Exercise 7.2

Answer 1E.

Consider the following integral:

$$\int \sin^2 x \cos^3 x\, dx$$

Now, evaluate the integral

$$\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x \cos^2 x \cos x\, dx$$
$$= \int \sin^2 x \left(1-\sin^2 x\right) \cos x\, dx$$

Substitute the values of $\sin x$ and $\cos x\, dx$ with t and dt respectively.

$$\int \sin^2 x \cos^3 x\, dx = \int t^2\left(1-t^2\right) dt$$
$$= \int \left(t^2 - t^4\right) dt$$
$$= \frac{t^3}{3} - \frac{t^5}{5} + c$$
$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C \qquad [\text{Replace } t = \sin x]$$

Hence, $\int \sin^2 x \cos^3 x\, dx = \boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C}$.

Answer 2E.

Given $\int \sin^3 \theta \cos^4 \theta\, d\theta$

We have to evaluate the given indefinite integral

We know that

$$\int \sin^{2k+1} x \cos^n x\, dx = \int \left(\sin^2 x\right)^k \cos^n x \sin x\, dx$$
$$= \int \left(1-\cos^2 x\right)^k \cos^n x \sin x\, dx$$

$$\int \sin^3 \theta \cos^4 \theta\, d\theta = \int \cos^4 \theta \left(\sin^2 \theta\right) \sin \theta\, d\theta$$
$$= \int \cos^4 \theta \left(1-\cos^2 \theta\right) \sin \theta\, d\theta$$

Put $\cos\theta = t \quad \Rightarrow \quad -\sin\theta\, d\theta = dt$

$$\int \sin^3\theta \cos^4\theta\, d\theta = \int t^4\left(1-t^2\right)(-dt)$$
$$= \int \left(t^6 - t^4\right) dt$$
$$= \frac{t^7}{7} - \frac{t^5}{5} + c$$
$$= \frac{\cos^7\theta}{7} - \frac{\cos^5\theta}{5} + c$$

Hence $\boxed{\int \sin^3\theta \cos^4\theta\, d\theta = \frac{\cos^7\theta}{7} - \frac{\cos^5\theta}{5} + c}$

Answer 3E.

Given $\int_0^{\pi/2} \sin^7\theta\left(\cos^5\theta\right) d\theta$

We have to evaluate the definite integral

We know that

$$\int \sin^m x \cos^{2k+1} x\, dx = \int \sin^m x\left(\cos^2 x\right)^k \cos x\, dx$$
$$= \int \sin^m x\left(1-\sin^2 x\right)^k \cos x\, dx$$

$$\int_0^{\pi/2} \sin^7\theta\left(\cos^5\theta\right) d\theta = \int_0^{\pi/2} \sin^7\theta \cos^4\theta \cos\theta\, d\theta$$
$$= \int_0^{\pi/2} \sin^7\theta\left(1-\sin^2\theta\right)^2 \cos\theta\, d\theta$$

Put $\sin\theta = t \quad \Rightarrow \quad \cos\theta\, d\theta = dt$

$\theta = 0 \quad \Rightarrow \quad t = 0$ and $\theta = \frac{\pi}{2} \quad \Rightarrow \quad t = 1$

$$\int_0^{\pi/2} \sin^7\theta \cos^5\theta\, d\theta = \int_0^1 t^7\left(1-t^2\right)^2 dt$$
$$= \int_0^1 t^7\left(1+t^4-2t^2\right) dt$$
$$= \left(t^7 + t^{11} - 2t^9\right) dt$$

$$= \left[\frac{t^8}{8} + \frac{t^{12}}{12} - \frac{2t^{10}}{10}\right]_0^1$$
$$= \frac{1}{8} + \frac{1}{12} - \frac{1}{5}$$
$$= \frac{15+10-24}{120}$$
$$= \frac{1}{120}$$

Hence $\boxed{\int_0^{\pi/2} \sin^7\theta\left(\cos^5\theta\right) d\theta = \frac{1}{120}}$

Answer 4E.

Given $\int_0^{\pi/2} \sin^5 x\,dx$

We have to evaluate the given definite integral

We know that

$$\int \sin^{2k+1} x \cos^n x\,dx = \int \left(\sin^2 x\right)^k \cos^n x \sin x dx$$
$$= \int \left(1-\cos^2 x\right)^k \cos^n x \sin x dx$$

Therefore,

$$\int_0^{\pi/2} \sin^5 x\,dx = \int_0^{\pi/2} \sin^4 x \sin x\,dx$$
$$= \int_0^{\pi/2} \left(1-\cos^2 x\right)^2 \sin x\,dx$$

Put $\cos x = t \quad \Rightarrow \quad -\sin x\,dx = dt$

$x = 0 \Rightarrow t = 1$

$x = \pi/2 \Rightarrow t = 0$

$$\int_0^{\pi/2} \sin^5 x\,dx = \int_1^0 \left(1-t^2\right)^2 (-dt)$$
$$= \int_0^1 \left(1+t^4-2t^2\right) dt$$

$$= \left[t + \frac{t^5}{5} - \frac{2t^3}{3}\right]_0^1$$
$$= 1 + \frac{1}{5} - \frac{2}{3}$$
$$= \frac{15+3-10}{15}$$
$$= \frac{8}{15}$$

Hence $\boxed{\int_0^{\pi/2} \sin^5 x\,dx = \frac{8}{15}}$

Answer 5E.

Consider the integral,

$\int \sin^2(\pi x)\cos^5(\pi x)\,dx$.

Need to evaluate the above integral.

Let $u = \pi x$.

Then,

$$du = \pi dx$$
$$dx = \frac{1}{\pi}dx$$

So the integral reduced as,

$$\begin{aligned}\int \sin^2(\pi x)\cos^5(\pi x)\,dx &= \int \sin^2 u \cos^5 u \frac{1}{\pi}du \\ &= \frac{1}{\pi}\int \sin^2 u \cos^5 u\,du \\ &= \frac{1}{\pi}\int \sin^2 u(\cos^2 u)^2 \cos u\,du \\ &= \frac{1}{\pi}\int \sin^2 u(1-\sin^2 u)^2 \cos u\,du\end{aligned}$$

$$= \frac{1}{\pi}\int \sin^2 u(1+\sin^4 u - 2\sin^2 u)\cos u\,du$$

$$= \frac{1}{\pi}\int (\sin^2 u + \sin^6 u - 2\sin^4 u)\cos u\,du \quad \text{...... (1)}$$

Use substitution method: Let $\sin u = t$

Then,

$$\cos(u)\,du = dt$$

So the integral (1) turns as,

$$\begin{aligned}\int \sin^2(\pi x)\cos^5(\pi x)\,dx &= \frac{1}{\pi}\int (\sin^2 u + \sin^6 u - 2\sin^4 u)\cos u\,du \\ &= \frac{1}{\pi}\int (t^2 + t^6 - 2t^4)\,dt \\ &= \frac{1}{\pi}\left[\int t^2 dt + \int t^6 dt - 2\int t^4 dt\right] \\ &= \frac{1}{\pi}\left[\frac{t^3}{3} + \frac{t^7}{7} - 2\left(\frac{t^5}{5}\right)\right] + C\end{aligned}$$

$$= \frac{1}{\pi}\left[\frac{\sin^3 u}{3} + \frac{\sin^7 u}{7} - \frac{2\sin^5 u}{5}\right] + C \quad \text{Use } \sin u = t$$

$$= \frac{1}{\pi}\left[\frac{\sin^3 \pi x}{3} + \frac{\sin^7 \pi x}{7} - \frac{2\sin^5 \pi x}{5}\right] + C$$

Hence, $\int \sin^2(\pi x)\cos^5(\pi x)\,dx = \boxed{\frac{1}{\pi}\left[\frac{\sin^3 \pi x}{3} + \frac{\sin^7 \pi x}{7} - \frac{2\sin^5 \pi x}{5}\right] + C}$.

Answer 6E.

We have to evaluate the integral $\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}}dx$. The first step in evaluating trigonometric functions is often to try to reduce the function to a lesser power that we know the integral of.

$$\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}}dx$$

$$= \int \frac{\sin^2(\sqrt{x})\sin(\sqrt{x})}{\sqrt{x}}dx$$

$$= \int \frac{(1-\cos^2\sqrt{x})\sin(\sqrt{x})}{\sqrt{x}}dx \text{(Using the trigonometric identity: } \sin^2 x + \cos^2 x = 1)$$

We can now separate this function into two integrals.

$$= \int \frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}dx - \int \frac{\cos^2\left(\sqrt{x}\right)\sin\left(\sqrt{x}\right)}{\sqrt{x}}dx$$

Looking at the first integral:

$$= \int \frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}dx \quad \left(\text{Taking } u = \sqrt{x} \text{ and } du = \frac{1}{2\sqrt{x}}dx\right)$$

$$= 2\int \sin u\,du$$

$$= -2\cos u + C_1$$

$$= -2\cos\left(\sqrt{x}\right) + C_1$$

Looking at the second integral:

$$= \int \frac{\cos^2\left(\sqrt{x}\right)\sin\left(\sqrt{x}\right)}{\sqrt{x}}dx \quad \left(\text{Taking } u = \sqrt{x} \text{ and } du = \frac{1}{2\sqrt{x}}dx\right)$$

$$= 2\int \cos^2(u)\sin(u)\,du \quad (\text{Taking } v = \cos u \text{ and } dv = -\sin u\,du)$$

$$= -2\int v^2 dv$$

$$= -\frac{2}{3}v^3 + C_2$$

$$= -\frac{2}{3}\cos^3\left(\sqrt{x}\right) + C_2$$

Finally, uniting both integrals, we get:

$$\boxed{= -2\cos\left(\sqrt{x}\right) + \frac{2}{3}\cos^3\left(\sqrt{x}\right) + C, C = C_1 - C_2}$$

Answer 7E.

Consider the integral $\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta.$

Need to evaluate the given integral.

Then

$$\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)d\theta \quad \left[\text{Since } \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)\right]$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\sin 2\left(\frac{\pi}{2}\right)}{2} - 0 - \frac{\sin 2(0)}{2}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\sin \pi}{2} - 0 - \frac{\sin(0)}{2}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{0}{2} - 0 - \frac{0}{2}\right]$$

$$= \frac{\pi}{4}$$

Hence the required value of the given integral is $\boxed{\frac{\pi}{4}}$.

Consider the following integral:

$$\int_0^{2\pi} \sin^2\left(\frac{1}{3}\theta\right) d\theta$$

Use the half-angle formula $\sin^2\left(\frac{1}{3}\theta\right) = \frac{1-\cos 2\left(\frac{1}{3}\theta\right)}{2}$.

$$\int_0^{2\pi} \sin^2\left(\frac{1}{3}\theta\right) d\theta = \frac{1}{2}\int_0^{2\pi}\left(1-\cos 2\left(\frac{1}{3}\theta\right)\right) d\theta$$

$$= \frac{1}{2}\left[\int_0^{2\pi} d\theta - \int_0^{2\pi} \cos 2\left(\frac{1}{3}\theta\right) d\theta\right]$$

$$= \frac{1}{2}[\theta]_0^{2\pi} - \frac{1}{2}\left[\frac{\sin 2\left(\frac{1}{3}\theta\right)}{\left(\frac{2}{3}\right)}\right]_0^{2\pi}$$

$$= \frac{1}{2}[\theta]_0^{2\pi} - \frac{1}{2}\frac{3}{2}\left[\sin 2\left(\frac{1}{3}\theta\right)\right]_0^{2\pi}$$

$$= \frac{1}{2}[2\pi - 0] - \frac{1}{2}\frac{3}{2}\left[\sin 2\left(\frac{1}{3}(2\pi)\right)\right]$$

$$= \pi - \frac{3}{4}\sin\frac{4\pi}{3}$$

Therefore, $\int_0^{2\pi} \sin^2\left(\frac{1}{3}\theta\right) d\theta = \boxed{\pi - \frac{3}{4}\sin\frac{4\pi}{3}}$

Answer 8E.

Consider the following integral:

$$\int_0^{2\pi} \sin^2\left(\frac{1}{3}\theta\right) d\theta$$

Use the half-angle formula $\sin^2\left(\frac{1}{3}\theta\right) = \frac{1-\cos 2\left(\frac{1}{3}\theta\right)}{2}$.

$$\int_0^{2\pi} \sin^2\left(\frac{1}{3}\theta\right) d\theta = \frac{1}{2}\int_0^{2\pi}\left(1-\cos 2\left(\frac{1}{3}\theta\right)\right) d\theta$$

$$= \frac{1}{2}\left[\int_0^{2\pi} d\theta - \int_0^{2\pi} \cos 2\left(\frac{1}{3}\theta\right) d\theta\right]$$

$$= \frac{1}{2}[\theta]_0^{2\pi} - \frac{1}{2}\left[\frac{\sin 2\left(\frac{1}{3}\theta\right)}{\left(\frac{2}{3}\right)}\right]_0^{2\pi}$$

$$= \frac{1}{2}[\theta]_0^{2\pi} - \frac{1}{2}\frac{3}{2}\left[\sin 2\left(\frac{1}{3}\theta\right)\right]_0^{2\pi}$$

$$= \frac{1}{2}[2\pi - 0] - \frac{1}{2}\frac{3}{2}\left[\sin 2\left(\frac{1}{3}(2\pi)\right)\right]$$

$$= \pi - \frac{3}{4}\sin\frac{4\pi}{3}$$

Therefore, $\int_0^{2\pi} \sin^2\left(\frac{1}{3}\theta\right) d\theta = \boxed{\pi - \frac{3}{4}\sin\frac{4\pi}{3}}$

Answer 9E.

Given $\int_0^{\pi} \cos^4(2t)\, dt$

We have to evaluate the given definite integral

Put

$2t = \theta$

$\Rightarrow 2dt = d\theta$

$\Rightarrow dt = \frac{1}{2}d\theta$

If $t = 0$ then $\theta = 0$

If $t = \pi$ *then* $\theta = 2\pi$

Therefore,

$$\int_0^{\pi} \cos^4(2t)\,dt = \int_0^{2\pi} \cos^4\theta \frac{1}{2}d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(\cos^2\theta\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{1}{8}\int_0^{2\pi} \left(1+\cos^2 2\theta + 2\cos 2\theta\right)d\theta$$

$$= \frac{1}{8}\left[\int_0^{2\pi} (1+2\cos 2\theta)d\theta + \int_0^{2\pi} \cos^2 2\theta d\theta\right]$$

$$= \frac{1}{8}\left\{\left[\theta + 2\frac{\sin 2\theta}{2}\right]_0^{2\pi} + \int_0^{2\pi}\left(\frac{1+\cos 4\theta}{2}\right)d\theta\right\}$$

$$= \frac{1}{8}(2\pi) + \frac{1}{16}\left[\theta + \frac{\sin 4\theta}{4}\right]_0^{2\pi}$$

$$= \frac{\pi}{4} + \frac{\pi}{8}$$

$$= \frac{3\pi}{8}$$

Hence $\boxed{\int_0^{\pi} \cos^4(2t)\,dt = \frac{3\pi}{8}}$

Answer 10E.

We have to evaluate the following integral

$$\int_0^{\pi} \sin^2 t \cos^4 t dt$$

$$\int_0^{\pi} \sin^2 t \cos^4 t dt = \int_0^{\pi} \left(1-\cos^2 t\right)\cos^4 t dt \quad \left(\text{Using: } \sin^2 t + \cos^2 t = 1\right)$$

$$= \int_0^{\pi} \left(\cos^4 t - \cos^6 t\right)dt$$

Separating and using $\cos^2 t = \frac{1}{2}(1+\cos 2t)$ in

$$\int_0^{\pi} \cos^4 t\, dt - \int_0^{\pi} \cos^6 t\, dt$$

Let's look at the first integral first:

$$\int_0^{\pi} \cos^4 t dt = \int_0^{\pi} \frac{1}{4}(1+\cos 2t)^2 dt \qquad \text{(Using the power reduction formula)}$$

$$= \frac{1}{4}\int_0^{\pi} \left(\cos^2 2t + 2\cos 2t + 1\right)dt$$

$$= \frac{1}{4}\int_0^{\pi} \left[\left(\frac{1}{2}(1+\cos 4t)\right) + 2\cos 2t + 1\right]dt \text{(Using the power reduction formula)}$$

$$=\frac{1}{4}\left[\frac{t}{2}+\frac{\sin 4t}{8}+\sin 2t+t\right]_0^{\pi}$$

$$=\frac{1}{4}\left[\frac{\pi}{2}+\pi\right]$$

$$=\frac{1}{4}\left[\frac{3\pi}{2}\right]$$

$$=\frac{3\pi}{8}$$

Looking now at the second integral:

$$\int_0^{\pi}\cos^6 t\,dt=\int_0^{\pi}\frac{1}{8}(1+\cos 2t)^3 dt \qquad \text{(Using the power reduction formula)}$$

$$=\frac{1}{8}\int_0^{\pi}(\cos^3 2t+3\cos^2 2t+3\cos 2t+1)dt$$

$$=\frac{1}{8}\int_0^{\pi}(\cos 2t\left(1-\sin^2 2t\right)+3\cos^2 2t+3\cos 2t+1)dt$$

$$=\frac{1}{8}\int_0^{\pi}\left[\cos 2t-\cos 2t\sin^2 2t+\frac{3}{2}+\frac{3}{2}\cos 4t+3\cos 2t+1\right]dt$$

(Using the power reduction formula)

$$=\left[\frac{1}{16}\sin 2t-\frac{1}{48}\sin^3 2t+\frac{3t}{16}-\frac{3}{64}\sin 4t+\frac{3}{16}\sin 2t+\frac{t}{8}\right]_0^{\pi}$$

$$=\frac{5\pi}{16}$$

Putting them both together:

$$\int_0^{\pi}\cos^4 t\,dt-\int_0^{\pi}\cos^6 t\,dt=\boxed{\frac{\pi}{16}}$$

Answer 11E.

Consider the integral $\int_0^{\pi/2}\sin^2 x\cos^2 x\,dx$

$$\int_0^{\pi/2}\sin^2 x\cos^2 x\,dx=\int_0^{\pi/2}(\sin x\cos x)^2\,dx$$

$$=\int_0^{\pi/2}\left(\frac{1}{2}2\sin x.\cos x\right)^2 dx$$

$$=\frac{1}{4}\int_0^{\pi/2}(\sin 2x)^2\,dx \quad \text{Since } \sin 2x=2\sin x\cos x$$

$$=\frac{1}{4}\int_0^{\pi/2}\sin^2 2x\,dx$$

$$=\frac{1}{4}\int_0^{\pi/2}\frac{1}{2}(1-\cos 4x)dx \quad \text{Since } 1-\cos 2x=2\sin^2 x$$

$$1-\cos 4x=2\sin^2(2x)$$

$$=\frac{1}{8}\int_0^{\pi/2}dx-\frac{1}{8}\int_0^{\pi/2}\cos 4x\,dx$$

Thus

$$\int_0^{\pi/2} \sin^2 x \cos^2 x dx = \frac{1}{8}[x]_0^{\pi/2} - \frac{1}{8}\left[\frac{\sin 4x}{4}\right]_0^{\pi/2} \quad \text{Since } \int \cos nx dx = \frac{\sin nx}{n} + C$$

$$= \frac{1}{8}\left(\frac{\pi}{2} - 0\right) - \frac{1}{32}\left(\sin 4\frac{\pi}{2} - \sin 0\right)$$

$$= \frac{\pi}{16} - \frac{1}{32}(\sin 2\pi - 0) \quad [\sin 0 = 0]$$

$$= \frac{\pi}{16} \quad [\sin 2\pi = 0]$$

Therefore, $\int_0^{\pi/2} \sin^2 x \cos^2 x dx = \boxed{\frac{\pi}{16}}$

Answer 12E.

Given $\int_0^{\pi/2} (2 - \sin\theta)^2 d\theta$

We have to evaluate the given definite integral

Now

$$\int_0^{\pi/2} (2 - \sin\theta)^2 d\theta = \int_0^{\pi/2} \left(4 + \sin^2\theta - 4\sin\theta\right) d\theta$$

$$= \int_0^{\pi/2} \left[4 + \frac{1 - \cos 2\theta}{2} - 4\sin\theta\right]$$

$$= \left[4\theta + \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) - 4(-\cos\theta)\right]_0^{\pi/2}$$

$$= \frac{4\pi}{2} + \frac{\pi}{4} - 4$$

$$= 2\pi - 4 + \frac{\pi}{4}$$

$$= \frac{9\pi}{4} - 4$$

Hence $\boxed{\int_0^{\pi/2} (2 - \sin\theta)^2 d\theta = \frac{9\pi}{4} - 4}$

Answer 13E.

Given $\int t \sin^2 t \, dt$

We have to evaluate the given definite integral

Now

$$\int t \sin^2 t \, dt = \int t\left(\frac{1 - \cos 2t}{2}\right) dt$$

$$= \frac{1}{2}\left[\int (t - t\cos 2t) dt\right]$$

$$= \frac{1}{2}\left[\frac{t^2}{2} - \int (t\cos 2t) dt\right]$$

$$= \frac{1}{2}\left[\frac{t^2}{2} - \left(t\frac{\sin 2t}{2} + \frac{\cos 2t}{4}\right)\right] + c$$

$$=\frac{1}{2}\left[\frac{t^2}{2}-\frac{t}{2}\sin 2t-\frac{1}{4}\cos 2t\right]+c$$

$$=\frac{t^2}{4}-\frac{t}{4}\sin 2t-\frac{1}{8}\cos 2t+c$$

Hence $\boxed{\int t\sin^2 t\,dt=\frac{t^2}{4}-\frac{t}{4}\sin 2t-\frac{1}{8}\cos 2t+c}$

Answer 14E.

Consider the following integral:

$$\int \cos\theta\cos^5(\sin\theta)\,d\theta$$

Substitute $u=\sin\theta$, then, $du=\cos\theta d\theta.$

$$\int \cos\theta\cos^5(\sin\theta)\,d\theta=\int \cos^5(u)\,du$$

$$=\int \cos^4(u)\cos(u)\,du$$

$$=\int \cos^2(u)\cos^2(u)\cos(u)\,du$$

$$=\int(1-\sin^2(u))(1-\sin^2(u))\cos(u)\,du \quad \text{Since, } \cos^2(u)=1-\sin^2(u)$$

$$=\int(1-\sin^2(u))^2\cos(u)\,du$$

$$=\int(1+\sin^4(u)-2\sin^2(u))\cos(u)\,du$$

$$=\int(\cos(u)+\sin^4(u)\cos(u)-2\sin^2(u)\cos(u))\,du$$

Substitute, $x=\sin(u)$ and $dx=\cos(u)du.$

$$=\int\cos(u)du+\int\sin^4(u)\cos(u)du-2\int\sin^2(u)\cos(u)du$$

$$=\sin(u)+\int x^4dx-2\int x^2dx$$

$$=\sin(u)+\frac{x^5}{5}-2\frac{x^3}{3}+C \quad \text{Since, } \int x^n dx=\frac{x^{n+1}}{n+1}+C$$

Replace, $u=\sin\theta.$

$$=\sin(\sin\theta)+\frac{x^5}{5}-2\frac{x^3}{3}+C$$

Replace, $x=\sin(u).$

$$=\sin(\sin\theta)+\frac{\sin^5(u)}{5}-2\frac{\sin^3(u)}{3}+C$$

Replace, $u=\sin\theta.$

$$=\sin(\sin\theta)+\frac{\sin^5(\sin\theta)}{5}-2\frac{\sin^3(\sin\theta)}{3}+C$$

Therefore, $\int \cos\theta\cos^5(\sin\theta)\,d\theta=\boxed{\sin(\sin\theta)+\frac{\sin^5(\sin\theta)}{5}-2\frac{\sin^3(\sin\theta)}{3}+C}.$

Answer 15E.

Consider the following integral:

$$\int \frac{\cos^5 \alpha}{\sqrt{\sin \alpha}} d\alpha$$

Evaluate the integral as shown below:

$$\begin{aligned}\int \frac{\cos^5 \alpha}{\sqrt{\sin \alpha}} d\alpha &= \int \frac{(\cos^2 \alpha)^2 \cos \alpha}{\sqrt{\sin \alpha}} d\alpha \\ &= \int \frac{(1-\sin^2 \alpha)^2 \cos \alpha}{\sqrt{\sin \alpha}} d\alpha\end{aligned}$$

Put $\sin \alpha = u$ so that, $\cos \alpha \, d\alpha = du$.

Substitute these values in above integral.

$$\begin{aligned}\int \frac{\cos^5 \alpha}{\sqrt{\sin \alpha}} d\alpha &= \int \frac{(1-u^2)^2}{\sqrt{u}} du \\ &= \int \frac{(1-2u^2+u^4)}{\sqrt{u}} du \\ &= \int \left(u^{-1/2} - 2u^{3/2} + u^{7/2}\right) du \\ &= 2u^{1/2} - 2\cdot\frac{2}{5}u^{5/2} + \frac{2}{9}u^{9/2} + C\end{aligned}$$

$$= \frac{2}{45}u^{1/2}\left(45 - 18u^2 + 5u^4\right) + C$$

$$= \boxed{\frac{2}{45}\sqrt{\sin \alpha}\left(45 - 18\sin^2 \alpha + 5\sin^4 \alpha\right) + C} \quad \text{Replace } u \text{ by } \sin \alpha.$$

Answer 16E.

Consider the integral

$$\int x \sin^3 x \, dx$$

To evaluate the definite integral

Write the integral as

$$\begin{aligned}\int x \sin^3 x \, dx &= \int x \sin^2 x \sin x \, dx \\ &= \int x\left(1-\cos^2 x\right)\sin x \, dx \qquad \left(\text{Since } 1-\cos^2 x = \sin^2 x\right) \\ &= \int x \sin x \, dx - \int x \cos^2 x \sin x \, dx\end{aligned}$$

$$\int x \sin^3 x dx = I_1 - I_2$$

Where

$$I_1 = \int x \sin x dx \quad \text{and} \quad I_2 = \int x \cos^2 x \sin x dx$$

First find I_1

$$I_1 = \int x \sin x dx$$

It is solved by using Integration by parts method

Integration by parts formula is

$$\int u dv = uv - \int v du$$

Here

$$u = x \quad dv = \sin x$$

Then

$$du = dx \quad v = \int \sin x = -\cos x$$

Then

$$\begin{aligned} I_1 &= \int x \sin x dx \\ &= x(-\cos x) - \int (-\cos x) dx \\ &= -x \cos x + \sin x + C_1 \end{aligned}$$

Now, to find I_2

$$\begin{aligned} I_2 &= \int x \cos^2 x \sin x dx \\ &= \int x \left(\frac{1 + \cos 2x}{2} \right) \sin x \, dx \end{aligned}$$

So

$$\int x \sin^3 x dx = -x \cos x + \sin x - \int x \left(\frac{1 + \cos(2x)}{2} \right) \sin x dx$$

$$= -x \cos x + \sin x - \frac{1}{2} \left[\int x \sin x \, dx + \int x \cos 2x \sin x \, dx \right]$$

The trigonometry identity is

$$2 \cos A \sin B = \sin(A + B) - \sin(A - B)$$

Here

$$\begin{aligned} \cos 2x \sin x &= \frac{1}{2}(2 \cos 2x \sin x) \qquad \text{Multiply \& Divide by 2} \\ &= \frac{1}{2}(\sin(2x + x) - \sin(2x - x)) \\ &\qquad (\text{Use } 2 \cos A \sin B = \sin(A + B) - \sin(A - B)) \\ &= \frac{1}{2}(\sin(3x) - \sin x) \end{aligned}$$

Becomes

$$\int x\sin^3 x dx = -x\cos x + \sin x - \frac{1}{2}(-x\cos x + \sin x) - \frac{1}{2}\int x\frac{1}{2}(\sin 3x - \sin x)dx$$

$$= -x\cos x + \sin x - \frac{1}{2}(-x\cos x + \sin x) - \frac{1}{2}\int \frac{x}{2}(\sin 3x - \sin x)dx$$

$$= \frac{1}{2}(-x\cos x + \sin x) - \frac{1}{4}\int x\sin 3x + \frac{1}{4}\int x\sin x\, dx$$

$$= \frac{1}{2}(\sin x - x\cos x) + \frac{1}{4}(\sin x - x\cos x) - \frac{1}{4}\left(x\left(-\frac{\cos 3x}{3}\right) + \frac{\sin 3x}{9}\right)$$

$$\left(\text{Replace } x \text{ by } 3x \text{ then } \int x\sin 3x dx = -\frac{x}{3}\cos 3x + \frac{\sin 3x}{9} + C\right)$$

$$= \frac{3}{4}(\sin x - x\cos x) + \frac{1}{4}\left(\frac{x}{3}\cos 3x - \frac{\sin 3x}{9}\right) + c$$

Hence $\boxed{\int x\sin^3 x\, dx = \frac{3}{4}(\sin x - x\cos x) + \frac{1}{4}\left(\frac{x}{3}\cos 3x - \frac{\sin 3x}{9}\right) + c}$

Answer 17E.

Consider the following integral:

$$\int \cos^2 x\tan^3 x dx$$

$$\int \cos^2 x\tan^3 x dx = \int \cos^2 x\frac{\sin^3 x}{\cos^3 x}dx$$

$$= \int \frac{\sin^3 x}{\cos x}dx$$

$$= \int \frac{\sin x\sin^2 x}{\cos x}dx$$

$$= \int \frac{\sin x(1-\cos^2 x)}{\cos x}dx \quad \text{since, } \sin^2 x = 1 - \cos^2 x$$

Substitute, $u = \cos x$ and $du = -\sin x dx$.

$$\int \cos^2 x\tan^3 x dx = \int \frac{\sin x(1-\cos^2 x)}{\cos x}dx$$

$$= -\int \frac{(1-u^2)}{u}du$$

$$= -\int \left(\frac{1}{u} - u\right)du$$

$$= -\ln|u| + \frac{u^2}{2} + C$$

Replace, $u = \cos x$.

$$\int \cos^2 x\tan^3 x dx = -\ln|u| + \frac{u^2}{2} + C$$

$$= -\ln|\cos x| + \frac{\cos^2 x}{2} + C$$

Therefore, $\int \cos^2 x\tan^3 x dx = \boxed{-\ln|\cos x| + \frac{\cos^2 x}{2} + C}$.

Answer 18E.

$$\int \cot^5\theta \sin^4\theta d\theta = \int \frac{\cos^5\theta}{\sin^5\theta}.\sin^4\theta d\theta$$
$$= \int \frac{\cos^5\theta}{\sin\theta} d\theta$$
$$= \int \frac{\left(\cos^2\theta\right)^2}{\sin\theta}\cos\theta d\theta$$
$$= \int \frac{\left(1-\sin^2\theta\right)^2}{\sin\theta}\cos\theta d\theta$$

Substitute $\sin\theta = t$

Differentiating both sides, we have $\cos\theta d\theta = dt$

Then $\int \cos^5\theta \sin^4\theta d\theta = \int \frac{\left(1-t^2\right)^2}{t} dt$

$$= \int \frac{1-2t^2+t^4}{t} dt$$
$$= \int \left(\frac{1}{t} - 2t + t^3\right) dt$$
$$= \ln|t| - \frac{2t^2}{2} + \frac{t^4}{4} + C$$
$$= \ln|t| - t^2 + \frac{t^4}{4} + C$$
$$= \boxed{\ln|\sin\theta| - \sin^2\theta + \frac{1}{4}\sin^4\theta + C}$$

Answer 19E.

Consider the following integral:

$$\int \frac{\cos x + \sin 2x}{\sin x} dx$$

Evaluate the integral.

$$\int \frac{\cos x + \sin 2x}{\sin x} dx = \int \left[\frac{\cos x}{\sin x} + \frac{\sin 2x}{\sin x}\right] dx$$
$$= \int \left[\cot x + \frac{2\sin x \cos x}{\sin x}\right] dx$$
$$= \int \left[\cot x + 2\cos x\right] dx$$
$$= \int \cot x dx + 2\int \cos x dx$$
$$= \ln|\sin x| + 2\sin x + C$$

Hence, $\boxed{\int \frac{\cos x + \sin 2x}{\sin x} dx = \ln|\sin x| + 2\sin x + C}$.

Answer 20E.

Consider the following integral:

$$\int \cos^2 x \sin 2x\, dx$$

The objective is to evaluate the integral.

Use the following substitution,

$$u = \cos^2 x,$$
$$du = -2\cos x \sin x dx$$
$$= -\sin(2x) dx,$$

And $\sin 2x dx = -du$

Substitute above values in the given integral, we get

$$\int \cos^2 x \sin(2x) dx = \int u(-du)$$
$$= -\int u du$$
$$= -\frac{u^2}{2} + C$$
$$= -\frac{\cos^4 x}{2} + C \qquad (\text{since } u = \cos^2 x)$$

Therefore, $\int \cos^2 x \sin(2x) dx = \boxed{-\frac{\cos^4 x}{2} + C}$.

Answer 21E.

Given $\int \tan x \sec^3 x\, dx$

We have to evaluate the given indefinite integral

Now

$$\int \tan x \sec^3 x\, dx = \int \sec^2 x \sec x \tan x\, dx$$

Put $\sec x = t \Rightarrow \sec x \tan x\, dx = dt$

$$\int \tan x \sec^3 x\, dx = \int t^2 dt$$
$$= \left[\frac{t^3}{3}\right] + c$$
$$= \frac{1}{3}\sec^3 x + c$$

Hence $\boxed{\int \tan x \sec^3 x\, dx = \frac{1}{3}\sec^3 x + c}$

Answer 22E.

Consider the integral

$$\int \tan^2 \theta \sec^4 \theta\, d\theta$$

To evaluate the indefinite integral

For the known trigonometric identity

$$\sec^2 \theta = 1 + \tan^2 \theta$$

$$\int \tan^m x \sec^{2k} x\, dx = \int \tan^m x \left(\sec^2 x\right)^{k-1} \sec^2 x\, dx$$
$$= \int \tan^m x \left(1 + \tan^2 x\right)^{k-1} \sec^2 x\, dx$$

Write the integral as

$$\int \tan^2 \theta \sec^4 \theta\, d\theta = \int \tan^2 \theta \sec^2 \theta \sec^2 \theta\, d\theta$$
$$\left(\text{Since } \sec^4 \theta = \sec^2 \theta \cdot \sec^2 \theta\right)$$
$$= \int \tan^2 \theta \left(1 + \tan^2 \theta\right) \sec^2 \theta\, d\theta$$

Put

$$\tan \theta = t$$

Take differentiations on both sides

$$\sec^2 \theta\, d\theta = dt$$

Now, the integral become

$$\int \tan^2\theta \sec^4\theta \, d\theta = \int t^2\left(1+t^2\right)dt$$

$$=\int\left(t^2+t^4\right)dt$$

$$=\frac{t^{2+1}}{2+1}+\frac{t^{4+1}}{4+1}+c \qquad \left(\text{Since } \int x^n dx=\frac{x^{n+1}}{n+1},\ n\neq -1\right)$$

$$=\frac{t^3}{3}+\frac{t^5}{5}+c$$

Substitute $t=\tan\theta$

$$\int \tan^2\theta \sec^4\theta \, d\theta = \frac{\tan^3\theta}{3}+\frac{\tan^5\theta}{5}+c$$

Hence

$$\boxed{\int \tan^2\theta \sec^4\theta \, d\theta = \frac{\tan^3\theta}{3}+\frac{\tan^5\theta}{5}+c}$$

Answer 23E.

We have
$$\int \tan^2 x\,dx = \int\left(\sec^2 x-1\right)dx$$
$$=\int \sec^2 x dx-\int dx$$
$$=\boxed{\tan x - x + C}$$

Answer 24E.

Consider the following integral:

$$\int\left(\tan^2 x+\tan^4 x\right)dx$$

Evaluate the integral as shown below:

$$\int\left(\tan^2 x+\tan^4 x\right)dx=\int \tan^2 x\left(1+\tan^2 x\right)dx$$
$$=\int \tan^2 x \sec^2 x\,dx$$

Put $\tan x = u$, so that, $\sec^2 x\,dx = du$.

Substitute these values in the above integral.

$$\int\left(\tan^2 x+\tan^4 x\right)dx=\int \tan^2 x \sec^2 x\,dx$$
$$=\int u^2\,du$$
$$=\frac{u^3}{3}+C$$
$$=\boxed{\frac{1}{3}\tan^3 x + C} \qquad \text{Replace } u \text{ by } \tan x.$$

Answer 25E.

Given $\int \tan^4 x \sec^6 x\,dx$

We have to evaluate the given indefinite integral

We know that

$$\int \tan^m x \sec^{2k} x\,dx = \int \tan^m x\left(\sec^2 x\right)^{k-1}\sec^2 x\,dx$$
$$=\int \tan^m x\left(1+\tan^2 x\right)^{k-1}\sec^2 x\,dx$$

Now

$$\int \tan^4 x \sec^6 x\,dx = \int \tan^4 x \sec^4 x\cdot\sec^2 x\,dx$$
$$=\int \tan^4 x\left(1+\tan^2 x\right)^2\sec^2 x\,dx$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$\int \tan^4 x \sec^6 x dx = \int t^4 \left(1+t^2\right)^2 dt$$
$$= \int t^4 \left(1+t^4+2t^2\right) dt$$
$$= \int \left(t^4+t^8+2t^6\right) dt$$
$$= \frac{t^5}{5}+\frac{t^9}{9}+\frac{2t^7}{7}+c$$
$$= \frac{\tan^5 x}{5}+\frac{\tan^9 x}{9}+\frac{2\tan^7 x}{7}+c$$

Hence $\boxed{\int \tan^4 x \sec^6 x dx = \frac{\tan^5 x}{5}+\frac{\tan^9 x}{9}+\frac{2\tan^7 x}{7}+c}$

Answer 26E.

We have $\int_0^{\pi/4} \sec^4 \theta . \tan^4 \theta . d\theta = \int_0^{\pi/4} \sec^2 \theta . \tan^4 \theta . \sec^2 \theta . d\theta$

$$= \int_0^{\pi/4} \left(1+\tan^2 \theta\right) \tan^4 \theta . \sec^2 \theta . d\theta$$

Substitute $\tan \theta = t$, $\left[\text{when } \theta = 0, t = 0 \text{ and } \theta = \frac{\pi}{4}, t = 1\right]$

Differentiating both sides $\sec^2 \theta d\theta = dt$

Then $\int_0^{\pi/4} \sec^4 \theta . \tan^4 \theta . d\theta = \int_0^1 \left(1+t^2\right) t^4 dt$

$$= \int_0^1 \left(t^4+t^6\right) dt$$
$$= \left[\frac{t^5}{5}+\frac{t^7}{7}\right]_0^1$$
$$= \frac{1}{5}+\frac{1}{7}-0$$
$$= \frac{7+5}{35}$$
$$= \boxed{\frac{12}{35}}$$

Answer 27E.

We have $\int_0^{\pi/3} \tan^5 x \sec^4 x dx = \int_0^{\pi/3} \tan^5 x \sec^2 x \sec^2 x dx$

$$= \int_0^{\pi/3} \tan^5 x \left(1+\tan^2 x\right) \sec^2 x dx$$

Let $\tan x = t$ when, $x = 0, t = 0$, when $x = \frac{\pi}{3}, t = \sqrt{3}$

Then $\sec^2 x dx = dt$

So $$\int_0^{\pi/3} \tan^5 x\left(1+\tan^2 x\right)\sec^2 x dx = \int_0^{\sqrt{3}} t^5\left(1+t^2\right)dt$$

$$= \int_0^{\sqrt{3}} \left(t^5+t^7\right)dt$$

$$= \left[\frac{t^6}{6}+\frac{t^8}{8}\right]_0^{\sqrt{3}}$$

$$= \left[\frac{\left(\sqrt{3}\right)^6}{6}+\frac{\left(\sqrt{3}\right)^8}{8}\right]$$

$$= \frac{27}{6}+\frac{81}{8}$$

$$= \frac{351}{24} = \frac{117}{8}$$

So $$\boxed{\int_0^{\pi/3} \tan^5 x\sec^4 x dx = \frac{117}{8}}$$

Answer 28E.

Given $\int \tan^5 x\sec^3 x dx$

We have to find the given indefinite integral

We know that

$$\int \tan^{2k+1} x\sec^n x\, dx = \int \left(\tan^2 x\right)^k \sec^{n-1} x\sec x\tan x\, dx$$

$$= \int \left(\sec^2 x-1\right)^k \sec^{n-1} x\sec x\tan x\, dx$$

Now

$$\int \tan^5 x\sec^3 x dx = \int \sec^2 x\tan^4 x\sec x\tan x dx$$

$$= \int \sec^2 x\left(-1+\sec^2 x\right)^2 \sec x\tan x dx$$

Put $\sec x = t \Rightarrow \sec x\tan x dx = dt$

$$\int \tan^5 x\sec^3 x dx = \int t^2\left(t^2-1\right)^2 dt$$

$$= \int t^2\left(t^4+1-2t^2\right)dt$$

$$= \int \left(t^6+t^2-2t^4\right)dt$$

$$= \frac{t^7}{7}+\frac{t^3}{3}-\frac{2t^5}{5}+c$$

$$= \frac{\sec^7 x}{7}+\frac{\sec^3 x}{3}-\frac{2\sec^5 x}{5}+c$$

Hence $$\boxed{\int \tan^5 x\sec^3 x dx = \frac{\sec^7 x}{7}+\frac{\sec^3 x}{3}-\frac{2\sec^5 x}{5}+c}$$

Answer 29E.

$$\int \tan^3 x\sec x dx = \int \tan^2 x\sec x\tan x dx$$

$$= \int \left(\sec^2 x-1\right)\sec x\tan x dx$$

Substitute $\sec x = t$, then $\sec x\tan x dx = dt$

Therefore

$$\int \tan^3 x\sec x dx = \int \left(t^2-1\right)dt$$

$$= \frac{t^3}{3}-t+C$$

$$= \boxed{\frac{1}{3}\sec^3 x-\sec x+C}$$

Answer 30E.

Given $\int_0^{\pi/4} \tan^4 t\,dt$

We have to evaluate the given definite integral

Now

$$\int_0^{\pi/4} \tan^4 t\,dt = \int_0^{\pi/4} \tan^2 x\left(\sec^2 x-1\right)dt$$
$$= \int_0^{\pi/4} \tan^2 x\sec^2 x\,dx - \int_0^{\pi/4} \tan^2 x\,dx$$
$$= \left[\frac{\tan^3 x}{3}\right]_0^{\pi/4} - \int_0^{\pi/4}\left(\sec^2 x-1\right)dx$$
$$= \frac{1}{3}\tan^3\frac{\pi}{4} - \left[\tan x - x\right]_0^{\pi/4}$$
$$= \frac{1}{3}\tan^3\frac{\pi}{4} - \tan\frac{\pi}{4} + \frac{\pi}{4}$$
$$= \frac{1}{3} - 1 + \frac{\pi}{4}$$
$$= \frac{\pi}{4} - \frac{2}{3}$$

Hence $\boxed{\int_0^{\pi/4} \tan^4 t\,dt = \frac{\pi}{4} - \frac{2}{3}}$

Answer 31E.

$$\int \tan^5 x\,dx = \int\left(\tan^2 x\right)^2 \tan x\,dx$$
$$= \int\left(\sec^2 x-1\right)^2 \tan x\,dx$$
$$= \int\left(\sec^4 x - 2\sec^2 x + 1\right)\tan x\,dx$$
$$= \int \sec^4 x\tan x\,dx - 2\int \sec^2 x\tan x\,dx + \int \tan x\,dx$$
$$= \int \sec^3 x\sec x\tan x\,dx - 2\int \sec x\sec x\tan x\,dx + \ln\left|\sec x\right| + C_1$$

Let $u = \sec x \rightarrow du = \sec\tan x dx$

Then $\int \tan^5 x\,dx = \int u^3 du - 2\int u\,du + \ln\left|\sec x\right| + C_1$

$$= \frac{u^4}{4} - \frac{2u^2}{2} + \ln\left|\sec x\right| + C_1$$
$$= \frac{1}{4}\sec^4 x - \sec^2 x + \ln\left|\sec x\right| + C_1$$
$$= \frac{1}{4}\sec^4 x - \left(1+\tan^2 x\right) + \ln\left|\sec x\right| + C_1$$
$$= \boxed{\frac{1}{4}\sec^4 x - \tan^2 x + \ln\left|\sec x\right| + C} \qquad \text{Where } C = C_1 - 1$$

Answer 32E.

We have to evaluate $\int \tan^2 x\sec x dx$.

We have $\int \tan^2 x\sec x dx = \int\left(\sec^2 x-1\right)\sec x dx$

$$= \int\left(\sec^3 x - \sec x\right)dx$$
$$= \int \sec^3 x dx - \int \sec x dx$$
$$= \int \sec x\sec^2 x dx - \int \sec x dx \qquad \ldots(1)$$

First we evaluate $\int \sec x dx$.

$\int \sec x dx = \int \sec x . \frac{\sec x + \tan x}{\sec x + \tan x} dx$, by multiplying and dividing with (secx+tanx)

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

Now we make the substitution $u = \sec x + \tan x$, so that $\left(\sec^2 x + \sec x \tan x\right) dx = du$.

Therefore

$$\int \sec x dx = \int \frac{du}{u}$$

$$= \ln u$$

$$= \ln\left|(\sec x + \tan x)\right| + C_1 \quad ....(2)$$

Now we evaluate $\int \sec x . \sec^2 x dx$.

Let $u = \sec x, dv = \sec^2 x$ so that du = secx tanx and v = tanx.

Now using integration by parts $\int u dv = uv - \int v du$, we get

$$\int \sec x \sec^2 x dx = \sec x . \tan x - \int \sec x . \tan x . \tan x dx$$

$$= \sec x . \tan x - \int \sec x . \tan^2 x dx + C_2 \quad ...(3)$$

Using (2) and (3) in (1) we get

We have $\int \tan^2 x \sec x dx = -\ln\left|(\sec x + \tan x)\right| + \sec x . \tan x - \int \sec x . \tan^2 x dx + C_2 + C_1$

Or $\quad \int \tan^2 x \sec x dx = \sec x \tan x - \int \tan^2 x \sec x dx - \ln\left|\sec x + \tan x\right|$

Or $\quad 2\int \tan^2 x \sec x dx = \sec x \tan x - \ln\left|\sec x + \tan x\right| + C$

Then $\quad \int \tan^2 x \sec x dx = \boxed{\frac{1}{2}\sec x \tan x - \frac{1}{2}\ln\left|\sec x + \tan x\right| + C}$,

where $C = C_1 + C_2$.

Answer 33E.

Consider the following integral:

$$\int x \sec x \tan x \, dx$$

Evaluate the integral.

Use the integration by parts formula to evaluate the integral.

Let $u = x;\ dv = \sec x \tan x \, dx$, then $du = dx; v = \sec x.$

The formula for integration by parts is as shown below:

$$\int u \, dv = uv - \int v \, du$$

Substitute the values in the above formula.

$$\int x \sec x \tan x \, dx = x \sec x - \int \sec x \, dx$$

$$= x \sec x - \ln\left|\sec x + \tan x\right| + C$$

Therefore, $\int x \sec x \tan x \, dx = \boxed{x \sec x - \ln\left|\sec x + \tan x\right| + C}$.

Answer 34E.

Consider the following integral:

$$\int \frac{\sin\phi}{\cos^3\phi}\,d\phi$$

Rewrite the integral as,

$$\begin{aligned}\int \frac{\sin\phi}{\cos^3\phi}\,d\phi &= \int \frac{\sin\phi}{\cos\phi\cos^2\phi}\,d\phi\\ &= \int\left(\frac{1}{\cos^2\phi}\right)\left(\frac{\sin\phi}{\cos\phi}\right)d\phi\\ &= \int \sec^2\phi\tan\phi\,d\phi\\ &= \int \sec\phi\left(\sec\phi\tan\phi\,d\phi\right)\end{aligned}$$

Evaluate the given integral.

Use substitution $u = \sec\phi$.

Then $du = \sec\phi\tan\phi\,d\phi$.

$$\begin{aligned}\int \sec\phi\left(\sec\phi\tan\phi\,d\phi\right) &= \int u\,du\\ &= \frac{u^2}{2}+C && \int x^n dx = \frac{x^{n+1}}{n+1}+C.\\ &= \frac{(\sec\phi)^2}{2}+C && \text{Since } u=\sec\phi.\\ &= \frac{\sec^2\phi}{2}+C\end{aligned}$$

Therefore, $\int \frac{\sin\phi}{\cos^3\phi}\,d\phi = \boxed{\frac{\sec^2\phi}{2}+C}$.

Answer 35E.

$$\begin{aligned}\int_{\pi/6}^{\pi/2}\cot^2 x\,dx &= \int_{\pi/6}^{\pi/2}\left(\csc^2 x-1\right)dx\\ &= \left[-\cot x - x\right]_{\pi/6}^{\pi/2}\\ &= -\cot\frac{\pi}{2}-\frac{\pi}{2}+\cot\frac{\pi}{6}+\frac{\pi}{6}\\ &= 0+\frac{\pi}{6}-\frac{\pi}{2}+\sqrt{3}\\ &= -\frac{2\pi}{6}+\sqrt{3}\\ &= \boxed{\sqrt{3}-\frac{\pi}{3}}\end{aligned}$$

Answer 36E.

$$\begin{aligned}\int_{\pi/4}^{\pi/2}\cot^3 x\,dx &= \int_{\pi/4}^{\pi/2}\cot^2 x.\cot x\,dx\\ &= \int_{\pi/4}^{\pi/2}\left(\csc^2 x-1\right).\cot x\,dx\\ &= \int_{\pi/4}^{\pi/2}\cot x\csc^2 x\,dx - \int_{\pi/4}^{\pi/2}\cot x\,dx \qquad \text{--- (1)}\end{aligned}$$

In $\int \cot x\csc^2 x\,dx$. Substitute $\cot x = t$, $-\csc^2 x\,dx = dt$

$$\int \cot x\csc^2 x\,dx = -\int t\,dt = -\frac{t^2}{2}+C = -\frac{1}{2}\cot^2 x + C$$

Therefore equation (1) becomes

$$\int_{\pi/4}^{\pi/2} \cot^3 x\,dx = \left[-\frac{1}{2}\cot^2 x\right]_{\pi/4}^{\pi/2} - \left[\ln|\sin x|\right]_{\pi/4}^{\pi/2}$$

$$= -\frac{1}{2}\left(\cot^2\frac{\pi}{2} - \cot^2\frac{\pi}{4}\right) - \left(\ln\left(\sin\frac{\pi}{2}\right) - \ln\left(\sin\frac{\pi}{4}\right)\right)$$

$$= 0 + \frac{1}{2} - \left(\ln 1 - \ln\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2} - \ln\sqrt{2}$$

$$= \boxed{\frac{1}{2}(1 - \ln 2)}$$

Answer 37E.

Given $\int_{\pi/4}^{\pi/2} \cot^5\phi \csc^3\phi\, d\phi$

We have to evaluate the given definite integral

Now

$$\int_{\pi/4}^{\pi/2} \cot^5\phi \csc^3\phi\, d\phi = \int_{\pi/4}^{\pi/2} \cot^4\phi \csc^2\phi \cot\phi \csc\phi\, d\phi$$

$$= \int_{\pi/4}^{\pi/2} \left(\csc^2\phi - 1\right)^2 \csc^2\phi \cot\phi \csc\phi\, d\phi$$

Put $\csc\phi = t \Rightarrow -\csc\phi\cot\phi\, d\phi = dt$

$\phi = \frac{\pi}{4} \quad\Rightarrow\quad t = \sqrt{2}$

$\phi = \frac{\pi}{2} \quad\Rightarrow\quad t = 1$

$$\int_{\pi/4}^{\pi/2} \cot^5\phi \csc^3\phi\, d\phi = \int_{\sqrt{2}}^{1} \left(t^2 - 1\right)^2 t^2 (-dt)$$

$$= -\int_{\sqrt{2}}^{1} \left(t^4 + 1 - 2t^2\right) t^2\, dt$$

$$= -\int_{\sqrt{2}}^{1} \left(t^6 + t^2 - 2t^4\right) dt$$

$$= -\left[\frac{t^7}{7} + \frac{t^3}{3} - \frac{2t^5}{5}\right]_{\sqrt{2}}^{1}$$

$$= -\left(\frac{1}{7} + \frac{1}{3} - \frac{2}{5}\right) + \left(\frac{\left(\sqrt{2}\right)^5}{7} + \frac{\left(\sqrt{2}\right)^3}{3} - \frac{2\left(\sqrt{2}\right)^4}{5}\right)$$

$$= \frac{22}{105}\sqrt{2} - \frac{8}{105}$$

Hence $\boxed{\int_{\pi/4}^{\pi/2} \cot^5\phi \csc^3\phi\, d\phi = \frac{22}{105}\sqrt{2} - \frac{8}{105}}$

Answer 38EE.

$$\int \csc^4 x \cot^6 x\,dx = \int \csc^2 x \cot^6 x \csc^2 x\,dx$$

$$= \int \left(1 + \cot^2 x\right) \cot^6 x \csc^2 x\,dx$$

Substitute $\cot x = t$

Then $-\csc^2 x\,dx = dt$

And so $\int \csc^4 x \cot^6 x\,dx = \int (1+t^2)t^6(-dt)$

$$= -\int (t^6 + t^8)\,dt$$

$$= -\frac{1}{7}t^7 - \frac{1}{9}t^9 + C$$

$$= \boxed{-\frac{1}{7}\cot^7 x - \frac{1}{9}\cot^9 x + C}$$

Answer 39E.

We have to evaluate $\int \csc x\,dx$

Multiply numerator and denominator by $(\csc x - \cot x)$

$$\int \csc x\,dx = \int \csc x.\frac{(\csc x - \cot x)}{(\csc x - \cot x)}dx$$

$$= \int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x}dx$$

Substitute $u = \csc - \cot x$

Then $du = -\csc x \cot x + \csc^2 x$

So $\int \csc x\,dx = \int \frac{du}{u}$

$$= \ln|u| + C$$

Thus $\int \csc x\,dx = \boxed{\ln|\csc x - \cot x| + C}$

Answer 40E.

$$\int_{\pi/6}^{\pi/3} \csc^3 x\,dx = \int_{\pi/6}^{\pi/3} \csc x.\csc^2 x\,dx$$

Let $u = \csc x$ $\quad dv = \csc^2 x\,dx$

$du = -\csc x \cot x\,dx$ $\quad v = -\cot x$

Substituting in the integration by parts formula $\int u dv = uv - \int v du$ we get

$$\int_{\pi/6}^{\pi/3} \csc^3 x\,dx = [-\csc x \cot x]_{\pi/6}^{\pi/3} - \int_{\pi/6}^{\pi/3} \csc x.\cot^2 x\,dx$$

$$= \left(-\csc\frac{\pi}{3}.\cot\frac{\pi}{3} + \csc\frac{\pi}{6}.\cot\frac{\pi}{6}\right) - \int_{\pi/6}^{\pi/3} \csc x\left(\csc^2 x - 1\right)dx$$

$$\left[\because \cot^2 x = \csc^2 x - 1\right]$$

$$= \left(-\frac{2}{\sqrt{3}}.\frac{1}{\sqrt{3}} + 2\sqrt{3}\right) - \int_{\pi/6}^{\pi/3} \csc^3 x.dx + \int_{\pi/6}^{\pi/3} \csc x.dx$$

$$\left[\because \csc\left(\frac{\pi}{3}\right) = \frac{2}{\sqrt{3}}, \cot\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}, \csc\left(\frac{\pi}{6}\right) = 2, \cot\left(\frac{\pi}{6}\right) = \sqrt{3}\right]$$

$$\Rightarrow 2\int_{\pi/6}^{\pi/3} \csc^3 x\,dx = -\frac{2}{3} + 2\sqrt{3} + \int_{\pi/6}^{\pi/3} \csc x.dx$$

Then $\int_{\pi/6}^{\pi/3} \csc^3 x\,dx = -\frac{1}{3} + \sqrt{3} + \frac{1}{2}\left[\ln|\csc x - \cot x|\right]_{\pi/6}^{\pi/3}$

$$\left[\because \int \csc x dx = \ln|\csc x - \cot x|\right]$$

$$= -\frac{1}{3} + \sqrt{3} + \frac{1}{2}\ln\left|\csc\frac{\pi}{3} - \cot\frac{\pi}{3}\right| - \frac{1}{2}\ln\left|\csc\frac{\pi}{6} - \cot\frac{\pi}{6}\right|$$

$$= -\frac{1}{3} + \sqrt{3} + \frac{1}{2}\ln\left|\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right| - \frac{1}{2}\ln\left|2 - \sqrt{3}\right|$$

$$= \boxed{-\frac{1}{3} + \sqrt{3} + \frac{1}{2}\ln\left(\frac{1}{\sqrt{3}}\right) - \frac{1}{2}\ln\left(2 - \sqrt{3}\right)}$$

$$\approx 1.7825$$

Answer 41E.

Consider the following integral:

$$\int \sin 8x \;\cos 5x \; dx$$

Evaluate the above integral.

This integral could be evaluated using integration by parts, but it's easier to use the identity in equation as follows:

Use the equation, $\sin A \cos B = \frac{1}{2}\left[\sin(A-B)+\sin(A+B)\right]$.

$$\int \sin 8x \cos 5x \, dx = \int \frac{1}{2}\left[\sin(8x-5x)+\sin(8x+5x)\right]dx$$
$$= \frac{1}{2}\int (\sin 3x + \sin 13x)\, dx$$
$$= \frac{1}{2}\int \sin 3x dx + \frac{1}{2}\int \sin 13x dx$$
$$= \frac{1}{2}\frac{(-\cos 3x)}{3} + \frac{1}{2}\frac{(-\cos 13x)}{13} + C$$
$$= -\frac{1}{6}\cos 3x - \frac{1}{26}\cos 13x + C$$

Therefore, $\int \sin 8x \;\cos 5x \; dx = \boxed{-\frac{1}{6}\cos 3x - \frac{1}{26}\cos 13x + C}$.

Answer 42E.

Consider the following integral:

$$\int \cos \pi x \cos 4\pi x \; dx$$

Evaluate the above integral.

This integral could be evaluated using integration by parts, but it's easier to use the identity in equation as shown below:

Use the equation, $\cos A \cos B = \frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$.

$$\int \cos \pi x \cos 4\pi x \, d\theta = \int \frac{1}{2}\left[\cos(\pi x - 4\pi x)+\cos(\pi x + 4\pi x)\right]dx$$
$$= \frac{1}{2}\int \cos 3\pi x \, dx + \frac{1}{2}\int \cos 5\pi x \, dx \qquad \left[\text{Since } \cos(-x)=\cos x\right]$$
$$= \frac{1}{2}\cdot\frac{\sin 3\pi x}{3\pi} + \frac{1}{2}\cdot\frac{\sin 5\pi x}{5\pi} + C$$
$$= \frac{\sin 3\pi x}{6\pi} + \frac{\sin 5\pi x}{10\pi} + C$$

Therefore, $\int \cos \pi x \cos 4\pi x \; dx = \boxed{\frac{\sin 3\pi x}{6\pi} + \frac{\sin 5\pi x}{10\pi} + C}$.

Answer 43E.

Consider the integral $\int \sin 5\theta \sin\theta \, d\theta$

Rewrite the integral

$$\int \sin 5\theta \sin\theta \, d\theta = \frac{1}{2}\int 2\sin 5\theta \sin\theta \, d\theta$$ Multiply and divide both sides with 2

$$= \frac{1}{2}\int \left[\cos(5\theta-\theta) - \cos(5\theta+\theta)\right] d\theta$$

Here use $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$$= \frac{1}{2}\int [\cos 4\theta - \cos 6\theta] d\theta$$

$$= \frac{1}{2}\int \cos 4\theta \, d\theta - \frac{1}{2}\int \cos 6\theta \, d\theta$$

$$= \frac{1}{2}\frac{\sin 4\theta}{4} - \frac{1}{2}\frac{\sin 6\theta}{6} + C$$

$$= \frac{1}{8}\sin 4\theta - \frac{1}{12}\sin 6\theta + C$$

Therefore

$$\int \sin 5\theta \sin\theta \, d\theta = \boxed{\frac{1}{8}\sin 4\theta - \frac{1}{12}\sin 6\theta + C}$$

Answer 44E.

We have
$$\int \frac{\cos x + \sin x}{\sin 2x} = \int \frac{\cos x + \sin x}{2\sin x \cos x} dx$$
$$= \int \left(\frac{\cos x}{2\sin x \cos x} + \frac{\sin x}{2\sin x \cos x} \right) dx$$
$$= \int \left(\frac{1}{2\sin x} + \frac{1}{2\cos x} \right) dx$$
$$= \frac{1}{2}\int \csc dx + \frac{1}{2}\int \sec x \, dx$$
$$= \boxed{\frac{1}{2}\ln|\csc x - \cot x| + \frac{1}{2}\ln|\sec x + \tan x| + C}$$

Answer 45E.

Given $\int_0^{\pi/6} \sqrt{1+\cos 2x} \, dx$

We have to evaluate the given definite integral

Now

$$\int_0^{\pi/6} \sqrt{1+\cos 2x} \, dx = \int_0^{\pi/6} \sqrt{2\cos^2 x} \, dx \qquad (\because \cos 2x = 2\cos^2 x - 1)$$
$$= \sqrt{2}\int_0^{\pi/6} \cos x \, dx$$
$$= \sqrt{2}[\sin x]_0^{\pi/6}$$
$$= \sqrt{2}\left[\sin\frac{\pi}{6} - \sin 0\right]$$
$$= \sqrt{2}\left[\frac{1}{2} - 0\right] \qquad \left(\because \sin\frac{\pi}{6} = \frac{1}{2}\right)$$
$$= \frac{\sqrt{2}}{2}$$
$$= \frac{1}{\sqrt{2}}$$

Hence $\boxed{\int_0^{\pi/6} \sqrt{1+\cos 2x} \, dx = \frac{1}{\sqrt{2}}}$

Answer 46E.

Given $\int_0^{\pi/4} \sqrt{1-\cos 4\theta}\, d\theta$

We have to evaluate the given definite integral

We know $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$\Rightarrow \quad 1-\cos 2\theta = 2\sin^2\theta$

$\Rightarrow \quad 1-\cos 4\theta = 2\sin^2 2\theta$

Therefore,

$$\int_0^{\pi/4} \sqrt{1-\cos 4\theta}\; d\theta = \int_0^{\pi/4} \sqrt{2\sin^2 2\theta}\; d\theta$$

$$= \sqrt{2}\int_0^{\pi/4} \sin 2\theta\; d\theta$$

$$= -\sqrt{2}\left[\frac{\cos 2\theta}{2}\right]_0^{\pi/4}$$

$$= -\frac{1}{\sqrt{2}}\left[\cos\frac{\pi}{2} - \cos 0\right]$$

$$= -\frac{1}{\sqrt{2}}(-1)$$

$$= \frac{1}{\sqrt{2}}$$

Hence $\boxed{\int_0^{\pi/4} \sqrt{1-\cos 4\theta}\, d\theta = \frac{1}{\sqrt{2}}}$

Answer 47E.

We have $\quad \int \dfrac{1-\tan^2 x}{\sec^2 x}dx = \int \dfrac{1-\left(\sec^2 x - 1\right)}{\sec^2 x}dx$

$$= \int \frac{1-\sec^2 x + 1}{\sec^2 x}dx$$

$$= \int \frac{2-\sec^2 x}{\sec^2 x}dx$$

$$= \int \left(\frac{2}{\sec^2 x} - \frac{\sec^2 x}{\sec^2 x}\right)dx$$

$$= \int \left(2\cos^2 x - 1\right)dx$$

$$= \int \cos 2x\, dx$$

$$= \boxed{\frac{\sin 2x}{2} + C}$$

Answer 48E.

We have to evaluate $\int \dfrac{dx}{\cos x - 1}$. We know that $\sin^2\alpha = \dfrac{1}{2}(1-\cos 2\alpha)$, by taking $2\alpha = x$ we get $\sin^2\left(\dfrac{x}{2}\right) = \dfrac{1}{2}(1-\cos x)$.

So $\int \dfrac{dx}{\cos x - 1} = \int \dfrac{dx}{-2\sin^2(x/2)}$

$$= -\frac{1}{2}\int \csc^2(x/2)\,dx \qquad \ldots(1)$$

Put $u = \dfrac{x}{2} \Rightarrow dx = 2du$, then

$$-\frac{1}{2}\int \csc^2(x/2)\,dx = -\frac{1}{2}\int \csc^2(u)\,2du$$

$$= \cot u + C$$

$$= \cot(x/2) + C$$

Therefore $\boxed{\int \frac{dx}{\cos x - 1} = \cot(x/2) + C}$

We can solve this result further

$$\cot(x/2) + C = \frac{\cos(x/2)}{\sin(x/2)} + C$$
$$= \frac{\cos^2(x/2)}{\cos(x/2)\sin(x/2)} + C$$
$$= \frac{2\cos^2(x/2)}{2\cos(x/2)\sin(x/2)} + C$$
$$= \frac{(1+\cos x)}{\sin x} + C$$
$$\left[\because 2\sin\frac{x}{2}\cos\frac{x}{2} = \sin x, 2\cos^2\frac{x}{2} = 1 + \cos x\right]$$
$$= \frac{1}{\sin x} + \frac{\cos x}{\sin x} + C$$
$$= \boxed{\csc x + \cot x + C}$$

Answer 49E.

Consider the integration

$$\int x\tan^2 x\,dx$$

To evaluate the indefinite integral

Here use Trigonometric identity

$$\tan^2\theta = \sec^2\theta - 1$$
$$\int x\tan^2 x\,dx = \int x\left(\sec^2 x - 1\right)dx$$
$$= \int x\sec^2 x\,dx - \int x\,dx$$
$$= I_1 - I_2$$

Where $I_1 = \int x\sec^2 xdx$ and $I_2 = \int xdx$

First solve I_1 by using Integration by parts method

Integration by parts method

$$\int udv = uv - \int vdu$$

Now

$$I_1 = \int x\sec^2 xdx$$

Here

$$u = x, \quad dv = \sec^2 x$$

Then

$$du = dx, \quad v = \int \sec^2 xdx = \tan x$$

Becomes

$$I_1 = \int x\sec^2 xdx$$
$$= x\tan x - \int \tan xdx$$
$$= x\tan x - \ln|\sec x| + C_1$$

And

$$I_2 = \int xdx$$
$$= \frac{x^2}{2} + C_2 \qquad \left(\text{Since } \int x^n dx = \frac{x^{n+1}}{n+1}, \ n \neq -1\right)$$

Then

$$\int x\tan^2 x\,dx = x\int \sec^2 x\,dx - \int\left(\int\left(\sec^2 x\,dx\right)\right)dx - \frac{x^2}{2}$$

$$= x\tan x - \int \tan x\,dx - \frac{x^2}{2}$$

$$= x\tan x - \ln|\sec x| - \frac{x^2}{2} + c$$

Hence $\boxed{\int x\tan^2 x\,dx = x\tan x - \ln|\sec x| - \frac{x^2}{2} + c}$

Answer 50E.

Given that $\int_0^{\pi/4} \tan^6 x \sec x\,dx = I$ ----(1)

Now, $\int_0^{\pi/4} \tan^8 x \sec x\,d = \int_0^{\pi/4} \left(\tan^7 x\right)\cdot\left(\tan x \sec x\,dx\right)$

Let us take $u = \tan^7 x \Rightarrow du = 7\tan^6 x\sec^2 x\,dx$

And $dv = \tan x\sec x\,dx \Rightarrow v = \sec x$

Now,

$$\int_0^{\pi/4} \tan^8 x\sec x\,dx = \left[\tan^7 x\cdot\sec x\right]_0^{\pi/4} - \int_0^{\pi/4}\left(7\tan^6 x\sec^2 x\,dx\right)\cdot\sec x$$

$$= \left[\tan^7\frac{\pi}{4}\cdot\sec\frac{\pi}{4} - \tan^7 0\cdot\sec 0\right] - \int_0^{\pi/4} 7\tan^6 x\left(1+\tan^2 x\right)\sec x\,dx$$

$$= \left[(1)^7\cdot\sqrt{2} - 0\right] - \int_0^{\pi/4}\left(7\tan^6 x\sec x + 7\tan^8 x\sec x\right)dx$$

$$= \sqrt{2} - 7\int_0^{\pi/4}\tan^6 x\sec x\,dx - 7\int_0^{\pi/4}\tan^8 x\sec x\,dx$$

$$\Rightarrow \int_0^{\pi/4}\tan^8 x\sec x\,dx + 7\int_0^{\pi/4}\tan^8 x\sec x\,dx = \sqrt{2} - 7I \qquad \text{Using (1)}$$

$$\Rightarrow 8\int_0^{\pi/4}\tan^8 x\sec x\,dx = \sqrt{2} - 7I$$

$$\Rightarrow \int_0^{\pi/4}\tan^8 x\sec x\,dx = \frac{\sqrt{2}}{8} - \frac{7}{8}I$$

Hence, $\boxed{\int_0^{\pi/4}\tan^8 x\sec x\,dx = \frac{\sqrt{2}}{8} - \frac{7}{8}I}$

Answer 51E.

Consider the integral $\int x\sin^2(x^2)dx$

Setting $x^2 = u \Rightarrow 2xdx = du$

Rewrite the integral

$$\int x\sin^2(x^2)dx = \int \frac{1}{2}\sin^2 u du$$

$$= \frac{1}{2}\int \sin^2 u du$$

$$= \frac{1}{2}\int\left(\frac{1}{2} - \frac{1}{2}\cos(2u)\right)du$$

$$= \frac{1}{2}\int \frac{1}{2}du + \frac{1}{2}\int -\frac{1}{2}\cos(2u)du$$

$$= \frac{1}{4}u - \frac{1}{4}\int \cos(2u)du$$

$$= \frac{1}{4}u - \frac{1}{4}\left[\frac{\sin(2u)}{2}\right]$$

$$= \frac{1}{4}u - \frac{1}{8}\sin(2u)$$

Replace x^2 for u in the above integration, to obtained

$$= \frac{1}{4}x^2 - \frac{1}{8}\sin(2x^2)$$

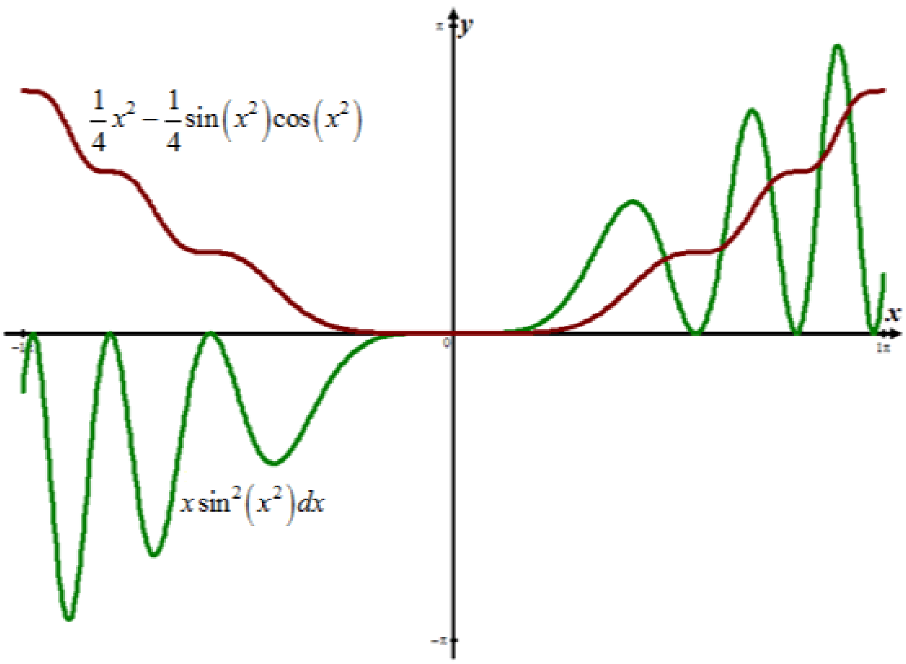
$$= \frac{1}{4}x^2 - \frac{1}{8}\left(2\sin(x^2)\cos(x^2)\right) \quad \text{Here use } \sin(2A) = 2\sin(A)\cos(A)$$

$$= \frac{1}{4}x^2 - \frac{1}{4}\sin(x^2)\cos(x^2) + C$$

Therefore

$$\int x\sin^2(x^2)dx = \boxed{\frac{1}{4}x^2 - \frac{1}{4}\sin(x^2)\cos(x^2) + C}$$

The required graph is



Answer 52E.

Given integral is $\int \sin^5 x \cos^3 x\,dx$

We have to evaluate the given indefinite integral

We know that

$$\int \sin^m x \cos^{2k+1} x\,dx = \int \sin^m x\left(\cos^2 x\right)^k \cos x\,dx$$
$$= \int \sin^m x\left(1-\sin^2 x\right)^k \cos x\,dx$$

Therefore,

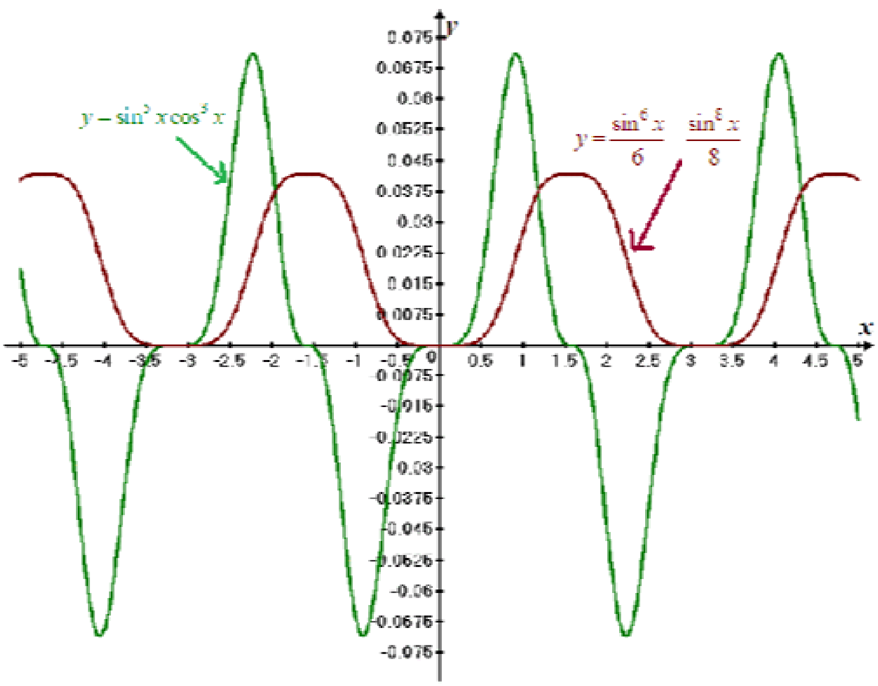
$$\int \sin^5 x \cos^3 x\,dx = \int \sin^5 x\left(1-\sin^2 x\right)\cos x\,dx$$
$$= \int \sin^5 x \cos x\,dx - \int \sin^7 x \cos x\,dx$$

Assume $\sin x = t \Rightarrow \cos x\,dx = dt$

$$\int \sin^5 x \cos^3 x\,dx = \int t^5 dt - \int t^7\,dt$$
$$= \frac{t^6}{6} - \frac{t^8}{8} + c$$
$$= \frac{(\sin x)^6}{6} - \frac{(\sin x)^8}{8} + c$$
$$= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + c$$

Hence $\boxed{\int \sin^5 x \cos^3 x\,dx = \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + c}$

The graphs of the integrand $\sin^5 x \cos^3 x$ and its antiderivative $\frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + c$ (with $c = 0$) are as shown below



From the graph, we observe that $\frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + c$ is antiderivative of $\sin^5 x \cos^3 x$

Answer 53E.

Consider the integral $\int \sin 3x \sin 6x \, dx$

Rewrite the integral

$$\int \sin 3x \sin 6x \, dx = \frac{1}{2}\int 2\sin 3x \sin 6x \, dx$$

Multiply and divide both sides with 2

$$= \frac{1}{2}\left[\int \cos(6x-3x) - \cos(6x+3x)\right]dx$$

Here use $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$$= \frac{1}{2}\left[\int \cos(3x) - \cos(9x)\right]dx$$

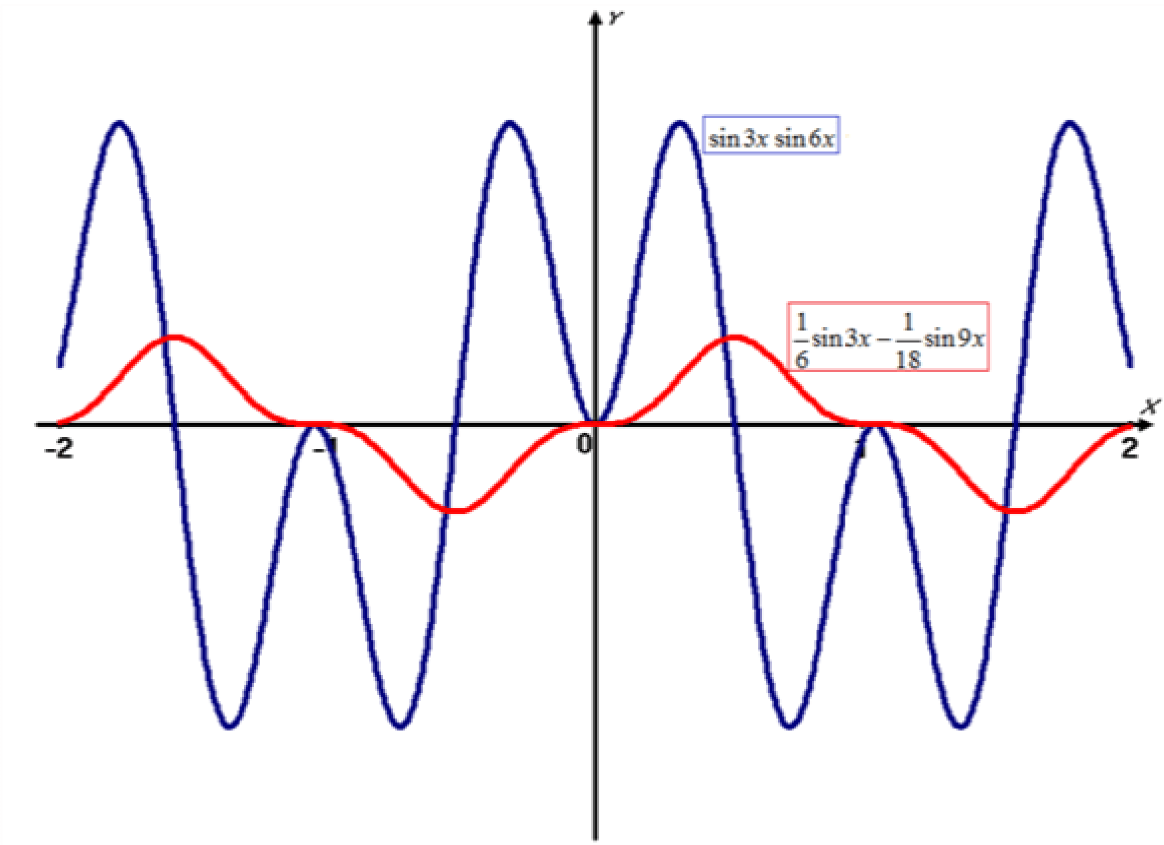
$$= \frac{1}{2}\int \cos(3x)dx - \frac{1}{2}\int \cos(9x)dx$$

$$= \frac{1}{6}\sin 3x - \frac{1}{18}\sin 9x + C$$

Therefore

$$\int \sin 3x \sin 6x \, dx = \boxed{\frac{1}{6}\sin 3x - \frac{1}{18}\sin 9x + C}$$

The required graph is



Answer 54E.

Consider the integral $\int \sec^4 \frac{x}{2}\, dx$

Setting $\frac{x}{2} = u$

Rewrite the integral

$$\int \sec^4 \frac{x}{2}\, dx = \int \sec^4\left(\frac{1}{2}x\right) dx$$

$$= \int 2\sec^4 u\; du$$

$$= 2\int \sec^4 u\; du$$

$$= 2\int \left(\sec^2 u + sce^2 u \tan^2 u\right) du$$

$$= 2\int \sec^2 u\; du + 2\int \sec^2 u \tan^2 u\, du$$

$$= 2\int 1 dul + 2\int \sec^2 u \tan^2 u\, du$$

$$= 2\tan(u) + 2\int \sec^2 u \tan^2 u\, du$$

$$= 2\tan(u) + 2\int ul^2\, dul$$

$$= 2\tan(u) + \frac{2}{3} ul^3$$

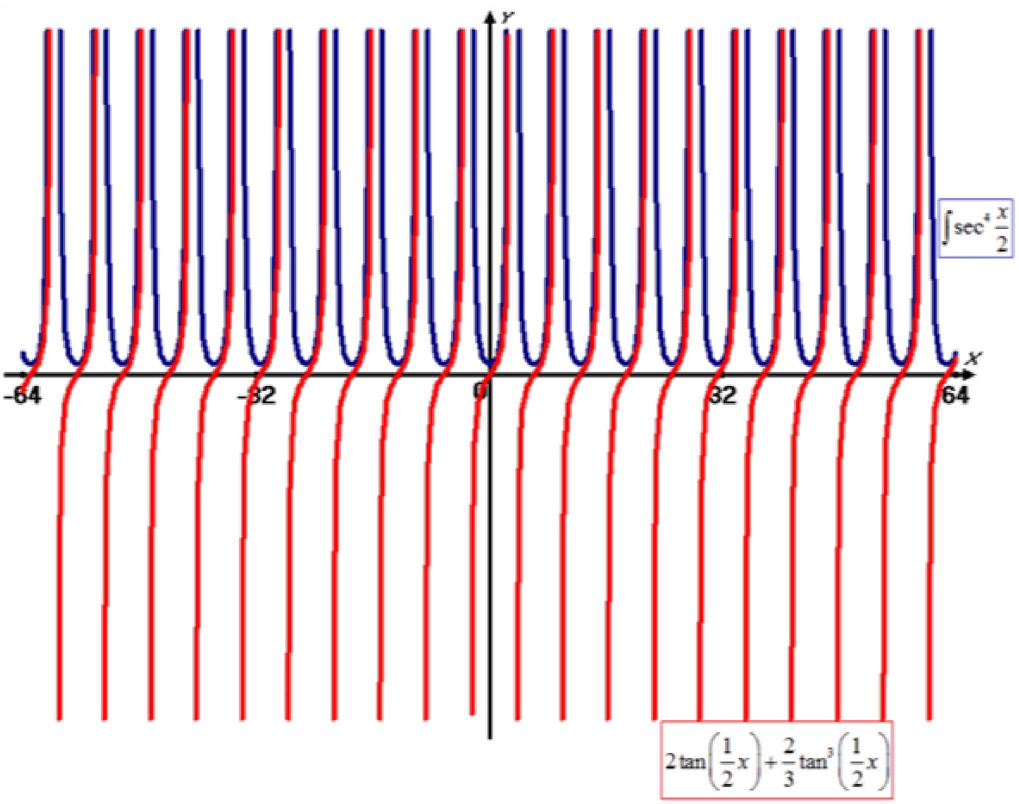
$$= 2\tan(u) + \frac{2}{3}\tan^3 u$$

$$= 2\tan\left(\frac{1}{2}x\right) + \frac{2}{3}\tan^3\left(\frac{1}{2}x\right)$$

Therefore

$$\int \sec^4 \frac{x}{2}\, dx = \boxed{2\tan\left(\frac{1}{2}x\right) + \frac{2}{3}\tan^3\left(\frac{1}{2}x\right)}$$

The required graph is



Answer 55E.

Average value of the function $f(x)=\sin^2 x\cos^3 x$ on the interval $[-\pi,\pi]$ is

$$\begin{aligned} f_{ave} &= \frac{1}{\pi-(-\pi)}\int_{-\pi}^{\pi}\sin^2 x\cos^3 x dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2 x\cos^3 x dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2 x\left(1-\sin^2 x\right).\cos x dx \qquad \text{--- (1)} \end{aligned}$$

Substitute $\sin x = t \Rightarrow \cos dx = dt$

Then $\int \sin^2 x\left(1-\sin^2 x\right)\cos x dx = \int t^2\left(1-t^2\right)dt$

$$\begin{aligned} &= \int\left(t^2-t^4\right)dt \\ &= \frac{t^3}{3}-\frac{t^5}{5}+C \\ &= \frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C \end{aligned}$$

From Equation (1)

$$f_{ave}=\frac{1}{2\pi}\left[\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x\right]_{-\pi}^{\pi}$$

Since $\sin\pi=\sin(-\pi)=0$

Thus average value $\boxed{f_{ave}=0}$

Answer 56E.

We have to evaluate $\int \sin x\cos x\, dx$

(A) Substitute $u=\cos x \Rightarrow du=-\sin x dx$

Thus $\int \sin x\cos x\, dx=\int u(-du)$

$$\begin{aligned} &= -\int u\, du \\ &= -\frac{u^2}{2}+C \\ &= \boxed{-\frac{1}{2}\cos^2 x+C_1} \end{aligned}$$

(B) Substitute $u=\sin x \Rightarrow du=\cos x dx$

Thus $\int \sin x\cos x\, dx=\int u\, du$

$$\begin{aligned} &= \frac{u^2}{2} \\ &= \boxed{\frac{1}{2}\sin^2 x+C_2} \end{aligned}$$

(C) Use the identity $\sin 2x = 2\sin x\cos x$

$$\begin{aligned} \int \sin x\cos x\, dx &= \int\frac{1}{2}\sin 2x dx \\ &= \frac{1}{2}\int \sin 2x\, dx \\ &= \frac{1}{2}\left(-\frac{\cos 2x}{2}\right)+C \\ &= \boxed{-\frac{1}{4}\cos 2x+C_3} \end{aligned}$$

(D) Integrate by parts

Take $u = \sin x \qquad dv = \cos x dx$

$du = \cos x\, dx \quad v = \sin x$

Then $\int \sin x \cos x\, dx = \sin x . \sin x - \int \sin x \cos x dx$

$2\int \sin x \cos x\, dx = \sin^2 x + C_4'$

$\int \sin x \cos x\, dx = \boxed{\frac{1}{2}\sin^2 x + C_4}$ Where $C_4 = C'_4/2$

Since $\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

So $-\frac{1}{4}\cos 2x = \frac{1}{2}\sin^2 x - \frac{1}{4} = -\frac{1}{2}\cos^2 x + \frac{1}{4}$

Thus the answers differ from each other by constants.

Answer 57E.

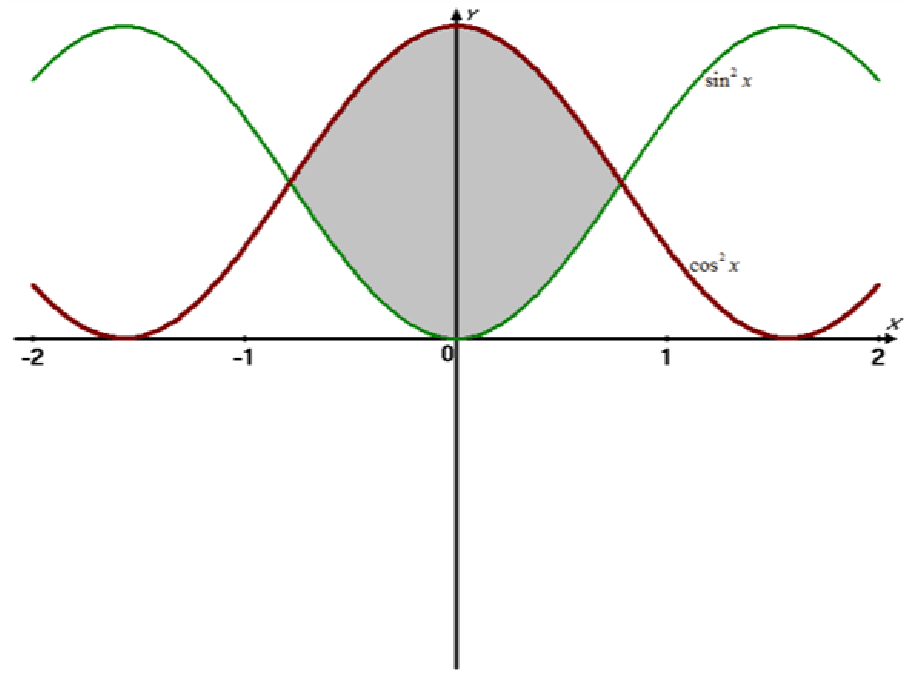
Consider

$y = \sin^2 x, y = \cos^2 x, -\pi/4 \le x \le \pi/4$

Remember that, area between the curves $y = f(x), y = g(x)$ between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)| dx$$

The region is sketched in the below graph, observe that $\cos^2 x \ge \sin^2 x$



Now the required area is

$$A = \int_{-\pi/4}^{\pi/4} |\cos^2 x - \sin^2 x| dx$$

$$= \int_{-\pi/4}^{\pi/4} \cos 2x\, dx$$

$$= 2\int_0^{\pi/4} \cos 2x\, dx$$ Since cos 2x is even function

$$= 2\left[\frac{\sin 2x}{2}\right]_0^{\pi/4}$$

$$= \sin 2(\pi/4) - 0$$

$$= \sin\frac{\pi}{2}$$

$$= \boxed{1}$$

Answer 58E.

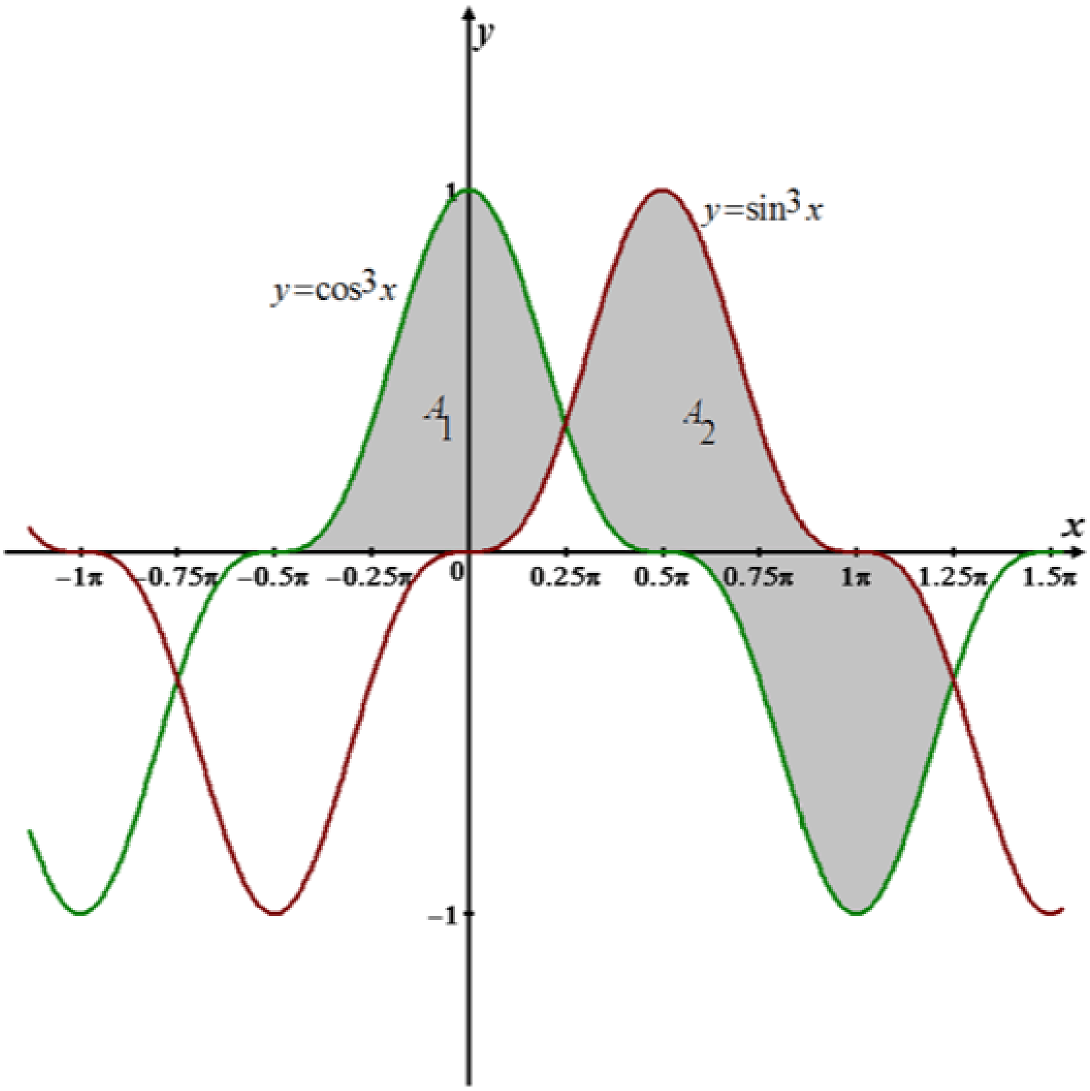
Consider

$y=\sin^3 x, y=\cos^3 x, -\pi/4 \le x \le 5\pi/4$

Remember that, area between the curves $y=f(x), y=g(x)$ between $x=a$ and $x=b$ is

$$A=\int_a^b |f(x)-g(x)|dx$$

In the region A_1, observe that $\cos^3 x \ge \sin^3 x$ and in the region A_2, observe that $\cos^3 x \le \sin^3 x$



Now the required area is

$$A=A_1+A_2$$

$$=\int_{-\pi/4}^{0} |\sin^3 x-\cos^3 x|dx+\int_0^{5\pi/4} |\cos^3 x-\sin^3 x|dx$$

First evaluate the integral $A_1=\int_{-\pi/4}^{0} |\sin^3 x-\cos^3 x|dx$

Now

$$\int_{-\pi/4}^{0} |\sin^3 x - \cos^3 x|\,dx = \int_{-\pi/4}^{0}\left(\frac{3\sin x - \sin 3x}{4} - \frac{\cos 3x + 3\cos x}{4}\right)dx$$

$$= \frac{1}{4}\int_{-\pi/4}^{0}(3\sin x - \sin 3x - \cos 3x - 3\cos x)\,dx$$

$$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3} - \frac{\sin 3x}{3} - 3\sin x\right]_{-\pi/4}^{0}$$

$$= \frac{1}{4}\left[\begin{matrix}-3\cos(-\pi/4) + \frac{\cos 3(-\pi/4)}{3} - \frac{\sin 3(-\pi/4)}{3} - 3\sin(-\pi/4) \\ +3\cos 0 - \frac{\cos 0}{3} + \frac{\sin 0}{3} + 3\sin 0\end{matrix}\right]$$

$$= \frac{1}{4}\left[\begin{matrix}-3\left(\frac{1}{\sqrt{2}}\right) + \frac{\left(-\frac{1}{\sqrt{2}}\right)}{3} + \frac{\left(\frac{1}{\sqrt{2}}\right)}{3} + 3\left(\frac{1}{\sqrt{2}}\right) \\ +3(1) - \frac{1}{3} + 0 + 0\end{matrix}\right]$$

$$= \frac{1}{4}\left[\frac{8}{3}\right]$$

$$= \frac{2}{3}$$

Thus, $A_1 = \frac{2}{3}$

Now evaluate $A_2 = \int_{0}^{5\pi/4} |\cos^3 x - \sin^3 x|\,dx$

$$\int_{0}^{5\pi/4} |\cos^3 x - \sin^3 x|\,dx = \int_{0}^{5\pi/4}\left(\frac{\cos 3x + 3\cos x}{4} - \frac{3\sin x - \sin 3x}{4}\right)dx$$

$$= \frac{1}{4}\int_{0}^{5\pi/4}(\cos 3x + 3\cos x - 3\sin x + \sin 3x)\,dx$$

$$= \frac{1}{4}\left[\frac{\sin 3x}{3} + 3\sin x + 3\cos x - \frac{\cos 3x}{3}\right]_{0}^{5\pi/4}$$

$$= \frac{1}{4}\left[\begin{matrix}\frac{\sin 3(5\pi/4)}{3} + 3\sin(5\pi/4) + 3\cos(5\pi/4) - \frac{\cos 3(5\pi/4)}{3} \\ -\frac{\sin 3(0)}{3} - 3\sin(0) - 3\cos(0) + \frac{\cos 3(0)}{3}\end{matrix}\right]$$

$$= \frac{1}{4}\left[\begin{matrix}\frac{\left(-\frac{1}{\sqrt{2}}\right)}{3} + 3\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(\frac{1}{\sqrt{2}}\right)}{3} - 3\left(\frac{1}{\sqrt{2}}\right) \\ -0 - 0 + 3 - \frac{1}{3}\end{matrix}\right]$$

$$= \frac{1}{4}\left[\frac{8}{3}\right]$$

$$= \frac{2}{3}$$

Thus, $A_2 = \frac{2}{3}$

Hence the area of the required region is

$$A = A_1 + A_2$$

$$= \frac{2}{3} + \frac{2}{3}$$

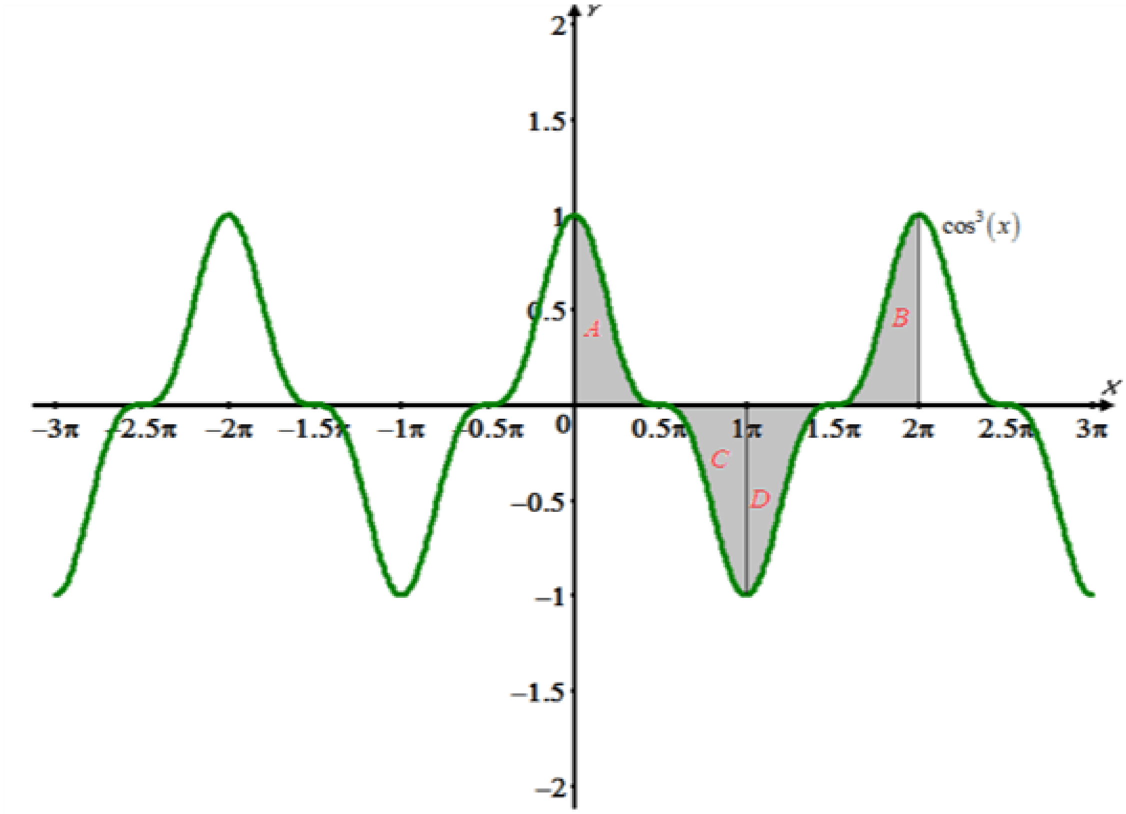
$$= \boxed{\frac{4}{3}}$$

Answer 59E.

Consider

$$\int_0^{2\pi} \cos^3(x)\,dx$$

The graph is represented as below



From the graph

$$A = \int_0^{\pi/2} \cos^3(x)\,dx$$

$$B = \int_{3\pi/2}^{2\pi} \cos^3(x)\,dx$$

$$C = \int_{\pi/2}^{\pi} \cos^3(x)\,dx$$

$$D = \int_{\pi}^{3\pi/2} \cos^3(x)\,dx$$

The area of the total region is

$$\text{Area} = A + B - C - D$$

$$= \int_0^{\pi/2} \cos^3(x)\,dx + \int_{3\pi/2}^{2\pi} \cos^3(x)\,dx - \int_{\pi/2}^{\pi} \cos^3(x)\,dx - \int_{\pi}^{3\pi/2} \cos^3(x)\,dx$$

$$= 0$$

Therefore from the graph, the integral is $\boxed{0}$

Answer 60E.

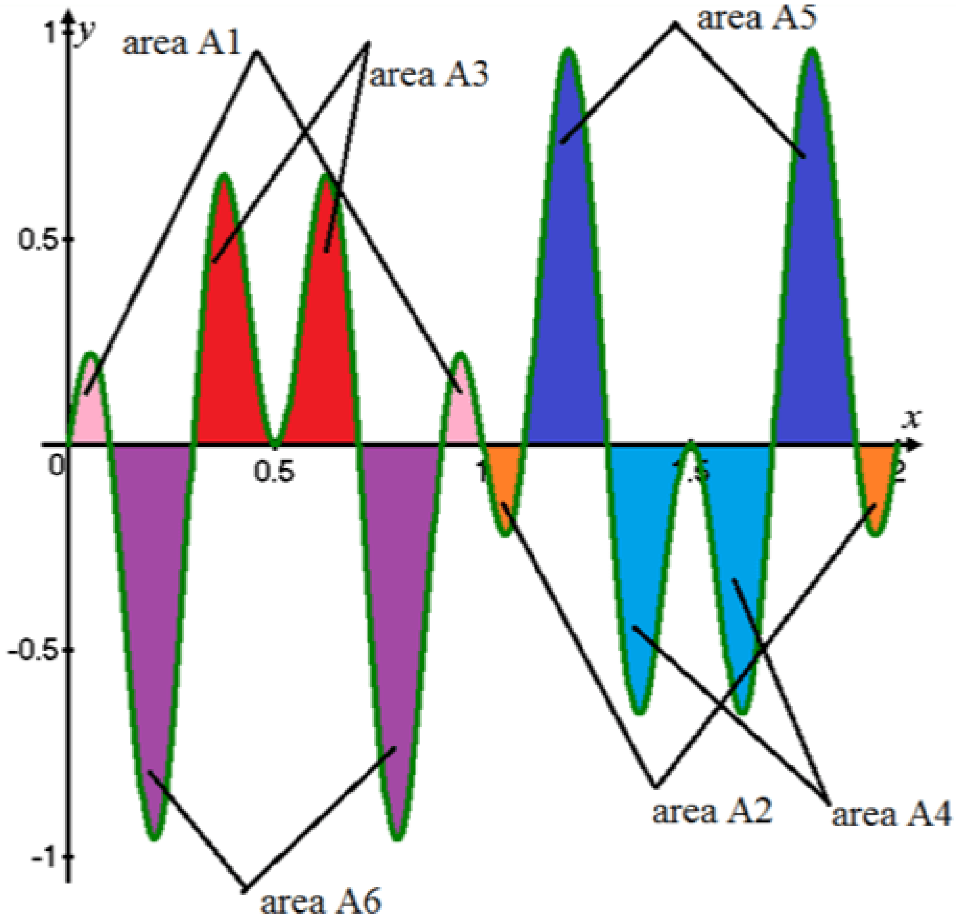
Consider the following definite integral

$$\int_0^2 \sin(2\pi x)\cdot\cos(5\pi x)\,dx$$

The integrand of this integral is $f(x)=\sin(2\pi x)\ \cos(5\pi x)$

The definite integral represents area under the curve

The graph of integrand on interval $[0,2]$ is as shown below



From figure the area A_1 represents area under the curve of two small size loops of above $x-\text{axis}$ and A_2 represents area under eh curve of two small size loops of below the $x-\text{axis}$, but magnitude of these two values are equal and has opposite direction

Since function has positive value on above x-axis and below x-axis it has negative sign

$$|A_1|=|A_2| \text{ and } A_1=-A_2$$

Similarly $|A_3|=|A_4|$ and $A_3=-A_4$

$$|A_5|=|A_6| \text{ and } A_5=-A_6$$

$$\int_0^2 \sin(2\pi x)\cdot\cos(5\pi x)\,dx = A_1+A_2+A_3+A_4+A_5+A_6$$

$$= A_1+(-A_1)+A_3+(-A_3)+A_5+(-A_5)$$

$$=0$$

The approximate value of definite integral is zero

The integrand can written as follows by using following identity

$$\sin A\cos B=\frac{1}{2}\left[\sin(A-B)+\sin(A+B)\right]$$

$$\begin{aligned}\sin(2\pi x)\cdot\cos(5\pi x)&=\frac{1}{2}\left[\sin(2\pi x-5\pi x)+\sin(2\pi x+5\pi x)\right]\\&=\frac{1}{2}\left[\sin(-3\pi x)+\sin(7\pi x)\right]\\&=\frac{1}{2}\left[-\sin(3\pi x)+\sin(7\pi x)\right]\end{aligned}$$

$$\begin{aligned}\int_0^2\sin(2\pi x)\cdot\cos(5\pi x)\,dx&=\int_0^2\frac{1}{2}\left[-\sin(3\pi x)+\sin(7\pi x)\right]dx\\&=\frac{1}{2}\int_0^2(\sin 7\pi x-\sin 3\pi x)\,dx\\&=\frac{1}{2}\left[\frac{-\cos 7\pi x}{7\pi}\right]_0^2-\frac{1}{2}\left[\frac{-\cos 3\pi x}{3\pi}\right]_0^2\\&=-\frac{1}{14\pi}\left[\cos 7\pi x\right]_0^2+\frac{1}{6\pi}\left[\cos 3\pi x\right]_0^2\end{aligned}$$

$$=-\frac{1}{14\pi}\left[\cos 14\pi-\cos 0\right]+\frac{1}{6\pi}\left[\cos 6\pi-\cos 0\right]$$

$$=-\frac{1}{14\pi}[1-1]+\frac{1}{6\pi}[1-1]$$

$$=0$$

Answer 61E.

Consider

$$y=\sin x,\ y=0,\ \frac{\pi}{2}\le x\le\pi$$

The region *R* is shown in the below figure A.

If we rotate about *x*-axis, we get the solid shown in the figure B.

When we slice through the point *x*, we get a disk with radius $\sin x$

The area of the cross section is

$$\begin{aligned}A(x)&=\pi(\sin x)^2\\&=\pi\sin^2 x\end{aligned}$$

And the volume of the approximating cylinder (a disk with thickness Δx) is

$$A(x)\Delta x=\pi\sin^2 x\Delta x$$

The figure A:

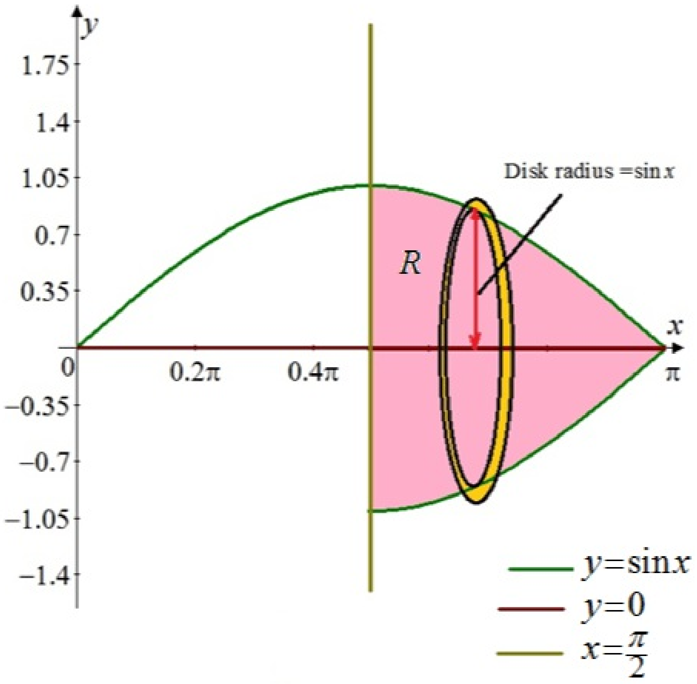
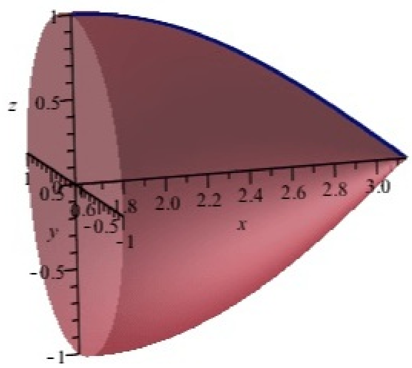


Figure B:



The solid lies between $x = \frac{\pi}{2}$ and $x = \pi$, so its volume is

$$V = \int_{\frac{\pi}{2}}^{\pi} \pi(\sin x)^2 dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} \pi \sin^2 x dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} \pi\left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{\pi}{2}\int_{\frac{\pi}{2}}^{\pi}(1-\cos 2x)dx$$

$$= \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{\pi}{2}\left[\pi - \frac{\sin 2\pi}{2} - \frac{\pi}{2} + \frac{\sin \pi}{2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \boxed{\frac{\pi^2}{4}}$$

Answer 62E.

Consider

$y = \sin^2 x,\ y = 0,\ 0 \le x \le \pi$

The region R is shown in the below figure A.

If we rotate about x-axis, we get the solid shown in the figure B.

When we slice through the point x, we get a disk with radius $\sin x$

The area of the cross section is

$A(x) = \pi\left(\sin^2 x\right)^2$

And the volume of the approximating cylinder (a disk with thickness Δx) is

$A(x)\Delta x = \pi\left(\sin^2 x\right)^2 \Delta x$

The figure A:

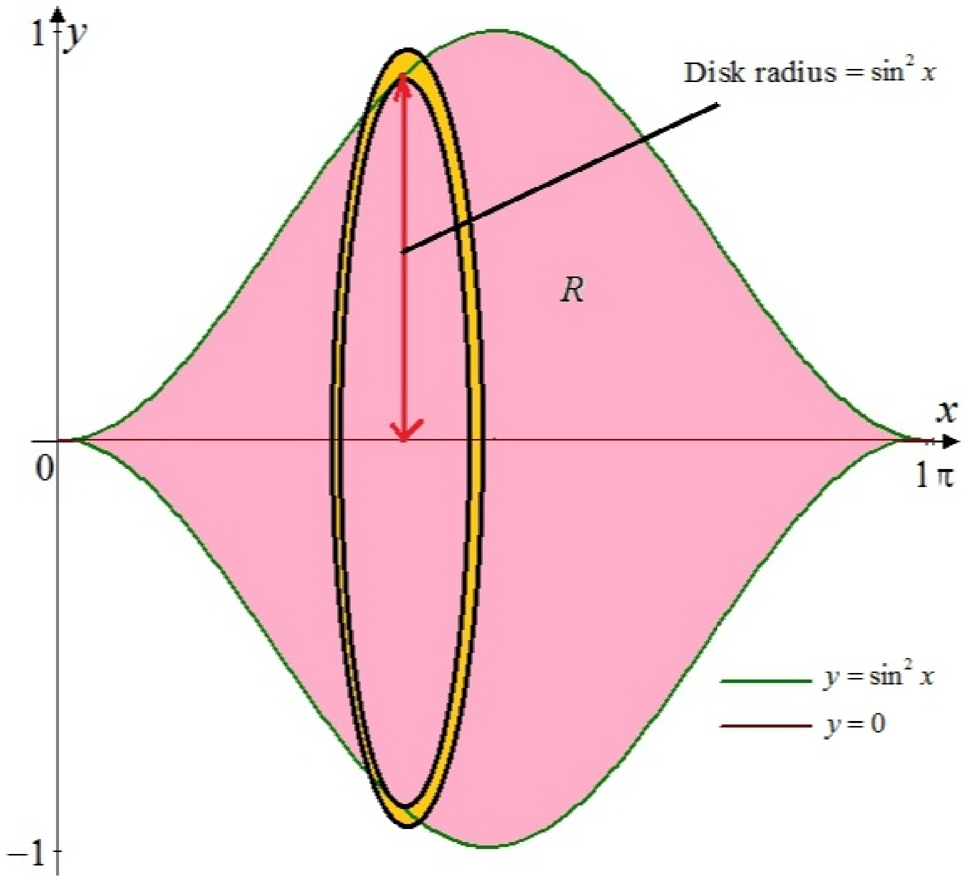
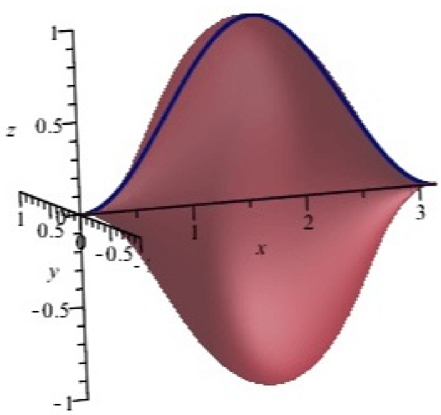


Figure B:



The solid lies between $x=0$ and $x=\pi$, so its volume is

$$V=\int_0^{\pi}\pi\left(\sin^2 x\right)^2 dx$$

$$=\int_0^{\pi}\pi\left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$=\frac{\pi}{4}\int_0^{\pi}(1-\cos 2x)^2 dx$$

$$=\frac{\pi}{4}\int_0^{\pi}\left(1+\cos^2 2x-2\cos 2x\right)dx$$

$$=\frac{\pi}{4}\int_0^{\pi}\left(1+\frac{1+\cos 2x}{2}-2\cos 2x\right)dx$$

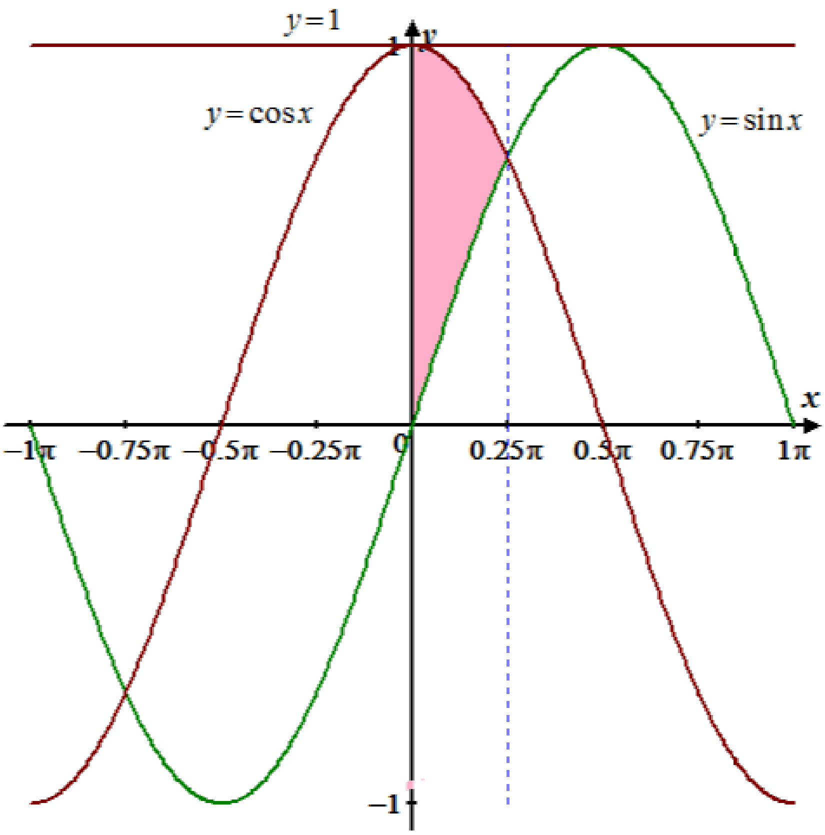
$$=\frac{3\pi}{8}\int_0^{\pi}(1-\cos 2x)dx$$

$$=\frac{3\pi}{8}\left[x-\frac{\sin 2x}{2}\right]_0^{\pi}$$

$$=\boxed{\frac{3}{8}\pi^2}$$

Answer 63E.

By using washer method to find the volume obtained by rotating the region bounded by the curves $y=\sin x$ and $y=\cos x$, $0\le x\le\frac{\pi}{4}$ about $y=1$.



The cross sectional area of washer is

$$A(x)=\pi\left[(\text{outer radius})^2-(\text{inner radius})^2\right]$$
$$=\pi\left[(1-\sin x)^2-(1-\cos x)^2\right]$$

From the figure outer and inner radii are as follows. Integrating between 0 and $\frac{\pi}{4}$.

$$V=\int_0^{\frac{\pi}{4}} A(x)dx$$

$$=\pi\int_0^{\frac{\pi}{4}}\left((1-\sin x)^2-(1-\cos x)^2\right)dx$$

$$=\pi\int_0^{\frac{\pi}{4}}\left(\left(1+\sin^2 x-2\sin x\right)-\left(1+\cos^2 x-2\cos x\right)\right)dx$$

$$=\pi\int_0^{\frac{\pi}{4}}\left(1+\sin^2 x-2\sin x-1-\cos^2 x+2\cos x\right)dx$$

$$=\pi\int_0^{\frac{\pi}{4}}\left(\sin^2 x-\cos^2 x-2\sin x+2\cos x\right)dx$$

$$=\pi\int_0^{\frac{\pi}{4}}\left(-\cos 2x-2\sin x+2\cos x\right)dx$$

$$=\pi\left[-\frac{\sin 2x}{2}-2(-\cos x)+2\sin x\right]_0^{\frac{\pi}{4}}$$

$$=\pi\left[-\frac{\sin 2x}{2}+2\cos x+2\sin x\right]_0^{\frac{\pi}{4}}$$

$$=\pi\left[-\frac{\sin 2\left(\frac{\pi}{4}\right)}{2}+2\cos\left(\frac{\pi}{4}\right)+2\sin\left(\frac{\pi}{4}\right)-\frac{\sin 2(0)}{2}-2\cos(0)-2\sin(0)\right]$$

$$=\pi\left[-\frac{1}{2}+2\left(\frac{1}{\sqrt{2}}\right)+2\left(\frac{1}{\sqrt{2}}\right)-0-2(1)-0\right]$$

$$=\pi\left[-\frac{1}{2}+2\sqrt{2}-2\right]$$

$$=\pi\left[2\sqrt{2}-\frac{5}{2}\right]$$

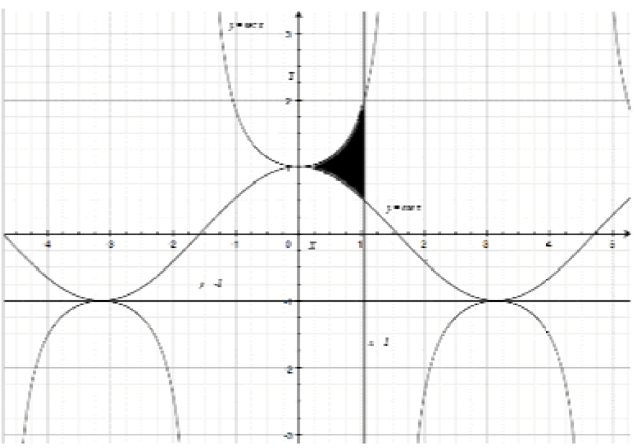
Hence the volume is $\boxed{\pi\left[2\sqrt{2}-\frac{5}{2}\right]}$.

Answer 64E.

We have to find the volume obtained by rotating the region bounded by the curves

$y=\sec x, y=\cos x, 0\le x\le\frac{\pi}{3}$ about $y=-1$

It would help to first graph the area that we are spinning to get our volume:



To spin it about $y=-1$ we must set up an integral, using disc method:

$$\text{Volume } =\pi\int\left(\text{Height}^2-\text{Washer}^2\right)dx$$

Filling this formula in, an accounting for the translation of the axis, we get:

$$\text{Volume } =\pi\int_0^{\pi/3}(\sec x+1)^2-(\cos x+1)^2\,dx$$
$$=\pi\int_0^{\pi/3}\left(\sec^2 x+2\sec x+1-\cos^2 x-2\cos x-1\right)dx$$
$$=\pi\int_0^{\pi/3}\sec^2 x+2\sec x-\cos^2 x-2\cos x dx$$

$$=\pi\int_0^{\pi/3}\sec^2 x+2\sec x-\left(\frac{1+\cos 2x}{2}\right)-2\cos x dx$$

$$=\pi\left[\tan x+2\ln\left|\sec x+\tan x\right|-\frac{x+\frac{1}{2}\sin(2x)}{2}-2\sin x\right]_0^{\pi/3}$$

$$=\pi\left[\left(\sqrt{3}+2\ln\left|2-\sqrt{3}\right|-\frac{\frac{\pi}{3}+\frac{\sqrt{3}}{4}}{2}-\sqrt{3}\right)-\left(0+\frac{4\sqrt{2}}{3}+2\ln(1+0)-0-0\right)\right]$$

$$=\boxed{\pi\left[\left(2\ln\left|2-\sqrt{3}\right|-\frac{4\pi+3\sqrt{3}}{24}\right)-\frac{4\sqrt{2}}{3}\right]}$$

Answer 65E.

Consider the velocity function

$$v(t)=\sin\omega t\ \cos^2\omega t$$

Required to find the position function $s=f(t)$ if $f(0)=0$.

The integral of velocity function is nothing but position function

$$s=\int_0^t v(t)\ dt$$
$$=\int_0^t \sin\omega t\ \cos^2\omega t\ dt$$

Let $u=\cos wt$ then $du=-\omega\ \sin\omega t\ dt$

So, $-\frac{du}{\omega}=\sin\omega t$

If $t=0$ then $u=1$ and $t=\cos\omega t$

The above integral becomes

$$\int_0^t \sin\omega t\ \cos^2\omega t\ dt=-\frac{1}{\omega}\int_1^{\cos\omega t}u^2\ du$$
$$=-\frac{1}{\omega}\left[\frac{u^3}{3}\right]_1^{\cos\omega t}$$
$$=\frac{1}{\omega}\left[1-\cos^3\omega t\right]$$

Therefore, position function is $s=\boxed{\frac{1}{\omega}\left[1-\cos^3\omega t\right]}$.

Answer 66E.

Consider the voltage of the equation

$E(t) = 155\sin(120\pi t)$

Alternating current varies from 155V to -155V and frequency of 60 cycles per second.

Required to calculate the RMS voltage of household current.

RMS voltage of the current is the root mean square voltage which is a square root of the average of $[E(t)]^2$ over a cycle.

Average of $[E(t)]^2$ is $\int_0^{2\pi} \frac{[E(t)]^2}{2\pi}\, dt$

Now,

$$\begin{aligned}
\int_0^{2\pi} \frac{[E(t)]^2}{2\pi}\, dt &= \int_0^{2\pi} \frac{[155\sin(120\pi t)]^2}{2\pi}\, dt \\
&= \frac{155^2}{2\pi}\int_0^{2\pi} \sin^2(120\pi t)\, dt \\
&= \frac{155^2}{4\pi}\int_0^{2\pi} (1-\cos 240\pi t)\, dt \quad \left[\text{Since } 1-\cos 2x = 2\sin^2 x\right] \\
&= \frac{155^2}{4\pi}\left[t - \frac{\sin 240\pi t}{240\pi}\right]_0^{2\pi}
\end{aligned}$$

$$= \frac{155^2}{4\pi}[2\pi - 0 - 0]$$
$$= \frac{155^2}{2}$$

Average of $[E(t)]^2$ is $= \frac{155^2}{2}$

Therefore, RMS voltage is $\sqrt{[E(t)]^2} = \boxed{\frac{155}{\sqrt{2}}}$

(b) If the amplitude of voltage is A with $V(t) = A\sin(120\pi t)$ then the RMS voltage is $\frac{A}{\sqrt{2}}$.

Electric stoves require an RMS voltage of 220 V.

Required to find the corresponding amplitude A.

Average of $[E(t)]^2$ is $\int_0^{2\pi} \frac{[E(t)]^2}{2\pi}\, dt$

Now,

$$\begin{aligned}
\int_0^{2\pi} \frac{[E(t)]^2}{2\pi}\, dt &= \frac{A^2}{2\pi}\int_0^{2\pi} \sin^2(120\pi t)\, dt \\
&= \frac{A^2}{2\pi}\int_0^{2\pi} (1-\cos 240\pi t)\, dt \left[\text{Since } 1-\cos 2x = 2\sin^2 x\right] \\
&= \frac{A^2}{4\pi}\left[t - \frac{\sin 240\pi t}{240\pi}\right]_0^{2\pi}
\end{aligned}$$

$$= \frac{A^2}{4\pi}[2\pi - 0 - 0]$$
$$= \frac{A^2}{2}$$

Average of $[E(t)]^2$ is $= \frac{A^2}{2}$

Therefore, RMS voltage is $\sqrt{[E(t)]^2} = \boxed{\frac{A}{\sqrt{2}}}$

So,

$$\begin{aligned} A &= \text{RMS voltage} \times \sqrt{2} \\ &= 220\sqrt{2} \end{aligned}$$

Therefore, amplitude A is $\boxed{220\sqrt{2}}$

Answer 67E.

Prove that $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0.$

Here, m and n are positive integers.

Use the formula, $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)].$

$$\begin{aligned} \int_{-\pi}^{\pi} \sin mx \cos nx \, dx &= \frac{1}{2}\int_{-\pi}^{\pi} [\sin(mx - nx) + \sin(mx + nx)]dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} [\sin(m-n)x + \sin(m+n)x]dx \\ &= -\frac{1}{2}\left[\frac{\cos(m-n)x}{m-n} + \frac{\cos(m+n)x}{m+n}\right]_{-\pi}^{\pi} \end{aligned}$$

Cosine is an even function. That is, $\cos(-x) = \cos x.$

Integrate from $-\pi$ to π to obtain the following:

$$\begin{aligned} \cos x - \cos(-x) &= \cos x - \cos x \\ &= 0 \end{aligned}$$

Therefore, the above integral becomes 0.

Thus, $\boxed{\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0}.$

Answer 68E.

Prove that $\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

Use the transformation formulas to prove the result.

$$\begin{aligned} \cos((m-n)x) &= \cos(mx - nx) \\ &= \cos(mx)\cos(nx) + \sin(mx)\sin(nx) \qquad \text{......(1)} \end{aligned}$$

And

$$\begin{aligned} \cos((m+n)x) &= \cos(mx + nx) \\ &= \cos(mx)\cos(nx) - \sin(mx)\sin(nx) \qquad \text{......(2)} \end{aligned}$$

And also

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

From (1) and (2)

$$\sin mx \sin nx = \frac{1}{2}\left[\cos\left((m-n)\right)x - \cos\left((m+n)x\right)\right]$$ and then if $m \neq n$

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos\left((m-n)x\right) - \cos\left((m+n)x\right)\right] dx$$

$$= \left[\frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi}$$

$$= \left[\frac{\sin(m-n)\pi}{2(m-n)} - \frac{\sin(m+n)\pi}{2(m+n)}\right] - \left[\frac{\sin(m-n)(-\pi)}{2(m-n)} - \frac{\sin(m+n)(-\pi)}{2(m+n)}\right]$$

$$= [0-0+0-0]$$

$$= \boxed{0}$$

If $n = m$ then

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos\left((n-n)x\right) - \cos\left((n+n)x\right)\right] dx$$

$$= \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos(0) - \cos\left((2n)x\right)\right] dx$$

$$= \int_{-\pi}^{\pi} \left[\frac{1-\cos(2nx)}{2}\right] dx$$

$$= \left[\frac{x}{2} - \frac{\sin(2nx)}{4n}\right]_{-\pi}^{\pi} \text{ where } n \text{ is a postive integer}$$

$$= \left[\frac{\pi}{2} - \frac{\sin(2n\pi)}{4n}\right] - \left[-\frac{\pi}{2} - \frac{\sin(-2n\pi)}{4n}\right]$$

$$= \frac{\pi}{2} - 0 + \frac{\pi}{2} - 0$$

$$= \boxed{\pi}$$

Answer 69E.

Prove that $\int_{-\pi}^{\pi} \cos mx \cos nx dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

Use the transformation formulas to prove the result.

$$\cos\left((m-n)x\right) = \cos(mx - nx)$$
$$= \cos(mx)\cos(nx) + \sin(mx)\sin(nx) \quad \text{......(1)}$$

And

$$\cos\left((m+n)x\right) = \cos(mx + nx)$$
$$= \cos(mx)\cos(nx) - \sin(mx)\sin(nx) \quad \text{......(2)}$$

And also

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

From (1) and (2)

$\cos mx\cos nx=\frac{1}{2}\left[\cos\left(\left(m-n\right)\right)x+\cos\left(\left(m+n\right)x\right)\right]$ and then if $m\neq n$

$$\int_{-\pi}^{\pi}\cos mx\cos nx dx=\int_{-\pi}^{\pi}\frac{1}{2}\left[\cos\left(\left(m-n\right)x\right)+\cos\left(\left(m+n\right)x\right)\right]dx$$

$$=\left[\frac{\sin\left(m-n\right)x}{2\left(m-n\right)}+\frac{\sin\left(m+n\right)x}{2\left(m+n\right)}\right]_{-\pi}^{\pi}$$

$$=\left[\frac{\sin\left(m-n\right)\pi}{2\left(m-n\right)}+\frac{\sin\left(m+n\right)\pi}{2\left(m+n\right)}\right]-\left[\frac{\sin\left(m-n\right)\left(-\pi\right)}{2\left(m-n\right)}+\frac{\sin\left(m+n\right)\left(-\pi\right)}{2\left(m+n\right)}\right]$$

$$=\left[0-0+0-0\right]$$

$$=\boxed{0}$$

If $n=m$ then

$$\int_{-\pi}^{\pi}\sin mx\sin nx dx=\int_{-\pi}^{\pi}\frac{1}{2}\left[\cos\left(\left(n-n\right)x\right)+\cos\left(\left(n+n\right)x\right)\right]dx$$

$$=\int_{-\pi}^{\pi}\frac{1}{2}\left[\cos\left(0\right)+\cos\left(\left(2n\right)x\right)\right]dx$$

$$=\int_{-\pi}^{\pi}\left[\frac{1+\cos\left(2nx\right)}{2}\right]dx$$

$$=\left[\frac{x}{2}+\frac{\sin\left(2nx\right)}{4n}\right]_{-\pi}^{\pi}\quad\text{where } n \text{ is a postive integer}$$

$$=\left[\frac{\pi}{2}-\frac{\sin\left(2n\pi\right)}{4n}\right]-\left[-\frac{\pi}{2}+\frac{\sin\left(-2n\pi\right)}{4n}\right]$$

$$=\frac{\pi}{2}+0+\frac{\pi}{2}+0$$

$$=\boxed{\pi}$$

Answer 70E.

Consider a finite Fourier series sum

$$f\left(x\right)=\sum_{n=1}^{N}a_n\sin nx$$
$$=a_1\sin x+a_2\sin 2x+a_3\sin 3x+\ ...\ +a_N\sin Nx$$

Show that the mth coefficient a_m is given by the formula

$$a_m=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(x\right)\sin mx\ dx \qquad ......(1)$$

First show orthogonality of the trigonometric sine monomials. Let m and n be unequal integers, then

Use the transformation formula

$$\sin A\sin B=\frac{1}{2}\left[\cos\left(A-B\right)-\cos\left(A+B\right)\right]$$

$$\int_{-\pi}^{\pi}\sin mx\sin nx dx=\int_{-\pi}^{\pi}\frac{1}{2}\left[\cos\left(\left(m-n\right)x\right)-\cos\left(\left(m+n\right)x\right)\right]dx$$

$$=\left[\frac{\sin\left(m-n\right)x}{2\left(m-n\right)}-\frac{\sin\left(m+n\right)x}{2\left(m+n\right)}\right]_{-\pi}^{\pi}$$

$$=\left[\frac{\sin\left(m-n\right)\pi}{2\left(m-n\right)}-\frac{\sin\left(m+n\right)\pi}{2\left(m+n\right)}\right]-$$
$$\left[\frac{\sin\left(m-n\right)\left(-\pi\right)}{2\left(m-n\right)}-\frac{\sin\left(m+n\right)\left(-\pi\right)}{2\left(m+n\right)}\right]$$

$$=\left[0-0+0-0\right]$$

$$=0$$

If $n = m$ then

$$\int_{-\pi}^{\pi} \sin^2 mx dx = \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos((m-m)x) - \cos((m+m)x)\right] dx$$

$$= \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos(0) - \cos((2m)x)\right] dx$$

$$= \int_{-\pi}^{\pi} \left[\frac{1-\cos(2mx)}{2}\right] dx$$

$$= \left[\frac{x}{2} - \frac{\sin(2mx)}{4m}\right]_{-\pi}^{\pi} \text{ where } m \text{ is a postive integer}$$

$$= \left[\frac{\pi}{2} - \frac{\sin(2m\pi)}{4m}\right] - \left[-\frac{\pi}{2} - \frac{\sin(-2m\pi)}{4m}\right]$$

$$= \frac{\pi}{2} - 0 + \frac{\pi}{2} - 0$$

$$= \boxed{\pi}$$

From the equation of (1) Taking RHS

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \sum_{n=1}^{N} a_n \sin nx \sin mx \, dx$$

$$= \frac{1}{\pi}\sum_{n=1}^{N} \int_{-\pi}^{\pi} a_n \sin nx \sin mx \, dx$$

$$= \frac{1}{\pi} a_m \int_{-\pi}^{\pi} \sin^2 mx \, dx$$

$$= \frac{1}{\pi} a_m \cdot \pi$$

$$= \boxed{a_m}$$