1.1 Logarithms

1.1.1 Definition

"The Logarithm of a given number to a given base is the index of the power to which the base must be raised in order to equal the given number."

If $a > 0$ and $\neq 1$, then logarithm of a positive number *N* is defined as the index *x* of that power of '*a*' which equals *N* *i.e.*, $\mathbf{\log_a N = x \text{ iff } a^x = N \Rightarrow a^{\log_a N} = N, a > 0, a \neq 1}$ and $\mathbf{N > 0}$

It is also known as fundamental logarithmic identity.

The function defined by $f(x) = \log_a x,\ a > 0, a \neq 1$ is called logarithmic function.

Its domain is $(0, \infty)$ and range is R. *a* is called the base of the logarithmic function.

When base is '*e*' then the logarithmic function is called natural or Napierian logarithmic function and when base is 10, then it is called common logarithmic function.

Note : ❑ The logarithm of a number is unique *i.e.* No number can have two different log to a given base.

❑ $\log_e a = \log_e 10 . \log_{10} a$ or $\log_{10} a = \dfrac{\log_e a}{\log_e 10} = 0.434 \log_e a$

1.1.2 Characteristic and Mantissa

(1) The integral part of a logarithm is called the characteristic and the fractional part is called mantissa.

$$\log_{10} N = \underset{\text{Characterstics}}{\underset{\downarrow}{\text{integer}}} + \underset{\text{Mantissa}}{\underset{\downarrow}{\text{fraction } (+ve)}}$$

(2) The mantissa part of log of a number is always kept positive.

(3) If the characteristics of $\log_{10} N$ be *n*, then the number of digits in *N* is (*n*+1)

(4) If the characteristics of $\log_{10} N$ be (– *n*) then there exists (*n* – 1) number of zeros after decimal part of *N*.

Example: 1 For $y = \log_a x$ to be defined '*a*' must be [IIT 1990]

(a) Any positive real number (b) Any number

(c) $\geq e$ (d) Any positive real number $\neq 1$

Solution: (d) It is obvious (Definition).

Example: 2 Logarithm of $32\sqrt[5]{4}$ to the base $2\sqrt{2}$ is

(a) 3.6 (b) 5 (c) 5.6 (d) None of these

Solution: (a) Let *x* be the required logarithm , then by definition $(2\sqrt{2})^x = 32\sqrt[5]{4}$

$(2.2^{1/2})^x = 2^5.2^{2/5}$; $\therefore$ $2^{\frac{3x}{2}} = 2^{5+\frac{2}{5}}$

Here, by equating the indices, $\frac{3}{2}x = \frac{27}{5}$, $\therefore x = \frac{18}{5} = 3.6$

1.1.3 Properties of Logarithms

Let *m* and *n* be arbitrary positive numbers such that $a > 0,\ a \neq 1,\ b > 0,\ b \neq 1$ then

(1) $\log_a a = 1,\ \log_a 1 = 0$

(2) $\log_a b.\log_b a = 1 = \log_a a = \log_b b \Rightarrow \log_a b = \frac{1}{\log_b a}$

(3) $\log_c a = \log_b a.\ \log_c b$ or $\log_c a = \frac{\log_b a}{\log_b c}$

(4) $\log_a (mn) = \log_a m + \log_a n$

(5) $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

(6) $\log_a m^n = n \log_a m$

(7) $a^{\log_a m} = m$

(8) $\log_a \left(\frac{1}{n}\right) = -\log_a n$

(9) $\log_{a^\beta} n = \frac{1}{\beta} \log_a n$

(10) $\log_{a^\beta} n^\alpha = \frac{\alpha}{\beta} \log_a n$, $(\beta \neq 0)$

(11) $a^{\log_c b} = b^{\log_c a}$, $(a, b, c > 0$ and $c \neq 1)$

Example: 3 The number $\log_2 7$ is [IIT 1990]

(a) An integer (b) A rational number (c) An irrational number (d) A prime number

Solution: (c) Suppose, if possible, $\log_2 7$ is rational, say p/q where *p* and *q* are integers, prime to each other.

Then, $\frac{p}{q} = \log_2 7 \quad \Rightarrow 7 = 2^{p/q} \quad \Rightarrow 2^p = 7^q$,

Which is false since L.H.S is even and R.H.S is odd. Obviously $\log_2 7$ is not an integer and hence not a prime number

Example: 4 If $\log_7 2 = m$, then $\log_{49} 28$ is equal to [Roorkee 1999]

(a) $2(1+2m)$ (b) $\frac{1+2m}{2}$ (c) $\frac{2}{1+2m}$ (d) $1+m$

Solution: (b) $\log_{49} 28 = \frac{\log 28}{\log 49} = \frac{\log 7 + \log 4}{2 \log 7} = \frac{\log 7}{2 \log 7} + \frac{\log 4}{2 \log 7} = \frac{1}{2} + \frac{1}{2}\log_7 4 = \frac{1}{2} + \frac{1}{2}.2\log_7 2 = \frac{1}{2} + \log_7 2 = \frac{1}{2} + m = \frac{1+2m}{2}$

Example: 5 If $\log_e\left(\frac{a+b}{2}\right) = \frac{1}{2}(\log_e a + \log_e b)$, then relation between *a* and *b* will be [UPSEAT 2000]

(a) $a = b$ (b) $a = \frac{b}{2}$ (c) $2a = b$ (d) $a = \frac{b}{3}$

Solution: (a) $\log_e\left(\frac{a+b}{2}\right) = \frac{1}{2}(\log_e a + \log_e b) = \frac{1}{2}\log_e(ab) = \log_e \sqrt{ab}$

$\Rightarrow \frac{a+b}{2} = \sqrt{ab} \Rightarrow a + b = 2\sqrt{ab} \Rightarrow \left(\sqrt{a} - \sqrt{b}\right)^2 = 0 \Rightarrow \sqrt{a} - \sqrt{b} = 0 \Rightarrow a = b$

Example: 6 If $\log_{10} 3 = 0.477$, the number of digits in 3^{40} is [IIT 1992]

(a) 18 (b) 19 (c) 20 (d) 21

Solution: (c) Let $y = 3^{40}$ is

Taking log both the sides, $\log y = \log 3^{40} \Rightarrow \log y = 40 \log 3 \Rightarrow \log y = 19.08$

$\therefore$ Number of digits in $y = 19 + 1 = 20$

Example: 7 Which is the correct order for a given number α in increasing order [Roorkee 2000]

(a) $\log_2 \alpha, \log_3 \alpha, \log_e \alpha, \log_{10} \alpha$ (b) $\log_{10} \alpha, \log_3 \alpha, \log_e \alpha, \log_2 \alpha$

(c) $\log_{10} \alpha, \log_e \alpha, \log_2 \alpha, \log_3 \alpha$ (d) $\log_3 \alpha, \log_e \alpha, \log_2 \alpha, \log_{10} \alpha$

Solution: (b) Since 10, 3, *e*, 2 are in decreasing order

Obviously, $\log_{10} \alpha, \log_3 \alpha, \log_e \alpha, \log_2 \alpha$ are in increasing order.

1.1.4 Logarithmic Inequalities

(1) If $a > 1, p > 1 \Rightarrow \log_a p > 0$

(2) If $0 < a < 1, p > 1 \Rightarrow \log_a p < 0$

(3) If $a > 1,\ 0 < p < 1 \Rightarrow \log_a p < 0$

(4) If $p > a > 1 \Rightarrow \log_a p > 1$

(5) If $a > p > 1 \Rightarrow 0 < \log_a p < 1$

(6) If $0 < a < p < 1 \Rightarrow 0 < \log_a p < 1$

(7) If $0 < p < a < 1 \Rightarrow \log_a p > 1$

(8) If $\log_m a > b \Rightarrow \begin{cases} a > m^b, & \text{if } m > 1 \\ a < m^b, & \text{if } 0 < m < 1 \end{cases}$

(9) $\log_m a < b \Rightarrow \begin{cases} a < m^b, & \text{if } m > 1 \\ a > m^b, & \text{if } 0 < m < 1 \end{cases}$

(10) $\log_p a > \log_p b \Rightarrow a \ge b$ if base *p* is positive and >1 or $a \le b$ if base *p* is positive and < 1 *i.e.,* $0 < p < 1$

In other words, if base is greater than 1 then inequality remains same and if base is positive but less than 1 then the sign of inequality is reversed.

Example: 8 If $x = \log_3 5,\ y = \log_{17} 25,$ which one of the following is correct **[W.B. JEE 1993]**

(a) $x < y$ (b) $x = y$ (c) $x > y$ (d) None of these

Solution: (c) $y = \log_{17} 25 = 2\log_{17} 5$

$\therefore \frac{1}{y} = \frac{1}{2}\log_5 17$

$\frac{1}{x} = \log_5 3 = \frac{1}{2}\log_5 9$

Clearly $\frac{1}{y} > \frac{1}{x}$, $\therefore x > y$

Example: 9 If $\log_{0.3}(x-1) < \log_{0.09}(x-1),$ then *x* lies in the interval

(a) $(2, \infty)$ (b) (– 2, –1) (c) (1, 2) (d) None of these

Solution: (a) $\log_{0.3}(x-1) < \log_{(0.3)^2}(x-1) = \frac{1}{2}\log_{0.3}(x-1)$

$\therefore \frac{1}{2}\log_{0.3}(x-1) < 0$

or $\log_{0.3}(x-1) < 0 = \log 1$ or $(x-1) > 1$ or $x > 2$

As base is less than 1, therefore the inequality is reversed, now $x>2 \Rightarrow x$ lies in $(2, \infty)$.

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