# 9.11 Triple Integral

Functions of three variables: $f(x,y,z)$, $g(x,y,z)$, ...

Triple integrals: $\iiint\limits_G f(x,y,z)dV$, $\iiint\limits_G g(x,y,z)dV$, ...

Riemann sum: $\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(u_i, v_j, w_k)\Delta x_i \Delta y_j \Delta z_k$

Small changes: $\Delta x_i$, $\Delta y_j$, $\Delta z_k$

Limits of integration: a, b, c, d, r, s

Regions of integration: G, T, S

Cylindrical coordinates: $r$, $\theta$, $z$

Spherical coordinates: $r$, $\theta$, $\varphi$

Volume of a solid: V

Mass of a solid: m

Density: $\mu(x,y,z)$

Coordinates of center of mass: $\bar{x}$, $\bar{y}$, $\bar{z}$

First moments: $M_{xy}$, $M_{yz}$, $M_{xz}$

Moments of inertia: $I_{xy}$, $I_{yz}$, $I_{xz}$, $I_x$, $I_y$, $I_z$, $I_0$

**1099.** Definition of Triple Integral
The triple integral over a parallelepiped $[a, b]\times[c, d]\times[r, s]$ is defined to be

$$\iiint\limits_{[a,\,b]\times[c,\,d]\times[r,\,s]} f(x,y,z)dV = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_j \to 0 \\ \max \Delta z_k \to 0}} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(u_i, v_j, w_k)\Delta x_i \Delta y_j \Delta z_k,$$

where $(u_i, v_j, w_k)$ is some point in the parallelepiped $(x_{i-1}, x_i)\times(y_{j-1}, y_j)\times(z_{k-1}, z_k)$, and $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, $\Delta z_k = z_k - z_{k-1}$.

**1100.** $\iiint\limits_G [f(x,y,z)+g(x,y,z)]dV = \iiint\limits_G f(x,y,z)dV + \iiint\limits_G g(x,y,z)dV$

**1101.** $$\iiint\limits_G [f(x,y,z)-g(x,y,z)]dV = \iiint\limits_G f(x,y,z)dV - \iiint\limits_G g(x,y,z)dV$$

**1102.** $$\iiint\limits_G kf(x,y,z)dV = k\iiint\limits_G f(x,y,z)dV,$$

where k is a constant.

**1103.** If $f(x,y,z)\geq 0$ and G and T are nonoverlapping basic regions, then

$$\iiint\limits_{G\cup T} f(x,y,z)dV = \iiint\limits_G f(x,y,z)dV + \iiint\limits_T f(x,y,z)dV.$$

Here $G\cup T$ is the union of the regions G and T.

**1104.** Evaluation of Triple Integrals by Repeated Integrals

If the solid G is the set of points $(x,y,z)$ such that $(x,y)\in R$, $\chi_1(x,y)\leq z\leq \chi_2(x,y)$, then

$$\iiint\limits_G f(x,y,z)dxdydz = \iint\limits_R \left[\int_{\chi_1(x,y)}^{\chi_2(x,y)} f(x,y,z)dz\right]dxdy,$$

where R is projection of G onto the xy-plane.

If the solid G is the set of points $(x,y,z)$ such that $a\leq x\leq b$, $\varphi_1(x)\leq y\leq \varphi_2(x)$, $\chi_1(x,y)\leq z\leq \chi_2(x,y)$, then

$$\iiint\limits_G f(x,y,z)dxdydz = \int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)}\left(\int_{\chi_1(x,y)}^{\chi_2(x,y)} f(x,y,z)dz\right)dy\right]dx$$

**1105.** Triple Integrals over Parallelepiped

If G is a parallelepiped $[a,\, b]\times[c,\, d]\times[r,\, s]$, then

$$\iiint\limits_G f(x,y,z)dxdydz = \int_a^b\left[\int_c^d\left(\int_r^s f(x,y,z)dz\right)dy\right]dx.$$

In the special case where the integrand $f(x,y,z)$ can be written as $g(x)h(y)k(z)$ we have

$$\iiint_G f(x,y,z)dxdydz = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right)\left(\int_r^s k(z)dz\right).$$

**1106.** Change of Variables

$$\iiint_G f(x,y,z)dxdydz =$$

$$= \iiint_S f[x(u,v,w), y(u,v,w), z(u,v,w)]\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|dxdydz,$$

where $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0$ is the jacobian of the transformations $(x,y,z) \to (u,v,w)$, and S is the pull-back of G which can be computed by $x = x(u,v,w)$, $y = y(u,v,w)$
$z = z(u,v,w)$ into the definition of G.

**1107.** Triple Integrals in Cylindrical Coordinates

The differential dxdydz for cylindrical coordinates is

$$dxdydz = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right|drd\theta dz = rdrd\theta dz.$$

Let the solid G is determined as follows:
$(x,y) \in R,\ \chi_1(x,y) \le z \le \chi_2(x,y)$,
where R is projection of G onto the xy-plane. Then

$$\iiint_G f(x,y,z)dxdydz = \iiint_S f(r\cos\theta, r\sin\theta, z)rdrd\theta dz$$

$$= \iint\limits_{R(r,\theta)} \left[ \int\limits_{\chi_1(r\cos\theta, r\sin\theta)}^{\chi_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)dz \right] rdrd\theta .$$

Here S is the pullback of G in cylindrical coordinates.

**1108.** Triple Integrals in Spherical Coordinates

The Differential dxdydz for Spherical Coordinates is

$$dxdydz = \left| \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} \right| drd\theta d\varphi = r^2 \sin\theta drd\theta d\varphi$$

$$\iiint\limits_G f(x,y,z)dxdydz =$$

$$= \iiint\limits_S f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) r^2 \sin\theta drd\theta d\varphi ,$$

where the solid S is the pullback of G in spherical coordinates. The angle $\theta$ ranges from 0 to $2\pi$, the angle $\varphi$ ranges from 0 to $\pi$.

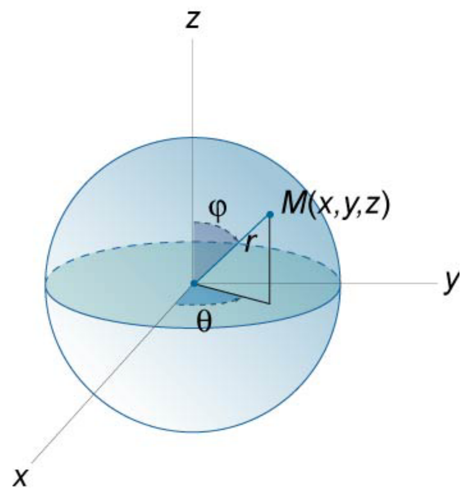


**Figure 202.**

**1109.** Volume of a Solid

$$V = \iiint_G dxdydz$$

**1110.** Volume in Cylindrical Coordinates

$$V = \iiint_{S(r,\theta,z)} r dr d\theta dz$$

**1111.** Volume in Spherical Coordinates

$$V = \iiint_{S(r,\theta,\varphi)} r^2 \sin\theta dr d\theta d\varphi$$

**1112.** Mass of a Solid

$$m = \iiint_G \mu(x,y,z) dV,$$

where the solid occupies a region $G$ and its density at a point $(x,y,z)$ is $\mu(x,y,z)$.

**1113.** Center of Mass of a Solid

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m},$$

where

$$M_{yz} = \iiint_G x\mu(x,y,z) dV,$$

$$M_{xz} = \iiint_G y\mu(x,y,z) dV,$$

$$M_{xy} = \iiint_G z\mu(x,y,z) dV$$

are the first moments about the coordinate planes $x = 0$, $y = 0$, $z = 0$, respectively, $\mu(x,y,z)$ is the density function.

**1114.** Moments of Inertia about the xy-plane (or $z = 0$), yz-plane ($x = 0$), and xz-plane ($y = 0$)

$$I_{xy} = \iiint_G z^2 \mu(x,y,z)\,dV,$$

$$I_{yz} = \iiint_G x^2 \mu(x,y,z)\,dV,$$

$$I_{xz} = \iiint_G y^2 \mu(x,y,z)\,dV.$$

**1115.** **Moments of Inertia about the x-axis, y-axis, and z-axis**

$$I_x = I_{xy} + I_{xz} = \iiint_G \left(z^2 + y^2\right)\mu(x,y,z)\,dV,$$

$$I_y = I_{xy} + I_{yz} = \iiint_G \left(z^2 + x^2\right)\mu(x,y,z)\,dV,$$

$$I_z = I_{xz} + I_{yz} = \iiint_G \left(y^2 + x^2\right)\mu(x,y,z)\,dV.$$

**1116.** **Polar Moment of Inertia**

$$I_0 = I_{xy} + I_{yz} + I_{xz} = \iiint_G \left(x^2 + y^2 + z^2\right)\mu(x,y,z)\,dV$$