Chapter

4

Circle and Systems of Circles

CONTENTS

G. Cramer

Cramer (1750 A.D.) made formal use of the two axes and gane the eqution of a circle as $(y - a)^2 + (b - x)^2 = r.r.$ He gave the best exposition of the analytic geometry of his time.

Kochanski gives an approximate method to find the length of the circumference of a circle. Jones introduces the Greek latter to represent the ratio of the circumference of a circle to its diameter in his Synopsis palmariorum matheseos (A new introduction to Mathematics).

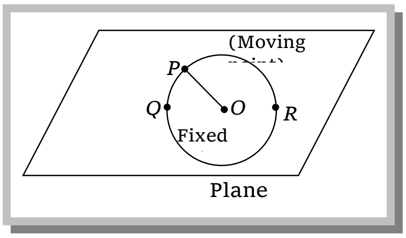
Feuerbach publishes his discoveries on the nine point circle of a triangle.

Nicholas of Cusa studies geometry and logic. He contributes to the study of infinity, the infinitely large and the infinitely small. He looks at the circle as the limit of regular polygons.

Circle and Systems of Circles

4.1 Definition

A circle is defined as the locus of a point which moves in a plane such that its distance from a fixed point in that plane always remains the same *i.e.* constant.



The fixed point is called the centre of the circle and the fixed distance is called the radius of the circle.

Note :
- If $r(r>0)$ is the radius of a circle, the diameter $d=2r$ is the maximum distance between any two points on the given circle.
- The length of the curve or perimeter (also called circumference) of circle $=2\pi r$ or πd.
- The area of circle $=\pi r^2$ or $\frac{\pi d^2}{4}$.
- Line joining any two points of a circle is called chord of circle.
- Curved section between any two points of a circle is called arc of circle.
- Angle subtended at the centre of a circle by any arc = arc/radius.
- Angle subtended at the centre of a circle by an arc is double of angle subtended at the circumference of a circle.

4.2 Standard forms of Equation of a Circle

(1) **General equation of a circle :** The general equation of a circle is $x^2+y^2+2gx+2fy+c=0$, where g, f, c are constant.

(i) Centre of the circle is $(-g,-f)$. *i.e.*, $(-\frac{1}{2}$ coefficient of x, $-\frac{1}{2}$ coefficient of $y)$

(ii) Radius of the circle is $\sqrt{g^2+f^2-c}$.

Note :
- The general equation of second degree $ax^2+by^2+2hxy+2gx+2fy+c=0$ represents a circle if $a=b\neq 0$ and $h=0$.
- Locus of a point P represent a circle if its distance from two points A and B is not equal *i.e.* $PA=kPB$ represent a circle if $k\neq 1$.
- Discussion on nature of the circle :
  - If $g^2+f^2-c>0$, then the radius of the circle will be real. Hence, in this case, it is possible to draw a circle on a plane.
  - If $g^2+f^2-c=0$, then the radius of the circle will be zero. Such a circle is known as point circle.

- If $g^2+f^2-c<0$, then the radius $\sqrt{g^2+f^2-c}$ of the circle will be an imaginary number. Hence, in this case, it is not possible to draw a circle.

❑ Special features of the general equation $x^2+y^2+2gx+2fy+c=0$ of the circle :

This equation has the following peculiarities :

- It is a quadratic equation in *x* and *y*.
- Here the co-efficient of x^2 = the co-efficient of y^2

In working out problems it is advisable to keep the co-efficient of x^2 and y^2 as unity.

- There is no term containing *xy, i.e.* the co-efficient of the term *xy* is zero.
- This equation contains three arbitrary constants. If we want to find the equation of a circle of which neither the centre nor the radius is known, we take the equation in the above form and determine the values of the constants *g, f, c* for the circle in question from the given geometrical conditions.

❑ Keeping in mind the above special features, we can say that the equation

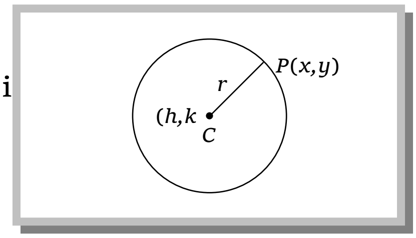
$ax^2+ay^2+2gx+2fy+c=0$ .....(i) also represents a circle.

This equation can also be written as $x^2+y^2+2\frac{g}{a}x+2\frac{f}{a}y+\frac{c}{a}=0$, dividing by $a\neq 0$.

Hence, the centre $=\left(\frac{-g}{a},\frac{-f}{a}\right)$ and radius $=\sqrt{\frac{g^2}{a^2}+\frac{f^2}{a^2}-\frac{c}{a}}$

(2) **Central form of equation of a circle :** The equation of a circle having centre (*h, k*) and radius *r* is

$$(x-h)^2+(y-k)^2=r^2$$



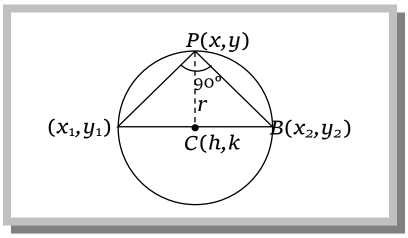
Note : ❑ If the centre is origin, then the equation of the circle is

$$x^2+y^2=r^2$$

❑ If *r* = 0, then circle is called point circle and its equation is $(x-h)^2+(y-k)^2=0$

(3) **Concentric circle :** Two circles having the same centre *C* (*h, k*) but different radii r_1 and r_2 respectively are called concentric circles. Thus the circles $(x-h)^2+(y-k)^2=r_1^2$ and $(x-h)^2+(y-k)^2=r_2^2$, $r_1\neq r_2$ are concentric circles. Therefore, the equations of concentric circles differ only in constant terms.

(4) **Circle on a given diameter :** The equation of the circle drawn on the straight line joining two given points (x_1,y_1) and (x_2,y_2) as diameter is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.



Note : ❑ If the coordinates of the end points of a diameter of a circle are given, we can also find the equation of the circle by finding the coordinates of the centre and radius. The centre is the mid-point of the diameter and radius is half of the length of the diameter.

(5) **Parametric coordinates**

(i) The parametric coordinates of any point on the circle $(x-h)^2+(y-k)^2=r^2$ are given by $(h+r\cos\theta, k+r\sin\theta)$, $(0\le\theta<2\pi)$

In particular, co-ordinates of any point on the circle $x^2+y^2=r^2$ are $(r\cos\theta, r\sin\theta)$, $(0\le\theta<2\pi)$

(ii) The parametric co-ordinates of any point on the circle $x^2+y^2+2gx+2fy+c=0$ are

$$x=-g+\sqrt{(g^2+f^2-c)}\cos\theta \text{ and } y=-f+\sqrt{(g^2+f^2-c)}\sin\theta, \quad (0\le\theta<2\pi)$$

(6) **Equation of a circle under given conditions:** The general equation of circle, *i.e.*, $x^2+y^2+2gx+2fy+c=0$ contains three independent constants *g, f* and *c*. Hence for determining the equation of a circle, three conditions are required.

(i) The equation of the circle through three non-collinear points $A(x_1,y_1), B(x_2,y_2), C(x_3,y_3)$:

Let the equation of circle be $x^2+y^2+2gx+2fy+c=0$ .....(i)

If three points (x_1,y_1), (x_2,y_2), (x_3,y_3) lie on the circle (i), their co-ordinates must satisfy its equation. Hence solving equations $x_1^2+y_1^2+2gx_1+2fy_1+c=0$ .....(ii)

$$x_2^2+y_2^2+2gx_2+2fy_2+c=0 \quad .....(iii)$$

$$x_3^2+y_3^2+2gx_3+2fy_3+c=0 \quad .....(iv)$$

g, f, c are obtained from (ii), (iii) and (iv). Then to find the circle (i).

Alternative method

(1) The equation of the circle through three non-collinear points $A(x_1,y_1), B(x_2,y_2), C(x_3,y_3)$ is

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix}=0$$

(2) From given three points taking any two as extremities of diameter of a circle $S = 0$ and equation of straight line passing through these two points is $L = 0$. Then required equation of circle is $S+\lambda L=0$, where λ is a parameter which can be found out by putting third point in the equation.

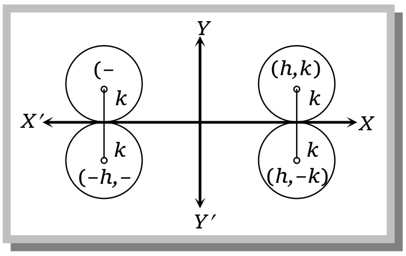
Note : ❑ **Cyclic quadrilateral :** If all the four vertices of a quadrilateral lie on a circle, then the quadrilateral is called a cyclic quadrilateral. The four vertices are said to be concylic.

4.3 Equation of a Circle in Some special cases

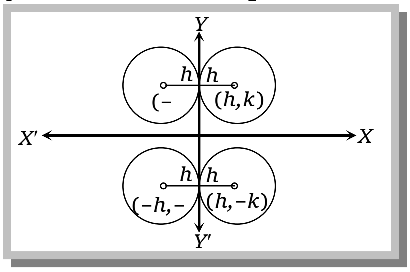
(1) If centre of the circle is (h,k) and it passes through origin then its equation is

$$(x-h)^2+(y-k)^2=h^2+k^2 \Rightarrow x^2+y^2-2hx-2ky=0$$

(2) If the circle touches x axis then its equation is (Four cases) $(x \pm h)^2 + (y \pm k)^2 = k^2$

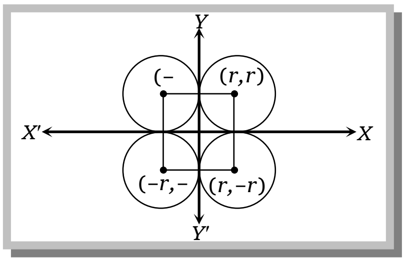


(3) If the circle touches y axis then its equation is (Four cases) $(x \pm h)^2 + (y \pm k)^2 = h^2$



(4) If the circle touches both the axes then its equation is (Four

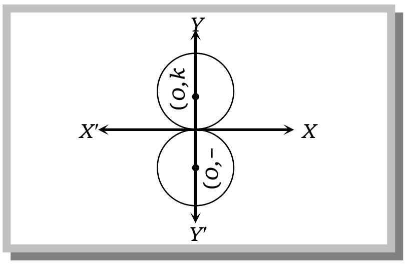
$(x \pm r)^2 + (y \pm r)^2 = r^2$



(5) If the circle touches x- axis at origin (Two cases)

$x^2 + (y \pm k)^2 = k^2$

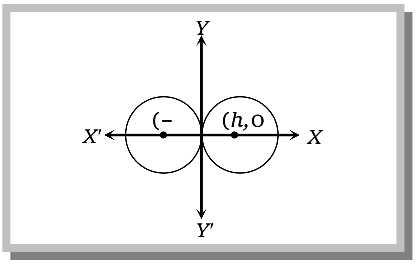
$\Rightarrow x^2 + y^2 \pm 2ky = 0$



(6) If the circle touches y-axis at origin (Two cases)

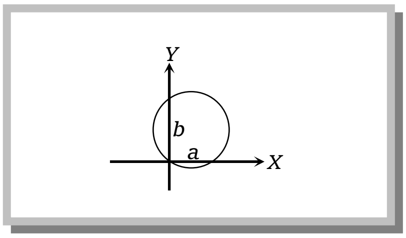
$(x \pm h)^2 + y^2 = h^2$

$\Rightarrow x^2 + y^2 \pm 2xh = 0$



(7) If the circle passes through origin and cut intercepts of a and b on axes, the equation of circle is (Four cases)

$x^2 + y^2 - ax - by = 0$ and centre is $(a/2, b/2)$



Note : ❑ **Circumcircle of a triangle :** If we are given sides of a triangle, then first we should find vertices then we can find the equation of the circle using general form.

Alternate : If equation of the sides are $L_1 = 0, L_2 = 0$ and $L_3 = 0$, then equation of circle is $(L_1.L_2)+\lambda(L_2.L_3)+\mu(L_3.L_1)=0$, where λ and μ are the constant which can be found out by the conditions, coefficient of x^2 = coefficient of y^2 and coefficient of xy = 0

❑ If the triangle is right angled then its hypotenuse is the diameter of the circle. So using diameter form we can find the equation.

❑ **Circumcircle of a square or a rectangle** : Diagonals of the square and rectangle will be diameters of the circumcircle. Hence finding the vertices of a diagonal, we can easily determine the required equation.

Alternate : If sides of a quadrilateral are $L_1 = 0, L_2 = 0, L_3 = 0$ and $L_4 = 0$. Then equation of circle is $L_1L_3 + \lambda L_2L_4 = 0,$ where λ is a constant which can be obtained by the condition of circle.

❑ If a circle is passing through origin then constant term is absent *i.e.* $x^2+y^2+2gx+2fy=0$

❑ If the circle $x^2+y^2+2gx+2fy+c=0$ touches X-axis, then $-f=\sqrt{g^2+f^2-c}$ or $g^2=c$

❑ If the circle $x^2+y^2+2gx+2fy+c=0$ touches Y-axis, then $-g=\sqrt{g^2+f^2-c}$ or $f^2=c$

❑ If the circle $x^2+y^2+2gx+2fy+c=0$ touches both axes, then $-g=-f=\sqrt{g^2+f^2-c}$ or $g^2=f^2=c$

Example: 1 A point P moves in such a way that the ratio of its distances from two coplanar points is always fixed number $(\neq 1)$. Then its locus is **[IIT 1970]**

(a) Straight line (b) Circle (c) Parabola (d) A pair of straight lines

Solution: (b) Let two coplanar points are (0, 0) and (a, 0) and coordinates of point P is (x, y).

Under given conditions, we get $\frac{\sqrt{x^2+y^2}}{\sqrt{(x-a)^2+y^2}}=\lambda$ (where λ is any number and $\lambda \neq 1$)

$\Rightarrow x^2+y^2=\lambda^2[(x-a)^2+y^2] \Rightarrow x^2+y^2+\left(\frac{\lambda^2}{\lambda^2-1}\right)(a^2-2ax)=0$, which is equation of a circle.

Example : 2 The lines $2x-3y=5$ and $3x-4y=7$ are the diameters of a circle of area 154 square units. The equation of the circle is

[IIT 1989; AIEEE 2003; DCE 2001]

(a) $x^2+y^2+2x-2y=62$ (b) $x^2+y^2-2x+2y=47$

(c) $x^2+y^2+2x-2y=47$ (d) $x^2+y^2-2x+2y=62$

Solution : (b) Centre of circle = Point of intersection of diameters,

On solving equations, $2x-3y=5$ and $3x-4y=7$, we get, $(x,y)=(1,-1)$

$\therefore$ Centre of circle = $(1,-1)$. Now area of circle = 154 $\Rightarrow \pi r^2=154 \Rightarrow r = 7$

Hence, the equation of required circle is $(x-1)^2+(y+1)^2=(7)^2 \Rightarrow x^2+y^2-2x+2y=47$.

Example : 3 The equation of a circle with origin as centre passing through the vertices of an equilateral triangle whose median is of length $3a$ is **[BIT Ranchi 1992; AIEEE 2002]**

(a) $x^2+y^2=9a^2$ (b) $x^2+y^2=16a^2$ (c) $x^2+y^2=a^2$ (d) None of these

Solution : (d) Since the triangle is equilateral, therefore centroid of the triangle is the same as the circumcentre and radius of the circum-circle $=\frac{2}{3}$ (median) $=\frac{2}{3}(3a)=2a$ [$\because$ Centroid divides median in ratio of 2 : 1]

Hence, the equation of the circum-circle whose centre is (0, 0) and radius $2a$ is $x^2+y^2=(2a)^2$

$\Rightarrow x^2+y^2=4a^2$

Example : 4 A circle of radius 5 units touches both the axes and lies in first quadrant. If the circle makes one complete roll on x-axis along the positive direction of x-axis, then its equation in the new position is

(a) $x^2+y^2+20\pi x-10y+100\pi^2=0$ (b) $x^2+y^2+20\pi x+10y+100\pi^2=0$

(c) $x^2+y^2-20\pi x-10y+100\pi^2=0$ (d) None of these

Solution : (d) The x-coordinate of the new position of the circle is 5 + circumferrence of the first circle $=5+10\pi$

The y-coordinate is 5 and the radius is also 5.

Hence, the equation of the circle in the new position is $(x-5-10\pi)^2+(y-5)^2=(5)^2$

$\Rightarrow x^2+25+100\pi^2-10x+100\pi-20\pi x+y^2+25-10y=25$

$\Rightarrow x^2+y^2-20\pi x-10x-10y+100\pi^2+100\pi+25=0$

Example : 5 The abscissae of A and B are the roots of the equation $x^2+2ax-b^2=0$ and their ordinates are the roots of the equation $y^2+2by-q^2=0$. The equation of the circle with AB as diameter is **[IIT 1984]**

(a) $x^2+y^2+2ax+2by-b^2-q^2=0$ (b) $x^2+y^2+2ax+by-b^2-q^2=0$

(c) $x^2+y^2+2ax+2by+b^2+q^2=0$ (d) None of these

Solution : (a) Let x_1, x_2 and y_1, y_2 be roots of $x^2+2ax-b^2=0$ and $y^2+2by-q^2=0$ respectively.

Then, $x_1+x_2=-2a,\ x_1x_2=-b^2$ and $y_1+y_2=-2b,\ y_1y_2=-q^2$

The equation of the circle with $A(x_1,y_1)$ and $B(x_2,y_2)$ as the end points of diameter is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

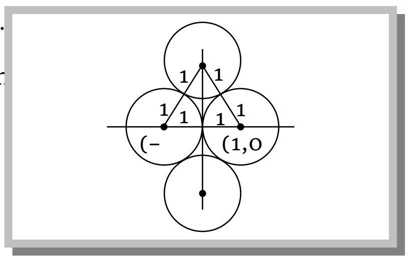
$x^2+y^2-x(x_1+x_2)-y(y_1+y_2)+x_1x_2+y_1y_2=0$; $x^2+y^2+2ax+2by-b^2-q^2=0$

Example : 6 The equation of a circle of radius 1 touching the circles $x^2+y^2-2|x|=0$ is

(a) $x^2+y^2+2\sqrt{3}x-2=0$ (b) $x^2+y^2-2\sqrt{3}y+2=0$

(c) $x^2+y^2+2\sqrt{3}y+2=0$ (d) $x^2+y^2+2\sqrt{3}x+2=0$

Solution : (b,c) The given circles are $x^2+y^2-2x=0,\ x>0,$ and $x^2+y^2+2x=0,\ x<0.$



From the figure, the centres of the required circles will be $(0, \sqrt{3})$ ar

$\therefore$ The equations of the circles are $(x-0)^2+(y\mp\sqrt{3})^2=1^2$.

$\Rightarrow x^2+y^2+3\mp 2\sqrt{3}y=1$

$\Rightarrow x^2+y^2\mp 2\sqrt{3}y+2=0$

Example : 7 If the line $x+2by+7=0$ is a diameter of the circle $x^2+y^2-6x+2y=0$, then $b=$ **[MP PET 1991]**

(a) 3 (b) – 5 (c) – 1 (d) 5

Solution : (d) Here the centre of circle (3, –1) must lie on the line $x+2by+7=0$. Therefore, $3-2b+7=0 \Rightarrow b=5$

Example : 8 The centre of the circle $r^2=2-4r\cos\theta+6r\sin\theta$ is

(a) (2, 3) (b) (– 2, 3) (c) (– 2, – 3) (d) (2, – 3)

Solution : (b) Let $r\cos\theta = x$ and $r\sin\theta = y$

Squaring and adding, we get $r^2=x^2+y^2$.

Putting these values in given equation, $x^2+y^2=2-4x+6y \Rightarrow x^2+y^2+4x-6y-2=0$

Hence, centre of the circle = (–2, 3)

Example : 9 The number of integral values of λ for which $x^2+y^2+\lambda x+(1-\lambda)y+5=0$ is the equation of a circle whose radius cannot exceed 5, is

(a) 14 (b) 18 (c) 16 (d) None of these

Solution : (c) Centre of circle $=\left(-\frac{\lambda}{2},-\frac{(1-\lambda)}{2}\right)$; Radius of circle $=\sqrt{\left(\frac{\lambda}{2}\right)^2+\left(\frac{1-\lambda}{2}\right)^2-5}\le 5$

$\Rightarrow 2\lambda^2-2\lambda-119\le 0$, $\therefore \frac{1-\sqrt{239}}{2}\le\lambda\le\frac{1+\sqrt{239}}{2}$

$\Rightarrow -7.2\le\lambda\le 8.2$ (Nearly). $\therefore \lambda=-7,-6,$ ....,7, 8. Hence number of integral values of λ is 16.

Example : 10 Let $f(x,y)=0$ be the equation of a circle. If $f(0,\lambda)=0$ has equal roots $\lambda=2,2$ and $f(\lambda,0)=0$ has roots $\lambda=\frac{4}{5},5,$ then the centre of the circle is

(a) $\left(2,\frac{29}{10}\right)$ (b) $\left(\frac{29}{10},2\right)$ (c) $\left(-2,\frac{29}{10}\right)$ (d) None of these

Solution : (b) $\because f(x,y)\equiv x^2+y^2+2gx+2fy+c=0$

Now, $f(0,\lambda)\equiv\lambda^2+2f\lambda+c=0$ and its roots are 2, 2. $\therefore 2+2=-2f,\ 2\times 2=c,$ *i.e.* $f=-2,\ c=4$

$f(\lambda,0)\equiv\lambda^2+2g\lambda+c=0,$ and its roots are $\frac{4}{5}$, 5.

$\therefore \frac{4}{5}+5=-2g,\ \frac{4}{5}\times 5=c,$ *i.e.*, $g=\frac{-29}{10},\ c=4$. Hence, centre of the circle $=(-g,-f)=\left(\frac{29}{10},2\right)$.

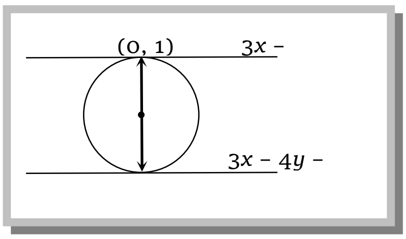
Example : 11 If the lines $3x-4y+4=0$ and $6x-8y-7=0$ are tangents to a circle, then the radius of the circle is

[IIT 1984; MP PET 1994, 2002; Rajasthan PET 1995, 97; Kurukshetra CEE 1998]

(a) 3/2 (b) 3/4 (c) 1/10 (d) 1/20

Solution : (b) Since both tangents are parallel to each other. The diameter of the circle is perpendicular distance between the parallel lines (tangents) $3x-4y+4=0$ and $3x-4y-\frac{7}{2}=0$ and so it is equal to $\frac{4}{\sqrt{9+16}}+\frac{7/2}{\sqrt{9+16}}=\frac{3}{2}$.

Hence radius of circle is $\frac{3}{4}$.



Alternative method : Perpendicular distance $=\frac{3(0)-4(1)-7/2}{5}=\frac{3}{2}$,

i.e., Diameter $=\frac{3}{2}$

Hence radius of circle is $\frac{3}{4}$.

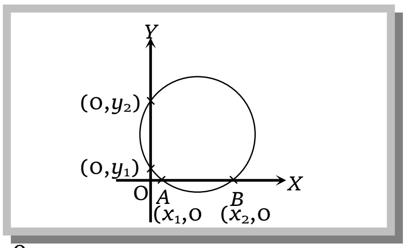
4.4 Intercepts on the Axes

The lengths of intercepts made by the circle $x^2+y^2+2gx+2fy+c=0$ with X and Y axes are $2\sqrt{g^2-c}$ and $2\sqrt{f^2-c}$ respectively.

Let the equation of circle be $x^2+y^2+2gx+2fy+c=0$ ......(i)

Length of intercepts on x-axis and y-axis are $|AB|=|x_2-x_1|$ and $|CD|=|y_2-y_1|$ respectively.

The circle intersects the x-axis, when y = 0, then $x^2+2gx+c=0$



Since the circle intersects the x-axis at $A(x_1,0)$ and $B(x_2,0)$.

Then $x_1+x_2=-2g,\ x_1x_2=c$

$\therefore\ |AB|=|x_2-x_1|=\sqrt{(x_2+x_1)^2-4x_1x_2}=2\sqrt{(g^2-c)}$

As the circle intersects the y-axis, when x = 0, then $y^2+2fy+c=0$

Since the circle intersects the y-axis at C (0, y_1) and D (0, y_2), then $y_1+y_2=-2f,\ y_1y_2=c$

$\therefore\ |CD|=|y_2-y_1|=\sqrt{(y_2+y_1)^2-4y_2y_1}=2\sqrt{(f^2-c)}$.

Note :

- If $g^2>c$, then the roots of the equation $x^2+2gx+c=0$ are real and distinct, so the circle $x^2+y^2+2gx+2fy+c=0$ meets the x-axis in two real and distinct points and the length of the intercept on x-axis is $2\sqrt{g^2-c}$.
- If $g^2=c$, then the roots of the equation $x^2+2gx+c=0$ are real and equal, so the circle touches x-axis and the intercept on x-axis is zero.
- If $g^2<c$, then the roots of the equation $x^2+2gx+c=0$ are imaginary, so the circle $x^2+y^2+2gx+2fy+c=0$ does not meet x-axis in real points.
- Similarly, the circle $x^2+y^2+2gx+2fy+c=0$ cuts the y-axis in real and distinct points, touches or does not meet in real points according as $f^2 >,=$ or $<c$.

4.5 Position of a point with respect to a Circle

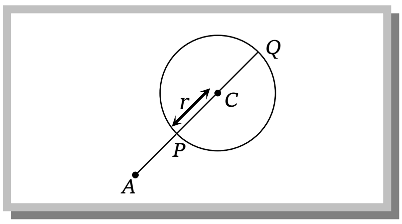
A point (x_1,y_1) lies outside, on or inside a circle $S\equiv x^2+y^2+2gx+2fy+c=0$ according as $S_1\equiv x_1^2+y_1^2+2gx_1+2fy_1+c$ is positive, zero or negative *i.e.,*

$S_1>0\Rightarrow$ Point is outside the circle.

$S_1=0\Rightarrow$ Point is on the circle.

$S_1<0\Rightarrow$ Point is inside the circle.

(1) **The least and greatest distance of a point from a circle :** Let S = 0 be a circle and $A(x_1,y_1)$ be a point. If the diameter of the circle is passing through the circle at P



$AP = AC - r =$ least distance ; $AQ = AC + r =$ greatest distance

where 'r' is the radius and C is the centre of the circle.

Example : 12 The number of points with integral coordinates that are interior to the circle $x^2 + y^2 = 16$ is

(a) 43 (b) 49 (c) 45 (d) 51

Solution : (c) The number of points is equal to the number of integral solutions (x, y) such that $x^2 + y^2 < 16$.

So, x, y are integers such that $-3 \le x \le 3,\ -3 \le y \le 3$ satisfying the inequation $x^2 + y^2 < 16$. The number of selections of values of x is 7, namely −3, −2, −1, 0, 1, 2, 3. The same is true for y. So the number of ordered pairs (x, y) is 7×7. But (3, 3), (3, −3), (−3, 3), (−3, −3) are rejected because they do not satisfy the inequation $x^2 + y^2 < 16$.

So the number of points is 45.

Example : 13 The range of values of a for which the point $(a, 4)$ is outside the circles $x^2 + y^2 + 10x = 0$ and $x^2 + y^2 - 12x + 20 = 0$ is

(a) $(-\infty, -8) \cup (-2, 6) \cup (6, +\infty)$ (b) (− 8, − 2)

(c) $(-\infty, -8) \cup (-2, +\infty)$ (d) None of these

Solution : (a) For circle, $x^2 + y^2 + 10x = 0$;

$a^2 + (4)^2 + 10a > 0 \Rightarrow a^2 + 10a + 16 > 0 \Rightarrow (a+8)(a+2) > 0 \Rightarrow a < -8$ or $a > -2$ .....(i)

For circle, $x^2 + y^2 - 12x + 20 = 0$; $a^2 + (4)^2 - 12a + 20 > 0 \Rightarrow a^2 - 12a + 36 > 0$

$\Rightarrow (a-6)^2 > 0 \Rightarrow a \in R \sim \{6\}$ .....(ii)

Taking common values from (i) and (ii), $a \in (-\infty, -8) \cup (-2, 6) \cup (6, +\infty)$.

4.6 Intersection of a Line and a Circle

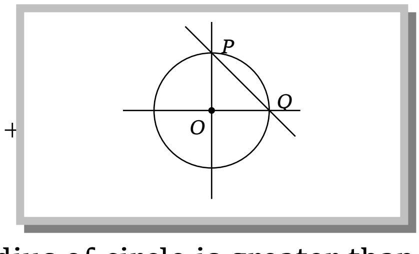
Let the equation of the circle be $x^2 + y^2 = a^2$ .....(i)

and the equation of the line be $y = mx + c$ .....(ii)

From (i) and (ii), $x^2 + (mx + c)^2 = a^2$ or $(1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0$ .....(iii)

Case I: When points of intersection are real and distinct. In this case (iii) has two distinct roots.

$\therefore\ B^2 - 4AC > 0$ or $4m^2c^2 - 4(1 + m^2)(c^2 - a^2) > 0$ or $a^2 > \dfrac{c^2}{1 + m^2}$



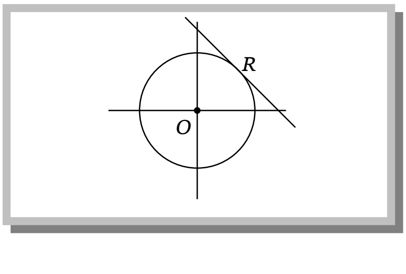
or $a > \dfrac{|c|}{\sqrt{(1 + m^2)}}$ = length of perpendicular from (0, 0) to $y = mx +$

$\Rightarrow$ a > length of perpendicular from (0, 0) to $y = mx + c$

Thus, a line intersects a given circle at two distinct points if radius of circle is greater than the length of perpendicular from centre of the circle to the line.

Case II: When the points of intersection are coincident in this case (iii) has two equal roots.

$\therefore\ B^2 - 4AC = 0 \Rightarrow 4m^2c^2 - 4(1 + m^2)(c^2 - a^2) = 0$



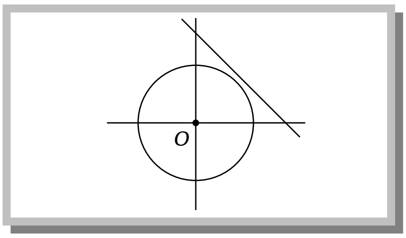
$\therefore\ a^2 = \dfrac{c^2}{1 + m^2}$ or $a = \dfrac{|c|}{\sqrt{(1 + m^2)}}$

a = length of perpendicular from the point (0, 0) to $y = mx + c$.

Thus, a line touches the circle if radius of circle is equal to the length of perpendicular from centre of the circle to the line.

Case III: When the points of intersection are imaginary. In this case (iii) has imaginary roots.

$\therefore \quad B^2 - 4AC < 0 \Rightarrow 4m^2c^2 - 4(1+m^2)(c^2 - a^2) < 0, \quad \therefore a^2 < \dfrac{c^2}{1+m^2}$



or $\quad a < \dfrac{|c|}{\sqrt{(1+m^2)}}$ = length of perpendicular from (0, 0) to $y = mx$

$\Rightarrow$ a < length of perpendicular from (0, 0) to $y = mx + c$

Thus, a line does not intersect a circle if the radius of circle is less than the length of perpendicular from centre of the circle to the line.

(1) **The length of the intercept cut off from a line by a circle :** The length of the intercept cut off from the line $y = mx + c$ by the circle $x^2 + y^2 = a^2$ is $2\sqrt{\dfrac{a^2(1+m^2) - c^2}{1+m^2}}$

(2) **Condition of tangency :** A line L = 0 touches the circle S = 0, if length of perpendicular drawn from the centre of the circle to the line is equal to radius of the circle *i.e.* $p = r$. This is the condition of tangency for the line L = 0.

Circle $x^2 + y^2 = a^2$ will touch the line $y = mx + c$ if $c = \pm a\sqrt{1+m^2}$

Again, (i) If $a^2(1+m^2) - c^2 > 0$ line will meet the circle at real and different points.

(ii) If $c^2 = a^2(1+m^2)$ line will touch the circle.

(iii) If $a^2(1+m^2) - c^2 < 0$ line will meet circle at two imaginary points.

Example : 14 If the straight line $y = mx$ is outside the circle $x^2 + y^2 - 20y + 90 = 0$, then **[Roorkee 1999]**

(a) $m > 3$ (b) $m < 3$ (c) $|m| > 3$ (d) $|m| < 3$

Solution : (d) If the straight line $y = mx$ is outside the given circle then perpendicular distance of line from centre of circle > radius of circle

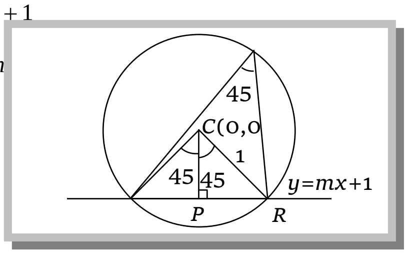
$\dfrac{10}{\sqrt{1+m^2}} > \sqrt{10} \quad \Rightarrow \quad (1+m^2) < 10 \Rightarrow m^2 < 9 \quad \Rightarrow |m| < 3$

Example : 15 If the chord $y = mx + 1$ of the circle $x^2 + y^2 = 1$ subtends an angle of measure 45° at the major segment of the circle then value of m is **[AIEEE 2002]**

(a) 2 (b) – 2 (c) ± 1 (d) None of these

Solution : (c) Given circle is $x^2 + y^2 = 1$, C (0,0) and radius = 1 and chord is $y = mx + 1$

$\cos 45^o = \dfrac{CP}{CR}$; CP = Perpendicular distance from (0, 0) to chord $y = mx + 1$

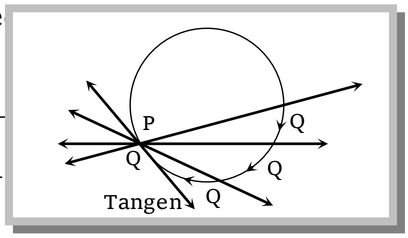


$CP = \dfrac{1}{\sqrt{m^2+1}}$ (CR = radius =1)

$\cos 45^o = \dfrac{\frac{1}{\sqrt{m^2+1}}}{1} \Rightarrow \dfrac{1}{\sqrt{2}} = \dfrac{1}{\sqrt{m^2+1}} \Rightarrow m^2 + 1 = 2 \Rightarrow m = \pm 1.$

4.7 Tangent to a Circle at a given Point

The limiting position of the line PQ, when Q moves towards P and ultimately coincides with P, is called the tangent to the circle at the point P. The point P is calle



(1) **Point form**

(i) The equation of tangent at (x_1, y_1) to circle $x^2 + y^2 = a^2$ is $xx_1 +$

(ii) The equation of tangent at (x_1, y_1) to circle $x^2 + y^2 + 2gx + 2fy +$

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Note :
- For equation of tangent of circle at (x_1, y_1), substitute xx_1 for x^2, yy_1 for $y^2, \frac{x + x_1}{2}$ for $x, \frac{y + y_1}{2}$ for y and $\frac{xy_1 + x_1 y}{2}$ for xy and keep the constant as such.
- This method of tangent at (x_1, y_1) is applied any conics of second degree. *i.e.*, equation of tangent of

  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at (x_1, y_1)

  is $axx_1 + h(xy_1 + x_1 y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$

(2) **Parametric form :** Since parametric co-ordinates of circle $x^2 + y^2 = a^2$ is $(a\cos\theta, a\sin\theta)$, then equation of tangent at $(a\cos\theta, a\sin\theta)$ is $x.a\cos\theta + y.a\sin\theta = a^2$ or $x\cos\theta + y\sin\theta = a$.

(3) **Slope form :** Let $y = mx + c$ is the tangent of the circle $x^2 + y^2 = a^2$.

$\therefore$ Length of perpendicular from centre of circle (0, 0) on line $(y = mx + c)$ = radius of circle

$\therefore \quad \frac{|c|}{\sqrt{1 + m^2}} = a \Rightarrow c = \pm a\sqrt{1 + m^2}$

Substituting this value of c in $y = mx + c$, we get $y = mx \pm a\sqrt{1 + m^2}$. Which are the required equations of tangents.

Note :
- The reason why there are two equations $y = mx \pm a\sqrt{1 + m^2}$ is that there are two tangents, both are parallel and at the ends of a diameter.
- The line $ax + by + c = 0$ is a tangent to the circle $x^2 + y^2 = r^2$ if and only if $c^2 = r^2(a^2 + b^2)$.
- The condition that the line $lx + my + n = 0$ touches the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(lg + mf - n)^2 = (l^2 + m^2)(g^2 + f^2 - c)$.
- Equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of slope is

  $y = mx + mg - f \pm \sqrt{(g^2 + f^2 - c)}\ \sqrt{(1 + m^2)}$

(4) **Point of contact :** If circle be $x^2 + y^2 = a^2$ and tangent in terms of slope be $y = mx \pm a\sqrt{(1 + m^2)}$,

Solving $x^2+y^2=a^2$ and $y=mx\pm a\sqrt{(1+m^2)}$ simultaneously, we get $x=\pm\frac{am}{\sqrt{(1+m^2)}}$ and

$y=\mp\frac{a}{\sqrt{(1+m^2)}}$

Thus, the co-ordinates of the points of contact are $\left(\pm\frac{am}{\sqrt{(1+m^2)}},\mp\frac{a}{\sqrt{(1+m^2)}}\right)$

Alternative method : Let point of contact be (x_1,y_1) then tangent at (x_1,y_1) of $x^2+y^2=a^2$ is $xx_1+yy_1=a^2$. Since $xx_1+yy_1=a^2$ and $y=mx\pm a\sqrt{(1+m^2)}$ are identical , $\therefore\ \frac{x_1}{m}=\frac{y_1}{-1}=\frac{+a^2}{\pm a\sqrt{1+m^2}}$

$\therefore\quad x_1=\pm\frac{am}{\sqrt{(1+m^2)}}$ and $y_1=\mp\frac{a}{\sqrt{(1+m^2)}}$

Thus, the co-ordinates of the points of contact are $\left(\pm\frac{am}{\sqrt{(1+m^2)}},\mp\frac{a}{\sqrt{(1+m^2)}}\right)$

Note :

- ❑ If the line $y = mx + c$ is the tangent to the circle $x^2+y^2=r^2$ then point of contact is given by $\left(-\frac{mr^2}{c},\frac{r^2}{c}\right)$
- ❑ If the line $ax+by+c = 0$ is the tangent to the circle $x^2+y^2=r^2$ then point of contact is given by $\left(-\frac{ar^2}{c},-\frac{br^2}{c}\right)$

Example : 16 The equations to the tangents to the circle $x^2+y^2-6x+4y=12$ which are parallel to the straight line 4x+3y+5=0, are **[ISM Dhanbad 1973, MP PET 1991]**

(a) $3x-4y-19=0,\ \ 3x-4y+31=0$ (b) $4x+3y-19=0,\ \ 4x+3y+31=0$

(c) $4x+3y+19=0,\ \ 4x+3y-31=0$ (d) $3x-4y+19=0,\ \ 3x-4y+31=0$

Solution : (c) Let equation of tangent be $4x+3y+k=0,$ then $\sqrt{9+4+12}=\left|\frac{4(3)+3(-2)+k}{\sqrt{16+9}}\right|$ $\Rightarrow 6+k=\pm 25\Rightarrow k=19$ and – 31

Hence the equations of tangents are $4x+3y+19=0$ and $4x+3y-31=0$

Example : 17 The equations of any tangents to the circle $x^2+y^2-2x+4y-4=0$ is

(a) $y=m(x-1)+3\sqrt{1+m^2}-2$ (b) $y=mx+3\sqrt{1+m^2}$

(c) $y=mx+3\sqrt{1+m^2}-2$ (d) None of these

Solution : (a) Equation of circle is $(x-1)^2+(y+2)^2=3^2$.

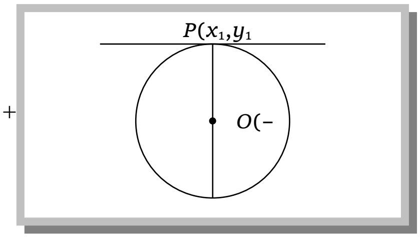
As any tangent to $x^2+y^2=3^2$ is given by $y=mx+3\sqrt{1+m^2}$

Any tangent to the given circle will be $y+2=m(x-1)+3\sqrt{1+m^2}\ \Rightarrow\ y=m(x-1)+3\sqrt{1+m^2}-2$

Example : 18 If a circle, whose centre is (–1, 1) touches the straight line $x + 2y + 12 = 0,$ then the coordinates of the point of contact are
[MP PET 1998]

(a) $\left(-\frac{7}{2}, -4\right)$ (b) $\left(-\frac{18}{5}, -\frac{21}{5}\right)$ (c) (2, –7) (d) (– 2, – 5)

Solution : (b) Let point of contact be $P(x_1, y_1)$.



This point lies on the given line , $\therefore x_1 + 2y_1 = -12$

Gradient of $OP = m_1 = \frac{y_1 - 1}{x_1 + 1}$, Gradient of $x + 2y +$

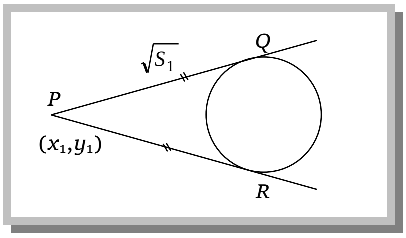
Both are perpendicular, $\therefore\ m_1 m_2 = -1$

$\Rightarrow \left(\frac{y_1 - 1}{x_1 + 1}\right)\left(\frac{-1}{2}\right) = -1 \Rightarrow y_1 - 1 = 2x_1 + 2 \Rightarrow 2x_1 - y_1 = -3$ ..... (ii)

On solving the equation (i) and (ii), $(x_1, y_1) = \left(\frac{-18}{5}, \frac{-21}{5}\right)$

4.8 Length of Tangent

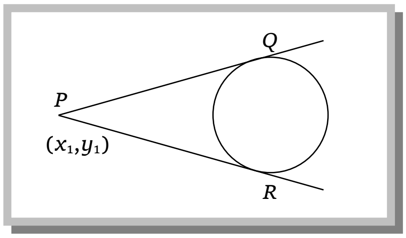
From any point, say $P(x_1, y_1)$ two tangents can be drawn to a circle which are real, coincident or imaginary according as P lies outside, on or inside the circle.



Le PQ and PR be two tangents drawn from $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Then $PQ = PR$ is called the length of tangent drawn from point P and is given by $PQ = PR = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{S_1}$

4.9 Pair of Tangents

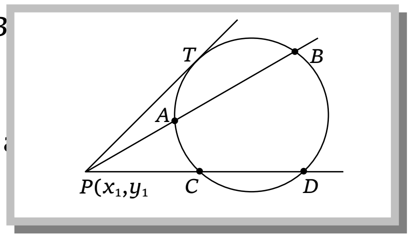
From a given point $P(x_1, y_1)$ two tangents PQ and PR can be drawn to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$. Their combined equation is $SS_1 = T^2$.



Where $S = 0$ is the equation of circle, $T = 0$ is the equation of tangent at (x_1, y_1) and S_1 is obtained by replacing x by x_1 and y by y_1 in S.

4.10 Power of Point with respect to a Circle

Let $P(x_1, y_1)$ be a point outside the circle and PAB and PCD drawn two secants. The power of $P(x_1, y_1)$ with respect to $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is equal to PA . PB



$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \Rightarrow S_1 = 0$

$\therefore$ Power remains constant for the circle *i.e.,* independent of A

$\therefore\ PA.PB = PC.CD = (PT)^2 = S_1 = (\sqrt{S_1})^2$

$\therefore\ PA.PB = (\sqrt{S_1})^2 =$ square of the length of tangent.

Note : ❑ If P is outside, inside or on the circle then PA . PB is +*ve*, –*ve* or zero respectively.

Important Tips

- *The length of the tangent drawn from any point on the circle* $x^2+y^2+2gx+2fy+c_1=0$ *to the circle* $x^2+y^2+2gx+2fy+c=0$ *is* $\sqrt{c-c_1}$.
- *If two tangents drawn from the origin to the circle* $x^2+y^2+2gx+2fy+c=0$ *are perpendicular to each other, then* $g^2+f^2=2c$.
- *If the tangent to the circle* $x^2+y^2=r^2$ *at the point* (a, b) *meets the coordinate axes at the points A and B and O is the origin, then the area of the triangle OAB is* $\frac{r^4}{2ab^-}$.
- *If* θ *is the angle subtended at* $P(x_1, y_1)$ *by the circle* $S=x^2+y^2+2gx+2fy+c=0$, *then* $\cot\frac{\theta}{2}=\frac{\sqrt{S_1}}{\sqrt{g^2+f^2-c}}$
- *The angle between the tangents from* (α, β) *to the circle* $x^2+y^2=a^2$ *is* $2\tan^{-1}\left(\frac{a}{\sqrt{\alpha^2+\beta^2-a^2}}\right)$.
- *If OA and OB are the tangents from the origin to the circle* $x^2+y^2+2gx+2fy+c=0$ *and C is the centre of the circle, then the area of the quadrilateral OACB is* $\sqrt{c(g^2+f^2-c)}$.

Example : 19 If the distances from the origin to the centres of three circles $x^2+y^2+2\lambda_i x-c^2=0$ $(i=1,2,3)$ are in *G.P.* then the lengths of the tangents drawn to them from any point on the circle $x^2+y^2=c^2$ are in

(a) *A.P.* (b) *G.P.* (c) *H.P.* (d) None of these

Solution : (b) The centres of the given circles are $(-\lambda_i, 0)$ $(i=1,2,3)$

The distances from the origin to the centres are λ_i $(i=1,2,3)$. It is given that $\lambda_2^2=\lambda_1\lambda_3$.

Let $P(h,k)$ be any point on the circle $x^2+y^2=c^2$, then, $h^2+k^2=c^2$

Now, L_i = length of the tangent from (h, k) to $x^2+y^2+2\lambda_i x-c^2=0$

$$=\sqrt{h^2+k^2+2\lambda_i h-c^2}=\sqrt{c^2+2\lambda_i h-c^2}=\sqrt{2\lambda_i h} \quad [\because h^2+k^2=c^2 \text{ and } i=1,2,3]$$

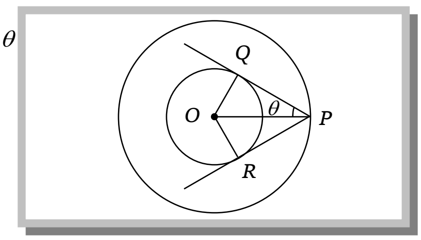
Therefore, $L_2^2=2\lambda_2 h=2h(\sqrt{\lambda_1\lambda_3})$ $[\because \lambda_2^2=\lambda_1\lambda_3]$

$=\sqrt{2h\lambda_1}\sqrt{2h\lambda_3}=L_1L_3$. Hence, L_1, L_2, L_3 are in *G.P.*

Example : 20 From a point on the circle $x^2+y^2=a^2$, two tangents are drawn to the circle $x^2+y^2=a^2\sin^2\alpha$. The angle between them is **[Rajasthan PET 2002]**

(a) α (b) $\frac{\alpha}{2}$ (c) 2α (d) None of these

Solution : (c) Let any point on the circle $x^2+y^2=a^2$ be $(a\cos t, a\sin t)$ and $\angle OPQ=\theta$



Now; PQ = length of tangent from P on the circle $x^2+y^2=a^2\sin^2\alpha$

$\therefore$ $PQ=\sqrt{a^2\cos^2 t+a^2\sin^2 t-a^2\sin^2\alpha}=a\cos\alpha$

$OQ =$ Radius of the circle $x^2 + y^2 = a^2 \sin^2 \alpha$

$OQ = a \sin \alpha$, $\therefore$ $\tan \theta = \dfrac{OQ}{PQ} = \tan \alpha \Rightarrow \theta = \alpha$; $\therefore$ Angle between tangents $\angle QPR = 2\alpha$.

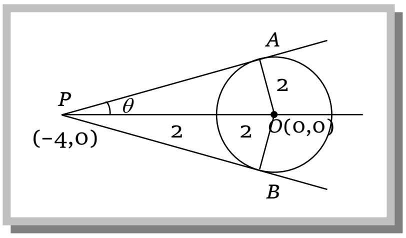
Alternative Method : We know that, angle between the tangent from (α, β) to the circle $x^2 + y^2 = a^2$ is $2 \tan^{-1}\left(\dfrac{a}{\sqrt{\alpha^2 + \beta^2 - a^2}}\right)$. Let point on the circle $x^2 + y^2 = a^2$ be $(a \cos t, a \sin t)$

Angle between tangent = $2 \tan^{-1}\left(\dfrac{a \sin \alpha}{\sqrt{a^2 \cos^2 t + a^2 \sin^2 t - a^2 \sin^2 \alpha}}\right) = 2 \tan^{-1}\left(\dfrac{a \sin \alpha}{a \cos \alpha}\right) = 2\alpha$

Example : 21 Two tangents to the circle $x^2 + y^2 = 4$ at the points *A* and *B* meet at *P* (– 4, 0). The area of quadrilateral *PAOB*, where *O* is the origin, is

(a) 4 (b) $6\sqrt{2}$ **(c)** $4\sqrt{3}$ (d) None of these

Solution : (c) Clearly, $\sin \theta = \dfrac{2}{4} = \dfrac{1}{2}$, $\therefore \theta = 30^o$. So area $(\Delta POA) = \dfrac{1}{2}.2.4.\sin 60^o$



$\therefore$ Area (quadrilateral *PAOB*) $= 2.\dfrac{1}{2}.2.4 \sin 60^o = 8.\dfrac{\sqrt{3}}{2} = 4\sqrt{3}.$

Trick : Area of quadrilateral $= r\sqrt{S_1} = 2\sqrt{12} = 4\sqrt{3}$

Example : 22 The angle between a pair of tangents drawn from a point *P* to the circle $x^2 + y^2 + 4x - 6y + 9 \sin^2 \alpha + 13 \cos^2 \alpha = 0$ is 2α. The equation of the locus of the point *P* is

[IIT 1996]

(a) $x^2 + y^2 + 4x - 6y + 4 = 0$ (b) $x^2 + y^2 + 4x - 6y - 9 = 0$

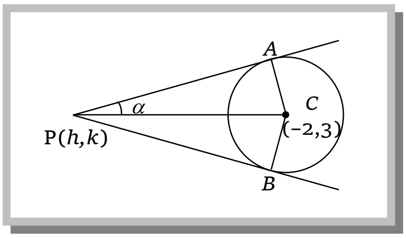
(c) $x^2 + y^2 + 4x - 6y - 4 = 0$ (d) $x^2 + y^2 + 4x - 6y + 9 = 0$

Solution : (d) The centre of the circle $x^2 + y^2 + 4x - 6y + 9 \sin^2 \alpha + 13 \cos^2 \alpha = 0$ is $C(-2, 3)$ and its radius is

$\sqrt{2^2 + (-3)^2 - 9 \sin^2 \alpha - 13 \cos^2 \alpha} = \sqrt{4 + 9 - 9 \sin^2 \alpha - 13 \cos^2 \alpha} = 2 \sin \alpha$

Let *P* (*h*, *k*) be any point on the locus. The $\angle APC = \alpha$. Also $\angle PAC = \pi / 2$ *i.e.* triangle *APC* is a right angle triangle.

Thus $\sin \alpha = \dfrac{AC}{PC} = \dfrac{2 \sin \alpha}{\sqrt{(h+2)^2 + (k-3)^2}}$



$\Rightarrow \sqrt{(h+2)^2 + (k-3)^2} = 2 \Rightarrow (h+2)^2 + (k-3)^2 = 4$

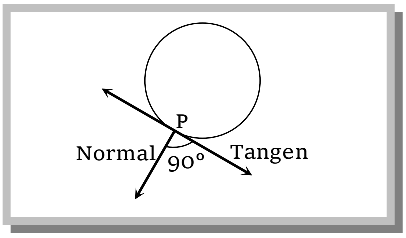
or $h^2 + k^2 + 4h - 6k + 9 = 0$

Thus the required equation of the locus is $x^2 + y^2 + 4x - 6y + 9 = 0$.

4.11 Normal to a Circle at a given Point

The normal of a circle at any point is a straight line, which is perpendicular to the tangent at the point and always passes through the centre of the circle.

(1) **Equation of normal:** The equation of normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at any point (x_1, y_1) is $y - y_1 = \dfrac{y_1 + f}{x_1 + g}(x - x_1)$ or $\dfrac{x - x_1}{x_1 + g} = \dfrac{y - y_1}{y_1 + f}$



Note : ❑ The equation of normal to the circle $x^2+y^2=a^2$ at any point (x_1,y_1) is $xy_1-x_1y=0$ or $\frac{x}{x_1}=\frac{y}{y_1}$

❑ The equation of any normal to the circle $x^2+y^2=a^2$ is $y=mx$ where m is the slope of normal.

❑ The equation of any normal to the circle $x^2+y^2+2gx+2fy+c=0$ is $y+f=m(x+g)$. where m is the slope of normal.

❑ If the line $y=mx+c$ is a normal to the circle with radius r and centre at (a, b) then $b=ma+c$.

(2) **Parametric form :** Since parametric co-ordinates of circle $x^2+y^2=a^2$ is $(a\cos\theta,\ a\sin\theta)$.

$\therefore$ Equation of normal at $(a\cos\theta,\ a\sin\theta)$ is $\frac{x}{a\cos\theta}=\frac{y}{a\sin\theta}$ or $\frac{x}{\cos\theta}=\frac{y}{\sin\theta}$

or $y=x\tan\theta$ or $y=mx$ where $m=\tan\theta$, which is slope form of normal.

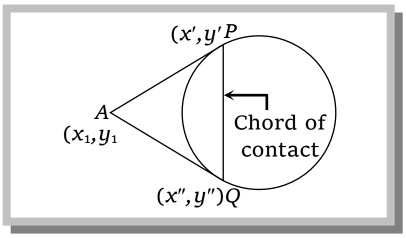
Example : 23 The line $lx+my+n=0$ is a normal to the circle $x^2+y^2+2gx+2fy+c=0$, if **[MP PET 1995]**

(a) $lg+mf-n=0$ (b) $lg+mf+n=0$ (c) $lg-mf-n=0$ (d) $lg-mf+n=0$

Solution : (a) Since normal always passes through centre of circle, therefore $(-g,-f)$ must lie on $lx+my+n=0$. Hence, $lg+mf-n=0$

4.12 Chord of Contact of Tangents

(1) **Chord of contact :** The chord joining the points of contact of the two tangents to a conic drawn from a given point, outside it, is called the chord of contact of tangents.



(2) **Equation of chord of contact :** The equation of the chord of contact of tangents drawn from a point (x_1,y_1) to the circle $x^2+y^2=a^2$ is $xx_1+yy_1=a^2$.

Equation of chord of contact at (x_1,y_1) to the circle $x^2+y^2+2gx+2fy+c=0$ is $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$.

It is clear from the above that the equation to the chord of contact coincides with the equation of the tangent, if point (x_1,y_1) lies on the circle.

The length of chord of contact $=2\sqrt{r^2-p^2}$; (p being length of perpendicular from centre to the chord)

Area of ΔAPQ is given by $\frac{a(x_1^2+y_1^2-a^2)^{3/2}}{x_1^2+y_1^2}$.

(3) **Equation of the chord bisected at a given point :** The equation of the chord of the circle $S\equiv x^2+y^2+2gx+2fy+c=0$ bisected at the point (x_1,y_1) is given by $T=S'$

i.e. $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c \; = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

Example : 24 Tangents are drawn from any point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$. If the chord of contact touches the circle $x^2 + y^2 = c^2$, $a > b$, then **[MP PET 1999, Rajasthan PET 1999]**

(a) *a, b, c* are in *A.P.* (b) *a, b, c* are in *G.P.* (c) *a, b, c* are in *H.P.* (d) *a, c, b* are in *G.P.*

Solution : (b) Chord of contact of any point $(a\cos\theta, a\sin\theta)$ on 1st circle with respect to 2nd circle is $ax\cos\theta + ay\sin\theta = b^2$

This chord touches the circle $x^2 + y^2 = c^2$,

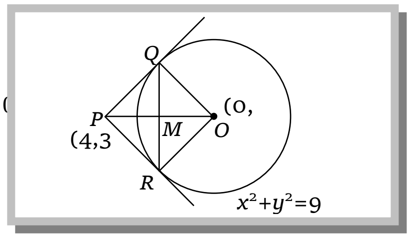
Hence, Radius = Perpendicular distance of chord from centre.

$c = \frac{b^2}{a\sqrt{\cos^2\theta + \sin^2\theta}} \Rightarrow b^2 = ac$ .Hence *a,b,c* are in G.P.

Example : 25 The area of the triangle formed by the tangents from the point (4, 3) to the circle $x^2 + y^2 = 9$ and the line joining their points of contact is **[IIT 1981, MP PET 1991]**

(a) $\frac{25}{192}$ *sq.units* (b) $\frac{192}{25}$ *sq.units* (c) $\frac{384}{25}$ *sq.units* (d) None of these

Solution : (b) The equation of the chord of contact of tangents drawn from *P* (4, 3) to $x^2 + y^2 = 9$ is $4x + 3y = 9$. The equation of *OP* is $y = \frac{3}{4}x$.



Now, *OM* = (length of the perpendicular from (0, 0) on $4x + 3y - 9 = 0$

$\therefore \; QR = 2.QM = 2\sqrt{OQ^2 - OM^2} = 2\sqrt{9 - \frac{81}{25}} = \frac{24}{5}$

Now, $PM = OP - OM = 5 - \frac{9}{5} = \frac{16}{5}$ .So, Area of $\Delta PQR = \frac{1}{2}\left(\frac{24}{5}\right)\left(\frac{16}{5}\right) = \frac{192}{25}$ *sq. units*

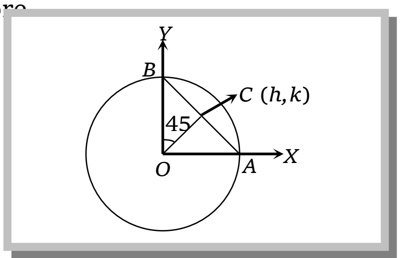
Example : 26 The locus of the middle points of those chords of the circle $x^2 + y^2 = 4$ which subtend a right angle at the origin is **[MP PET 1990; IIT 1984; Rajasthan PET 1997; DCE 2000, 01]**

(a) $x^2 + y^2 - 2x - 2y = 0$ (b) $x^2 + y^2 = 4$ (c) $x^2 + y^2 = 2$ (d) $(x-1)^2 + (y-2)^2 = 5$

Solution : (c) Let the mid-point of chord is (*h, k*). Also radius of circle is 2. Therefore

$\frac{OC}{OB} = \cos 45^o \Rightarrow \frac{\sqrt{h^2 + k^2}}{2} = \frac{1}{\sqrt{2}} \Rightarrow h^2 + k^2 = 2$

Hence locus is $x^2 + y^2 = 2$



Example : 27 If two distinct chords, drawn from the point (*p, q*) on the circle $x^2 + y^2 = px + qy$ (where *p, q* ≠ 0) are bisected by the *x*-axis, then **[IIT 1999]**

(a) $p^2 = q^2$ (b) $p^2 = 8q^2$ (c) $p^2 < 8q^2$ (d) $p^2 > 8q^2$

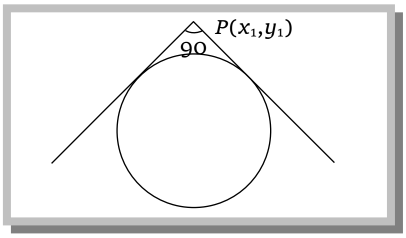
Solution : (d) Let (*h*, 0) be a point on *x*-axis, then the equation of chord whose mid-point is (*h*, 0) will be $xh - \frac{1}{2}p(x+h) - \frac{1}{2}q(y+0) = h^2 - ph$. This passes through (*p, q*), hence $ph - \frac{1}{2}p(p+h) - \frac{1}{2}q.q = h^2 - ph$

$\Rightarrow ph - \frac{1}{2}p^2 - \frac{1}{2}ph - \frac{1}{2}q^2 = h^2 - ph \Rightarrow h^2 - \frac{3}{2}ph + \frac{1}{2}(p^2 + q^2) = 0$; $\because$ h is real, hence $B^2 - 4AC > 0$

$\therefore \frac{9}{4}p^2 - 4.\frac{1}{2}(p^2 + q^2) > 0 \Rightarrow 9p^2 - 8(p^2 + q^2) > 0 \Rightarrow p^2 - 8q^2 > 0 \Rightarrow p^2 > 8q^2$

4.13 Director Circle

The locus of the point of intersection of two perpendicular tangents to a circle is called the Director circle.



Let the circle be $x^2 + y^2 = a^2$, then equation of the pair of tangents to a circle from a point is $(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$. If this represents a pair of perpendicular lines, coefficient of x^2 + coefficient of $y^2 = 0$ *i.e.* $(x_1^2 + y_1^2 - a^2 - x_1^2) + (x_1^2 + y_1^2 - a^2 - y_1^2) = 0 \Rightarrow x_1^2 + y_1^2 = 2a^2$

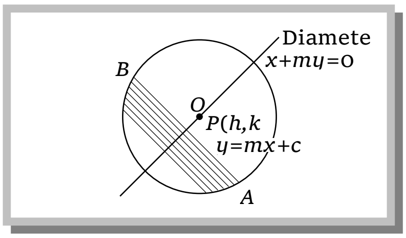
Hence the equation of director circle is $x^2 + y^2 = 2a^2$.

Obviously director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the given circle.

Note : ❑ Director circle of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$.

4.14 Diameter of a Circle

The locus of the middle points of a system of parallel chords of a circle is called a diameter of the circle.



The equation of the diameter bisecting parallel chords $y = mx + c$ (c is a parameter) of the circle $x^2 + y^2 = a^2$ is $x + my = 0$.

Note : ❑ The diameter corresponding to a system of parallel chords of a circle always passes through the centre of the circle and is perpendicular to the parallel chords.

Example : 28 A foot of the normal from the point (4, 3) to a circle is (2, 1) and a diameter of the circle has the equation $2x - y = 2$. Then the equation of the circle is

(a) $x^2 + y^2 + 2x - 1 = 0$ (b) $x^2 + y^2 - 2x - 1 = 0$ (c) $x^2 + y^2 - 2y - 1 = 0$ (d) None of these.

Solution : (b) The line joining (4, 3) and (2, 1) is also along a diameter. So, the centre is the intersection of the diameters $2x - y = 2$ and $y - 3 = (x - 4)$. Solving these, the centre = (1, 0)

$\therefore$ Radius = Distance between (1, 0) and (2, 1) = $\sqrt{2}$.

$\therefore$ Equation of circle $(x-1)^2 + y^2 = (\sqrt{2})^2 \Rightarrow x^2 + y^2 - 2x - 1 = 0$

Example : 29 The diameter of the circle $x^2 + y^2 - 4x + 2y - 11 = 0$ corresponding to a system of chords parallel to the line $x - 2y + 1 = 0$

(a) $x - 2y + 3 = 0$ (b) $2x - y + 3 = 0$ (c) $2x + y - 3 = 0$ (d) None of these

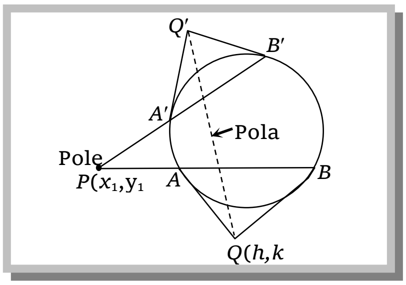
Solution : (c) The centre of the given circle is (2, –1) the equation of the line perpendicular to chord $x - 2y + 1 = 0$ is $2x + y + k = 0$

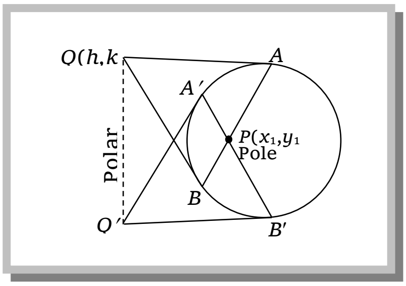
Since the line passes through the point (2, –1) therefore k = – 3. The equation of diameter is $2x + y - 3 = 0$.

4.15 Pole and Polar

Let $P(x_1, y_1)$ be any point inside or outside the circle. Draw chords AB and $A'B'$ passing through P. If tangents to the circle at A and B meet at Q (h, k), then locus of Q is called the polar of P with respect to circle and P is called the pole and if tangents to the circle at A' and B' meet at Q', then the straight line QQ' is polar with P as its pole.

If circle be $x^2 + y^2 = a^2$ then AB is the chord of contact of Q (h, k), $hx + ky = a^2$ is its equation. But $P(x_1, y_1)$ lies on AB, $\therefore\ hx_1 + ky_1 = a^2$.





Hence, locus of Q (h, k) is $xx_1 + yy_1 = a^2$, which is polar of $P(x_1, y_1)$ with respect to the circle $x^2 + y^2 = a^2$.

(1) **Coordinates of pole of a line :** The pole of the line $lx + my + n = 0$ with respect to the circle $x^2 + y^2 = a^2$. Let pole be (x_1, y_1), then equation of polar with respect to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 - a^2 = 0$, which is same as $lx + my + n = 0$

Then $\frac{x_1}{l} = \frac{y_1}{m} = -\frac{a^2}{n}$, $\therefore\ x_1 = -\frac{a^2 l}{n}$ and $y_1 = -\frac{a^2 m}{n}$. Hence, the required pole is $\left(-\frac{a^2 l}{n}, -\frac{a^2 m}{n}\right)$.

(2) **Properties of pole and polar**

(i) If the polar of $P(x_1, y_1)$ w.r.t. a circle passes through $Q(x_2, y_2)$ then the polar of Q will pass through P and such points are said to be conjugate points.

(ii) If the pole of the line $ax + by + c = 0$ w.r.t. a circle lies on another line $a_1 x + b_1 y + c_1 = 0$; then the pole of the second line will lie on the first and such lines are said to be conjugate lines.

(iii) The distance of any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ from the centre of a circle is proportional to the distance of each from the polar of the other.

(iv) If O be the centre of a circle and P any point, then OP is perpendicular to the polar of P.

(v) If O be the centre of a circle and P any point, then if OP (produced, if necessary) meet the polar of P in Q, then OP . OQ = (radius)^2.

Note :

- ❑ Equation of polar is like as equation of tangent *i.e.*, T = 0 (but point different)
- ❑ Equation of polar of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with respect to (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
- ❑ If the point P is outside the circle then equation of polar and chord of contact will coincide. In this case the polar cuts the circle at two points.

- If the point P is on the circle then equation of polar, chord of contact and tangent at P will coincide. So in this case the polar touches the circle.
- If the point P is inside the circle (not its centre) then only its polar will exist. In this case the polar is outside the circle. The polar of the centre lies at infinity.
- If a triangle is like that its each vertex is a pole of opposite side with respect to a circle then it is called self conjugate triangle.

Example : 30 The polar of the point $\left(5, -\frac{1}{2}\right)$ with respect to circle $(x-2)^2+y^2=4$ is **[Rajasthan PET 1996]**

(a) $5x-10y+2=0$ (b) $6x-y-20=0$ (c) $10x-y-10=0$ (d) $x-10y-2=0$

Solution : (b) The polar of the point $\left(5, -\frac{1}{2}\right)$ is $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

$\Rightarrow 5x-\frac{1}{2}y-2(x+5)+0+0=0 \Rightarrow 3x-\frac{y}{2}-10=0 \Rightarrow 6x-y-20=0$.

Example : 31 The pole of the straight line $9x+y-28=0$ with respect to circle $2x^2+2y^2-3x+5y-7=0$ is **[Rajasthan PET 1990, 99; MNR 1984; UPSEAT 2000]**

(a) (3, 1) (b) (1, 3) (c) (3, – 1) (d) (– 3, 1)

Solution : (c) Equation of given circle is $x^2+y^2-\frac{3}{2}x+\frac{5}{2}y-\frac{7}{2}=0$

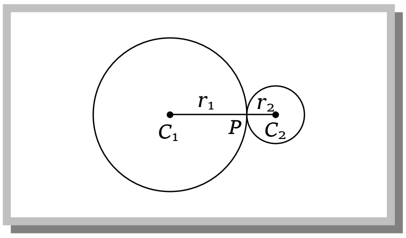
$\Rightarrow \left(x-\frac{3}{4}\right)^2+\left(y+\frac{5}{4}\right)^2-\frac{9}{16}-\frac{25}{16}-\frac{7}{2}=0 \Rightarrow \left(x-\frac{3}{4}\right)^2+\left(y+\frac{5}{4}\right)^2-\frac{45}{8}=0$

Put $X=x-\frac{3}{4}$ and $Y=y+\frac{5}{4}$, we get the equation of circle $X^2+Y^2-\frac{45}{8}=0$ and the line $9X+Y-\frac{45}{2}=0$

Hence pole $=\left[\frac{9\times\frac{45}{8}}{\frac{45}{2}}, \frac{1\times\frac{45}{8}}{\frac{45}{2}}\right]=\left(\frac{9}{4}, \frac{1}{4}\right)$. But, $x=\frac{9}{4}+\frac{3}{4}=3$ and $y=\frac{1}{4}-\frac{5}{4}=-1$, hence the pole is (3, – 1).

4.16 Two Circles touching each other

(1) **When two circles touch each other externally :** Then distance between their centres = Sum of their radii *i.e.,* $|C_1C_2|=r_1+r_2$

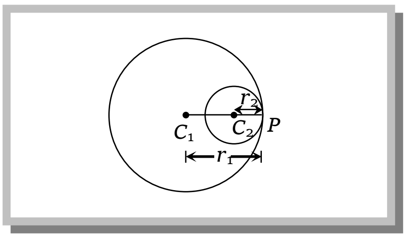


In such cases, the point of contact P divides the line joining C_1 and C_2 internally in the ratio $r_1:r_2 \Rightarrow \frac{C_1P}{C_2P}=\frac{r_1}{r_2}$

If $C_1\equiv(x_1,y_1)$ and $C_2\equiv(x_2,y_2)$, then co-ordinate of P is

$$\left(\frac{r_1x_2+r_2x_1}{r_1+r_2}, \frac{r_1y_2+r_2y_1}{r_1+r_2}\right)$$

(2) **When two circles touch each other internally :** Then distance between their centres = Difference of their radii *i.e.,* $|C_1C_2|=r_1-r_2$



In such cases, the point of contact P divides the line joining C_1 and C_2 externally in the ratio $r_1:r_2 \Rightarrow \frac{C_1P}{C_2P}=\frac{r_1}{r_2}$

If $C_1\equiv(x_1,y_1)$ and $C_2\equiv(x_2,y_2)$, then co-ordinate of P is

$$\left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right)$$

4.17 Common Tangents to Two circles

Different cases of intersection of two circles :

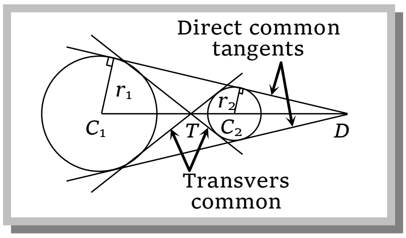
Let the two circles be $(x - x_1)^2 + (y - y_1)^2 = r_1^2$ .....(i)

and $(x - x_2)^2 + (y - y_2)^2 = r_2^2$ .....(ii)

with centres $C_1(x_1, y_1)$ and $C_2(x_2, y_2)$ and radii r_1 and r_2 respectively. Then following cases may arise :

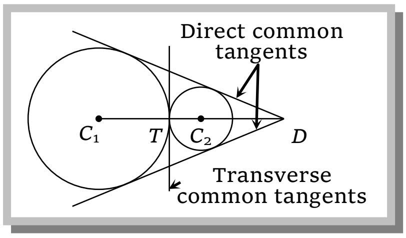
Case I : When $|C_1C_2| > r_1 + r_2$ *i.e.,* the distance between the centres is greater than the sum of radii.

In this case four common tangents can be drawn to the two circles, in which two are direct common tangents and the other two are transverse common tangents.



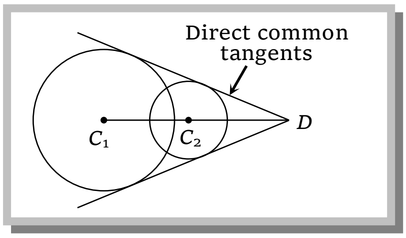
Case II : When $|C_1C_2| = r_1 + r_2$ *i.e.,* the distance between the centres is equal to the sum of radii.

In this case two direct common tangents are real and distinct while the transverse tangents are coincident.



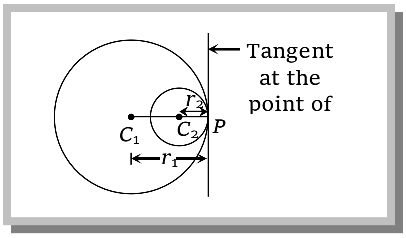
Case III : When $|C_1C_2| < r_1 + r_2$ *i.e.,* the distance between the centres is less than sum of radii.

In this case two direct common tangents are real and distinct while the transverse tangents are imaginary.



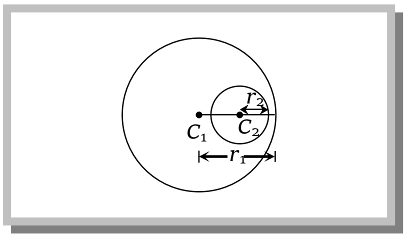
Case IV : When $|C_1C_2| = |r_1 - r_2|$, *i.e.,* the distance between the centres is equal to the difference of the radii.

In this case two tangents are real and coincident while the other two tangents are imaginary.



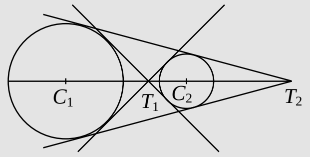
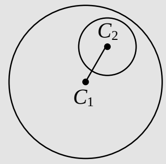
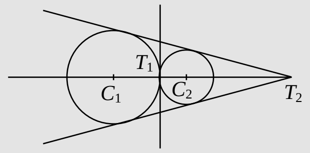
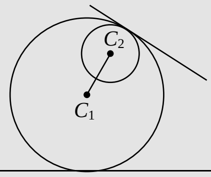
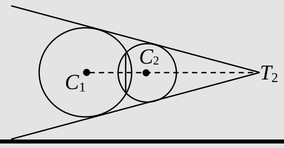
Case V : When $|C_1C_2| < |r_1 - r_2|$, *i.e.,* the distance between the centres is less than the difference of the radii.

In this case, all the four common tangents are imaginary.



Note :

- **Points of intersection of common tangents :** The points T_1 and T_2 (points of intersection of indirect and direct common tangents) divides C_1C_2 internally and externally in the ratio $r_1 : r_2$ respectively.
- Equation of the common tangents at point of contact is $S_1 - S_2 = 0$.
- If the circle $x^2 + y^2 + 2gx + c^2 = 0$ and $x^2 + y^2 + 2fy + c^2 = 0$ touch each other, then $\frac{1}{g^2} + \frac{1}{f^2} = \frac{1}{c^2}$.

	Condition	Position	Diagram	No. of common tangents
(i)	$C_1C_2 > r_1 + r_2$	Do not intersect or one outside the other		4
(ii)	$C_1C_2 < \lvert r_1 - r_2 \rvert$	One inside the other		0
(iii)	$C_1C_2 = r_1 + r_2$	External touch		3
(iv)	$C_1C_2 = \lvert r_1 - r_2 \rvert$	Internal touch		1
(v)	$\lvert r_1 - r_2 \rvert < C_1C_2 < r_1 + r_2$	Intersection at two real points		2

Example : 32 If circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch each other, then **[MNR 1987]**

(a) $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$ (b) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$ (c) $\frac{1}{a} + \frac{1}{b} = c^2$ (d) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c}$

Solution : (d) $C_1 = (-a, 0)$, $r_1 = \sqrt{a^2 - c}$; $C_2 = (0, -b)$, $r_2 = \sqrt{b^2 - c}$; $C_1C_2 = \sqrt{a^2 + b^2}$

$\because$ Circles touch each other, therefore $r_1 + r_2 = C_1C_2$

$\Rightarrow \sqrt{a^2 - c} + \sqrt{b^2 - c} = \sqrt{a^2 + b^2} \Rightarrow a^2b^2 - b^2c - a^2c = 0$

Multiplying by $\frac{1}{a^2b^2c^2}$, we get $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c}$.

Example : 33 If two circles $(x-1)^2 + (y-3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points, then

[IIT 1989; Karnataka CET 2002; DCE 2000, 01; AIEEE 2003]

(a) $2 < r < 8$ (b) $r = 2$ (c) $r < 2$ (d) $r > 2$

Solution : (a) When two circles intersect each other, then difference between their radii < Distance between their centres

$\Rightarrow r - 3 < 5 \Rightarrow r < 8$ …..(i)

Sum of their radii > Distance between their centres

$\Rightarrow r + 3 > 5 \Rightarrow r > 2$ …..(ii)

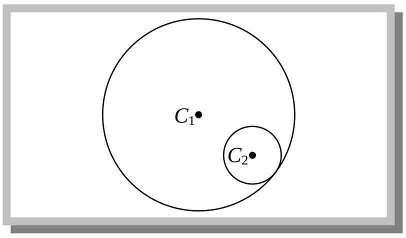
Hence by (i) and (ii), $2 < r < 8$.

Example : 34 The equation of the circle having the lines $x^2 + 2xy + 3x + 6y = 0$ as its normals and having size just sufficient to contain the circle $x(x-4)+y(y-3)=0$ is **[Roorkee 1990]**

(a) $x^2 + y^2 + 3x - 6y - 40 = 0$ (b) $x^2 + y^2 + 6x - 3y - 45 = 0$

(c) $x^2 + y^2 + 8x + 4y - 20 = 0$ (d) $x^2 + y^2 + 4x + 8y + 20 = 0$

Solution : (b) Given pair of normals is $x^2 + 2xy + 3x + 6y = 0$ or $(x+2y)(x+3)=0$



$\therefore$ Normals are $x + 2y = 0$ and $x + 3 = 0$

The point of intersection of normals $x + 2y = 0$ and $x + 3 = 0$ is the centre of required circle, we get centre $C_1 = (-3, 3/2)$ and other circle is

$x(x-4)+y(y-3)=0$ or $x^2 + y^2 - 4x - 3y = 0$ .....(i)

Its centre $C_2 = (2, 3/2)$ and radius $r = \sqrt{4 + \frac{9}{4}} = \frac{5}{2}$

Since the required circle just contains the given circle (i), the given circle should touch the required circle internally from inside.

Therefore, radius of the required circle $= |C_1 - C_2| + r = \sqrt{(-3-2)^2 + \left(\frac{3}{2} - \frac{3}{2}\right)^2} + \frac{5}{2} = 5 + \frac{5}{2} = \frac{15}{2}$

Hence, equation of required circle is $(x+3)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{15}{2}\right)^2$ or $x^2 + y^2 + 6x - 3y - 45 = 0$.

Example : 35 The equation of the circle which touches the circle $x^2 + y^2 - 6x + 6y + 17 = 0$ externally and to which the lines $x^2 - 3xy - 3x + 9y = 0$ are normals, is **[Roorkee 1994]**

(a) $x^2 + y^2 - 6x - 2y - 1 = 0$ (b) $x^2 + y^2 + 6x + 2y + 1 = 0$

(c) $x^2 + y^2 - 6x - 6y + 1 = 0$ (d) $x^2 + y^2 - 6x - 2y + 1 = 0$

Solution : (d) Joint equations of normals are $x^2 - 3xy - 3x + 9y = 0 \Rightarrow x(x-3y) - 3(x-3y) = 0 \Rightarrow (x-3)(x-3y) = 0$

$\therefore$ Given normals are $x - 3 = 0$ and $x - 3y = 0$, which intersect at centre of circle whose coordinates are (3, 1).

The given circle is $C_1 = (3, -3)$, $r_1 = 1$; $C_2 = (3, 1)$, $r_2 = ?$

If the two circles touch externally, then $C_1C_2 = r_1 + r_2 \Rightarrow 4 = 1 + r_2 \Rightarrow r_2 = 3$

$\therefore$ Equation of required circle is $(x-3)^2 + (y-1)^2 = (3)^2 \Rightarrow x^2 + y^2 - 6x - 2y + 1 = 0$

Example : 36 The number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x - 8y = 24$ is **[IIT 1998]**

(a) 0 (b) 1 (c) 3 (d) 4

Solution : (b) Circles $S_1 \equiv x^2 + y^2 = (2)^2$ and $S_2 \equiv (x-3)^2 + (y-4)^2 = (7)^2$

$\therefore$ Centres $C_1 = (0,0)$, $C_2 = (3,4)$ and radii $r_1 = 2$, $r_2 = 7$

$C_1C_2 = \sqrt{(3)^2 + (4)^2} = 5$, $r_2 - r_1 = 7 - 2 = 5$

$\therefore$ $C_1C_2 = r_2 - r_1$ *i.e.* circles touch internally. Hence there is only one common tangent.

Example : 37 There are two circles whose equations are $x^2 + y^2 = 9$ and $x^2 + y^2 - 8x - 6y + n^2 = 0$, $n \in Z$. If the two circles have exactly two common tangents, then the number of possible values of n is

(a) 2 (b) 8 (c) 9 (d) None of these

Solution : (c) For $x^2 + y^2 = 9$, the centre = (0, 0) and the radius = 3

For $x^2 + y^2 - 8x - 6y + n^2 = 0$. The centre = (4, 3) and the radius $= \sqrt{(4)^2 + (3)^2 - n^2}$

$\therefore$ $4^2 + 3^2 - n^2 > 0$ or $n^2 < 5^2$ or $-5 < n < 5$.

Circles should cut to have exactly two common tangents.

So, $r_1 + r_2 > C_1C_2$, $\therefore$ $3 + \sqrt{25 - n^2} > \sqrt{(4)^2 + (3)^2}$ or $\sqrt{25 - n^2} > 2$ or $25 - n^2 > 4$

$\therefore$ $n^2 < 21$ or $-\sqrt{21} < n < \sqrt{21}$

Therefore, common values of n should satisfy $-\sqrt{21} < n < \sqrt{21}$.

But $n \in Z$, So, $n = -4, -3, ... 3, 4$. $\therefore$ Number of possible values of n = 9.

4.18 Common chord of two Circles

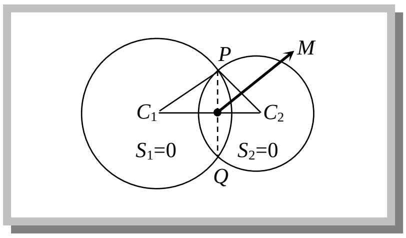
(1) **Definition :** The chord joining the points of intersection of two given circles is called their common chord.

(2) **Equation of common chord :** The equation of the common chord of two circles

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \text{....(i)}$$

and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ ....(ii)

is $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$ *i.e.* $S_1 - S_2 = 0$.



(3) **Length of the common chord :** $PQ = 2(PM) = 2\sqrt{C_1P^2 - C_1M^2}$

Where $C_1P =$ radius of the circle $S = 0$ and $C_1M =$ length of the perpendicular from the centre C_1 to the common chord *PQ*.

Note : ❑ The length of the common chord is $2\sqrt{r_1^2 - p_1^2} = 2\sqrt{r_2^2 - p_2^2}$ where p_1 and p_2 are the lengths of perpendicular drawn from the centre to the chord.

❑ While using the above equation of common chord the coefficient of x^2 and y^2 in both equation should be equal.

❑ Two circle touches each other if the length of their common chord is zero.

❑ Maximum length of the common chord = diameter of the smaller circle.

Example : 38 If the common chord of the circles $x^2 + (y - \lambda)^2 = 16$ and $x^2 + y^2 = 16$ subtend a right angle at the origin, then λ is equal to

(a) 4 (b) $4\sqrt{2}$ (c) $\pm 4\sqrt{2}$ (d) 8

Solution : (c) The common chord of given circles is $S_1 - S_2 = 0$

$\Rightarrow x^2 + (y - \lambda)^2 - 16 - \{x^2 + y^2 - 16\} = 0$ *i.e.*, $y = \frac{\lambda}{2}$ $(\because \lambda \neq 0)$

The pair of straight lines joining the origin to the points of intersection of $y = \frac{\lambda}{2}$ and $x^2 + y^2 = 16$ is $x^2 + y^2 = 16\left(\frac{2y}{\lambda}\right)^2$

$\Rightarrow \lambda^2 x^2 + (\lambda^2 - 64)y^2 = 0$. These lines are at right angles if $\lambda^2 + \lambda^2 - 64 = 0$, *i.e.*, $\lambda = \pm 4\sqrt{2}$.

Example : 39 Which of the following is a point on the common chord of the circles $x^2 + y^2 + 2x - 3y + 6 = 0$ and $x^2 + y^2 + x - 8y - 13 = 0$ **[Karnataka CET 2003]**

(a) (1, –2) (b) (1, 4) (c) (1, 2) (d) (1, – 4)

Solution : (d) Given circles are, $S_1 \equiv x^2 + y^2 + 2x - 3y + 6 = 0$ ..... (i) and $S_2 \equiv x^2 + y^2 + x - 8y - 13 = 0$ ..... (ii)

$\therefore$ Equation of common chord is $S_1 - S_2 = 0$

$\Rightarrow x + 5y + 19 = 0$, and out of the four given points only point (1, – 4) satisfies it.

Example : 40 If the circle $x^2 + y^2 = 4$ bisects the circumference of the circle $x^2 + y^2 - 2x + 6y + a = 0$, then *a* equals **[Rajasthan PET 1999]**

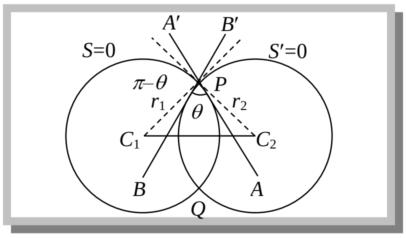
(a) 4 (b) – 4 (c) 16 (d) – 16

Solution : (c) The common chord of given circles is $S_1 - S_2 = 0 \Rightarrow 2x - 6y - 4 - a = 0$ .....(i)

Since, $x^2 + y^2 = 4$ bisects the circumferences of the circle $x^2 + y^2 - 2x + 6y + a = 0$, therefore (i) passes through the centre of second circle *i.e.* (1, – 3). $\therefore$ 2 + 18 – 4 – *a* = 0 ⇒ *a* = 16.

4.19 Angle of Intersection of Two Circles

The angle of intersection between two circles $S = 0$ and $S' = 0$ is defined as the angle between their tangents at their point of intersection.

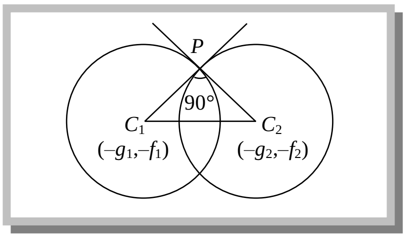


If $S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

are two circles with radii r_1, r_2 and d be the distance between their centres then the angle of intersection θ between them is given by $\cos\theta = \dfrac{r_1^2 + r_1^2 - d^2}{2r_1r_2}$ or $\cos\theta = \dfrac{2(g_1g_2 + f_1f_2) - (c_1 + c_2)}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_2^2 + f_2^2 - c_2}}$

(1) **Condition of Orthogonality :** If the angle of intersection of the two circles is a right angle $(\theta = 90^o)$, then such circles are called orthogonal circles and condition for their orthogonality is $2g_1g_2 + 2f_1f_2 = c_1 + c_2$



Note : ❑ When the two circles intersect orthogonally then the length of tangent on one circle from the centre of other circle is equal to the radius of the other circle.

❑ Equation of a circle intersecting the three circles $x^2 + y^2 + 2g_ix + 2f_iy + c_i = 0\ (i = 1, 2, 3)$ orthogonally is $\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ -c_1 & g_1 & f_1 & -1 \\ -c_2 & g_2 & f_2 & -1 \\ -c_3 & g_3 & f_3 & -1 \end{vmatrix} = 0$

Example : 41 A circle passes through the origin and has its centre on $y = x$. If it cuts $x^2 + y^2 - 4x - 6y + 10 = 0$ orthogonally, then the equation of the circle is **[EAMCET 1994]**

(a) $x^2 + y^2 - x - y = 0$ (b) $x^2 + y^2 - 6x - 4y = 0$ (c) $x^2 + y^2 - 2x - 2y = 0$ (d) $x^2 + y^2 + 2x + 2y = 0$

Solution : (c) Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ....(i)

This passes through (0, 0), therefore $c = 0$

The centre $(-g, -f)$ of (i) lies on $y = x$, hence $g = f$.

Since (i) cuts the circle $x^2 + y^2 - 4x - 6y + 10 = 0$ orthogonally, therefore $2(-2g - 3f) = c + 10$

$\Rightarrow\ -10g = 10 \Rightarrow g = f = -1$ $(\because g = f$ and $c = 0)$. Hence the required circle is $x^2 + y^2 - 2x - 2y = 0$.

Example : 42 The centre of the circle, which cuts orthogonally each of the three circles $x^2 + y^2 + 2x + 17y + 4 = 0$, $x^2 + y^2 + 7x + 6y + 11 = 0$ and $x^2 + y^2 - x + 22y + 3 = 0$ is **[MP PET 2003]**

(a) (3, 2) (b) (1, 2) (c) (2, 3) (d) (0, 2)

Solution : (a) Let the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ …..(i)

Circle (i) cuts orthogonally each of the given three circles. Then according to condition $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

$2g + 17f = c + 4$ …..(ii)

$7g + 6f = c + 11$ …..(iii)

$-g + 22f = c + 3$ …..(iv)

On solving (ii), (iii) and (iv), $g = -3,\ f = -2$. Therefore, the centre of the circle $(-g, -f) = (3, 2)$

Example : 43 The locus of the centre of a circle which cuts orthogonally the circle $x^2+y^2-20x+4=0$ and which touches $x=2$ is **[UPSEAT 2001]**

(a) $y^2=16x+4$ (b) $x^2=16y$ (c) $x^2=16y+4$ (d) $y^2=16x$

Solution : (d) Let the circle be $x^2+y^2+2gx+2fy+c=0$ …..(i)

It cuts the circle $x^2+y^2-20x+4=0$ orthogonally

$\therefore \quad 2(-10g+0\times f)=c+4 \Rightarrow -20g=c+4$ …..(ii)

Circle (i) touches the line $x=2$; $\therefore \quad x+0y-2=0$

$\therefore \left|\frac{-g+0-2}{\sqrt{1}}\right|=\sqrt{g^2+f^2-c} \Rightarrow (g+2)^2=g^2+f^2-c \Rightarrow 4g+4=f^2-c$ …..(iii)

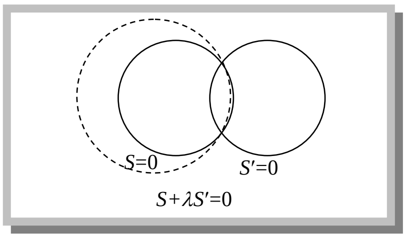
Eliminating c from (ii) and (iii), we get $-16g+4=f^2+4 \Rightarrow f^2+16g=0$.

Hence the locus of $(-g,-f)$ is $y^2-16x=0 \Rightarrow y^2=16x$.

4.20 Family of Circles

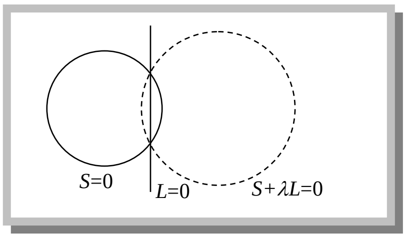
(1) The equation of the family of circles passing through the point of intersection of two given circles $S=0$ and $S'=0$ is given as

$$S+\lambda S'=0 \qquad \text{(where } \lambda \text{ is a parameter, } \lambda\neq -1)$$



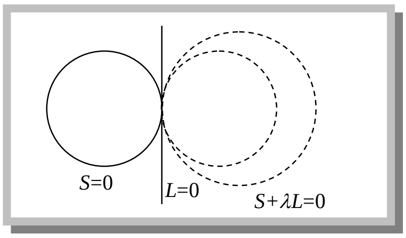
(2) The equation of the family of circles passing through the point of intersection of circle $S=0$ and a line $L=0$ is given as

$$S+\lambda L=0 \qquad \text{(where } \lambda \text{ is a parameter)}$$



(3) The equation of the family of circles touching the circle $S=0$ and the line $L=0$ at their point of contact P is

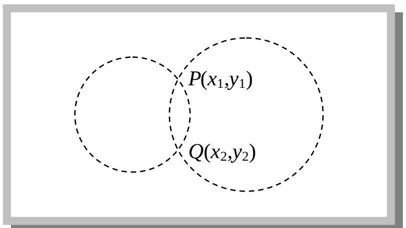
$$S+\lambda L=0 \qquad \text{(where } \lambda \text{ is a parameter)}$$



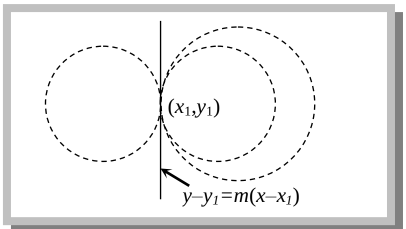
(4) The equation of a family of circles passing through two given points $P(x_1,y_1)$ and $Q(x_2,y_2)$ can be written in the form

$$\Rightarrow (x-x_1)(x-x_2)+(y-y_1)(y-y_2)+\lambda\begin{vmatrix} x & y & 1\\ x_1 & y_1 & 1\\ x_2 & y_2 & 1\end{vmatrix}=0$$

(where λ is a parameter)



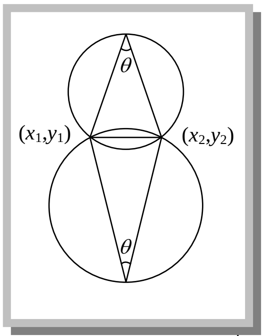
(5) The equation of family of circles, which touch $y-y_1=m(x-x_1)$ at (x_1,y_1) for any finite m is $(x-x_1)^2+(y-y_1)^2+\lambda\{(y-y_1)-m(x-x_1)\}=0$



And if m is infinite, the family of circles is

$(x-x_1)^2+(y-y_1)^2+\lambda(x-x_1)=0$ (where λ is a parameter)

(6) Equation of the circles given in diagram is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)\pm\cot\theta\{(x-x_1)(y-y_2)-(x-x_2)(y-y_1)\}=0$



Example : 44 The equation of the circle through the points of intersection of $x^2+y^2-1=0$, $x^2+y^2-2x-4y+1=0$ and touching the line $x+2y=0$ is **[Roorkee 1989]**

(a) $x^2+y^2+x+2y=0$ (b) $x^2+y^2-x+20=0$ (c) $x^2+y^2-x-2y=0$ (d) $2(x^2+y^2)-x-2y=0$

Solution : (c) Family of circles is $x^2+y^2-2x-4y+1+\lambda(x^2+y^2-1)=0$

$$x^2+y^2-\frac{2}{1+\lambda}x-\frac{4}{1+\lambda}y+\frac{1-\lambda}{1+\lambda}=0$$

Centre is $\left[\frac{1}{1+\lambda},\frac{2}{1+\lambda}\right]$ and radius $=\sqrt{\left(\frac{1}{1+\lambda}\right)^2+\left(\frac{2}{1+\lambda}\right)^2-\left(\frac{1-\lambda}{1+\lambda}\right)}=\sqrt{\frac{4+\lambda^2}{(1+\lambda)^2}}$

Since it touches the line $x+2y=0$, Hence Radius = perpendicular distance from centre to the line $=\left|\frac{\frac{1}{1+\lambda}+\frac{4}{1+\lambda}}{\sqrt{1^2+2^2}}\right|$

$$=\sqrt{\frac{4+\lambda^2}{(1+\lambda)^2}}=\frac{\sqrt{4+\lambda^2}}{1+\lambda}\Rightarrow\sqrt{5}=\sqrt{4+\lambda^2}\Rightarrow\lambda=\pm 1$$

$\lambda=-1$ cannot be possible in case of circle, so $\lambda=1$. $\therefore$ Equation of circle is $x^2+y^2-x-2y=0$

Example : 45 The equation of the circle through the points of intersection of the circles $x^2+y^2-6x+2y+4=0$, $x^2+y^2+2x-4y-6=0$ and with its centre on the line $y=x$

(a) $7x^2+7y^2+10x-10y-12=0$ (b) $7x^2+7y^2-10x-10y-12=0$

(c) $7x^2+7y^2-10x+10y-12=0$ (d) $7x^2+7y^2+10x+10y+12=0$

Solution : (b) Equation of any circle through the points of intersection of given circles is

$(x^2+y^2-6x+2y+4)+\lambda(x^2+y^2+2x-4y-6)=0\Rightarrow x^2(1+\lambda)+y^2(1+\lambda)-2x(3-\lambda)+2y(1-2\lambda)+(4-6\lambda)=0$

or, $x^2+y^2-\frac{2x(3-\lambda)}{(1+\lambda)}+\frac{2y(1-2\lambda)}{(1+\lambda)}+\frac{(4-6\lambda)}{(1+\lambda)}=0$ …..(i)

Its centre $=\left\{\frac{3-\lambda}{1+\lambda},\frac{2\lambda-1}{1+\lambda}\right\}$ lies on the line $y=x$. Then $\frac{2\lambda-1}{1+\lambda}=\frac{3-\lambda}{1+\lambda}\Rightarrow 2\lambda-1=3-\lambda\ \{\because\ \lambda\neq-1\}$

$\Rightarrow 3\lambda=4\Rightarrow\lambda=\frac{4}{3}$

Substituting the value of $\lambda=\frac{4}{3}$ in (i), we get the required equation as $7x^2+7y^2-10x-10y-12=0$.

4.21 Radical Axis

The radical axis of two circles is the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal.

Consider, $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ …..(i) and $S' \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ …..(ii)

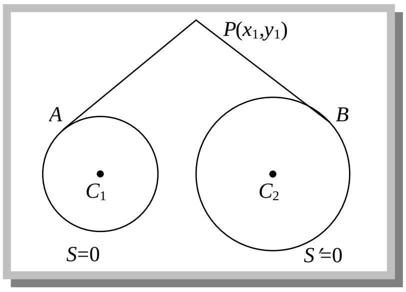
Let $P(x_1, y_1)$ be a point such that $|PA| = |PB|$

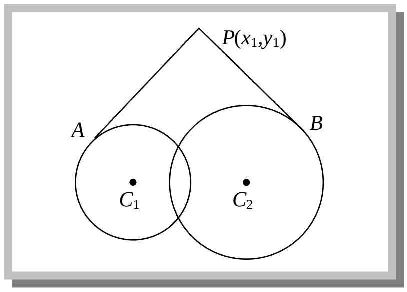
$\Rightarrow \sqrt{(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)} = \sqrt{(x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1)}$

On squaring, $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1$

$\Rightarrow 2(g - g_1)x_1 + 2(f - f_1)y_1 + c - c_1 = 0$

$\therefore$ Locus of $P(x_1, y_1)$ is $2(g - g_1)x + 2(f - f_1)y + c - c_1 = 0$

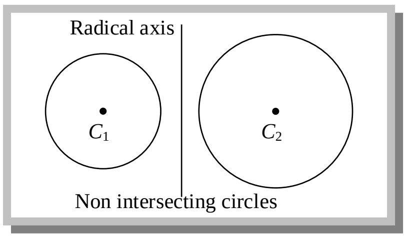


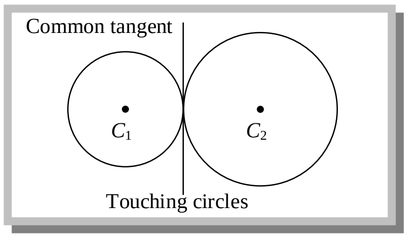


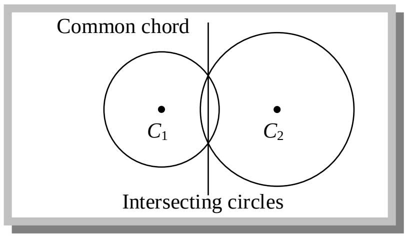
which is the required equation of radical axis of the given circles. Clearly this is a straight line.

(1) **Some properties of the radical axis**

(i) The radical axis and common chord are identical : Since the radical axis and common chord of two circles $S = 0$ and $S' = 0$ are the same straight line $S - S' = 0$, they are identical. The only difference is that the common chord exists only if the circles intersect in two real points, while the radical axis exists for all pair of circles irrespective of their position (Except when one circle is inside the other).





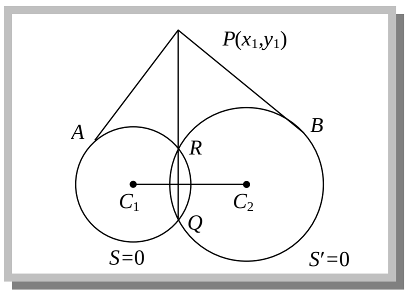


(ii) The radical axis is perpendicular to the straight line which joins the centres of the circles :

Consider, $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ …..(i)

and $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ …..(ii)

Since $C_1 \equiv (-g, -f)$ and $C_2 \equiv (-g_1, -f_1)$ are the centres of the circles (i) and (ii), then slope of the straight line $C_1C_2 = \dfrac{-f_1 + f}{-g_1 + g} = \dfrac{f - f_1}{g - g_1} = m_1$ (say)

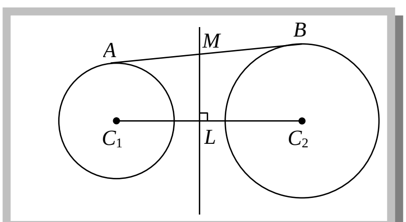


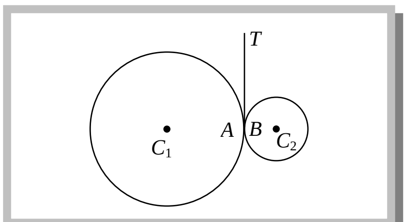
Equation of the radical axis is, $2(g - g_1)x + 2(f - f_1)y + c - c_1 = 0$

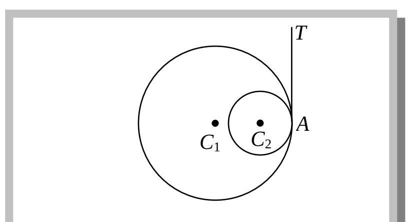
Slope of radical axis is $-\dfrac{(g - g_1)}{(f - f_1)} = m_2$ (say). $\therefore$ $m_1m_2 = -1$

Hence C_1C_2 and radical axis are perpendicular to each other.

(iii) The radical axis bisects common tangents of two circles : Let AB be the common tangent. If it meets the radical axis LM in M then MA and MB are two tangents to the circles. Hence $MA = MB$, since length of tangents are equal from any point on radical axis. Hence radical axis bisects the common tangent AB.







If the two circles touch each other externally or internally then *A* and *B* coincide. In this case the common tangent itself becomes the radical axis.

(iv) The radical axis of three circles taken in pairs are concurrent : Let the equations of three circles be

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \text{.....(i)}$$

$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad \text{.....(ii)}$$

$$S_3 \equiv x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0 \quad \text{.....(iii)}$$

The radical axis of the above three circles taken in pairs are given by

$$S_1 - S_2 \equiv 2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0 \quad \text{.....(iv)}$$

$$S_2 - S_3 \equiv 2x(g_2 - g_3) + 2y(f_2 - f_3) + c_2 - c_3 = 0 \quad \text{.....(v)}$$

$$S_3 - S_1 \equiv 2x(g_3 - g_1) + 2y(f_3 - f_1) + c_3 - c_1 = 0 \quad \text{.....(vi)}$$

Adding (iv), (v) and (vi), we find *L.H.S.* vanished identically. Thus the three lines are concurrent.

(v) If two circles cut a third circle orthogonally, the radical axis of the two circles will pass through the centre of the third circle

or

The locus of the centre of a circle cutting two given circles orthogonally is the radical axis of the two circles.

Let $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ .....(i)

$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad \text{.....(ii)}$$

$$S_3 \equiv x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0 \quad \text{.....(iii)}$$

Since (i) and (ii) both cut (iii) orthogonally, $\therefore \; 2g_1g_3 + 2f_1f_3 = c_1 + c_3$ and $2g_2g_3 + 2f_2f_3 = c_2 + c_3$

Subtracting, we get $2g_3(g_1 - g_2) + 2f_3(f_1 - f_2) = c_1 - c_2$ .....(iv)

Now radical axis of (i) and (ii) is $S_1 - S_2 = 0$ or $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$

Since it will pass through the centre of circle (iii)

$\therefore \; -2g_3(g_1 - g_2) - 2f_3(f_1 - f_2) + c_1 - c_2 = 0$ or $2g_3(g_1 - g_2) + 2f_3(f_1 - f_2) = c_1 - c_2$ .....(v)

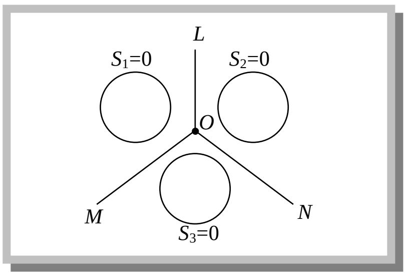
which is true by (iv).

Note : ❑ Radical axis need not always pass through the mid point of the line joining the centres of the two circles.

4.22 Radical Centre

The radical axes of three circles, taken in pairs, meet in a point, which is called their radical centre. Let the three circles be $S_1 = 0$ .....(i) , $S_2 = 0$ .....(ii) , $S_3 = 0$ .....(iii)

Let *OL*, *OM* and *ON* be radical axes of the pair sets of circles



$\{S_1 = 0, S_2 = 0\}, \{S_3 = 0, S_1 = 0\}$ and $\{S_2 = 0, S_3 = 0\}$ respectively.

Equation of *OL*, *OM* and *ON* are respectively

$S_1 - S_2 = 0$ .....(iv) , $S_3 - S_1 = 0$ .....(v), $S_2 - S_3 = 0$ .....(vi)

Let the straight lines (iv) and (v) *i.e.*, *OL* and *OM* meet in *O*. The equation of any straight line passing through *O* is $(S_1 - S_2) + \lambda(S_3 - S_1) = 0$ where λ is any constant

For $\lambda = 1$, this equation become $S_2 - S_3 = 0$, which is by (vi), equation of *ON*.

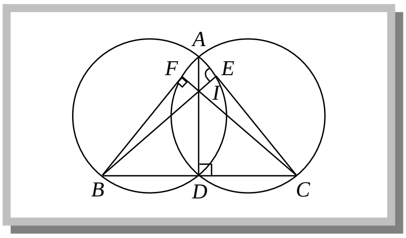
Thus the third radical axis also passes through the point where the straight lines (iv) and (v) meet.

In the above figure *O* is the radical centre.

(1) Properties of radical centre

(i) Co-ordinates of radical centre can be found by solving the equations

$$S_1 = S_2 = S_3 = 0$$



(ii) The radical centre of three circles described on the sides of a triangle as diameters is the orthocentre of the triangle :

Draw perpendicular from *A* on *BC*. $\therefore$ $\angle ADB = \angle ADC = \pi/2$

Therefore, the circles whose diameters are *AB* and *AC* passes through *D* and *A*. Hence *AD* is their radical axis. Similarly the radical axis of the circles on *AB* and *BC* as diameter is the perpendicular line from *B* on *CA* and radical axis of the circles on *BC* and *CA* as diameter is the perpendicular line from *C* on *AB*. Hence the radical axis of three circles meet in a point. This point *I* is radical centre but here radical centre is the point of intersection of altitudes *i.e.*, *AD*, *BE* and *CF*. Hence radical centre = orthocentre.

(iii) The radical centre of three given circles will be the centre of a fourth circle which cuts all the three circles orthogonally and the radius of the fourth circle is the length of tangent drawn from radical centre of the three given circles to any of these circles.

Let the fourth circle be $(x-h)^2 + (y-k)^2 = r^2$, where (*h*, *k*) is centre of this circle and *r* be the radius. The centre of circle is the radical centre of the given circles and *r* is the length of tangent from (*h*, *k*) to any of the given three circles.

Example : 46 The gradient of the radical axis of the circles $x^2 + y^2 - 3x - 4y + 5 = 0$ and $3x^2 + 3y^2 - 7x + 8y + 11 = 0$ is **[MP PET 2000]**

(a) $\frac{1}{3}$ (b) $-\frac{1}{10}$ (c) $-\frac{1}{2}$ (d) $-\frac{2}{3}$

Solution : (b) Equation of radical axis is $S_1 - S_2 = 0$

$S_1 \equiv x^2 + y^2 - 3x - 4y + 5 = 0$, $S_2 \equiv x^2 + y^2 - \frac{7}{3}x + \frac{8y}{3} + \frac{11}{3} = 0$

$\therefore$ Radical axis is $-2x - 20y + 4 = 0$.

Hence, gradient of radical axis = $-\frac{1}{10}$

Example : 47 The equations of three circles are $x^2 + y^2 - 12x - 16y + 64 = 0$, $3x^2 + 3y^2 - 36x + 81 = 0$ and $x^2 + y^2 - 16x + 81 = 0$. The coordinates of the point from which the length of tangents drawn to each of the three circles is equal **[Rajasthan PET 2002]**

(a) $\left(\frac{33}{4}, 2\right)$ (b) (2, 2) (c) $\left(2, \frac{33}{4}\right)$ (d) None of these

Solution : (d) The required point is the radical centre of the three given circles

Now, $S_1 - S_2 = 0 \Rightarrow -16y + 37 = 0$, $S_2 - S_3 = 0 \Rightarrow 4x - 54 = 0$ and $S_3 - S_1 = 0 \Rightarrow -4x + 16y + 17 = 0$

Solving these equations, we get $x = \frac{54}{4},\ \ y = \frac{37}{16} \quad \Rightarrow \quad x = \frac{27}{2},\ \ y = \frac{37}{16}$. Hence the required point is $\left(\frac{27}{2}, \frac{37}{16}\right)$.

Example : 48 The equation of the circle, which passes through the point $(2a, 0)$ and whose radical axis is $x = \frac{a}{2}$ with respect to the circle $x^2 + y^2 = a^2$, will be **[Rajasthan PET 1999]**

(a) $x^2 + y^2 - 2ax = 0$ (b) $x^2 + y^2 + 2ax = 0$ (c) $x^2 + y^2 + 2ay = 0$ (d) $x^2 + y^2 - 2ay = 0$

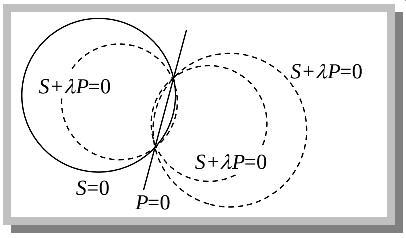
Solution : (a) Equation of radical axis is $x = \frac{a}{2} \Rightarrow 2x - a = 0$

Equation of required circle is $x^2 + y^2 - a^2 + \lambda(2x - a) = 0$

$\because$ It is passes through the point $(2a, 0)$, $\therefore\ 4a^2 - a^2 + \lambda(4a - a) = 0 \Rightarrow \lambda = -a$

$\therefore$ Equation of circle is $x^2 + y^2 - a^2 - 2ax + a^2 = 0 \Rightarrow x^2 + y^2 - 2ax = 0$

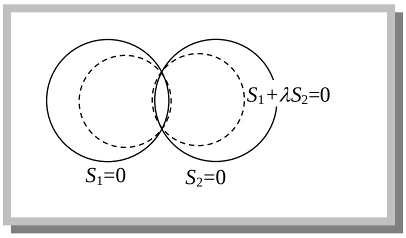
4.23 Co-Axial System of Circles

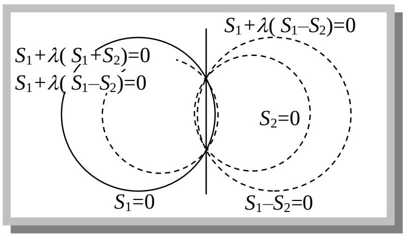


A system (or a family) of circles, every pair of which have the same radical axis, are called co-axial circles.

(1) The equation of a system of co-axial circles, when the equation of the radical axis and of one circle of the system are $P \equiv lx + my + n = 0$ and $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ respectively, is $S + \lambda P = 0$ (λ is an arbitrary constant).

(2) The equation of a co-axial system of circles, where the equation of any two circles of the system are





$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

Respectively, is $S_1 + \lambda(S_1 - S_2) = 0$, $(\lambda \neq -1)$ or $S_2 + \lambda_1(S_1 - S_2) = 0$, $(\lambda_1 \neq -1)$

Other form $S_1 + \lambda S_2 = 0$, $(\lambda \neq -1)$

(3) The equation of a system of co-axial circles in the simplest form is $x^2 + y^2 + 2gx + c = 0$, where g is variable and c is a constant.

4.24 Limiting Points

Limiting points of a system of co-axial circles are the centres of the point circles belonging to the family (Circles whose radii are zero are called **point circles**).

(1) **Limiting points of the co-axial system :** Let the circle is $x^2 + y^2 + 2gx + c = 0$ .....(i)

where g is variable and c is constant.

$\therefore$ Centre and the radius of (i) are $(-g, 0)$ and $\sqrt{(g^2 - c)}$ respectively. Let $\sqrt{g^2 - c} = 0 \Rightarrow g = \pm\sqrt{c}$

Thus we get the two limiting points of the given co-axial system as $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$

Clearly the above limiting points are real and distinct, real and coincident or imaginary according as $c >, =, < 0$

(2) **System of co-axial circles whose two limiting points are given :** Let (α, β) and (γ, δ) be the two given limiting points. Then the corresponding point circles with zero radii are

$(x - \alpha)^2 + (y - \beta)^2 = 0$ and $(x - \gamma)^2 + (y - \delta)^2 = 0$

or $x^2+y^2-2\alpha x-2\beta y+\alpha^2+\beta^2=0$ and $x^2+y^2-2\gamma x-2\delta y+\gamma^2+\delta^2=0$

The equation of co-axial system is $(x^2+y^2-2\alpha x-2\beta y+\alpha^2+\beta^2)+\lambda(x^2+y^2-2\gamma x-2\delta y+\gamma^2+\delta^2)=0$

where $\lambda \neq -1$ is a variable parameter.

$\Rightarrow \quad x^2(1+\lambda)+y^2(1+\lambda)-2x(\alpha+\gamma\lambda)-2y(\beta+\delta\lambda)+(\alpha^2+\beta^2)+\lambda(\gamma^2+\delta^2)=0$

or $\quad x^2+y^2-\dfrac{2(\alpha+\gamma\lambda)}{(1+\lambda)}x-2\dfrac{(\beta+\delta\lambda)}{(1+\lambda)}y+\dfrac{(\alpha^2+\beta^2)+\lambda(\gamma^2+\delta^2)}{(1+\lambda)}=0$

Centre of this circle is $\left(\dfrac{(\alpha+\gamma\lambda)}{(1+\lambda)}, \dfrac{(\beta+\delta\lambda)}{(1+\lambda)}\right)$ …..(i)

For limiting point, radius $=\sqrt{\dfrac{(\alpha+\gamma\lambda)^2}{(1+\lambda)^2}+\dfrac{(\beta+\delta\lambda)^2}{(1+\lambda)^2}-\dfrac{(\alpha^2+\beta^2)+\lambda(\gamma^2+\delta^2)}{(1+\lambda)}}=0$

After solving, find λ. Substituting value of λ in (i), we get the limiting point of co-axial system.

(3) **Properties of limiting points**

(i) The limiting point of a system of co-axial circles are conjugate points with respect to any member of the system : Let the equation of any circle be $x^2+y^2+2gx+c=0$ ….(i)

Limiting points of (i) are $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$. The polar of the point $(\sqrt{c}, 0)$ with respect to (i) is

$x\sqrt{c}+y.0+g(x+\sqrt{c})+c=0$ or $x\sqrt{c}+g(x+\sqrt{c})+c=0$ or $(x+\sqrt{c})(g+\sqrt{c})=0$ or $x+\sqrt{c}=0$ and it clearly passes through the other limiting point $(-\sqrt{c}, 0)$. Similarly polar of the point $(-\sqrt{c}, 0)$ with respect to (i) also passes through $(\sqrt{c}, 0)$. Hence the limiting points of a system of co-axial circles are conjugate points.

(ii) Every circle through the limiting points of a co-axial system is orthogonal to all circles of the system :

Let the equation of any circle be $x^2+y^2+2gx+c=0$ …..(i)

where g is a parameter and c is constant. Limiting points of (i) are $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$

Now let $x^2+y^2+2g'x+2f'y+c'=0$ …..(ii)

be the equation of any circle. If it passes through the limiting points of (i), then $c+2g'\sqrt{c}+c'=0$ and $c-2g'\sqrt{c}+c'=0$. Solving, we get $c'=-c$ and $g'=0$

From (ii), $x^2+y^2+2f'y-c=0$ …..(iii)

where c is constant and f' is variable. Applying the condition of orthogonality on (i) and (iii) *i.e.,* $2g_1g_2+2f_1f_2=c_1+c_2$ we find that $2\times g\times 0+2\times 0\times f'=c-c$ *i.e.,* $0=0$

Hence condition is satisfied for all values of g' and f'.

Example : 49 The point (2, 3) is a limiting point of a co-axial system of circles of which $x^2+y^2=9$ is a member. The coordinates of the other limiting point is given by **[MP PET 1993]**

(a) $\left(\dfrac{18}{13}, \dfrac{27}{13}\right)$ (b) $\left(\dfrac{9}{13}, \dfrac{6}{13}\right)$ (c) $\left(\dfrac{18}{13}, -\dfrac{27}{13}\right)$ (d) $\left(-\dfrac{18}{13}, -\dfrac{9}{13}\right)$

Solution : (a) Equation of circle with (2, 3) as limiting point is $(x-2)^2+(y-3)^2=0$

or $(x^2+y^2-9)-4x-6y+22=0$ or $(x^2+y^2-9)-\lambda(2x+3y-11)=0$ represents the family of co-axial circles.

$c=\left(\lambda, \dfrac{3\lambda}{2}\right)$, $\quad r=\sqrt{\lambda^2+\dfrac{9\lambda^2}{4}-11\lambda+9}$. For limiting points $r=0 \Rightarrow 13\lambda^2-44\lambda+36=0 \quad \Rightarrow \lambda=\dfrac{18}{13}, 2$

$\therefore$ The limiting points are (2, 3) and $\left[\frac{18}{13}, \frac{3}{2}\left(\frac{18}{13}\right)\right]$ or $\left(\frac{18}{13}, \frac{27}{13}\right)$.

Example : 50 In the co-axial system of circle $x^2 + y^2 + 2gx + c = 0$ where g is a parameter, if $c > 0$. Then the circles are **[Karnataka CET 1999]**

(a) Orthogonal (b) Touching type (c) Intersecting type (d) Non intersecting type

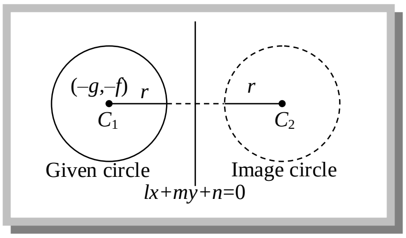
Solution : (d) The equation of a system of circle with its centre on the axis of x is $x^2 + y^2 + 2gx + c = 0$. Any point on the radical axis is $(0, y_1)$

Putting, $x = 0$, $y = \pm\sqrt{-c}$

If c is positive ($c > 0$), we have no real point on radical axis, then circles are said to be non-intersecting type.

4.25 Image of the Circle by the Line Mirror

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ and line mirror $lx + my + n = 0$. In this condition, radius of circle remains unchanged but centre changes. Let the centre of image circle be (x_1, y_1).



Slope of C_1C_2 × slope of $lx + my + n = -1$ …..(i)

and mid point of $C_1(-g, -f)$ and $C_2(x_1, y_1)$ lie on $lx + my + n = 0$

i.e., $l\left(\frac{x_1 - g}{2}\right) + m\left(\frac{y_1 - f}{2}\right) + n = 0$ …..(ii)

Solving (i) and (ii), we get (x_1, y_1)

$\therefore$ Required image circle is $(x - x_1)^2 + (y - y_1)^2 = r^2$, where $r = \sqrt{(g^2 + f^2 - c)}$

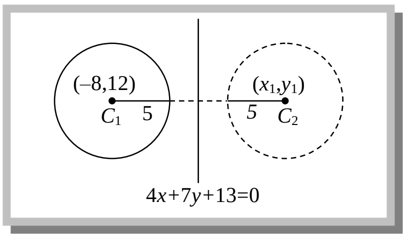
Example : 51 The equation of the image of the circle $x^2 + y^2 + 16x - 24y + 183 = 0$ by the line mirror $4x + 7y + 13 = 0$ is

(a) $x^2 + y^2 + 32x - 4y + 235 = 0$ (b) $x^2 + y^2 + 32x + 4y - 235 = 0$

(c) $x^2 + y^2 + 32x - 4y - 235 = 0$ (d) $x^2 + y^2 + 32x + 4y + 235 = 0$

Solution : (d) The given circle and line are $x^2 + y^2 + 16x - 24y + 183 = 0$ …..(i) and $4x + 7y + 13 = 0$ …..(ii)

Centre and radius of circle (i) are (– 8, 12) and 5 respectively. Let the centre of the image circle be (x_1, y_1).



Then slope of C_1C_2 × slope of $4x + 7y + 13 = -1$

$\Rightarrow \left(\frac{y_1 - 12}{x_1 + 8}\right) \times \left(-\frac{4}{7}\right) = -1$ or $4y_1 - 48 = 7x_1 + 56$

or $7x_1 - 4y_1 + 104 = 0$ …..(iii)

and mid point of C_1C_2 *i.e.*, $\left(\frac{x_1 - 8}{2}, \frac{y_1 + 12}{2}\right)$ lie on $4x + 7y + 13 = 0$,

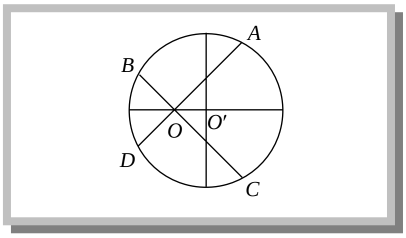
then $4\left(\frac{x_1 - 8}{2}\right) + 7\left(\frac{y_1 + 12}{2}\right) + 13 = 0$ or $4x_1 + 7y_1 + 78 = 0$ …..(iv)

Solving (iii) and (iv), we get $(x_1, y_1) = (-16, -2)$

$\therefore$ Equation of the image circle is $(x + 16)^2 + (y + 2)^2 = 5^2$ or $x^2 + y^2 + 32x + 4y + 235 = 0$

4.26 Some Important Results

(1) **Concyclic points :** If A, B, C, D are concyclic then $OA . OD = OC . OB$, where O' be the centre of the circle.



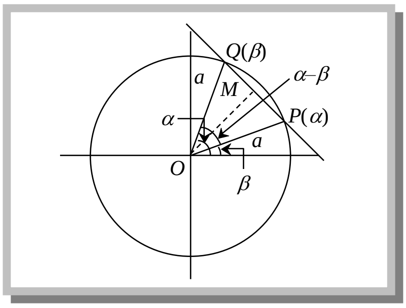
(2) Equation of the straight line joining two points α and β on the circle

$$x^2 + y^2 = a^2$$

Required equation is $x\cos\left(\frac{\alpha + \beta}{2}\right) + y\sin\left(\frac{\alpha + \beta}{2}\right) = a\cos\left(\frac{\alpha - \beta}{2}\right)$

(3) The point of intersection of the tangents at the point P (α) and Q (β) on the circle $x^2+y^2=a^2$ is

$$\left(\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \frac{a\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right)$$



(4) Maximum and Minimum distance of a point from the circle :

Let any point $P(x_1, y_1)$ and circle $x^2+y^2+2gx+2fy+c=0$ .....(i)

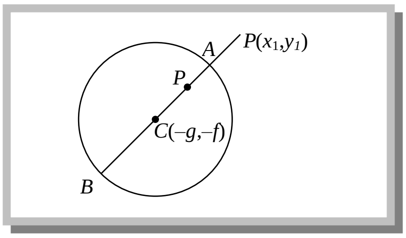
The centre and radius of the circle are

$C(-g,-f)$ and $\sqrt{(g^2+f^2-c)}$ respectively.

The maximum and minimum distance from $P(x_1, y_1)$ to the circle (i) are

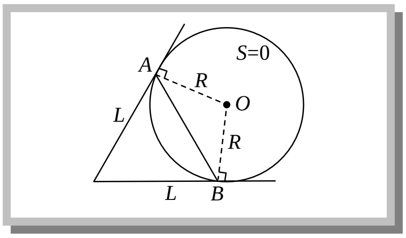
$PB = CB + PC = r + PC$ and $PA = |CP - CA| = |PC - r|$ (P inside or outside)

where $r=\sqrt{(g^2+f^2-c)}$



(5) Length of chord of contact is $AB = \dfrac{2LR}{\sqrt{(R^2+L^2)}}$

and area of the triangle formed by the pair of tangents and its chord of contact is $= \dfrac{RL^3}{R^2+L^2}$



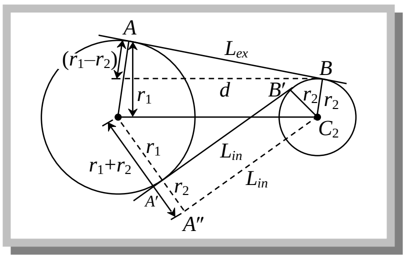
Where R is the radius of the circle and L is the length of tangent from $P(x_1, y_1)$ on $S=0$. Here $L=\sqrt{S_1}$.

(6) Length of an external common tangent and internal common tangent to two circles is given by

Length of external common tangent $L_{ex}=\sqrt{d^2-(r_1-r_2)^2}$

and length of internal common tangent

$L_{in}=\sqrt{d^2-(r_1+r_2)^2}$ [Applicable only when $d>(r_1+r_2)$]

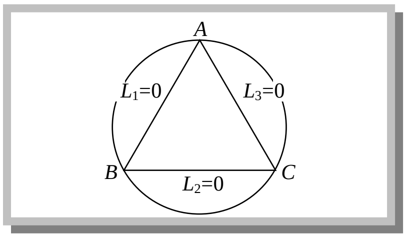


where d is the distane between the centres of two circles *i.e.*, $|C_1C_2|=d$ and r_1 and r_2 are the radii of two circles.

(7) Family of circles circumscribing a triangle whose sides are given by $L_1=0$; $L_2=0$ and $L_3=0$ is given by $L_1L_2+\lambda L_2L_3+\mu L_3L_1=0$ provided coefficient of $xy=0$ and coefficient of $x^2=$coefficient of y^2.

Equation of the circle circumscribing the triangle formed by the lines $a_rx+b_ry+c_r=0$, where r = 1, 2, 3, is

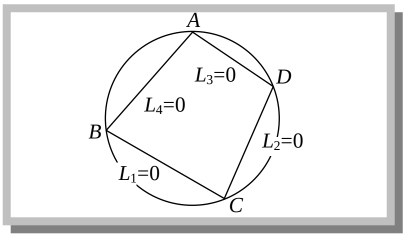
$$\begin{vmatrix} \dfrac{a_1^2+b_1^2}{a_1x+b_1y+c_1} & a_1 & b_1 \\ \dfrac{a_2^2+b_2^2}{a_2x+b_2y+c_2} & a_2 & b_2 \\ \dfrac{a_3^2+b_3^2}{a_3x+b_3y+c_3} & a_3 & b_3 \end{vmatrix} = 0$$



(8) Equation of circle circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0,\ L_2 = 0,\ L_3 = 0$ and $L_4 = 0$ is given by

$$L_1 L_3 + \lambda L_2 L_4 = 0$$

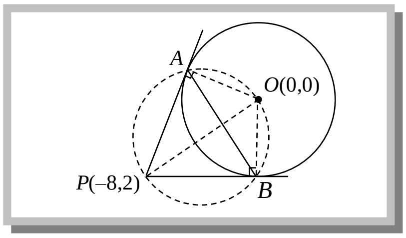
provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$



(9) Equation of the circle circumscribing the triangle *PAB* is

$$(x - x_1)(x + g) + (y - y_1)(y + f) = 0$$

where $O(-g, -f)$ is the centre of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$



(10) Locus of mid point of a chord of a circle $x^2 + y^2 = a^2$, which subtends an angle α at the centre is $x^2 + y^2 = (a\cos\alpha/2)^2$

(11) The locus of mid point of chords of circle $x^2 + y^2 = a^2$, which are making right angle at centre is $x^2 + y^2 = \frac{a^2}{2}$

(12) The locus of mid point of chords of circle $x^2 + y^2 + 2gx + 2fy + c = 0$, which are making right angle at origin is $x^2 + y^2 + gx + fy + c/2 = 0$.

(13) The area of triangle , which is formed by co-ordinate axes and the tangent at a point (x_1, y_1) of circle $x^2 + y^2 = a^2$ is $a^4 / 2x_1 y_1$

(14) If a point is outside, on or inside the circle then number of tangents from the points is 2, 1 or none.

(15) A variable point moves in such a way that sum of square of distances from the vertices of a triangle remains constant then its locus is a circle whose centre is the centroid of the triangle.

(16) If the points where the line $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ meets the coordinate axes are concyclic then $a_1 a_2 = b_1 b_2$.

Example : 52 If $\left(m_i, \frac{1}{m_i}\right)$, $i = 1, 2, 3, 4$ are concylic points, then the value of $m_1.m_2.m_3.m_4$ is **[Karnataka CET 2002]**

(a) 1 (b) -1 (c) 0 (d) None of these

Solution : (a) Let the equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Since the point $\left(m_i, \frac{1}{m_i}\right)$ lies on this circle

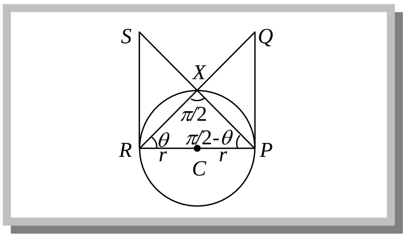
$$\therefore\ m_i^2 + \frac{1}{m_i^2} + 2gm_i + \frac{2f}{m_i} + c = 0 \Rightarrow m_i^4 + 2gm_i^3 + cm_i^2 + 2fm_i + 1 = 0$$

Clearly its roots are m_1, m_2, m_3 and m_4, $\therefore$ $m_1.m_2.m_3.m_4$ = product of roots $= \frac{1}{1} = 1$

Example : 53 Let *PQ* and *RS* be tangents at the extremities of the diameter *PR* of a circle of radius *r*. If *PS* and *RQ* intersect at a point *X* on the circumference of the circle, then 2*r* equals **[IIT Screening 2001]**

(a) $\sqrt{PQ\,.\,RS}$ (b) $\dfrac{PQ+RS}{2}$ (c) $\dfrac{2\,PQ\,.\,RS}{PQ+RS}$ (d) $\dfrac{\sqrt{PQ^2+RS^2}}{2}$

Solution : (a) $\tan\theta = \dfrac{PQ}{PR} = \dfrac{PQ}{2r}$



Also $\tan\left(\dfrac{\pi}{2}-\theta\right) = \dfrac{RS}{2r}$

i.e. $\cot\theta = \dfrac{RS}{2r}$

$\therefore\ \tan\theta\,.\cot\theta = \dfrac{PQ\,.\,RS}{4r^2}$

$\Rightarrow\ 4r^2 = PQ\,.\,RS \ \Rightarrow\ 2r = \sqrt{(PQ)\,(RS)}$

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