Chapter 13

Probability

Conditional Probability and its Properties

Conditional Probability and Multiplication Theorem

Let A and B be two events such that $P(A) > 0$. Then $P(B|A)$ denote the conditional probability of B given that A has occurred. Since A is known to have occurred, it becomes the new sample space replacing the original S. From this we led to the definition. $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$ which is called conditional probability of B given A.

$\Rightarrow P(A \cap B) = P(A)\, P(B \mid A)$ which is called compound probability or multiplication theorem. It says the probability that both A and B occur is equal to the probability that A occur times the probability that B occurs given that A has occurred.

Note : For any three events A_1, A_2, A_3 we have $P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2 \mid A_1)\, P(A_3 \mid (A_1 \cap A_2))$

Ex.1 Two dice are thrown. Find the probability that the numbers appeared has a sum of 8 if it is known that the second dice always exhibits 4.

Sol. Let A be the event of occurrence of 4 always on the second die

$= \{(1,4), (2, 4), (3, 4)\ (4,4), (5, 4), (6, 4)\}$;

$\therefore\ n(A) = 6$

and B be the event of occurrence of such numbers on both dice whose sum is $8 = \{(4, 4)\}$.

Thus, $A \cap B = A \cap \{(4, 4)\} = \{(4, 4)\}$

$\therefore\ n(A\ Ç\ B) = 1$

$$\therefore\ P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{1}{6} \quad \text{or} \quad \frac{P(A \cap B)}{P(A)} = \frac{1/36}{6/36} = \frac{1}{6}$$

Ex.2 A bag contains 3 red, 6 white and 7 blue balls. Two balls are drawn one by one. What is the probability that first ball is white and second ball is blue when first drawn ball is not replaced in the bag ?

Sol. Let A be the event of drawing first ball white and B be the event of drawing second ball blue.

Here A and B are dependent events.

$P(A) = \frac{6}{16}$, $P(B|A) = \frac{7}{15}$

$\Rightarrow P(AB) = P(A).P(B/A) = \frac{6}{16} \times \frac{7}{15} = \frac{7}{40}$

Ex.3:A bag contains 4 red and 4 blue balls. Four balls are drawn one by one from the bag, then find the probability that the drawn balls are in alternate colour.

Sol. E_1 : Event that first drawn ball is red, second is blue and so on.

E_2 : Event that first drawn ball is blue, second is red and so on.

$\therefore P(E_1) = \frac{4}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5}$ and

$P(E_2) = \frac{4}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5}$

$\Rightarrow P(E) = P(E_1) + P(E_2) = 2 \times \frac{4}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{3}{5} = \frac{6}{35}$

Ex 4: In a survey in a class, the probability for a person to watch videos is 0.8 and the probability for a person to be a topper, if given that he watches videos is 0.99. find the probability for a person to be both topper and watches videos.

Solution:

let Event E denotes the event that a person watches Examfear videos

let Event F denotes the event that a person is topper

then P(E) = the probability that a person watches Examfear videos =0.8

and P(F|E)= the probability that a person is a topper if he watches Examfear videos=0.99

then P (E ∩ F)= the probability that a person is both topper and also watches Examfear videos

then according to Multiplication Theorem on probability

$P(E \cap F) = P(E)\, P(F \mid E)$

$= 0.8 \times 0.99$

$= 0.792$

∴ The probability that a person is both topper and also watches videos = 0.792

Independent Events and Their Important Properties

Independent Events

Two events A & B are said to be independent if occurrence or non occurrence of one does not effect the probability of the occurrence or non occurence of other.

(a) If the occurrence of one event affects the probability of the occurrence of the other event then the events are said to be dependent or Contingent. For two independent events A and B

$P(A \cap B) = P(A)$

. P(B). Often this is taken as the definition of independent events.

(b) Three events A, B & C are independent if & only if all the following conditions hold ;

$P(A \cap B) = P(A).\, P(B)$; $P(B \cap C) = P(B).\, P(C)$

$P(C \cap A) = P(C).\, P(A)$ & $P(A \cap B \cap C) = P(A).\, P(B).\, P(C)$

i.e., they must be pairwise as well as mutually independent.

Similarly for n events A1, A2, A3, ... An to be independent, the number of these conditions is equal to nC2 + nC3 + ... + nCn = 2n - n - 1.

Note : Independent events are not in general mutually exclusive & vice versa.

Mutually exclusiveness can be used when the events are taken from the same experiment & independence can be used when the events are taken from different experiments.

Ex.1 The probability that an anti aircraft gun can hit an enemy plane at the first, second and third shot are 0.6, 0.7 and 0.1 respectively. The probability that the gun hits the plane is

Sol. Let the events of hitting the enemy plane at the first, second and third shot are respectively A, B and C. Then as given P(A) = 0.6, P(B) = 0.7, P(C) = 0.1

Since events A, B, C are independent, so

$$P(A \cup B \cup C) + P(\overline{A \cap B \cap C}) = 1$$

$\Rightarrow$

$$P(A \cup B \cup C) = 1 - P(\overline{A} \cap \overline{B} \cap \overline{C})$$

Required probability $= P(A \cup B \cup C) = 1 - P(\overline{A})\, P(\overline{B})\, P(\overline{C})$

$= 1 - (1 - 0.6)\,(1 - 0.7)\,(1 - 0.1)) = 1 - (0.4)(0.3)(0.9) = 1 - 0.108 = 0.892$

Ex.2 If two events A and B are such that

$P(\overline{A}) = 0.3$

, P(B) = 0.4 and

$P(A\overline{B}) = 0.5$

then

$P(B \mid (A \cup \overline{B}))$

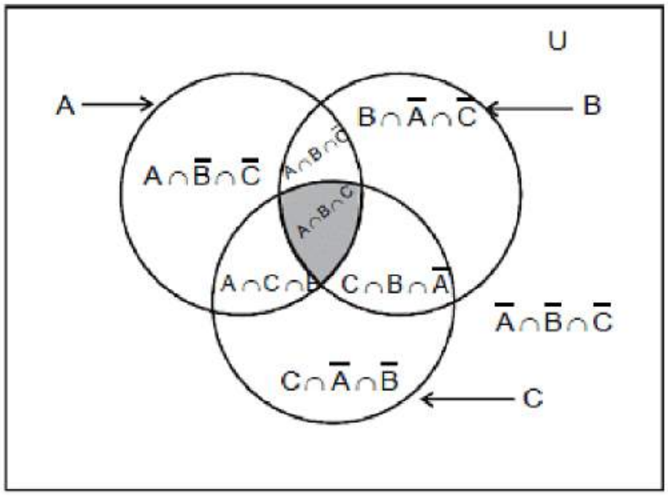
equals

Sol.

We have $P(B \mid A \cup \bar{B}) = \dfrac{P[B \cap (A \cup \bar{B})]}{P(A \cup \bar{B})} = \dfrac{P[(B \cap A) \cup (B \cap \bar{B})]}{P(A) + P(\bar{B}) - P(A \cap B)}$

$$= \frac{P(AB)}{P(A) + P(\bar{B}) - P(AB)} = \frac{P(A) - P(A\bar{B})}{P(A) + P(\bar{B}) - P(AB)} = \frac{0.7 - 0.5}{0.7 + 0.6 - 0.5} = \frac{0.2}{0.8} = \frac{1}{4}$$

Probability of Three Events



For any three events A, B and C we have

(a) P(A or B or C) = P(A) + P(B) + P(C) – P(A∩B) – P(B ∩ C) – P(C ∩ A) + P(A ∩ B ∩ C)

(b) P (at least two of A, B, C occur) = P(B ∩ C) + P (C ∩ A) + P(A ∩ B) – 2P (A ∩ B ∩ C)

(c) P (exactly two of A, B, C occur) = P(B ∩ C) + P (C ∩ A) + P(A ∩ B) – 3P (A ∩ B ∩ C)

(d) P (exactly one of A, B, C occurs) = P(A) + P(B) + P(C) – 2P (B ∩ C) – 2P (C ∩ A) – 2P (A ∩ B) + 3P (A ∩ B ∩ C)

Bayes' Theorem

Bayes' Theorem

If an event A can occur only with one of the n mutually exclusive and exhaustive events B_1. B_2,......B_n & the probabilities $P(A|B_1)$, $P(A|B_2)$ .... $P(A|B_n)$ are known

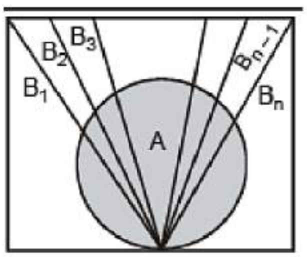
then $P(B_1 \mid A) = \frac{P(B_i).P(A \mid B_i)}{\sum_{i=1}^{n} P(B_i).P(A \mid B_i)}$

Explanation :

$A \equiv$ event what we have ; $B_1 \equiv$ event what we want ;

B_2, B_3, ... B_n, are alternative event.

Now, $P(AB_i) = P(A). P(B_i \mid A) = P(B_i). P(A \mid B_i)$



$$P(B_i \mid A) = \frac{P(B_i).P(A \mid B_i)}{P(A)} = \frac{P(B_i).P(A \mid B_i)}{\sum_{i=1}^{n} P(AB_i)}$$

$$\text{or } P(B_i \mid A) = \frac{P(B_i).P(A \mid B_i)}{\Sigma P(B_i).P(A \mid B_i)}$$

Ex.1 Given three identical boxes I, II and III, each containing two coins, In box I, both coins are gold coins, in box II, both are silver coins and in the box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold ?

Sol. Let E_1, E_2 and E_3 be the events that boxes I, II and III are chosen, respectively.

Then $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$. Also, let A be the event that 'the coin drawn is of gold'

Then $P(A \mid E_1) = P$ (a gold coin from box I) $= \frac{2}{2} = 1$

$P(A \mid E_2)$ = P (a gold coin from box II) = 0

$P(A \mid E_3)$ = P (a gold coin from box III) = $\frac{1}{2}$

Now, the probability that the other coin in the box is of gold

= the probability that gold coin is drawn from the box I = $P(E_1 \mid A)$

By Baye's theorem, $P(E_1 \mid A)$

$$= \frac{P(E_1)P(A \mid E_1)}{P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2) + P(E_3)P(A \mid E_3)} = \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2}} = \frac{2}{3}$$

Ex.2 In a factory which manufactures bolts, machines A, B and C manufacture respectively 25%, 35% and 40% of the bolts. Of their outputs, 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it is manufactured by the machine B ?

Sol. Let events B_1, B_2, B_3 be the following :

B_1 : the bolt is manufactured by machine A

B_2 : the bolt is manufactured by machine B

B_3 : the bolt is manufactured by machine C

Clearly, B_1, B_2, B_3 are mutually exclusive and exhaustive events and hence, they represents a partition of the sample space.

Let the event E be ' the bolt is defective. The event E occurs with B_1 or with B_2 or with B_3.

Given that $P(B_1) = 25\% = 0.25$, $P(B_2) = 0.35$ and $P(B_3) = 0.40$

Again $P(E \mid B_1)$ = Probability that the bolt drawn is defective given that it is manufactured by machine A = 5% = 0.05. Similarly, $P(E \mid B_1) = 0.04$, $P(E \mid B_3) = 0.02$

Hence, by Bayes' Theorem, we have $P(B_2 \mid E)$

$$= \frac{P(B_2)P(E \mid B_2)}{P(B_1)P(E \mid B_1) + P(B_2)P(E \mid B_2) + P(B_3)P(E \mid B_3)}$$

$$= \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 + 0.02} = \frac{0.0140}{0.0345} = \frac{28}{69}$$

Probability Through Statistical (Stochatic) Tree Diagram

These tree diagrams are generally drawn by economist and give a simple approach to solve a problem.

Ex.3 A bag initially contains 1 red ball and 2 blue balls. A trial consists of selecting a ball at random noting its colour and replacing it together with an additional ball of the same colour. Given that three trials are made, draw a tree diagram illustrating the various probabilities. Hence or otherwise, find the probability that

(a) atleast one blue ball is drawn

(b) exactly one blue ball is drawn

(c) Given that all three balls drawn are of the same colour find the probability that they are all red.

Sol.

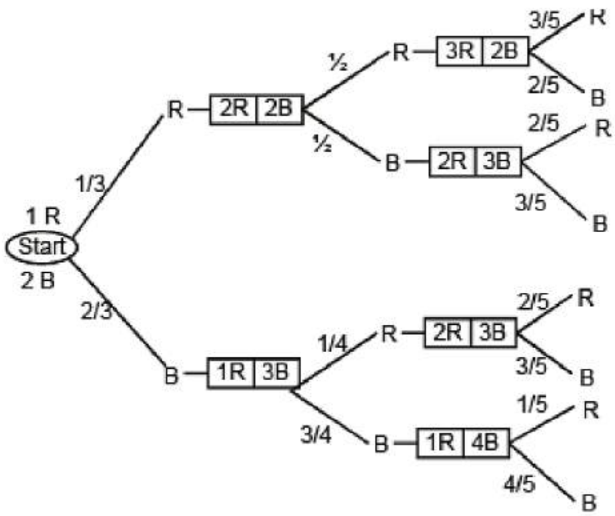
$$P(A) = 1 - P(RRR) = 1 - \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{3}{5} = 1 - \frac{1}{10} = \frac{9}{10}$$

$$P\text{ (exactly one Blue)} = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{5} + \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{5} + \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{2}{5}$$

$$= \frac{1}{15} + \frac{1}{15} + \frac{1}{15} = \frac{3}{15} = \frac{1}{5}$$

$$P(C) = P\left(\frac{RRR}{(RRR \cup BBB)}\right) = \frac{P(RRR)}{P(RRR) + P(BBB)}$$

$$= \frac{\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{3}{5}}{\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{3}{5} + \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5}} = \frac{\frac{1}{10}}{\frac{1}{10} + \frac{4}{10}} = \frac{1}{5}$$



Coincidence Testimony

If p_1 and p_2 are the probabilities of speaking the truth of two independent witnesses A and B then

P (their combined statement is true) = $\frac{p_1p_2}{p_1p_2+(1-p_1)(1-p_2)}$.
In this case it has been assumed that we have no knowledge of the event except the statement made by A and B. However if P is the probability of the happening of the event before their statement then P (their combined statement is true)
$= \frac{pp_1p_2}{pp_1p_2+(1-p)(1-p_1)(1-p_2)}$

Here it has been assumed that the statement given by all the independent witnesses can be given in two ways only, so that if all the witnesses tell falsehoods they agree in telling the same falsehood. If this is not the case and c is the change of their coincidence testimony then th

probability that the statement is true = $p\ p_1\ p_2$

probability that the statement is false = $(1 - p).c\ (1 - p_1)\ (1 - p_2)$

However chance of coincidence testimony is taken only if the joint statement is not contradicted by any witness.

Ex.4 A speaks truth in 75% case and B in 80% cases. What is the probability that they contradict each other in stating the same fact ?

Sol. There are two mutually exclusive cases in which they contradict each other. i.e. $\overline{A}B$ and $A\overline{B}$

Hence required probability = $P(A\overline{B}+\overline{A}B)=P(A\overline{B})+P(\overline{A}B)$

= $P(A)P(\overline{B})+P(\overline{A})P(B)=\frac{3}{4}\cdot\frac{1}{5}+\frac{1}{4}\cdot\frac{4}{5}=\frac{7}{20}$

Probability Distribution of a Random Variable

Probability Distribution

(a) A Probability Distribution spells out how a total probability of 1 is distributed over several values of a random variable.

(b) Mean of any probability distribution of a random variable is given by $\mu=\frac{\Sigma p_i x_i}{\Sigma p_i}=\Sigma p_i x_i$(Since $\Sigma p_i=1$)

(c) Variance of a random variable is given by $\sigma^2=\Sigma(x_i-\mu)^2.p_i \Rightarrow \sigma^2=\Sigma p_i x_i^2-\mu^2$ ($\because$ SD $=+\sqrt{\sigma^2}$)

(d) The probability distribution for a binomial variate 'X' is given by ; P (X = r) $= {}^nC_r\, p^r q^{n-r}$. The recurrence formula $\frac{P(r+1)}{P(r)}=\frac{n-r}{r+1}\cdot\frac{p}{q}$, is very helpful for quickly computing P(1), P(2), P(3) etc. if P(0) is known.

(e) Mean of BPD = np ; variance of BPD = npq.

(f) If p represents a persons chance of success in any venture and 'M' the sum of money which he will receive in case of success, then his expectations or probable value = pM

expectations = pM

Geometrical Probability

The following statements are axiomatic :

(a) If a point is taken at random on a given straight line AB, the chance that it falls on a particular segment PQ of the line is PQ/AB

(b) If a point is taken at random on the area S which includes an area s, the chance that the points falls on σ is σ/S.

Other Definitions Of Probability

(a) Axiomatic probability : Axiomatic approach is another way of describing probability of an event, in this approach some axioms or rules are depicted to assign probabilities.

Let S be the sample space of a random experiment. The probability P is a real valued function whose domain is the power set of S and range is the interval [0, 1] satisfying the following axioms:

(i) For any event E, $P(E) \geq 0$

(ii) $P(S) = 1$

(iii) If E and F are mutually exclusive events, the $P(E \cup F) = P(E) + P(F)$

It follows from (iii) that $P(\varphi) = 0$

Let S be a sample space containing outcomes $\omega_1, \omega_2, \ldots\ldots\ldots, \omega_n$ i.e, $S = \{\omega_1, \omega_2, \ldots\ldots\ldots \omega_n\}$
It follows from the axiomatic definition of probability that :

(i) $0 \leq P(\omega_i) \leq 1$ for each $\omega_i \in S$

(ii) $P(\omega_1) + P(\omega_2) + \ldots\ldots\ldots + P(\omega_n) = 1$

(iii) For any event A, $P(A) = \Sigma P(\omega_i), \omega_i \in A$

(b) Empirical probability : A method which can be adopted in the example given above is to throw the dart several times (each throw is a trial) and count the number of times you hit the bull's-eye (a success) and the number of times you miss (a failure). Then an empirical value of the probability that you hit the bull's - eye with any one throw is $\frac{\text{number of successes}}{\text{number of successes+number of failures}}$

If the number of throws is small this does not give a particular good estimate but for a large number of throws the result is more reliable.

When the probability of the occurrence of an event A cannot be worked out exactly, an empirical value can be found by adopting the approach described above, that is :

(i) making a large number of trials (i.e. set up an experiment in which the event may, or may not, occur and note the outcome)

(ii) counting the number of times the event does occur, i.e. the number of successes,

(iii) calculating the value of $\frac{\text{number of successes}}{\text{number of trials (i.e. successes+ failures)}} = \frac{r}{n}$

The probability of then event A occurring is defined as $P(A) = \lim_{n \to \infty}\left(\frac{r}{n}\right)$

$n \to \infty$ mean that the number of trials is large (but what should be taken as `large' depends on the problem).

Important Points

(a) If $A_1 \leq A_2$ then $P(A_1) \leq P(A_2)$ and $P(A_2 - A_1) = P(A_2) - P(A_1)$

(b) If $A = A_1 \cup A_2 \cup \ldots\ldots \cup A_n$ where $A_1, A_2, \ldots\ldots A_n$ are mutually exclusive events then $P(A) = P(A_1) + P(A_2) + \ldots\ldots P(A_n) = 1$

(c) Let A & B are two events corresponding to sample space S then $P(S|A) = P(A|A) = 1$

(d) Let A and B are two events corresponding to sample space S and F is any other event s.t. $P(F) \neq 0$ then $P((A \cup B)|F) = P(A|F) + P(B|F) - (P(A \cap B)|F))$

(e) $P(\overline{A}|B) = 1 - P(A|B)$

(f) $P(A \cap B) \leq P(A), P(B) \leq P(A \cup B) \leq P(A) + P(B)$

Ex 1. A, B, C in order cut a pack of cards, replacing them after each cut, on the condition that the first who cuts a spade shall win a prize; find their respective chances.

Sol. Let p be the chance of cutting a spade and q be the chance of not cutting a spade from a pack of 52 cards. Then $p = \frac{^{13}C_1}{^{52}C_1} = \frac{1}{4}$ and $q = 1 - \frac{1}{4} = \frac{3}{4}$

Now A will win a prize if he cuts spade at 1st, 4th, 7th, 10th turns, etc. Note that A will get a second chance if A, B, C all fail to cut a spade once and then A cuts a spade at the 4th turn.

Similarly he will cut a spade at the 7th turn when A, B, C fail to cut spade twice, etc.

Hence A's chance of winning the prize = $p+q^3p+q^6p+q^9p+\ldots\ldots=\frac{p}{1-q^3}=\frac{\frac{1}{4}}{1-\left(\frac{3}{4}\right)^3}=\frac{16}{37}$

Similarly B's chance $=(pq+q^4p+q^7p+\ldots\ldots)=q(p+q^3p+q^6p+\ldots\ldots)=\frac{3}{4}\cdot\frac{16}{37}=\frac{12}{37}$

and C's chance $=\frac{3}{4}$ of B's chance $=\frac{3}{4}\cdot\frac{12}{37}=\frac{9}{37}$

Ex.2 (a) If p and q are chosen randomly from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, with replacement, determine the probability that the roots of the equation $x^2 + px + q = 0$ are real.

(b) Each coefficient in the equation $ax^2 + bx + c = 0$ is determined by throwing at ordinary die. Find the probability that the equation will have equal roots.

Sol. (a) If roots of $x^2 + px + q = 0$ are real, then $p^2 - 4q \geq 0$ ....(i)

Both p, q belongs to set $S=\{1, 2, 3, \ldots\ldots.10\}$ when p = 1, no value of q from S will satisfy (i)

p = 2 ,q = 1 will satisfy, 1 value

p = 3 ,q = 1, 2 ,2 value

p = 4 , q = 1, 2, 3, 4 ,4 value

p = 5 , q = 1, 2, 3, 4, 5, 6, 6 value

p = 6 ,q = 1, 2, 3, 4, 5, 6, 7, 8, 9, ,9 value

For p = 7, 8, 9, 10 all the ten values of q will satisfy.

Sum of these selections in 1 + 2 + 4 + 6 + 9 + 10 + 10 + 10 = 62

But the total number of selections of p and q without any order is 10 × 10 = 100

Hence the required probability is $=\frac{62}{100}=0.62$

(b) Roots equal $\Rightarrow$ $b^2 - 4ac = 0$ $\therefore$ $\left(\frac{b}{2}\right)^2 = ac$ ....(i)

Each coefficient is an integer, so we consider the following cases :

$b = 1$, $\therefore$ $\frac{1}{4} = ac$ No integral values of a and c

$b = 2$, $1 = ac$ $\therefore$ (1, 1)

$b = 3$, $9/2 = ac$ No integral values of a and c

$b = 4$, $4 = ac$ $\therefore$ (1, 4), (2, 2), (4, 1)

$b = 5$, $25/2 = ac$ No integral values of a and c

$b = 6$,$9 = ac$ $\therefore$ (3, 3)

Thus we have 5 favourable way for b = 2, 4, 6,

Total number of equations is 6.6.6 = 216 $\therefore$ Required probability is $\frac{5}{216}$

Ex.3 In a test an examinee either guesses or copies or knows the answer to a multiple choice question with four choices. The probability that he makes a guess is and the probability that he copies the answer is $\frac{1}{6}$. The probability that his answer is correct given that he copied it, is $\frac{1}{8}$. Find the probability that he knew the answer to the question given that he correctly answered it.

Sol. Let A_1 be the event that the examinee guesses that answer ; A_2 the event that he copies the answer and A_3 the event that he knows the answer. Also let A be the event that he answers correctly. Then as given, we have $P(A_1) = \frac{1}{3}, P(A_2) = \frac{1}{6}, P(A_3) = 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}$

[We have assumed here that the events A_1, A_2 and A_3 are mutually exclusive and totally exhaustive.]

Now $P(A \mid A_1) = \frac{1}{4}, P(A \mid A_2) = \frac{1}{8}$ (as given)

Again it is reasonable to take the probability of answering correctly given that he knows the answer as 1, that is $P(A | A_3) = 1$. We have to find $P(A_3 | A)$.

By Baye's theorem, we have $P(A_3 | A) = \frac{P(A_3)P(A | A_3)}{P(A_1)P(A | A_1) + P(A_2)P(A | A_2) + P(A_3)P(A | A_3)}$

$= \frac{(1/2)}{(1/3)(1/4) + (1/6)(1/8) + 1/2).1} = \frac{24}{29}$

Ex.4 A lot contains 50 defective and 50 non-defective bulbs. Two bulbs are drawn at random, one at a time, with replacement. The events A, B, C are defined as

A = {The first bulb is defective}

B = {The second bulb is non-defective}

C = {The two bulbs are both defective or both non-defective}

Determine whether

(i) A, B, C are pairwise independent,

(ii) A, B, C are independent.

Sol. We have $P(A) = \frac{50}{100}.1 = \frac{1}{2}$; $P(B) = 1.\frac{50}{100} = \frac{1}{2}$; $P(C) = \frac{50}{100}.\frac{50}{100} + \frac{50}{100}.\frac{50}{100} = \frac{1}{2}$

$A \cap B$ is the event that first bulb is defective and second is non-defective.

$\therefore P(A \cap B) = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$

$A \cap C$ is the event that first bulb is defective and second is non-defective.

$\therefore P(A \cap C) = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$

Similarly $P(B \cap C) = \frac{1}{4}$. Thus we have $P(A \cap B) = P(A).P(B); P(A \cap C) = P(A).P(C); P(B \cap C) = P(B).P(C)$

$\therefore$ A, B and C are pairwise independent. There is no element in $A \cap B \cap C$

$\therefore \quad P(A \cap B \cap C) = 0 \therefore \quad P(A \cap B \cap C) \neq P(A).P(B).P(C)$

Hence A, B and C are not mutually independent.

Binomial Distribution

Binomial Probability Distribution

Suppose that we have an experiment such as tossing a coin or die repeatedly or choosing a marble from an urn repeatedly. Each toss or selection is called a trial. In any single trial there will be a probability associated with a particular event such as head on the coin, 4 on the die, or selection of a red marble. In some cases this probability will not change from one trial to the next (as in tossing a coin or die.) Such trials are then said to be independent and are often called Bernoulli trials after James Bernoulli who investigated them at the end of the seventeenth century.

Let p be the probability that an event will happen in any single Bernoulli trial (called the probability of success). Then q = 1 - p is the probability that the event will fail to happen in any single trial (called the probability of failure). The probability that the event will happen exactly x times in n trials (i.e., n successes and n - x failures will occur) is given by the probability function.

$$f(x) = P(X = x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x} \qquad ..(i)$$

where the random variable X denotes the number of successes in n trials and x = 0, 1, ...n.

Example : The probability of getting exactly 2 heads in 6 tosses of a fair coin is

$$P(X = 2) = \binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2} = \frac{6!}{2!\,4!}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2} = \frac{15}{64}$$

The discrete probability function (i) is often called the binomial distribution since for x = 0, 1, 2, ...n, it corresponds to successive terms in the binomial expansion

$$(q+p)^n = q^n + \binom{n}{1} q^{n-1} p + \binom{n}{2} q^{n-2} p^2 + \dots\dots\dots + p^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$$

The special case of a binomial distribution with n = 1 is also called the Bernoulli distribution.

Ex.1 If a fair coin is tossed 10 times, find the probability of

(i) exactly six heads

(ii) atleast six heads

(iii) atmost six heads

Sol. The repeated tosses of a coin are Bernoulli trials. Let X denotes the number of heads in an experiment of 10 trials. Clearly, X has the binomial distribution with n = 10 and p = 1/2

Therefore $P(X = x) = {}^nC_x q^{n-x} p^x$, x = 0, 1, 2, ...n

Here n = 10, p =1/2, q=1=1-p = 1/2.

Therefore $P(X = x) = {}^{10}C_x\left(\frac{1}{2}\right)^{10-x}\left(\frac{1}{2}\right)^x = {}^{10}C_x\left(\frac{1}{2}\right)^{10}$

Now **(i)** $P(X = 6) = {}^{10}C_6\left(\frac{1}{2}\right)^{10} = \frac{10!}{6! \times 4! 2^{10}} = \frac{105}{512}$

(ii) P (atleast six heads) = $P(X \geq 6) = P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$

$$= {}^{10}C_6\left(\frac{1}{2}\right)^{10} + {}^{10}C_7\left(\frac{1}{2}\right)^{10} + {}^{10}C_8\left(\frac{1}{2}\right)^{10} + {}^{10}C_9\left(\frac{1}{2}\right)^{10} + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$$

$$= \left[\left(\frac{10!}{6! \times 4!}\right) + \left(\frac{10!}{7! \times 3!}\right) + \left(\frac{10!}{8! \times 2!}\right) + \left(\frac{10!}{9! \times 1!}\right) + \left(\frac{10!}{10!}\right)\right]\frac{1}{2^{10}} = \frac{193}{512}$$

(iii) P (at most six heads) = $P(X \leq 6)$

$= P(X = 0) + P(x = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)$

$$= \left(\frac{1}{2}\right)^{10} + {}^{10}C_1\left(\frac{1}{2}\right)^{10} + {}^{10}C_2\left(\frac{1}{2}\right)^{10} + {}^{10}C_3\left(\frac{1}{2}\right)^{10} + {}^{10}C_4\left(\frac{1}{2}\right)^{10} + {}^{10}C_5\left(\frac{1}{2}\right)^{10} + {}^{10}C_6\left(\frac{1}{2}\right)^{10}$$

$$= \frac{848}{1024} = \frac{53}{64}$$

Ex.2 A coin is tossed 7 times. Each time a man calls head. The probability that he wins the toss on more than three occasions is

Sol. The man has to win at least 4 times then required probability

$$= {}^7C_4\left(\frac{1}{2}\right)^4.\left(\frac{1}{2}\right)^3 + {}^7C_5.\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^2 + {}^7C_6\left(\frac{1}{2}\right)^6.\frac{1}{2} + {}^7C_7\left(\frac{1}{2}\right)^7$$

$$= ({}^7C_4 + {}^7C_5 + {}^7C_6 + {}^7C_7).\frac{1}{2^7} = \frac{64}{2^7} = \frac{1}{2}$$

Ex.3 A man takes a step forward with probability 0.4 and backward with probability 0.6. Find the probability that at the end of eleven steps he is one step away from the starting point.

Sol. Since the man is one step away from starting point mean that either

(i) man has taken 6 steps forward and 5 steps backward.

(ii) man has taken 5 steps forward and 6 steps backward.

Taking, movement 1 step forward as success and 1 step backward as failure.

p = Probability of success = 0.4 and q = Probability of failure = 0.6

Required Probability $= P[X = 6 \text{ or } X = 5] = P[X = 6] + P(X = 5) = {}^{11}C_6 p^6 q^5 + {}^{11}C_5 p^5 q^6$

$$= \frac{11.10.9.8.7}{1.2.3.4.5}\{(0.4)^6(0.6)^5 + (0.4)^5(0.6)^6\}$$

$$= \frac{11.10.9.8.7}{1.2.3.4.5}(0.24)^5$$

Hence the required probability = 0.37