Differential Equations

Order of D.E:– The order of highest ordered derivative occuring in the diff eq^n is know as order of the Differential equation.

Degree of D.E. :– The degree of a diff eq^n is the degree of highest ordered derivative occuring in it when the derivative are free from fractional powers.

eg
$$\left(1+\frac{d^2y}{dx^2}\right)^{8/3} = \frac{dy}{dx} \Rightarrow \left(1+\frac{d^2y}{dx^2}\right)^{8} = \left(\frac{dy}{dx}\right)^{3}$$

order = 2, degree = 8

eg
$$\left(\frac{d^3y}{dx^3}\right)^{3/2} = \left(\frac{d^3y}{dx^3}\right)^{2/3} \Rightarrow \left(\frac{d^3y}{dx^3}\right)^{\frac{3}{2}-\frac{2}{3}} = 1, \left(\frac{d^3y}{dx^3}\right)^{5/6} = 1$$

$$\Rightarrow \left(\frac{d^3y}{dx^3}\right)^{5} = 1$$

order = 3
degree = 5

Formation of Differential equation:–

1. write the eq^n of given family of curves.
2. Differentiate as number of time as number of arbitrary Constant occuring in it.
3. Eliminate the arbitrary constant from the given eq^n by using the eq^n obtained in step 1, & step 2.

Eg Find the D.E. of family of Circle of the form $x^2+y^2+2gx=0$, where g is arbitrary Const.

Sol^n

Given eq^n

$$x^2+y^2+2gx=0 \quad -(1)$$

$$2x+2y\frac{dy}{dx}+2g=0$$

$$2g=-2x-2y\frac{dy}{dx} \quad -(2)$$

Sub (2) in (1) we get

$$x^2+y^2+x\left(-2x-2y\frac{dy}{dx}\right)=0$$

The required D.E.

$$y^2-x^2-2xy\frac{dy}{dx}=0$$

Eg^n Find the D.E. of family of Curves $y=A\cos x+B\sin x$

$$y=A\cos x+B\sin x$$

$$\frac{dy}{dx}=A(-\sin x)+B\cos x$$

$$\frac{d^2y}{dx^2}=-A\cos x-B\sin x=-y$$

$$\frac{d^2y}{dx^2}+y=0$$

Ques Find the D.E of family of curves of the form

$$y = Ax + Bx^2$$

$$y = Ax + Bx^2$$

The D.E is $\rightarrow$
$$\begin{vmatrix} y & x & x^2 \\ y' & 1 & 2x \\ y'' & 0 & 2 \end{vmatrix} = 0$$

$$\Rightarrow \quad y''(2x^2 - x^2) + 2(y - xy') = 0$$

$$x^2y'' - 2xy' + 2y = 0$$

* Apply only when Algebric/Exponential function

Variable - Seprable :-

use when $\frac{dy}{dx} = f(x) \cdot F(y)$

Eg. $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

$$\frac{dy}{dx} = \frac{e^x}{e^y} + \frac{x^2}{e^y}$$

$$e^y dy = (e^x + x^2)\,dx$$

$$\int e^y dy = \int e^x dx + \int x^2 dx$$

$$e^y = e^x + \frac{x^3}{3} + c$$

Solve $\dfrac{dy}{dx} = \dfrac{x(2\ln x+1)}{\sin y + y\cos y}$

$$\frac{dy}{dx} = \frac{x(2\ln x+1)}{\sin y + y\cos y}$$

$$\Rightarrow (\sin y + y\cos y)\,dy = \{x(2\ln x) + 1\}\,dx$$

$$\Rightarrow \int \sin y\,dy + \int y\cos y\,dy = \int 2x\ln x\,dx + \int x\,dx$$

$$\Rightarrow -\cos y + [y\sin y + \cos y] = x^2\ln x - \frac{x^2}{2} + \frac{x^2}{2}$$

$\therefore \int y\cos y\,dy = ?$ $\quad \int u\,dv = uv - \int v\,du$

$u = y \Rightarrow du = dy$

$\Rightarrow y\sin y - \int \sin y\,dy \qquad dv = \cos y\,dy$

$\Rightarrow y\sin y - \cos y \qquad v = \sin y$

$\Rightarrow \int 2x\ln x\,dx \qquad \cancel{u = 2x \Rightarrow du = 2dx}$

$\cancel{dv = \ln x\,dx}$

$u = \ln x, \quad dv = 2x\,dx$

$du = \frac{1}{x} \qquad v = x^2$

$\Rightarrow x^2\ln x - \int x^2 \cdot \frac{1}{x}\,dx \Rightarrow x^2\ln x - \frac{x^2}{2}$

$$\Rightarrow y\sin y = x^2\ln x + c$$

Reducible to Variable Seprable! —

$$\frac{dy}{dx} = f(x, y)$$

Eg
$$\frac{dy}{dx} = \sin(x+y) \quad -①$$

let $x+y = t$

$$1 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1 \quad ②$$

sub ② in ①

$$\frac{dt}{dx} - 1 = \sin t$$

$$\frac{dt}{dx} = 1 + \sin t$$

$$dt = dx + \sin t \cdot dx$$

$$\int \frac{dt}{1+\sin t} = \int dx$$

$$\int \frac{1-\sin t}{(1+\sin t)(1-\sin t)} dt = \int dx$$

$$\int \frac{1-\sin t}{\cos^2 t} dt = \int dx$$

$$\int (\sec^2 t - \sec t \tan t)\, dt = \int dx$$

$$\tan t - \sec t = x + C$$

$$\Rightarrow \tan(x+y) - \sec(x+y) = x + C$$

Question $\frac{dy}{dx} = (4x+y+1)^2$

$$\frac{dy}{dx} = (4x+y+1)^2 \quad -(1)$$

$$4x+y+1 = t$$

$$4+\frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 4 \quad -(2)$$

Sub in (1)

$$\frac{dt}{dx} - 4 = t^2$$

$$\int \frac{dt}{4+t^2} = \int dx$$

$$\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) = x + c$$

$$\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c$$

Question $\frac{dy}{dx} = \left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right)$

let $y = vx \Rightarrow x\frac{dv}{dx} + v = v + \tan(v)$

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$\frac{dv}{\tan v} = \frac{dx}{x}$$

~~ln cot~~

$$\ln(\sin v) = \ln x + \ln c$$

$$\sin\left(\frac{y}{x}\right) = cx$$

$\underline{\underline{*}}$ $$\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$$

where f_1 & f_2 are homogeneous fu^{n}s of same degree.

put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Eg. $$\frac{dy}{dx} = \frac{x^2y}{x^3+y^3} \quad -(1)$$

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad -(2)$$

Put in above eq^n

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2(vx)}{x^3+v^3x^3}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{1+v^3}$$

$$x\frac{dv}{dx} = -\frac{v^4}{1+v^3}$$

$$\frac{1+v^3}{v^4}\,dv = -\frac{dx}{x}$$

$$\int\left(\frac{1}{v^4} + \frac{v^3}{v^4}\right)dv = -\frac{dx}{x}$$

$$\int \left(v^{-4} + \frac{1}{v} \right) dv = \int -\frac{dx}{x}$$

$$\frac{v^{-3}}{-3} + \ln v = -\ln x + c$$

$$-\frac{1}{3}\frac{1}{v^3} + \underbrace{\ln v + \ln x}_{\ln(vx)} = c$$

$$v = y/x$$

$$\Rightarrow \quad -\frac{1}{3}\left(\frac{x}{y}\right)^3 + \ln y = 0$$

~~Question~~

Linear Diff. eqn :- A diff equation is said to linear diff. equation if the degree of dependent variable and it differential coefficient occuring with first degree and they are not multiplied togther.

Eg $\dfrac{dy}{dx} - xy = x^3$

degree of $y = 1$

degree of $\dfrac{dy}{dx} = 1$

dependent variable y

& they are not multiplied

Eg $\dfrac{dy}{dx} + y(x+y) = x^3$

$y \rightarrow 2$

Non linear

$\dfrac{dy}{dx} \rightarrow 1$

The standard form of a L.D.E. is given by

$$\frac{dy}{dx} + Py = Q \quad \text{where } P \,\&\, Q \text{ are function of } x$$

$$\boxed{I.F. = e^{\int P\,dx}}$$

Solution is $\boxed{y(I.F.) = \int Q(I.F.)\,dx + C}$

Ques $\frac{dy}{dx} + 2xy = e^{-x^2}$, $y(0) = 1$

$P = 2x$, $Q = e^{-x^2}$

$$I.F. = e^{\int 2x\,dx} = e^{x^2}$$

Soln → $y(e^{x^2}) = \int e^{-x^2} \cdot e^{x^2}\,dx + c$

$y(e^{x^2}) = x + c \qquad y(0) = 1$

$1 \times e^0 = 0 + c \Rightarrow c = 1$

$$ye^{x^2} = x + 1$$

Ques $x^2 \frac{dy}{dx} + 2xy = \frac{2\ln x}{x}$, $y(1) = 0$, $y(e) = ?$

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{2\ln x}{x^3}$$

$$P = \frac{2}{x} \qquad Q = \frac{2\ln x}{x^3}$$

$$\text{I.F.} = e^{\int P\,dx}$$

$$\text{I.F} = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2$$

so $\Rightarrow$ $y(\text{IF}) = \int Q(\text{I.F.})\,dx + c$

$$y e^{2\ln x} = \int \frac{2\ln x}{x^3} e^{2\ln x}\,dx + c$$

$$y e^{2\ln x} = \int \frac{2\ln x}{x^3} \cdot x^2\,dx + c$$

$$y(x^2) = \int 2\ln x \cdot \frac{1}{x}\,dx + c$$

$$y(x^2) = (\ln x)^2 + c$$

$y(1) = 0 \Rightarrow 0 = (\ln 1)^2 + c \quad \therefore \quad c = 0$

$y(e) = ?$

$$yx^2 = (\ln x)^2$$

$$y = \left(\frac{\ln x}{x^2}\right)^2 =$$

$$y(e) = \left(\frac{\ln e}{e}\right)^2 = 1/e^2$$

Ques. If $y'-x \neq 0$ then the solution of

$$y'(y'+y) = x(x+y), \quad y(0) = 2 \quad \text{is}$$

Ans

$$y'(y'+y) = x(x+y)$$

$$(y')^2 + y'y = x^2 + xy$$

$$(y')^2 - x^2 = xy - y'y$$

$$(y'+x)(y'-x) = y(x-y')$$

$$y'+x = -y$$

$$y' + y = -x$$

$$\frac{dy}{dx} + 1.y = -1.x$$

$P = 1, \quad Q = -x$

I.F. $\Rightarrow e^{\int 1.dx} = e^x$

$$\int xe^x\,dx$$

$\oplus\, x \ominus 1 \oplus 0$

$e^x \quad e^x \quad e^x$

$\Rightarrow xe^x - e^x$

$$\Rightarrow y(e^x) = \int -x.e^x\,dx + C$$

$$ye^x = e^x(1-x) + C$$

$$\cancel{(y+1)e^x = C}$$

$$\cancel{(3)e^0 = C \Rightarrow C = 3}$$

$$\Rightarrow y = 1 - x + ce^{-x}$$

$$y(0) = 2$$

$\cancel{y(0) = 2}$

$\cancel{y \to 2}$

$\cancel{x \to 0}$

$$2 = 1 + ce^{-0}$$

$$c = \underline{1}$$

$$y = 1 - x + e^{-x}$$

Question $\dfrac{dy}{dx} - xy = x^3y^2$ —(1)

$\Rightarrow \quad y^{-2}\dfrac{dy}{dx} - x/y = x^3$ —(2)

$-\dfrac{1}{y} = t$ —(3)

$+\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dt}{dx}$ —(4)

$\Rightarrow \quad \dfrac{dt}{dx} + xt = x^3$ —(5)

Sub (3) & (4) in (2)

$\dfrac{dt}{dx} + xt = x^3$ which is L.D.E

$P = x, \quad Q = x^3$

$I.F. = e^{\int P dx} = e^{\int x dx} = e^{x^2/2}$

$y(IF) = \int Q(I.F)dx + C$

$t(e^{x^2/2}) = \int x^3 . e^{x^2/2} dx + c$

$-\dfrac{1}{y} e^{x^2/2} = \int x . x^2 e^{x^2/2} dx + c$

$\Rightarrow \int x . x^2 . e^{x^2/2} dx$

Let
$x^2/2 = v$
$x\,dx = dv$

$= \int v\, e^{v} dv$

$= (ve^v - e^v)$

$$\Rightarrow -\frac{1}{y}.e^{x^2/2} = 2ve^{v} - 2.e^{v} + c$$

$$-\frac{1}{y}e^{x^2/2} = 2\left(\frac{x^2}{2}\right)e^{x^2/2} - 2e^{x^2/2} + c$$

$$\Rightarrow \left(-\frac{1}{y}\right) = (x^2-2) + ce^{-x^2/2}$$

Question $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{x.2\sin y\cos y}{\cos^2 y} = x^3$$

$$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3$$

let $\tan y = z$

$$\sec^2 y\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \frac{dz}{dx} + 2xz = x^3$$

$P = 2x$ $\quad Q = x^3$

$$I.F. = e^{\int P dx} = e^{\int 2x dx} = e^{x^2}$$

$$y(e^{x^2}) = \int x^3.e^{x^2}dx = \int x.x^2e^{x^2}dx$$

let $x^2 = t$

$xdx = dt/2$

$$\Rightarrow \tan y \cdot e^{x^2} = \int t \cdot e^t \frac{dt}{2}$$

$$\tan y \cdot e^{x^2} = \frac{1}{2}(t e^t - e^t) \; c$$

$$\tan y \cdot e^{x^2} = \frac{1}{2}\left(x^2 e^{x^2} - x^2\right) + c$$

$$\tan y = \frac{1}{2}\left(x^2 - 1\right) + c e^{-x^2}$$

Ans

Exact differential equation

The diff. eq^n $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be an diff eq^n if $\frac{dM}{dy} = \frac{dN}{dx}$

The solution is

$$\int\limits_{y \text{ is const.}} M\,dx + \int (\text{terms in } N \text{ free from } x)\,dy = c$$

eg $(xy^2 + x)\,dx + (x^2y - y)\,dy = 0$

$M = xy^2 + x \qquad N = x^2y - y$

$\frac{dM}{dy} = 2xy \qquad \frac{dN}{dx} = 2xy$

$\frac{dM}{dy} = \frac{dN}{dx}$ So eq^n is exact.

then the solution is

$$\underset{\substack{\text{treat } y \\ \text{const.}}}{\int M\,dx} + \int (\text{term in } N \text{ free from } x)\, dy = c$$

$$\int (xy^2 + x)\, dx + \int (-y)\, dy = c$$

$$\frac{y^2 x^2}{2} + \frac{x^2}{2} - \frac{y^2}{2} = c$$

$$x^2y^2 + x^2 - y^2 = c \qquad \underline{Ans}$$

Question $(e^y + 1) \cos x\, dx + (e^y \sin x)\, dy = 0$

$$M = (e^y + 1) \cos x \qquad N = e^y \sin x$$

$$\frac{dM}{dy} = e^y \cos x \qquad \frac{dN}{dx} = e^y \cos x$$

$$\frac{dM}{dy} = \frac{dN}{dx} \qquad \text{exact diff}$$

$$\int M\,dx + \int (\text{term in } N \text{ free from } x)\, dy = c$$

$$\int (e^y \cos x + \cos x)\, dx + \int 0\, dy = c$$

$$(e^y + 1) \sin x = c$$

Inexact diff eqⁿ

* If M and N are homogeneous function of same degree then $I.F. = \frac{1}{Mx+Ny}$

* If M and N are non homogeneous but yo

$$\left.\begin{array}{l} M = y f_1(x,y) \\ N = x f_2(x,y) \end{array}\right\} \text{then } I.F. = \frac{1}{Mx - Ny}$$

* If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = F(x)/\text{constant}$ then $I.F. = e^{\int F(x)\,dx}$

* If $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = F(y)/\text{cont.}$ then $I.F. = e^{\int F(y)\,dy}$

Question $(x^2+y^2+2x)dx + 2y\,dy = 0$ ——— (1)

$M = x^2+y^2+2x$ $\qquad N = 2y$

$\frac{dM}{dy} = 2y$ $\qquad \frac{dN}{dx} = 0$

$\frac{dM}{dy} \neq \frac{dN}{dx}$ $\qquad \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-0}{2y} = 1$ (cons'

So $I.F. = e^{\int 1.dx} = e^x$

① × I.F.

$\Rightarrow \quad e^x(x^2+y^2+2x)\,dx + e^x(2y)\,dy = 0 \quad$ —②

$M_1 = e^x(x^2+y^2+2x) \qquad N_1 = e^x 2y$

$\frac{dM_1}{dy} = 2e^x y \qquad \frac{\partial N_1}{\partial x} = 2e^x y$

$\frac{dM_1}{dy} \neq \frac{dN_1}{\partial x}$ eq② is exact.

So

$\int M_1\,dx + \int (x \text{ free term in } N_1)\,dy = c$

$\int e^x(x^2+y^2+2x)\,dx + \int 0\,dy = c$

$\int (e^x x^2 + e^x y^2 + e^x 2x)\,dx + \int 0 = c$

$\int x^2 e^x\,dx + \int y^2 e^x\,dx + \int e^x 2x\,dx = 0$

$x^2 e^x - \cancel{2x e^x} + \cancel{2e^x} + y^2 e^x + \cancel{2x e^x} - \cancel{2e^x} = 0$

$x^2 e^x + y^2 e^x = 0$

$(x^2+y^2)e^x = c$

Question $(y - xy^2)\,dx - (x + x^2y)\,dy = 0$

$M = y - xy^2 \qquad N = x + x^2y$

$\frac{dM}{dy} = 1 - 2xy \qquad \frac{dN}{dx} = 1 + 2xy$

rule ②

$$I.F. = \frac{1}{Mx - Ny}$$

$$I.F. = \frac{1}{(y-xy^2)x - y(x+x^2y)} = \frac{1}{2xy}$$

Multiple eqⁿ ① with I.F.

$$\frac{1}{2xy}(y - xy^2)\,dx - \frac{1}{2xy}(x+x^2y)\,dy = 0$$

$$\left(\frac{1}{2x} - \frac{y}{2}\right)dx - \left(\frac{1}{2y} + \frac{x}{2}\right)dy = 0$$

$M_1 = \frac{1}{2x} - \frac{y}{2} \qquad N_1 = \frac{-1}{2y} - \frac{x}{2}$

$\frac{dM_1}{dy} = -1 \qquad \frac{dN_1}{dx} = -1$

solⁿ is

$$\int\left(\frac{1}{2x} - \frac{y}{2}\right)dx + \int\left(-\frac{1}{2y}\right)dy = 0$$

$$\frac{1}{2}\ln x - \frac{yx}{2} - \frac{1}{2}\ln y = c$$

$$\ln(x/y) - xy = 2C$$

Orthogonal Trajectory :-

Procedure :-

1. Find the D.E. of given family of curves.
2. Replace $\frac{dy}{dx}$ by $\left(-\frac{dx}{dy}\right)$ to get DE of orthogonal one.

   { Replace $\frac{1}{r}\frac{dr}{d\theta}$ by $\left(-\frac{r\,d\theta}{dr}\right)$ if it is in polar form }
3. Solve the resultant diff eqn to get orthogonal trajectory.

Ques find O.T. of family of curves of the form

$$y = k(x-1) \quad \text{--- (1)}$$

Soln eqn of given family

$$y = k(x-1)$$

$$\frac{dy}{dx} = k \quad \text{--- (2)}$$

Put (2) in (1)

So $y = \frac{dy}{dx}(x-1)$

Now replace $\frac{dy}{dx} = \frac{-dx}{dy}$

$$y = -\frac{dx}{dy}(x-1)$$

$$\int y\,dy = \int (1-x)\,dx$$

$$\frac{y^2}{2} = x - \frac{x^2}{2} + c$$

$$x^2 + y^2 - 2x = 2c$$

Question Find O.T. of Curves of the form

$$r = a(1+\cos\theta)$$

sol^n

$$r = a(1+\cos\theta) \quad -(1)$$

$$\frac{dr}{d\theta} = a(-\sin\theta)$$

$$a = \frac{1}{-\sin\theta}\left(\frac{dr}{d\theta}\right) \quad -(2)$$

Sub. (2) in (1)

$$r = \frac{1}{-\sin\theta}\left(\frac{dr}{d\theta}\right)(1+\cos\theta)$$

Now replace $\frac{1}{r}\frac{dr}{d\theta} = \frac{-r\,d\theta}{dr}$

$$\cancel{1r} = \frac{1}{+\sin\theta}\left(+r\frac{d\theta}{dr}\right)(1+\cos\theta)$$

$$dr = r\left(\frac{1+\cos\theta}{\sin\theta}\right)d\theta$$

$$\frac{2\cos^2\theta/2}{2\sin\theta/2\cos\theta/2}\,d\theta = \frac{dr}{r}$$

$$\int \cot\theta/2\,d\theta = \int \frac{1}{r}\,dr$$

$$\frac{\ln(\theta/2)}{\theta/2} = \ln r + \ln c$$

$$2\ln\left(\sin\theta/2\right) = \ln(rc)$$

$$\sin^2\theta/2 = rc$$

$$r = \frac{1}{2c}(1-\cos\theta) \quad \text{required } \underline{D.E.}$$

Higher odered diff. equation :-

Linear diff eq^n with constant Coefficient :-

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_0 y = f(x)$$

if $f(x) = 0$ then it is homogeneous L.D.E. $\boxed{G.S. = C.F.}$

if $f(x) \neq 0$ then it is non homogeneous L.D.E.

$\boxed{G.S. = C.F. + P.I}$

G.S. - General sol^n
C.F. - Complementary function
P.I. - particular integral.

Root of auxillary equation		Corresponding Complimentary eq^n
① Root distinct	m_1, m_2, m_3	$c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x}$
② Real equal	m, m, m	$(c_1 + c_2x + c_3x^2)\,e^{mx}$
③ imaginary	$\alpha \pm i\beta$	$e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$
④ Imaginary	$\alpha \pm i\beta, \alpha \pm i\beta$	$e^{\alpha x}\left((c_1+c_2x)\cos\beta x + (c_3+c_4x)\sin\beta x\right)$

Que:- $$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

$$(D^2 - 5D + 6)\,y = 0$$

AE $\quad D^2 - 5D + 6 = 0$

$m_1 \,\&\, m_2 = 2, 3$ real and distinct

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{3x}$$

solution is

$$y = c_1 e^{2x} + c_2 e^{3x}$$

Ques $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

$$(D^2 - 4D + 4)\,y = 0$$

AE $\quad D^2 - 4D + 4 = 0$

$m_1 \,\&\, m_2 = 2, 2$ real & equal

$$\text{C.F.} = (c_1 + c_2 x)\,e^{2x}$$

the solution is

$$y = (c_1 + c_2 x)\,e^{2x}$$

Question $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

$(D^2 - 2D + 2)\, y = 0$

A.E. $(\cancel{D^2 - 2D + 2}) = 0$

$m^2 - 2m + 2 = 0$

$m = \frac{2 \pm \sqrt{4 - 4\times 2}}{2}$

$m = 1 \pm i$

C.F. $= e^x(c_1 \cos x + c_2 \sin x)$

solⁿ is $y = e^x(c_1 \cos x + c_2 \sin x)$

Ques If e^{2x} and e^{3x} are two independent solution of a section order D.E. then D.E is.

Ans The solution is

$y = c_1 e^{2x} + c_2 e^{3x}$

* Coefficient of arbitary Const called independent solⁿ

root are 2, 3

the D.E is $((D-2)(D-3))\, y = 0$

$(D^2 - 5D + 6)\, y = 0$.

non - homogenous

Type I $\quad$ $e^{\alpha x}$ form

$\boxed{\text{Replace } D=\alpha \text{ in } F(D)}$

eg $\quad (D^2-6D+8)y = e^{6x}$

A.E. is. $\quad m^2-6m+8=0$

$m_1 = 2, 4$

C.F. $= c_1 e^{2x} + c_2 e^{3x}$

P.I. $= \dfrac{1}{(D^2-6D+8)} e^{6x}$

replace $D = 6$

P.I. $= \dfrac{e^{6x}}{36-36+8} = \dfrac{e^{6x}}{8}$

G.S. $=$ C.F. + P.I

G.S. $= c_1 e^{2x} + c_2 e^{4x} + \dfrac{e^{6x}}{8}$

eg $\quad (D^2-6D+9) = e^{3x}$

A.E. $\quad m^2-6m+9=0$

$m = 3, 3$

C.I. $= (c_1+c_2x)e^{3x}$

$$P.I. = \frac{1}{D^2-6D+9} e^{3x}$$

D=3
Cas fails

$$P.I. = x \cdot \frac{1}{2D-6} e^{3x}$$

D = 3
Case fails

$$P.I. = x \cdot x \cdot \frac{1}{2} e^{3x} = \frac{x^2 e^{3x}}{2}$$

so $\quad G.S = C.F. + P.I.$

$$y = (c_1 + c_2 x) e^{3x} + \frac{x^2 e^{3x}}{2}$$

$$y = \left(c_1 + c_2 x + \frac{x^2}{2}\right) e^{3x}$$

Ques $\quad (D^2 + 9) y = 2^x$

A.E. $\quad m^2 + 9 = 0 \Rightarrow m = \pm 3i$

$$C.F. = \cancel{e^{3x}} (c_1 \cos 3x + c_2 \sin 3x)$$

$$P.I = \frac{1}{D^2+9} 2^x = \frac{1}{D^2+9} e^{\ln 2^x} = \frac{1}{D^2+9} e^{x(\ln 2)}$$

$$P.I. = \frac{1}{D^2+9} e^{x \ln 2}$$

$$\underline{D = \ln 2}$$

$$P.I. = \frac{e^0 \, 2^x}{(\ln 2)^2 + 9} = \frac{2^x}{9.48}$$

$$G.S. = C.I. + P.I.$$

$$y = e^{3x}\left(C_1 \cos 3x + C_2 \sin 3x\right) + \frac{2^x}{9.48}$$

<u>Ques</u> $(D^2+4)y = 16$

<u>A.E.</u> $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$$C.F = e^{0x}\left(C_1 \cos 2x + C_2 \sin 2x\right)$$

$$C.F = \left(C_1 \cos 2x + C_2 \sin 2x\right)$$

$$P.I. = \frac{1}{D^2+4} \cdot 16 = \frac{1}{D^2+4} 16 \cdot e^{0x}$$

$$\underline{D = 0}$$

$$P.I. = \frac{16}{4} = 4$$

$$G.S = C.F. + P.I$$

$$y = C_1 \cos 2x + C_2 \, 2x + 4$$

Type 2 $\cos\alpha x$ or $\sin\alpha x$

$$\boxed{\text{Hint: Replace } D^2 \text{ by } -(\alpha^2)}$$

Ques $(D^2+9)y = \sin 2x$

$m = \pm 3i$

$C.F. = (c_1 \cos 3x + c_2 \sin 3x)$

$P.I. = \frac{1}{D^2+9} \sin 2x$

$D^2 = -(2^2) = -4$

$P.I = \frac{1}{-4+9} \sin 2x = \frac{\sin 2x}{5}$

So $y = (c_1 \cos 3x + c_2 \sin 3x) + \frac{\sin 2x}{5}$

Ques $(D^2+4)y = \cos 2x$

$m = \pm 2i$ $C.F. = c_1 \cos 2x + c_2 \sin 2x$

$P.I. = \frac{1}{D^2+4} \cos 2x$ $D^2=-4$ fail

$$\boxed{\frac{1}{D} \to \int}$$

$P.I. = x \cdot \frac{1}{2D} \cos 2x$

$P.I. = \frac{x}{2} \int \cos 2x = \frac{x}{2}\left(\frac{\sin 2x}{2}\right) = \frac{x \sin 2x}{4}$

$y = c_1 \cos 2x + c_2 \sin 2x + \frac{x \sin 2x}{4}$

Question $(D^2-4D+3)\,y = \cos x$

A.E. $m^2 - 4m + 3 = 0$

$m = 1,\ 3$

C.F. $= c_1 e^{x} + c_2 e^{3x}$

P.I. $= \dfrac{\cos x}{D^2 - 4D + 3}$ $\qquad D^2 = -1$

P.I. $= \dfrac{\cos x}{(2-4D)}$

P.I. $= \dfrac{(2+4D)\cos x}{(2-4D)(2-4D)}$

P.I. $= \dfrac{(2+4D)\cos x}{4-16D^2}$ $\qquad \underline{D^2 = -1}$

P.I. $= \dfrac{1}{+20}\left(2\cos x + 4D(\cos x)\right)$

P.I. $= \dfrac{+1}{20}\left(2\cos x - 4\sin x\right)$

So
$$y = c_1 e^{x} + c_2 e^{3x} + \frac{1}{20}(2\cos x - 4\sin x)$$

Type III $\quad x^k$

Hint :- write $F(D)$ in the form of $(1+t)^{-1}$ or $(1-t)^{-1}$

eg $\quad (D^2-3D+2)y = x^2$

A.E. $\quad m^2 - 3m + 2 = 0$

$m = 1, 2$

$C.F = c_1 e^{x} + c_2 e^{2x}$

$$P.I. = \frac{1}{(D^2-3D+2)} x^2 = \frac{1}{2-3D+D^2} x^2$$

$$P.I. = \frac{1}{(1-D)(2-D)} x^2 = \left(\frac{1}{1-D} - \frac{1}{2-D}\right) x^2 .$$

$$= (1-D)^{-1} x^2 - \frac{1}{2}(1-D/2)^{-1} x^2$$

$$P.I = (\underbrace{1+D+D^2}+D^3+ - - -)x^2 - \frac{1}{2}\left(\underbrace{1+\frac{D}{2}+\frac{D^2}{4}}+\frac{D^3}{8}+ -\right)x^2$$

$$P.I. = x^2 + D.x^2 + D^2.x^2 - \frac{1}{2}\left(x^2 + D.x^2 + D^2.x^2\right)$$

$$P.I. = x^2 + 2x + 2 - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$$

$$PI = x^2/2 + 3x/2 + 7/4$$

$$y = c_1 e^{x} + c_2 e^{2x} + \frac{x^2}{2} + \frac{3x}{2} + 7/4$$

Question $(D^2-4D+4)y = x^2$

$m = 2, 2$

$C.F. = (c_1+c_2x)e^{2x}$

$$P.I. = \frac{1}{(D^2-4D+4)} \cdot x^2$$

$$P.I. = \frac{1}{(2-D)^2(2-D)} x^2 = \frac{x^2}{(2-D)^2}$$

$$P.I. = \frac{x^2}{4(1-D/2)^2} = \frac{1}{4}(1-D/2)^{-2}x^2$$

$$P.I. = \frac{1}{4}\left(\underbrace{1 + 2.\frac{D}{2} + 3.\frac{D^2}{4}} + 4\frac{D^3}{8} + \cdots\right)x^2$$

$$P.I. = \frac{1}{4}\left(x^2 + \frac{2}{2} \times 2x + \frac{3}{4} \times 2\right)$$

$$P.I. = \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8}$$

So $y = (c_1+c_2x)e^{2x} + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8}$

$$\boxed{(1-t)^{-2} = 1+2t+3t^2+4t^3 \cdots}$$

Type IV $e^{\alpha x}\cos\beta x$ (or) $e^{\alpha x}\sin\beta x$ (or) $e^{\alpha x}x^{k}$

Hint $$\frac{1}{f(D)}e^{\alpha x}V = e^{\alpha x}\left\{\frac{1}{f(D+\alpha)}V\right\}$$

where $V = \cos\beta x / \sin\beta x / x^{k}$

Quest $$(D^2-5D+6)y = e^{2x}\cos x$$

$$m = 2, 3$$

$$C.F = c_1e^{2x}+c_2e^{3x}$$

$$P.I. = \frac{1}{(D^2-5D+6)}e^{2x}\cos x$$

$$P.I. = e^{2x}\left\{\frac{1}{(D+2)^2-5(D+2)+6}\cos x\right\} = e^{2x}\left\{\frac{1}{D^2-D}\cos x\right\}$$

$$P.I. = e^{2x}\left\{\frac{\cos x}{D^2-D}\right\} \quad D^2=-1$$

$$P.I. = e^{2x}\left(\frac{1}{-1-D}\cos x\right) = \frac{e^{2x}}{-1}\left(\frac{1}{1+D}\cos x\right)$$

$$P.I. = -e^{2x}\left(\frac{(1-D)}{1-D^2}\cos x\right) \quad D^2=-1$$

$$P.I. = \frac{-e^{2x}}{2}\left((1-D)\cos x\right) = \frac{e^{2x}}{-2}(\cos x+\sin x)$$

$$y = c_1e^{2x}+c_2e^{3x}-\frac{e^{2x}}{2}(\cos x+\sin x)$$

Question $(D^2-6D+9)y = e^{3x}x^2$

$$m^2-6m+9=0$$

$$m = 3, 3$$

$$C.F. = (c_1+c_2x)e^{3x}$$

$$P.I. = \frac{1}{D^2-6D+9} e^{3x}x^2$$

replace $D = D+\alpha = (D+3)$.

$$P.I. = \frac{1}{(D+3)^2-6(D+3)+9} e^{3x}x^2 \quad \frac{e^{3x}x^2}{D^2+\cancel{6D}+\cancel{9}-\cancel{6D}-\cancel{18}+\cancel{9}}$$

$$P.I. = e^{3x}\frac{x^2}{D^2}$$

$$P.I. = e^{3x}\left\{\frac{x^3}{3D}\right\} = e^{3x}\left\{\frac{x^4}{12}\right\}$$

So $y = (c_1+c_2x)^{3x} + e^{3x}\frac{x^4}{12}$

Type V $\quad x\sin\alpha x$ (or) $x\cos\alpha x$

Hint:-

$$\frac{1}{f(D)}xV = x\cdot\frac{1}{f(D)}V - \frac{f'(D)}{(f(D))^2}V$$

where $V = \sin\alpha x$ (or) $\cos\alpha x$

Question $\quad (D^2+9) = x\sin 2x$

$m^2+9=0 \quad ,\Rightarrow m = \pm 3i$

$$C.F. = e^{0x}(c_1\cos 3x + c_2\sin 3x)$$

$$C.F. = c_1\cos 3x + c_2\sin 3x$$

$$P.I. = \frac{1}{D^2+9}x\sin 2x$$

$$P.I. = x.\frac{1}{D^2+9}(\sin 2x) - \frac{2D}{(D^2+9)^2}\sin 2x$$

replace $D^2 = -4$

$$P.I. = x\frac{1}{5}\sin 2x - \frac{2D(\sin 2x)}{25}$$

$$P.I. = \frac{x\sin 2x}{5} - \frac{4}{25}\cos 2x$$

So $\quad y = c_1\cos 3x + c_2\sin 3x + \frac{x\sin 2x}{5} - \frac{4\cos 2x}{25}$

Question $(D^2+4)y = x\cos 2x$

$m^2+4 \quad m = \pm 2i$

C.F. $= C_1 \cos 2x + C_2 \sin 2x$

P.I. $= \dfrac{1}{D^2+4}(x\cos 2x)$

$\cancel{D^2 = -4}$
$\cancel{\text{case}}$ fails

P.I. $= x\dfrac{1}{D^2+4}\cos 2x - \dfrac{2D}{(D^2+4)^2}\cos 2x$

$D^2 = -4$
case fails

Diff

P.I. $= x.x\dfrac{1}{2D}\cos 2x - \dfrac{x.2D\cos 2x}{2(D^2+4).2D}$

$D^2 = -4$
case fa

P.I. $= \dfrac{x^2}{2}\dfrac{1}{D}(\cos 2x) - \dfrac{x}{2}.x\dfrac{\cos 2x}{2D}$

P.I $= \dfrac{x^2}{2}\dfrac{1}{D}\cos 2x - \dfrac{x}{4}.x\dfrac{1}{D}\cos 2x$

P.I $= \dfrac{x^2}{2}\int \cos 2x\,dx - \dfrac{x^2}{4}\int \cos 2x\,dx$

P.I. $= \dfrac{x^2 \sin 2x}{8}$

$y = C_1\cos 2x + C_2 \sin 2x + \dfrac{x^2 \sin 2x}{8}$

Variation of parameters:-

$$F(D)\,y = R(x)$$

$$C.F. = c_1\,u(x) + c_2\,v(x)$$

$$P.I. = A\,u(x) + B\,v(x) \quad \text{where}$$

$$A = -\int \frac{v(x)\,R(x)}{u\frac{dv}{dx} - v\frac{du}{dx}}$$

$$B = \int \frac{u(x)\,R(x)}{u\frac{dv}{dx} - v\frac{du}{dx}}$$

Quest $(D^2+9)\,y = \sec 3x$

A.E. $m^2+9=0$, $m = \pm 3i$

$$C.F. = c_1 \cos 3x + c_2 \sin 3x$$

$$A = -\int \frac{\sin 3x \sec 3x}{3\cos 3x \cos 3x + 3\sin 3x \sin 3x}$$

$$A = -\frac{1}{3}\int \tan 3x = +\frac{1}{3}\,\frac{\ln(\cos 3x)}{3}$$

$$B = \int \frac{\cos 3x \sec 3x}{\cos 3x \cos 3x \cdot 3 - \sin 3x \sin 3x\,(-3)} = \frac{1}{3}\int \cos 3x \sec 3x\,dx$$

$$B = \frac{1}{3}\int dx = \frac{1}{3}x$$

$$PI = A\,u(x) + B\,v(x)$$

$$PI = \frac{(\cos 3x)\ln(\cos 3x)}{9} + \frac{x}{3}\sin 3x$$

So $$y = c_1\cos 3x + c_2\sin 3x + \frac{\cos 3x \ln(\cos 3x)}{9} + \frac{x}{3}\sin 3x$$

Ans

Question $$(D^2+1)y = \operatorname{cosec} x$$

$$m = \pm 1$$

$$CF = c_1\cos x + c_2 \sin x$$

$$u = \cos x \qquad v = \sin x$$

$$\frac{du}{dx} = -\sin x \qquad \frac{dv}{dx} = \cos x$$

$$u\frac{dv}{dx} - v\frac{du}{dx} = \cos^2 x + \sin^2 x = 1$$

$$A = -\int \frac{v(x)R(x)}{u\frac{dv}{dx} - v\frac{du}{dx}} = -\int \frac{\sin x \operatorname{cosec} x}{1}\,dx = -\int dx = x$$

$$B = \int \frac{\cos x \cdot \operatorname{cosec} x}{1} = \int \cot x = \ln(\sin x)$$

$$P.I. = -x\cos x + \sin x \ln(\sin x)$$

So $$y = c_1\cos x + c_2\sin x - x\cos x + \sin x \ln(\sin x)$$

Cauchy - Eulers diff eq^n

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = f(x)$$

$$\left(x^n D^n + a_1 x^{n-1} D^{n-1} + a_2 x^{n-2} D^{n-2} + \cdots + a_n\right) y = f(x)$$

$$x = e^z \iff z = \ln x$$

$$x \frac{d}{dx} = xD = \theta = \frac{d}{dz}$$

$$x^2 D^2 = \theta(\theta - 1)$$

$$x^3 D^3 = \theta(\theta-1)(\theta-2)$$

Ques1 $(x^2D^2 + xD + 1)y = x^2$

$$(\theta(\theta-1) + \theta + 1)\, y = (e^z)^2$$

$$(\theta^2 + 1)\, y = e^{2z}$$

A.E $m^2+1 = 0$

$m = \pm i$

C.F. $= c_1 \cos z + c_2 \sin z$

P.I. $= \frac{1}{\theta^2+1} e^{2z}$ $\qquad \theta = 2$

$$P.I. = \frac{1}{4+1} e^{2z} = \frac{e^{2z}}{5}$$

$$G.S = C.F + P.I.$$

$$y = c_1 \cos z + c_2 \cos z + \frac{e^{2z}}{5}$$

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x) + \frac{x^2}{5}$$

Question $(x^3 D^3 + 2x^2 D^2 + 2)y = 0$

$$(\theta(\theta-1)(\theta-2) + 2\theta(\theta-1) + 2)y = 0$$

$$(\theta(\theta^2 - 3\theta + 2) + 2\theta^2 - 2\theta + 2)y = 0$$

$$(\theta^3 - 3\theta^2 + \cancel{2\theta} + 2\theta^2 - \cancel{2\theta} + 2)y = 0$$

$$(\theta^3 - \theta^2 + 2)y = 0$$

A.E $\quad m^3 - m^2 + 2 = 0$

$m = -2$ $\quad (m+1)(m^2 - 2m + 2) = 0$

$$m = -1,\ 1 \pm i$$

$$C.F. = c_1 e^{-z} + e^{z}(c_2 \cos z + c_3 \sin z)$$

Sol^n $\quad y = c_1 x^{-1} + x(c_2 \cos(\ln x) + c_3 \sin(\ln x))$

Partial Diff eqⁿs :-

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} = F$$

(i) if $B^2 - 4AC = 0$ then the P.D.E. is parabolic

(ii) if $B^2 - 4AC < 0$ then the P.D.E. is Eliptic

(iii) if $B^2 - 4AC > 0$ then the P.D.E. is hyperbolic.

Ex 1-D Heat eq^n

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{du}{dt}$$

$$a^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$$

$A = a^2$, $B = 0$, $C = 0$

$B^2 - 4AC = 0 - 4(a^2)(0)$

$= 0$

parabolic

1-D wave eq^n

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$

$$a^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

$A = a^2$, $B = 0$, $C = -1$

$0^2 - 4(a^2)(-1) = 4a^2 > 0$

hyperbola.

2-D Heat eq^n (Also known as laplace eq^n)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$A = 1$, $B = 0$, $C = 1$

$B^2 - 4AC = 0 - 4.1.1$

$= -4 < 0$

Eleptic

Question :- Solve the P.D.E.

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} \quad , u(0,y) = 8e^{-3y} \quad \text{by}$$

method of sepration of variables.

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} \quad - (1)$$

let $u(x,y) = XY \quad - (2)$

$$\frac{\partial u}{\partial x} = X'Y \quad - (3) \qquad \frac{\partial u}{\partial y} = XY' \quad - (4)$$

sub eq (2) (3) (4) in (1)

$$X'Y = 4XY'$$

$$\frac{X'}{X} = 4\frac{Y'}{Y} = k \text{ (says}$$

$$X' - kX = 0 \quad \& \quad Y' - \frac{k}{4}Y = 0$$

$$X = c_1 e^{kx} \quad - (5) \qquad Y = c_2 e^{\frac{k}{4}y} \quad - (6)$$

sub (5) & (6) in (2)

$$u(x,y) = ce^{kx} e^{\frac{k}{4}y}$$

Now $u(0,y) = ce^{\frac{k}{4}y}$ $\qquad c = c_1 c_2$

Given that $u(0,y) = 8e^{-3y}$

$c = 8$, $-3 = k/4 \Rightarrow k = -12$

So

$$u(x,y) = 8e^{-12x}e^{-3y}$$

$$u(x,y) = 8e^{-3(y+4x)}$$ Ans.