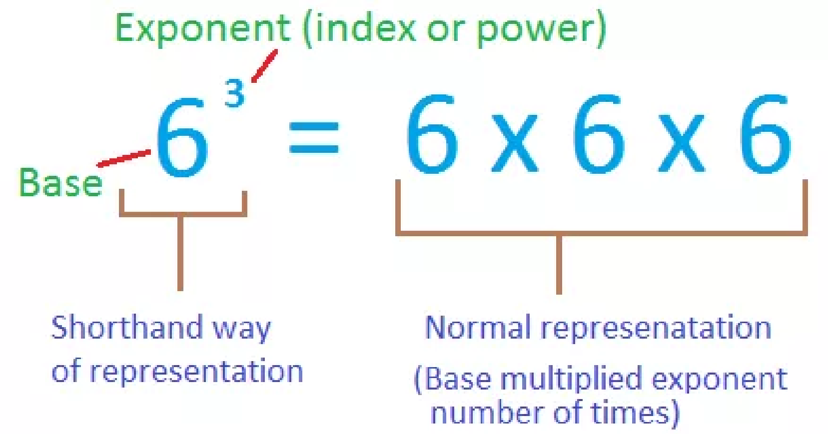
## 2. Exponents

### What is an Exponent?

### Exponents

The repeated addition of numbers can be written in short form (product form).




**Examples:**

| S.No. | Statements | Repeated Addition | Products Form |
|---|---|---|---|
| (i) | 4 times 2 | $2+2+2+2$ | $4 \times 2$ |
| (ii) | 5 time – 1 | $(-1)+(-1)+(-1)+(-1)+(-1)$ | $5 \times (-1)$ |
| (iii) | 3 times $\frac{-2}{3}$ | $\left(\frac{-2}{3}\right)+\left(\frac{-2}{3}\right)+\left(\frac{-2}{3}\right)$ | $3 \times \left(\frac{-2}{3}\right)$ |
| (iv) | 2 times 1 | $1+1$ | $2 \times 1$ |

Also, we can write the repeated multiplication of numbers in a short form known as exponential form. For example, when 5 is multiplied by itself for two times, we write the product $5 \times 5$ in exponential form as $5^2$ which is read as 5 raised to the power two.

Similarly, if we multiply 5 by itself for 6 times, the product $5 \times 5 \times 5 \times 5 \times 5 \times 5$ is written in exponential form as $5^6$ which is read as 5 raised to the power 6.
In $5^6$, the number 5 is called the base of $5^6$ and 6 is called the **exponent of the base**.

In general, we write,
An exponential number as $b^a$, where b is the base and a is the exponent.
The notation of writing the multiplication of a number by itself several times is called the **exponential notation** or **power notation**.

Thus, in general we find that :
If 'a' is a rational number then 'n' times the product of 'a' by itself is given as $a \times a \times a \times a$ ….. , n times and is denoted by $a^n$, where 'a' is called the base and n is called the exponent of $a^n$.

**Examples**

1. Write the following statements as repeated multiplication and complete the table:

| S.No. | Statements | Repeated Multiplication | Short form |
|---|---|---|---|

| | | | |
|---|---|---|---|
| (i) | 3 multiplied by 3 for 6 times | $3 \times 3 \times 3 \times 3 \times 3 \times 3 = 729$ | $3^6$ |
| (ii) | 2 multiplied by 2 for 3 times | $2 \times 2 \times 2$ | $2^3$ |
| (iii) | 1 multiplied by 1 for 7 times | $1 \times 1 \times 1 \times 1 \times \ 1 \times 1 \times 1$ | $1^7$ |

2. Write the base and exponent of following numbers. And also write in expanded form:

| S.No. | Numbers | Base | Exponent | Expanded Form | Value |
|---|---|---|---|---|---|
| (i) | $3^4$ | 3 | 4 | $3 \times 3 \times 3 \times 3$ | 81 |
| (ii) | $2^5$ | 2 | 5 | $2 \times 2 \times 2 \times 2 \times 2$ | 32 |
| (iii) | $3^3$ | 3 | 3 | $3 \times 3 \times 3$ | 27 |
| (iv) | $2^2$ | 2 | 2 | $2 \times 2$ | 4 |
| (v) | $1^7$ | 1 | 7 | $1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1$ | 1 |

## Exponents of Negative Integers

When the exponent of a negative integer is odd, the resultant is a negative number, and when the power of a negative number is even, the resultant is  a positive number.When the exponent of a negative integer is odd, the resultant is a negative number, and when the power of a negative number is even, the resultant is  a positive number.
or (a negative integer) an odd number = a negative integer.
(a negative integer) an even number  = a positive integer.

**Examples:**

**Ex.1** Express 144 in the powers of prime factors.

**Solution:**

$144 = 16 \times 9 = 2 \times 2 \times 2 \times 2 \times 3 \times 3$
Here 2 is multiplied four times and 3 is multiplied 2 times to get 144.
$\therefore\ 144 = 2^4 \times 3^2$

**Ex.2** Which one is greater : $3^5$ or $5^3$ ?

**Solution:**

$3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 9 \times 9 \times 3$
$= 81 \times 3 = 243$
and $5^3 = 5 \times 5 \times 5 = 25 \times 5 = 125$
Clearly, $243 > 125$
$\therefore\ 3^5 > 5^3$

## What are Laws of Exponents?

## Laws of Exponents

### Law-1:

If a is any non-zero integer and m and n are whole numbers, then

$\mathbf{a^m \times a^n = a^{m+n}}$

**Eg :**

$$\text{(i)}\quad 3^4 \times 3^2 = \underbrace{(3\times3\times3\times3)}_{\substack{\text{4 times multiplication}\\ \text{of 3 by itself}}} \times \underbrace{(3\times3)}_{\substack{\text{2 times multiplication}\\ \text{of 3 by itself}}}$$

$$= \underbrace{3\times3\times3\times3\times3\times3}_{\substack{\text{6 times multiplication}\\ \text{of 3 by itself}}} = 3^6 = 3^{4+2}$$

$$\text{Thus, } 3^4 \times 3^2 = 3^{4+2}$$

$$\text{(ii)}\quad 2^3 \times 2^5 = \underbrace{(2\times2\times2)}_{\substack{\text{3 times multiplication}\\ \text{of 2 by itself}}} \times \underbrace{(2\times2\times2\times2\times2)}_{\substack{\text{5 times multiplication}\\ \text{of 2 by itself}}}$$

$$= \underbrace{2\times2\times2\times2\times2\times2\times2\times2}_{\substack{\text{8 times multiplication}\\ \text{of 2 by itself}}}$$

$$= 2^8 = 2^{3+5}$$

$$\text{Thus, } 2^3 \times 2^5 = 2^{3+5}$$

Therefore, in general, we write,

$$a^m \times a^n = \underbrace{(a\times a\times a\times a\times \ldots)}_{\substack{\text{m times multiplication}\\ \text{of 'a' by itself}}} \times \underbrace{(a\times a\times a\times a\times \ldots)}_{\substack{\text{n times multiplication}\\ \text{of 'a' by itself}}}$$

$$= a \times a \times a \times a \times a \times \ldots, (m+n) \text{ times} = a^{m+n}.$$

**Law-2:**

If a and b are non-zero integers and m is a positive integer, then

$\mathbf{a^m \times b^m = (a \times b)^m}$

**Eg :**

$5^3 \times 3^3 = (5 \times 5 \times 5) \times (3 \times 3 \times 3)$
$= (5 \times 3) \times (5 \times 3) \times (5 \times 3)$
$= 15 \times 15 \times 15 = (15)^3$
So, $5^3 \times 3^3 = (5 \times 3)^3 = (15)^3$
Here, we find that 15 is the product of bases 5 and 3.
Also, if a and b are non-zero integers, then
$a^5 \times b^5 = (a \times a \times a \times a \times a) \times (b \times b \times b \times b \times b)$
$= (a \times b)\ (a \times b)\ (a \times b)\ (a \times b)\ (a \times b) = (ab)^5$

**Law-3:**

If a is a non-zero integer and m and n are two whole numbers such that m > n, then

$$a^m \div a^n = a^{m-n} \text{ and for } m < n \qquad \text{For example, } 2^5 \div 2^7 = \frac{2\times2\times2\times2\times2}{2\times2\times2\times2\times2\times2\times2}$$

$$a^m \div a^n = (a)^{m-n} = \frac{1}{a^{n-m}} \qquad = \frac{1}{2\times2} = \frac{1}{2^2} = \frac{1}{2^{7-5}}$$

When an exponential form is divided by another exponential form whose bases are same, then the resultant is an exponential form with same base but the exponent is the difference of the exponent of the divisor from the exponent of the dividend.

**Law-4:**

Division of exponential forms with the same exponents and different base:
If a and b are any two non-zero integers, have same exponent m then for $a^m \div b^m$, we write

$$\frac{a^m}{b^m} = \frac{a\times a\times a\times ..., \text{m times}}{b\times b\times b\times ..., \text{m times}}$$

$$= \left(\frac{a}{b}\right)\times\left(\frac{a}{b}\right)\times\left(\frac{a}{b}\right)\times ..., \text{m times} = \left(\frac{a}{b}\right)^m$$

For examples

(i) $$2^6 \div 3^6 = \frac{2^6}{3^6} = \frac{2\times2\times2\times2\times2\times2}{3\times3\times3\times3\times3\times3}$$

$$= \frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3} = \left(\frac{2}{3}\right)^6$$

$$\text{Hence, } 2^6 \div 3^6 = \left(\frac{2}{3}\right)^6$$

(ii) $$(-2)^4 \div b^4 = \frac{(-2)^4}{b^4} = \frac{-2\times-2\times-2\times-2}{b\times b\times b\times b}$$

$$= \frac{-2}{b}\times\frac{-2}{b}\times\frac{-2}{b}\times\frac{-2}{b} = \left(-\frac{2}{b}\right)^4$$

$$\text{Hence, } (-2)^4 \div b^4 = \left(-\frac{2}{b}\right)^4$$

**Law-5:**

If ‘a’ be any non-zero integer and m and n any two positive integers then

$[(a)^m]^n = a^{mn}$

**Eg :**

$(2^2)^3 = 2^2 \times 2^2 \times 2^2 = 2^{2+2+2} = 2^6 = 2^{2\times3}$
$(2^7)^2 = 2^7 \times 2^7 = 2^{7+7} = 2^{14} = 2^{7\times2}$

**Law-6:**

**Law of zero Exponent:**

We know that

$$2^6 \div 2^6 = \frac{2^6}{2^6} = \frac{2\times2\times2\times2\times2\times2}{2\times2\times2\times2\times2\times2} = 1$$

By using Law-3 of exponents, we have
$2^6 \div 2^6 = 2^{6-6} = 2^0$
Thus, $2^0 = 1$
In general $a^m \div a^m = a^{m-m} = a^0$ and also

$$\frac{a^m}{a^m} = \frac{a\times a\times a\times a\times a\times a\times\ldots., \text{m times}}{a\times a\times a\times a\times a\times a\times\ldots., \text{m times}} = 1$$

Hence, $a^0 = 1$
Any non-zero integer raised to the power 0 always results into 1.

## Use Of Exponents In Expressing Large Numbers

We know that
$100 = 10 \times 10 = 10^2$,
$1000 = 10 \times 10 \times 10 = 10^3$,
$10000 = 10 \times 10 \times 10 \times 10 = 10^4$
We can write a number followed by large number of zeroes in powers of 10.
For example, we can write the speed of light in vacuum = 300,000,000 m/s
= 3 × 1,00,000,000 m/s = $3 \times 10^8$ m/s
= $30 \times 10^7$ m/s = $300 \times 10^6$ m/s
Similarly,
the age of universe = 8,000,000,000 years (app.)
= $8 \times 10^9$ years (app.)
We can also express the age of universe as
$80 \times 10^8$ years or $800 \times 10^7$ years, etc.
But generally the number which preceded the power of 10 should be less than 10. Such a notation is called **standard or scientific** notation.
So $8 \times 10^9$ years is the standard form of the age of the universe.
Similarly, the standard form of the speed of light is $3 \times 10^8$ m/s.

**Eg:** Write the following numbers in standard form :

(i) 4340000
(ii) 173000
(iii) 140000

**Solution:**

(i) It is clear that 4340000 = 434 × 10000
Also, $4340000 = 4.34 \times 10^6$
$\because$ $434 = 4.34 \times 100 = 4.34 \times 10^2$
(ii) Also, $173000 = 1.73 \times 10^5$
(iii) Also, $140000 = 1.4 \times 10^5$

## Laws of Exponents Problems with Solutions

**1. Write in exponential form :**

(i) $(5 \times 7)^6$ (ii) $(-7n)^5$

**Solution:**

(i) $(5 \times 7)^6$
$= (5 \times 7)(5 \times 7)(5 \times 7)(5 \times 7)(5 \times 7)(5 \times 7)$
$= (5 \times 5 \times 5 \times 5 \times 5 \times 5)(7 \times 7 \times 7 \times 7 \times 7 \times 7) = 5^6 \times 7^6$
Hence, $(5 \times 7)^6 = 5^6 \times 7^6$
(ii) $(-7n)^5 = (-7n)(-7n)(-7n)(-7n)(-7n)$
$= (-7 \times -7 \times -7 \times -7 \times -7)(n \times n \times n \times n \times n)$
$= (-7)^5 \times (n)^5$

**2. Write the following in expanded form :**

(i) $\left(-\frac{7}{9}\right)^3$ (ii) $\left(\frac{5}{8}\right)^6$

**Solution:**

$$\text{(i) } \left(-\frac{7}{9}\right)^3 = \frac{-7}{9} \times \frac{-7}{9} \times \frac{-7}{9}$$

$$= \frac{-7 \times -7 \times -7}{9 \times 9 \times 9} = \frac{(-7)^3}{9^3}$$

$$\text{(ii) } \left(\frac{5}{8}\right)^6 = \frac{5}{8} \times \frac{5}{8} \times \frac{5}{8} \times \frac{5}{8} \times \frac{5}{8} \times \frac{5}{8}$$

$$= \frac{5 \times 5 \times 5 \times 5 \times 5 \times 5}{8 \times 8 \times 8 \times 8 \times 8 \times 8} = \frac{5^6}{8^6}$$

**3. Find the value of :**

(i) $(3^0 - 2^0) \times 5^0$
(ii) $2^0 \times 3^0 \times 4^0$
(iii) $(6^0 - 2^0) \times (6^0 + 2^0)$

**Solution:**

(i) We have, $(3^0 - 2^0) \times 5^0$
Therefore, $(1 - 1) \times 1 = 0 \times 1 = 0$
[Since $3^0 = 1$, $2^0 = 1$]
(ii) We have, $2^0 \times 3^0 \times 4^0 = (1 \times 1 \times 1) = 1$
(iii) We have, $(6^0 - 2^0) \times (6^0 + 2^0)$
$= (1 - 1) \times (1 + 1)$
$= 0 \times 2 = 0.$

## Review of Exponents

1. Exponents are the mathematician's shorthand.
2. In general, the format for using exponents is:
   $(\text{base})^{\text{exponent}}$
   where the exponent tells you how many of the **base** are being multiplied together.
3. Consider: 2 • 2 • 2 is the same as $2^3$, since there are **three 2's** being multiplied together.
   Likewise, 5 • 5 • 5 • 5 = 54, because there are **four 5's** being multiplied together.
4. Exponents are also referred to as "powers".
   For example, $2^3$ can be read as "two cubed" or as "two raised to the third power".

| Laws of Exponents | | |
|---|---|---|
| product | $a^m \cdot a^n = a^{m+n}$ | $2^2 \cdot 2^3 = (2\cdot2)(2\cdot2\cdot2) = 2^5$ |
| quotient | $\frac{a^m}{a^n} = a^{m-n}$ | $\frac{2^3}{2^2} = \frac{2\cdot\cancel{2}\cdot\cancel{2}}{\cancel{2}\cdot\cancel{2}} = 2^{3-1} = 2$ |
| power | $(a^m)^n = a^{m\cdot n}$ | $(2^2)^3 = (2\cdot2\cdot2)(2\cdot2\cdot2)(2\cdot2\cdot2) = 2^6$ |
| inverse | $a^{-1} = \frac{1}{a}$ | $2^{-1} = \frac{1}{2}$ (this is a definition) |
| zero power | $a^0 = 1$ | Why? We need $a^m a^n = a^{m+n}$ when m = 0. In order for this law to be satisfied when m = 0, we have $a^n = a^{m+n} = a^{0+n} = a^0 a^n$, so $a^0$ must be 1. |

### Exponents of Negative Values

When we multiply **negative** numbers together, we must utilize parentheses to switch to exponent notation.

$$(-3)(-3)(-3)(-3)(-3)(-3) = (-3)^6$$

BEWARE!! $-3^6$ is NOT the same as $(-3)^6$

The missing parentheses mean that **$-3^6$** will multiply **six 3's** together first (by order of operations), and then **take the negative of that answer.**

$(-3)^6 = 729$ but $-3^6 = -729$

**so be careful with negative values and exponents !**

**Note: Even powers of negative numbers** allow for the negative values to be arranged in pairs. This pairing guarantees that the answer will always be **positive.**

$$\begin{aligned}(-5)^6 &= (-5)\cdot(-5) \;\cdot\; (-5)\cdot(-5) \;\cdot\; (-5)\cdot(-5) \quad \leftarrow \text{All pairs.}\\ &= 25 \;\cdot\; 25 \;\cdot\; 25\\ &= 15625 \quad \text{(a positive answer)}\end{aligned}$$

**Odd powers of negative numbers,** however, always leave one factor of the negative number not paired. This one lone negative term guarantees that the answer will always be **negative.**

$$\begin{aligned}(-5)^5 &= (-5)\cdot(-5) \;\cdot\; (-5)\cdot(-5) \;\cdot\; (-5) \quad \leftarrow \text{One lone, un-paired, negative.}\\ &= 25 \;\cdot\; 25 \;\cdot\; (-5)\\ &= -3125 \quad \text{(a negative answer)}\end{aligned}$$

## Zero and Negative Exponents

- Zero Exponent: any nonzero expression to the zero power is 1. $a^0 = 1, a \neq 0$ $\quad 5^0 = 1$
- Negative Exponent: any term to a negative power is the reciprocal of that term with a positive power

$$a^{-n} = \frac{1}{a^n}, a \neq 0 \qquad 2^{-3} = \frac{1}{2^3}$$

$$\frac{1}{a^{-n}} = a^n, a \neq 0 \qquad \frac{1}{2^{-4}} = 2^4$$

## Zero Exponents

The number zero may be used as an exponent.
**The value of any expression raised to the zero power is 1.**
(Except zero raised to the zero power is undefined.)

| $Base^0$ | Value |
|---|---|
| $2^0 =$ | 1 |
| $(-6)^0 =$ | 1 |
| $4^0 =$ | 1 |
| $-8^0 =$ | -1<br>Raise to the zero power first: $8^0=1$<br>then take the negative. |
| $0^0 =$ | undefined |

## Negative Exponents

Negative numbers as exponents have a special meaning.
The rule is as follows:

$$\text{base}^{\text{negative exponent}} = \frac{1}{\text{base}^{\text{positive exponent}}}$$

**For example:**

| Negative Exponent | Positive Exponent |
|---|---|
| $4^{-1} =$ | $\frac{1}{4^1}$ |
| $7^{-3} =$ | $\frac{1}{7^3}$ |
| $(-5)^{-2} =$ | $\frac{1}{(-5)^2}$ |

## Exponents and Units

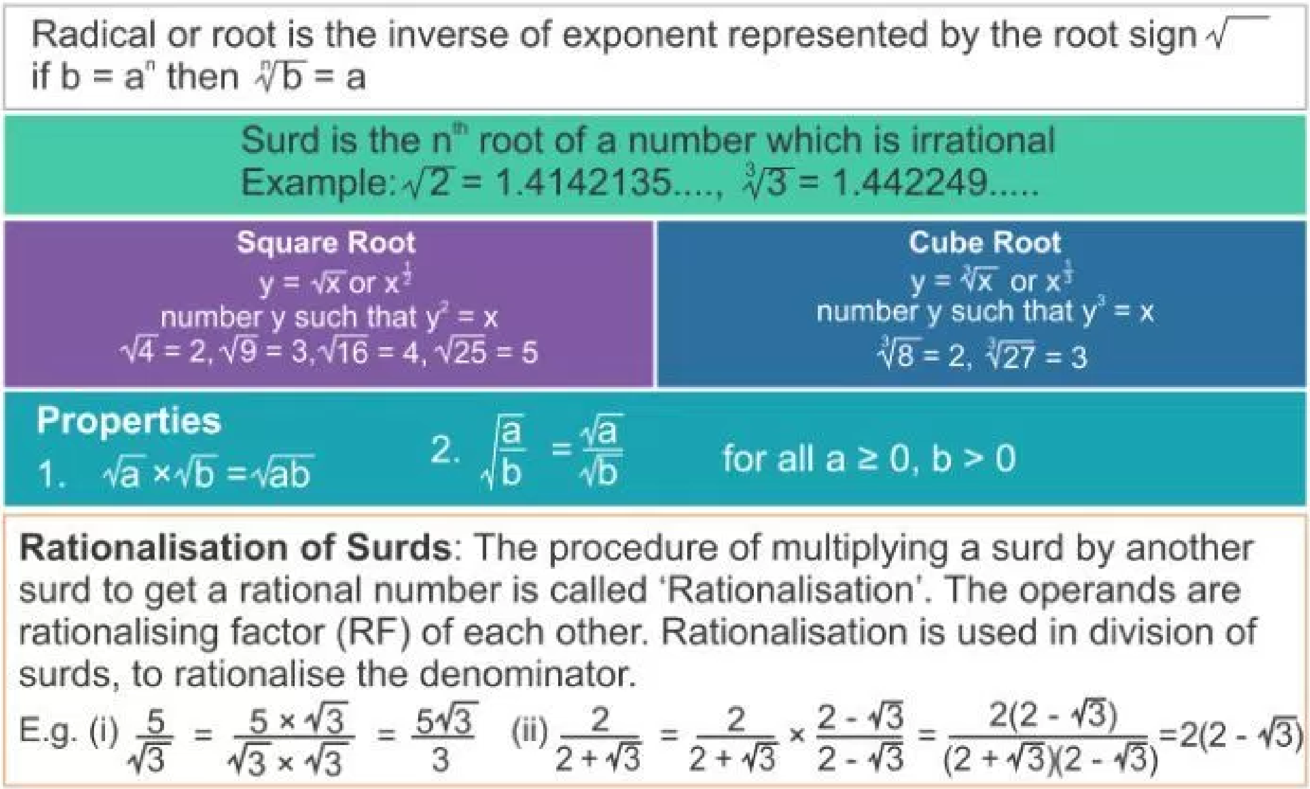
When working with units and exponents (or powers), remember to adjust the units appropriately.
$(36 \text{ ft})^3 = (36 \text{ ft}) \bullet (36 \text{ ft}) \bullet (36 \text{ ft})$
$= (36 \bullet 36 \bullet 36) (\text{ft} \bullet \text{ft} \bullet \text{ft})$
$= 46656 \text{ ft}^3$

Exponents can be very useful for evaluating expressions. It is also useful to learn how to use your calculator when working with exponents.

## Surds

### Definition of surd

Any root of a number which can not be exactly found is called a **surd**.




Let a be a rational number and n is a positive integer. If the $n^{th}$ root of x i.e., $x^{1/n}$ is irrational, then it is called surd of order n.

Order of a surd is indicated by the number denoting the root.

For example, $\sqrt{7}, \sqrt[3]{9}, 11^{3/5}, \sqrt[n]{3}$ are surds of second, third, fifth and $n^{th}$ order respectively.

A second order surd is often called a **quadratic surd**, a surd of third order is called a **cubic surd**.

### Types of surds

1. **Simple surds:** A surd consisting of a single term. For example 2√3, 6√5, √5 etc.
2. **Pure and mixed surds:** A surd consisting of wholly of an irrational number is called pure surd. A surd consisting of the product of a rational number and an irrational number is called a mixed surd.
3. **Compound surds:** An expression consisting of the sum or difference of two or more surds.
4. **Similar surds:** If the surds are different multiples of the same surd, they are called similar surds.
5. **Binomial surds:** A compound surd consisting of two surds is called a binomial surd.
6. **Binomial quadratic surds:** Binomial surds consisting of pure (or simple) surds of order two i.e., the surds of the form a√b ± c√d or a ± b√c are called binomial quadratic surds.
   Two binomial quadratic surds which differ only in the sign which connects their terms are said to be conjugate or complementary to each other. The product of a binomial quadratic surd and its conjugate is always rational.
   For example: The conjugate of the surd 2√7 + 5√3 is the surd 2√7 − 5√3.

## Properties of quadratic surds

1. The square root of a rational number cannot be expressed as the sum or difference of a rational number and a quadratic surd.
2. If two quadratic surds cannot be reduced to others, which have not the same irrational part, their product is irrational.
3. One quadratic surd cannot be equal to the sum or difference of two others, not having the same irrational part.
4. If $a + \sqrt{b} = c + \sqrt{d}$ where a and c are rational, and $\sqrt{b}$, $\sqrt{d}$ are irrational, then $a = c$ and $b = d$.

## Rationalisation factors

If two surds be such that their product is rational, then each one of them is called rationalising factor of the other.
Thus each of $2\sqrt{3}$ and $\sqrt{3}$ is a rationalising factor of each other. Similarly $\sqrt{3} + \sqrt{2}$ and $\sqrt{3} - \sqrt{2}$ are rationalising factors of each other, as $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 1$, which is rational.
To find the factor which will rationalize any given binomial surd:

**Case I:** Suppose the given surd is $\sqrt[p]{a} - \sqrt[q]{b}$
Let $a^{1/p} = x$, $b^{1/q} = y$ and let n be the L.C.M. of p and q. Then $x^n$ and $y^n$ are both rational.
Now $x^n - y^n$ is divisible by $x - y$ for all values of n, and $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + y^{n-1})$.
Thus the rationalizing factor is $x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + y^{n-1}$ and the rational product is $x^n - y^n$.

**Case II:** Let the given surd be $\sqrt[p]{a} + \sqrt[q]{b}$
Let have the same meaning as in Case I.
(1) If n is even, then $x^n - y^n$ is divisible by $x + y$ and $x^n - y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots - y^{n-1})$.
Thus the rationalizing factor is $x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots - y^{n-1}$ and the rational product is $x^n - y^n$.
(2) If n is odd, $x^n + y^n$ is divisible by $x + y$ and $x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots + y^{n-1})$
Thus the rationalizing factor is and the rational product is $x^n + y^n$.

## Square roots of a +√b and a + √b + √c + √d where √b , √c , √d are Surds

Let $\sqrt{(\sqrt{a} + \sqrt{b})} = \sqrt{x} + \sqrt{y}$ where $x, y > 0$ are rational numbers.

Then squaring both sides we have, $a + \sqrt{b} = x + y + 2\sqrt{x}\sqrt{y}$

$\Rightarrow a = x + y,\ \sqrt{b} = 2\sqrt{xy} \Rightarrow b = 4xy$

So, $(x - y)^2 = (x + y)^2 - 4xy = a^2 - b$

After solving we can find *x* and *y*.

Similarly square root of $a - \sqrt{b}$ can be found by taking $\sqrt{(a - \sqrt{b})} = \sqrt{x} - \sqrt{y},\ x > y$

**To find square root of *a* + √*b* + √*c* + √*d* :**

Let $\sqrt{(a + \sqrt{b} + \sqrt{c} + \sqrt{d})} = \sqrt{x} + \sqrt{y} + \sqrt{z}, (x, y, z > 0)$ and

take $\sqrt{(a + \sqrt{b} - \sqrt{c} - \sqrt{d})} = \sqrt{x} + \sqrt{y} - \sqrt{z}$ .

Then by squaring and equating, we get equations in x, y, z. On solving these equations, we can find the required square roots.

(1) If $a^2 - b$ is not a perfect square, the square root of $a + \sqrt{b}$ is complicated *i.e.*, we can't find the value of $\sqrt{(a + \sqrt{b})}$ in the form of a compound surd.

(2) If $\sqrt{(a + \sqrt{b})} = \sqrt{x} + \sqrt{y}, x > y$ then $\sqrt{(a - \sqrt{b})} = \sqrt{x} - \sqrt{y}$

(3) $\sqrt{a + \sqrt{(b)}} = \sqrt{\left(\frac{a + \sqrt{a^2 - b}}{2}\right)} + \sqrt{\left(\frac{a - \sqrt{a^2 - b}}{2}\right)}$

(4) $\sqrt{a - \sqrt{(b)}} = \sqrt{\left(\frac{a + \sqrt{a^2 - b}}{2}\right)} - \sqrt{\left(\frac{a - \sqrt{a^2 - b}}{2}\right)}$

(5) If $a$ is a rational number and $\sqrt{b}, \sqrt{c}, \sqrt{d}$ are surds then

(i) $\sqrt{a + \sqrt{b} + \sqrt{c} + \sqrt{d}} = \sqrt{\frac{bd}{4c}} + \sqrt{\frac{bc}{4d}} + \sqrt{\frac{cd}{4b}}$

(ii) $\sqrt{a - \sqrt{b} - \sqrt{c} + \sqrt{d}} = \sqrt{\frac{bd}{4c}} + \sqrt{\frac{cd}{4b}} + \sqrt{\frac{bc}{4d}}$

(iii) $\sqrt{a - \sqrt{b} - \sqrt{c} + \sqrt{d}} = \sqrt{\frac{bc}{4d}} - \sqrt{\frac{bd}{4c}} - \sqrt{\frac{cd}{4b}}$

**Cube root of a binomial quadratic surd**

If $(a + \sqrt{b})^{1/3} = x + \sqrt{y}$ then $(a - \sqrt{b})^{2/3} = x - \sqrt{y}$, where $a$ is a rational number and $b$ is a surd.

Procedure of finding $(a + \sqrt{b})^{1/3}$ is illustrated with the help of an example :

Taking $(37 - 30\sqrt{3})^{1/3} = x + \sqrt{y}$ we get on cubing both sides,

$37 - 30\sqrt{3} = x^3 + 3xy - (3x^2 + y)\sqrt{y}$

$\therefore x^3 + 3xy = 37$

$(3x^2 + y)\sqrt{y} = 30\sqrt{3} = 15\sqrt{12}$

As $\sqrt{3}$ can not be reduced, let us assume $y = 3$ we get $3x^2 + y = 3x^2 + 3 = 30$ . $\therefore x = 3$ ,

which doesn't satisfy $x^3 + 3xy = 37$ .

Again taking $y = 12$, we get $3x^2 + 12 = 15$ , $\therefore x = 1$

$x = 1, y = 12$ satisfy $x^3 + 3xy = 37$

$\therefore \sqrt[3]{37 - 30\sqrt{3}} = 1 - \sqrt{12} = 1 - 2\sqrt{3}$

## Equations involving surds

While solving equations involving surds, usually we have to square, on squaring the domain of the equation extends and we may get some extraneous solutions, and so we must verify the solutions and neglect those which do not satisfy the equation.
Note that from $ax = bx$, to conclude $a = b$ is not correct. The correct procedure is $x(a - b)=0$ i.e. $x = 0$ or $a = b$. Here, necessity of verification is required.