of the integral

$$\int\limits_0^x E(x)\,dx=\int\limits_0^1 E(x)\,dx+\int\limits_1^2 E(x)\,dx+\ldots+\int\limits_{n-1}^n E(x)\,dx+\int\limits_n^x E(x)\,dx.$$

6.8.6. The antiderivative $F_1(x)$ will lead to the correct result and $F_2(x)$ to the wrong one, since this function is discontinuous in the interval $[0, \pi]$.

6.8.7. $F(x)=y_0+\int\limits_{x_0}^x f(t)\,dt$. *Hint.* Any antiderivative $F(x)$ can be represented in the form $F(x)=\int\limits_{x_0}^x f(t)\,dt+C$. Putting $x=x_0$, find $C=y_0$.

6.8.8. $\xi=\frac{1}{2}\ln\frac{e^{2b}-e^{2a}}{2b-2a}$.

6.8.9. The function is defined on the interval $[-1, 1]$, it is odd, and increasing; convex on the interval $[-1, 0]$ and concave on the interval $[0, 1]$; the point $[0, 0]$ is a point of inflection.

6.8.10. *Hint.* The function

$$f(x)=\begin{cases} x^x & \text{at } 0<x\leqslant 1 \\ 1 & \text{at } x=0 \end{cases}$$

is continuous on the interval, it reaches the least value $m=e^{-\frac{1}{e}}\approx 0.692$ at $x=\frac{1}{e}$ and the greatest value $M=1$ at $x=0$ and at $x=1$.

6.8.11. *Hint.* Integrate the inequality $\frac{2}{\pi} \leqslant \frac{\sin x}{x} \leqslant 1$.

6.8.12. *Hint.* Integrate the inequality

$$\sqrt{x \sin x} > \sqrt{x^2 - \frac{x^4}{6}} = x\sqrt{1 - \frac{x^2}{6}} \text{ at } 0 \leqslant x \leqslant \frac{\pi}{6}$$

and write Schwarz-Bunyakovsky inequality

$$\int_0^{\frac{\pi}{2}} \sqrt{x \sin x}\, dx \leqslant \sqrt{\int_0^{\frac{\pi}{2}} x\, dx \int_0^{\frac{\pi}{2}} \sin x\, dx} = \sqrt{\frac{\pi^2}{8}} = \frac{\pi}{2\sqrt{2}}.$$

6.8.14. *Hint.* Apply the Schwarz-Bunyakovsky inequality in the form

$$\left[\int_a^b \sqrt{f(x) \cdot \frac{1}{f(x)}}\, dx\right]^2 \leqslant \int_a^b f(x)\, dx \int_a^b \frac{1}{f(x)}\, dx.$$

6.8.15. *Hint.* Make the substitution $\arctan x = \frac{t}{2}$.

6.8.16. *Hint.* If $f(x)$ is an even function, then $F(x) = \int_0^x f(t)\, dt$ is an odd function, since

$$F(-x) = \int_0^{-x} f(t)\, dt = -\int_0^x f(-z)\, dz = -F(x) \quad (t = -z).$$

And if $f(x)$ is an odd function, then $F(x) = \int_0^x f(t)\, dt$ is an even function, since

$$F(-x) = \int_0^{-x} f(t)\, dt = -\int_0^x f(-z)\, dz = F(x) \quad (t = -z);$$

all the remaining antiderivatives have the form $F(x) + C$ and, therefore, are also even functions.

6.8.17. *Hint.* The derivative of the integral I with respect to a equals zero: $\frac{dI}{da} = f(a+T) - f(a) = 0$.

Chapter VII

7.1.4. (a) $\ln 2$; (b) $\frac{2}{3}(2\sqrt{2} - 1)$; (c) $\frac{3}{4}$; (d) 1; (e) $\frac{1}{2}$.

7.2.2. (a) $\frac{1}{2}$; (b) $\frac{1}{2} + \frac{1}{2} \ln \frac{2}{e+1} \approx 0.283$. **7.2.5.** $\frac{\pi}{4}$. **7.2.10.** $\frac{2i}{h\sqrt{d^2+h^2}}$.

7.2.13. (a) $\mu = \frac{5}{3}$; (b) $\mu = \ln 2$; (c) $\mu = \frac{8}{\ln 3} + 2$. **7.2.15.** $\frac{2h}{3}$. **7.2.16.** $\frac{2I_0}{\pi}$.

7.3.4. $\frac{35}{6}$. **7.3.6.** $\frac{2}{3} + \frac{5}{2} \arcsin \frac{3}{5}$. **7.3.11.** $\frac{8}{15}$. **7.3.13.** 9. **7.3.16.** $\frac{1}{m+1}$.

7.3.19. $\frac{64}{3}$. **7.3.20.** $\frac{8}{3}$. **7.3.21.** $2\pi-(2\sqrt{3})\ln(2+\sqrt{3})$. **7.3.22.** 0.75π.
7.3.23. $\frac{128}{15}$. **7.3.24.** $\frac{1}{3}$. **7.3.25.** $\frac{4}{3}$. **7.3.26.** $\frac{8}{15}$. **7.3.27.** $\frac{1}{12}$. **7.3.28.** $\frac{91}{30}$.
7.4.6. $\frac{8}{5}$. **7.4.8.** $0.75\pi ab$. *Hint.* The curve is symmetrical about the coordinate axes and intersects them at the points $x=\pm a$, $y=\pm b$.

7.4.9. (a) $\frac{8}{15}$. *Hint.* The curve is symmetrical about the x-axis, intersecting it twice at the origin at $t=\pm 1$. The loop is situated in the second and third quadrants; (b) $\frac{8}{15}$. *Hint.* The points of self-intersection of the curve are found in the following way: $y=tx(t)$, therefore $y(t_1)=t_1x(t_1)=t_2x(t_2)$ at $t_1\neq t_2$ and $x(t_1)=x(t_2)$, only *if* $x(t_1)=x(t_2)=0$, i.e. $t_1=0$; $t_2=2$; (c) $\frac{8\sqrt{3}}{5}$.

7.4.10. $0.25\pi ab$. *Hint.* The curve is symmetrical with respect to both axes of coordinates and passes twice through the origin forming two loops. Therefore, it is sufficient to compute a quarter of the desired area corresponding to the variation of t from 0 to $\frac{\pi}{2}$ and multiply the obtained result by 4.

7.4.11. $\frac{3c^4\pi}{8ab}$. *Hint.* The curve resembles an astroid extended in the vertical direction.

7.5.2. (a) $\frac{3\pi}{2}$; (b) $\frac{\pi a^2}{4}$. *Hint.* The curve is a circle of radius $\frac{a}{2}$ passing through the pole and symmetrical about the polar axis, $-\frac{\pi}{2}\leqslant\varphi\leqslant\frac{\pi}{2}$.

7.5.6. $2a^2\left(\frac{5\pi}{8}-1\right)$. **7.5.8.** (a) $\frac{\pi a^2}{8}$; (b) $\frac{\pi a^2}{8}$. **7.5.9.** $a^2\left(\frac{7\pi}{12}-\sqrt{3}\right)$.
7.5.10. $\frac{\pi a^2}{32}$. *Hint.* The curve passes through the pole forming two loops located symmetrically about the y-axis in the first and fourth quadrants. It is sufficient to calculate the area enclosed by one loop corresponding to variation of φ from 0 to $\frac{\pi}{2}$ and double the result thus obtained.

7.5.11. $\frac{5}{32}\pi a^2$. *Hint.* The curve passes through the pole, it is symmetrical about the polar axis and situated in the first and fourth quadrants. It is sufficient to calculate the area of the upper portion of the figure which corresponds to variation of φ from 0 to $\frac{\pi}{2}$ and double the result thus obtained.

7.5.12. $a^2\left(1+\frac{\pi}{6}-\frac{\sqrt{3}}{2}\right)$.

7.5.13. $\frac{\pi a^2}{2}$. *Hint.* The curve is symmetrical about the coordinate axes and intersects them only at the origin, forming four loops—one in each quadrant (a *four-leaved rose*). Therefore, it is sufficient to find the area of one loop corresponding to the variation of φ from 0 to $\frac{\pi}{2}$ and multiply the result by 4.

7.5.14. $\sqrt{2}\,\pi a^2$. *Hint.* The curve is symmetrical about the axes of coordinates and the bisectors of the coordinate angles; it cuts off equal intercepts on the axes. The origin is an isolated point. It is sufficient to compute the area of one-eighth of the figure corresponding to variation of φ from 0 to $\frac{\pi}{4}$ and multiply the result by 8.

7.6.2. $9\frac{2}{3}\pi$. *Hint.* A plane perpendicular to the x-axis at the point x will cut the sphere along a circle of radius $r=\sqrt{16-x^2}$, therefore the cross-sectional area $S(x)=\pi(16-x^2)$.

7.6.5. $0.5\pi a^2 h$. *Hint.* The area of a triangle situated at a distance x from the centre of the circle is equal to $h\sqrt{a^2-x^2}$.

7.6.10. $2\pi^2 a^2 b$. **7.6.11.** $\frac{8}{7}$ (see Problem 7.3.9). **7.6.14.** $5\pi^2 a^3$. **7.6.16.** (a) $2\pi ab\left(1+\frac{1}{3c^2}\right)$; (b) $\frac{16}{3}a$; (c) $\frac{1}{2}abk^2\pi$. **7.6.17.** $\frac{2}{3}a^3\tan\alpha$. **7.6.18.** (a) 12π; (b) $\frac{16}{15}\pi$; (c) $\frac{64}{5}\pi$; (d) π^2; (e) $\frac{64}{3}\pi$; (f) $\frac{4}{3}\pi a^3$. **7.6.19.** $\frac{\pi a^3}{20}$. **7.6.20.** $\frac{\pi^2}{12}$. **7.6.21.** $\frac{1}{4}\pi a^3\left(e^{\frac{2c}{a}}-e^{-\frac{2c}{a}}\right)+\pi a^2 c=\frac{\pi a^3}{2}\sinh\frac{2c}{a}+\pi a^2 c$. **7.6.22.** $\frac{\pi}{20}(6\pi+5\sqrt{3})$. *Hint.* The abscissas of the points of intersection are: $x_1=-\frac{\pi}{3}$; $x_2=\frac{\pi}{3}$. **7.6.23.** $\frac{19}{48}\pi$. **7.6.24.** $\frac{127}{7}\pi$. **7.6.25.** $\frac{16\pi c^6}{105ab^2}$. *Hint.* Represent the evolute of the ellipse parametrically as follows: $x=\frac{c^2}{a}\cos^3 t$; $y=-\frac{c^2}{b}\sin^3 t$, where $c=\sqrt{a^2-b^2}$. **7.6.26.** $\frac{4}{3}\pi a^3$. **7.6.27.** $\frac{\pi^2 a^3}{4\sqrt{2}}$; $\frac{\pi a^3}{4}\left[\sqrt{2}\ln(1+\sqrt{2})-\frac{2}{3}\right]$. *Hint.* Pass over to polar coordinates. **7.6.28.** $\frac{4}{21}\pi a^3$. **7.7.2.** $\frac{112}{27}$. **7.7.4.** $\ln\frac{e^b-e^{-b}}{e^a-e^{-a}}$. **7.7.8.** (a) $\sqrt{6}+\ln(\sqrt{2}+\sqrt{3})$; (b) $2\ln(2-\sqrt{3})$. *Hint.* $x_1=-\frac{\pi}{2}$; $x_2=\frac{\pi}{3}$; (c) $\frac{2\sqrt{3}}{3}$. **7.7.9.** $\frac{a(a+2)}{2}$. **7.7.10.** $10\left(\frac{67}{27}+\sqrt{5}\right)$. **7.8.2.** $8a$. **7.8.5.** $\frac{13}{3}$. *Hint.* The curve intersects the axes at $t_1=0$ and $t_2=\sqrt[4]{8}$. **7.8.7.** $4\sqrt{3}$. **7.8.8.** $16a$. **7.8.9.** $8\pi a$. *Hint.* See Fig. 79. **7.8.10.** $\frac{4(a^3-b^3)}{ab}$. **7.8.11.** $\frac{\pi^3}{3}$. **7.8.12.** At $t=\frac{2\pi}{3}$ the point $M\left[a\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right), \frac{3a}{2}\right]$. **7.9.5.** $1.5\pi a$. **7.9.9.** $\frac{5}{12}+\ln\frac{3}{2}$. **7.9.10.** $2\sqrt{2}\pi a$. *Hint.* The curve $\rho=2\sqrt{2}\,a\cos\left(\varphi-\frac{\pi}{4}\right)$ is a circle.

7.9.11. $p\left[\sqrt{2}+\ln(1+\sqrt{2})\right]$. **7.10.3.** (a) $\frac{14\pi}{3}$; (b) $\frac{62\pi}{3}$. **7.10.5.** $2\pi\left(1+\frac{4\pi}{3\sqrt{3}}\right)$. **7.10.8.** $\frac{\pi}{2}$. **7.10.14.** $(34\sqrt{17}-2)\frac{\pi}{9}$. **7.10.15.** $2\pi\left[\sqrt{2}+\ln(1+\sqrt{2})\right]$. **7.10.16.** $\frac{56}{3}\pi a^2$. **7.10.17.** $\frac{2\sqrt{2}}{5}\pi(e^{\pi}-2)$. **7.10.18.** 29.6π. **7.10.19.** $4\pi^2 a^2$. **7.10.20.** $\frac{128}{5}\pi a^2$. **7.11.7.** $16a^2$ where a is the radius of the cylinders base. **7.11.8.** 1.5π. **7.11.10.** (a) $\frac{8}{15}$; (b) $\frac{7}{50}-\frac{1}{4}\arctan\frac{1}{2}$. **7.11.11.** (a) $\frac{\pi a^2}{4}$; (b) $\frac{p^2}{6}(3+4\sqrt{2})$; (c) $\frac{1}{8}(5\pi+6\sqrt{3})$. **7.11.13.** $\frac{a}{2}(2\ln 3-1)$. **7.11.14.** $\frac{\sqrt{2}}{3}(5\sqrt{5}-2\sqrt{2})$. **7.11.17.** $2\pi\frac{\sqrt{3}}{15}$. **7.11.18.** $\pi a^2\sqrt{pq}$. **7.11.19.** $\pi ab\left(\frac{l^3}{3c^2}-l+\frac{2c}{3}\right)$. **7.11.20.** $\frac{\pi abh}{3}$. **7.11.21.** 12π. **7.11.22.** $\left(\frac{4\sqrt{3}-6}{9}\right)\pi b^2 a$. **7.11.23.** $\frac{4}{21}\pi a^3$. **7.11.24.** (a) $\pi\left[(\sqrt{5}-\sqrt{2})+(\sqrt{2}+1)\frac{\sqrt{5}-1}{2}\right]$; (b) $\frac{4\pi a^2}{243}\left(21\sqrt{13}+2\ln\frac{3+\sqrt{13}}{2}\right)$; (c) $2\pi rh$. **7.12.2.** $\frac{2}{3}\gamma R^3$. **7.12.4.** $\frac{\pi R^4}{4}$. **7.12.9.** $\frac{MR^2\omega^2}{4}$. **7.12.11.** $\pi abhd$. **7.12.12.** πrdh^2. **7.12.13.** $\frac{1}{12}\pi R^2 H$. **7.13.3.** $0.25\pi R^3$. **7.13.7.** $M_x=\frac{1}{3}(5\sqrt{5}-1)$; $M_y=\frac{9}{8}\sqrt{5}+\frac{1}{16}\ln(2+\sqrt{5})$. **7.13.8.** $M_x=\frac{b}{2}\sqrt{a^2+b^2}$; $M_y=\frac{a}{2}\sqrt{a^2+b^2}$. **7.13.9.** $\sqrt{2}+\ln(1+\sqrt{2})$. **7.13.10.** 0.15. **7.13.11.** $I_x=\frac{ab^3}{12}$; $I_y=\frac{a^3b}{12}$. **7.13.12.** $\frac{(a+3b)h^3}{12}$. **7.13.16.** $x_c=y_c=0.4a$. **7.13.19.** $x_c=y_c=\frac{a}{5}$. **7.13.26.** $x_c=R\frac{\sin\alpha}{\alpha}$; $y_c=0$. **7.13.28.** $x_c=\frac{5a}{8}$; $y_c=0$. **7.13.29.** $x_c=-\frac{0.2(2e^{2\pi}-e^{\pi})}{e^{\pi}-e^{\frac{\pi}{2}}}$; $y_c=\frac{0.2a(e^{2\pi}-2e^{\pi})}{e^{\pi}-e^{\frac{\pi}{2}}}$. **7.13.30.** $4.5\pi a^3$. **7.13.31.** $x_c=0$; $y_c=\frac{4R}{3\pi}$. **7.14.1.** $\left|\frac{m-n}{m+n}\right|$; $4\left|\frac{m-n}{m+n}\right|$ if both m and n are even; $2\left|\frac{m-n}{m+n}\right|$ if both m and n are odd; $\left|\frac{m-n}{m+n}\right|$ if m and n are of different evenness. *Hint.* The curves $y^m=x^n$ and $y^n=x^m$ have two common points (0, 0) and (1, 1) in the first quadrant. The area of the figure situated in the first quadrant is equal to $\left|\int_0^1\left(x^{\frac{n}{m}}-x^{\frac{m}{n}}\right)dx\right|$. Depending on evenness and oddness of m and n this figure is mapped symmetrically either about the coordinate axes (m, n even) or about the origin (m, n odd). If m and n are of different evenness, then the curves enclose only the area lying in the first quadrant.

7.14.3. *Hint.* Take advantage of the formula for computing the area in polar coordinates.

7.14.4. *Hint.* Since the figures are of equal area, the function $S(x)$ appearing in the formula for the volume $V=\int_a^b S(x)\,dx$ is the same and, consequently, the values of the integrals are also equal.

7.14.5. *Hint.* The formula follows directly from Simpson's formula

$$\int_0^h f(x)\,dx=\frac{h}{6}\left[f(0)+4f\left(\frac{h}{2}\right)+f(h)\right],$$

for a sphere $S(x)=\pi(r^2-x^2)$; for a cone $S(x)=\frac{\pi r^2x^2}{h^2}$; for a paraboloid of revolution $S(x)=2\pi px$ and so on.

7.14.6. *Hint.* Divide the curvilinear trapezoid into strips Δx wide and write an expression for the element of volume $\Delta V=2\pi\, xy\,\Delta x$.

7.14.8. *Hint.* Use the formula for calculating the length of a curve represented parametrically.

7.14.9. $\ln\frac{\pi}{2}$. *Hint.* The point $(t=1)$ nearest to the origin with a vertical tangent corresponds to $t=\frac{\pi}{2}$.

7.14.13. $2\pi\frac{\sqrt{3}}{15}$. **7.14.14.** $\sqrt{2}\cdot z$. **7.14.16.** (a) $0.5\ln(x+y)$; (b) $\frac{\pi}{4}-0.5\arcsin x$.

Chapter VIII

8.1.2. (b) $\frac{1}{2}\ln 2$; (c) 1; (d) $1-\ln 2$; (e) π; (f) $\frac{1}{2}$.

8.1.6. (a) It diverges. *Hint.* $\frac{\ln(x^2+1)}{x}>\frac{1}{x}$ for $x>\sqrt{e-1}$; (b) converges; (c) diverges. *Hint.* $\frac{2+\cos x}{\sqrt{x}}>\frac{1}{\sqrt{x}}$; (d) converges; (e) diverges.

8.1.17. (a) 0. *Hint.* Represent the integral as the sum of two items: $\int_0^\infty\frac{\ln x}{1+x^2}dx=\int_0^1\frac{\ln x}{1+x^2}dx+\int_1^\infty\frac{\ln x}{1+x^2}dx$. Make the substitution $x=\frac{1}{t}$ in the second summand and show that $\int_1^\infty\frac{\ln x}{1+x^2}dx=-\int_0^1\frac{\ln x}{1+x^2}dx$; (b) $\frac{m!}{2}$.

8.2.2. (a) $9a^{\frac{2}{3}}$; (b) it diverges; (c) diverges; (d) $6\sqrt[3]{2}$; (e) $\frac{\pi}{3}$; (f) converges for $p<1$ and diverges for $p\geqslant 1$.

8.2.7. (a) It converges; (b) diverges; (c) converges; (d) converges; (e) diverges; (f) converges. **8.2.11.** (a) It diverges; (b) $2\sqrt{\ln 2}$; (c) $\frac{51}{7}$.

8.2.14. (a) It converges; (b) diverges; (c) diverges; (d) converges; (e) converges. **8.3.7.** (a) $\frac{\pi}{2}$; (a) 2π. **8.3.8.** $3\pi a^2$. **8.3.9.** $\frac{1}{2}$. **8.3.10.** $\frac{4\pi}{3}$.

8.3.14. mgR. *Hint.* The law of attraction of a body by the Earth is determined by the formula $f=\frac{mg\,R^2}{r^2}$, where m is the mass of the body, r is the distance between the body and the centre of the Earth, R is the radius of the Earth.

8.3.15. e_1. *Hint.* Electric charges interact with a force $\frac{e_1e_2}{r^2}$, where e_1 and e_2 are the magnitudes of the charges and r is the distance between them.

8.4.1. *Hint.* Represent the integral in the form of the sum

$$\int_1^{+\infty}\frac{dx}{x^p\ln^q x}=\int_1^{a}\frac{dx}{x^p\ln^q x}+\int_a^{+\infty}\frac{dx}{x^p\ln^q x}\quad (a>1)$$

and apply special tests for convergence, taking into consideration that in the first integral $\ln x = \ln[1+(x-1)] \sim x-1$ as $x \to 1$, and in the second integral the logarithmic function increases slower for $q < 0$ than any power function.

8.4.2. *Hint.* Making the substitution $x^q = t$, reduce the given integral to the form $\pm \frac{1}{q} \int_0^{+\infty} t^{\frac{p+1}{q}-1} \sin t\, dt$. Represent the integral $\int_0^{+\infty} t^{\frac{p+1}{q}-1} \sin t\, dt$ as the sum $\int_0^1 \frac{\sin t}{t^\alpha} dt + \int_1^{+\infty} \frac{\sin t}{t^\alpha} dt$, where $\alpha = 1 - \frac{p+1}{q}$, and show that the integral converges absolutely for $1 < \alpha < 2$ and conditionally for $0 < \alpha \leqslant 1$. Note that at $\frac{p+1}{q} = 0$ the integral is reduced to the conditionally converging integral $\int_0^{+\infty} \frac{\sin t}{t} dt$, and at $\frac{p+1}{q} = -1$ to the diverging integral $\int_0^{+\infty} \frac{\sin t}{t^2} dt$.

8.4.3. *Hint.* Represent the given integral as the sum $\int_0^{1/2} x^{p-1}(1-x)^{q-1} dx + \int_{1/2}^1 x^{p-1}(1-x)^{q-1} dx$ and apply the special comparison test.

8.4.4. *Hint.* If $|\alpha| \neq |\beta|$, then $\int_0^T \sin \alpha x \cdot \sin \beta x\, dx$ is bounded.

8.4.5. *Hint.* By substituting $t = x^2$ the integral is reduced to the Euler gamma-function.

8.4.6. *Hint.* $\int_a^\infty \frac{f(\alpha x) - f(\beta x)}{x} dx = \int_{a\alpha}^\infty \frac{f(x)}{x} dx - \int_{a\beta}^\infty \frac{f(x)}{x} dx = \int_{a\alpha}^{a\beta} \frac{f(x)}{x} dx = A \ln \frac{\beta}{\alpha} + \int_{\alpha a}^{\beta a} \frac{f(x)-A}{x} dx$. Applying the generalized mean value theorem, show that the last integral tends to zero as $a \to 0$.

8.4.7. *Hint.* Take the function $f(x) = e^{-x}$ for the first integral, the function $f(x) = \cos x$ for the second and take advantage of the results of Problem **8.4.6.**

8.4.8. It converges for $m < 3$ and diverges for $m \geqslant 3$. *Hint.* Take advantage of the equivalence of $1 - \cos x$ and $\frac{x^2}{2}$ as $x \to 0$.

8.4.9. *Hint.* Represent $\int_0^\pi \frac{dx}{(\sin x)^k}$ as the sum of two integrals $\int_0^{\frac{\pi}{2}} \frac{dx}{(\sin x)^k} + \int_{\frac{\pi}{2}}^{\pi} \frac{dx}{(\sin x)^k}$; reduce the second integral to the first one by making the substitution $x = \pi - t$ and take advantage of the equivalence of $\sin x$ and x as $x \to 0$.

8.4.10. *Hint.* $\int_0^{\infty} \frac{\sin x(1-\cos x)}{x^s}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x(1-\cos x)}{x^s}\,dx + \int_{\frac{\pi}{2}}^{\infty} \frac{\sin x(1-\cos x)}{x^s}\,dx$. The integrand of the first summand on the right side is an infinitely large quantity of order $s-3$ as $x \to 0$. By the special comparison test the first integral converges absolutely for $s-3<1$, i.e. $s<4$, and diverges for $s \geqslant 4$. The second integral in the right side converges absolutely for $s>1$, since the function $\sin x(1-\cos x)$ is bounded. But if $0<s\leqslant 1$, the second integral converges conditionally as the difference of two conditionally converging integrals $\int_{\frac{\pi}{2}}^{\infty} \frac{\sin x}{x^s}\,dx$ and $\int_{\frac{\pi}{2}}^{\infty} \frac{\sin x \cdot \cos x}{x^s}\,dx$ (see Problem **8.1.13**).

8.4.11. *Hint.* Integral (2) can diverge. For example, let

$$\varphi(x) = \begin{cases} 1, & 2n\pi \leqslant x \leqslant (2n+1)\pi, \\ -1, & (2n+1)\pi < x < (2n+2)\pi. \end{cases}$$

The integral $\int_0^{\infty} \frac{\sin x}{x}\,dx$ converges (see Problem **8.1.13**). But $\int_0^{\infty} \frac{\sin x}{x}\varphi(x)\,dx = \int_0^{\infty} \frac{|\sin x|}{x}\,dx$ diverges (see the same problem). But if the integral $\int_a^{\infty} f(x)\,dx$ converges absolutely, then the integral $\int_a^{\infty} f(x)\varphi(x)\,dx$ also converges absolutely: if $|\varphi(x)|<C$, then $|f(x)\varphi(x)|<C|f(x)|$, and it remains to use the comparison theorem.

8.4.12 *Hint.* Transform the integral $I(x)$ into $I(x) = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}-x} \ln \sin z\,dz$ by the substitution $y=\frac{\pi}{2}-z$. Taking into account that $\sin z = 2\sin\frac{z}{2}\cdot\cos\frac{z}{2}$, reduce the above to the sum of three integrals.

8.4.13. *Hint.* Putting $u=\ln\cos x$, $\cos 2nx\,dx = dv$, integrate by parts and get the equality $I_n = \frac{1}{2n}\int_0^{\frac{\pi}{2}} \sin 2nx \frac{\sin x}{\cos x}\,dx$, $n \neq 0$. Since

$$\sin 2nx = \sin(2n-2)x\cdot\cos 2x + \sin 2x\cdot\cos(2n-2)x,$$

$$I_n=\frac{1}{2n}\left[-\int_0^{\frac{\pi}{2}}\sin(2n-2)x\frac{\sin x}{\cos x}dx+\right.$$

$$\left.+\int_0^{\frac{\pi}{2}}\sin(2n-2)x\cdot\sin 2x\,dx+2\int_0^{\frac{\pi}{2}}\sin^2 x\cdot\cos(2n-2)x\,dx\right].$$

Check by direct calculation that for $n\geqslant 2$ the second and the third summands equal zero. Therefore, for $n\geqslant 2$

$$I_n=-\frac{1}{2n}\int_0^{\frac{\pi}{2}}\sin(2n-2)x\frac{\sin x}{\cos x}dx=-\frac{n-1}{n}I_{n-1}.$$

Since $I_1=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin 2x\frac{\sin x}{\cos x}dx=\frac{\pi}{4}$ we have $I_2=-\frac{1}{2}\cdot\frac{\pi}{4}$; $I_3=\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{\pi}{4}=\frac{\pi}{3\cdot 4}$

and by induction, $I_n=(-1)^{n-1}\frac{\pi}{4n}$.