7. Linear Equations

Equations of Straight Lines

When working with straight lines, there are several ways to arrive at an equation which represents the line.

General Form of Equation of a Line

The "General Form" of the equation of a straight line is:

Ax + By + C = 0

FORMS OF EQUATIONS OF STRAIGHT LINE

- Lines Parallel to Axes : Equation of straight line parallel to x-axis at a distance 'a' is $y = a$ and equation of straight line parallel to y-axis at a distance 'b' is $x = b$.
- Point-Slope Form : The equation of the straight line passing through the point (x_1, y_1) and having slope m is : $y - y_1 = m(x - x_1)$.
- Two-Point Form : The equation of the straight line passing through two point P (x_1, y_1) and Q (x_2, y_2) is :

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}\ x - x_1$$

or

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Remember:	**Slope** is found by using the formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$	**Slope** is also expressed as rise/run.

Equation Forms of Straight Lines

Slope Intercept Form	Point Slope Form
Use this form when you know the **slope** and the y-**intercept** (where the line crosses the y-axis).	Use this form when you know a **point** on the line and the **slope** (or can determine the slope).
$y = mx + b$	$y - y_1 = m(x - x_1)$
m = slope b = y-intercept (where line crosses the y-axis.)	m = slope (x_1, y_1) = any point on the line

Horizontal Lines	Vertical Lines
$y = 3$ (or any number) **Lines that are horizontal have a slope of zero.** Horizontal lines have "run", but no "rise". The rise/run formula for slope always yields zero since the **rise = 0.** Since the slope is zero, we have $y = mx + b$ $y = 0 \cdot x + 3$ $y = 3$ This equation also describes what is happening to the y-coordinates on the line. In this case the y-coordinates are always 3.	$x = -2$ (or any number) **Lines that are vertical have no slope (it does not exist).** Vertical lines have "rise", but no "run". The rise/run formula for slope always has a zero denominator and is undefined. The equations for these lines describe what is happening to the x-coordinates. In this example, the x-coordinates are always equal to -2.

Examples:

Examples using Slope-Intercept Form:	Examples using Point-Slope Form:
1. Find the slope and y-intercept for the equation $2y = -6x + 8$. First solve for "y =": $y = -3x + 4$ Remember the form: $y = mx + b$ **Answer:** the slope (m) is -3 the y-intercept (b) is 4	3. Given that the slope of a line is -3 and the line passes through the point (-2,4), write the equation of the line. The slope: $m = -3$ The point $(x_1, y_1) = (-2,4)$ Remember the form: $y - y_1 = m(x - x_1)$ Substitute: $y - 4 = -3(x - (-2))$ ANS. $y - 4 = -3(x + 2)$ If asked to express the answer in "y =" form: $y - 4 = -3x - 6$ $y = -3x - 2$
2. Find the equation of the line whose slope is 4 and the coordinates of the y-intercept are (0,2). In this problem $m = 4$ and $b = 2$. Remember the form: $y = mx + b$ and that b is where the line crosses the y-axis. Substitute: $y = 4x + 2$	4. Find the slope of the line that passes through the points (-3,5) and (-5,-8). First, find the slope: $m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{5-(-8)}{-3-(-5)} = \frac{13}{2} = 6.5$ Use either point: (-3,5) Remember the form: $y - y_1 = m(x - x_1)$ Substitute: $y - 5 = 6.5(x - (-3))$ $y - 5 = 6.5(x + 3)$ Ans.

Linear equation in one variable is an equation which can be written in the form of $ax + b = 0$, where a and b are real-number constants and $a \neq 0$.

Ex.

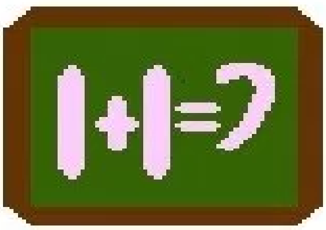
$x + 7 = 12$

Equation is a mathematical sentence indicating that two expressions are equal. The symbol "=" is used to indicate equality.

Ex.

$2x + 5 = 9$ is a conditional equation since its truth or falsity depends on the value of x

$2 + 9 = 11$ is identity equation since both of its sides are identical to the same number 11.



Solution Set of a Linear Equation

Example

$4x + 2 = 10$ this statement is either true of false

If $x = 1$, then $4x + 2 = 10$ is false because $4(1) + 2$ is $\neq 10$

If $x = 2$, then $4x + 2 = 10$ is true because $4(2) + 2 = 10$

A statement of equality of two algebraic expressions, which involve one or more unknown quantities is known as an equation.
A linear equation is an equation which involves linear polynomials.
A value of the variable which makes the two sides of the equation equal is called the solution of the equation.
Same quantity can be added/subtracted to/from both the sides of an equation without changing the equality.
Both the sides of an equation can be multiplied/divided by the same non-zero number without changing the equality.

GENERAL FORM OF LINEAR EQUATION IN TWO VARIABLES

$ax + by + c = 0$, $a \neq 0$, $b \neq 0$ or any one from a & b can zero.

General Form Of Linear Equation In Two Variables Example Problems With Solutions

Example 1: Express the following linear equations in general form and identify coefficients of x, y and constant term.

Solution:

S.No.	Equation	General form	Coeff. of x, y, constant
(1)	$3x - 2y = 5$	$3x - 2y - 5 = 0$	3, –2, –5
(2)	$\frac{3}{7}x - 2 + y = 0$	$\frac{3}{7}x + y - 2 = 0$	$\frac{3}{7}$, 1, –2
(3)	$5y = 2x + 7$	$2x - 5y + 7 = 0$	2, –5, 7
(4)	$18y - 72x = 8$	$72x - 18y + 8 = 0$	72, –18, 8
(5)	$3.\bar{7}x - y - \frac{1}{7} = 0$	$3.\bar{7}x - y - \frac{1}{7} = 0$	$3.\bar{7}, -1, -\frac{1}{7}$
(6)	$y = 5$	$0x + y - 5 = 0$	0, 1, –5
(7)	$\frac{x}{7} = 5$	$\frac{x}{7} + 0.y - 5 = 0$	$\frac{1}{7}$, 0, –5
(8)	$2x + 3 = 0$	$2x + 0y + 3 = 0$	2, 0, 3

Make linear equation by the following statements:

Example 2: The cost of 2kg of apples and 1 kg of grapes on a day was found to be 160. After a month, the cost of 4 kg of apples and 2 kg of grapes is 300. Represent the situation algebraically.

Solution: Let cost of per kg apples & grapes are x & y respectively then by Ist condition:
$2x + y = 160$ ……(i)
& by IInd condition: $4x + 2y = 300$ …..(ii)

Example 3: The coach of a cricket team buys 3 bats and 6 balls for 3900. Later, she buys another bat and 3 more balls of the same kind for 1300. Represent this situation algebraically.

Solution: Let cost of a bat and a ball are x & y respectively. According to questions
$3x + 6y = 3900$ …..(i)
& $x + 3y = 1300$ …..(ii)

Example 4: 10 students of class IX took part in a Mathematics quiz. If the number of girls is 4 more than the number of boys.

Solution: Let no. of boys and girls are x & y then according to question
$x + y = 10$ ……(i)

& $y = x + 4$ ……(ii)

Example 5: Half the perimeter of a rectangular garden, whose length is 4 m more than its width, is
36 m.

Solution: Let length & breadth are x m and y m.
∴ according to question 1/2 perimeter = 36
$1/2\ [2(l + b)] = 36$
$\Rightarrow x + y = 36$ …..(i)
also length = 4 + breadth
$x = 4 + y$ …..(ii)

Example 6: The difference between two numbers is 26 and one number is three times the other.

Solution: Let the numbers are x and y & x > y
$\therefore x - y = 26$ ……(i)
and $x = 3y$ ……(ii)

Example 7: The larger of two supplementary angles exceeds the smaller by 18 degrees.

Solution: Sol. Let the two supplementary angles are x and y & x > y
Then $x + y = 180°$ ……(i)
and $x = y + 18°$ ……(ii)

Example 8: A fraction becomes 9/11, if 2 is added to both the numerator and the denominator. If, 3 is added to both the numerator and the denominator it becomes 5/6.

Solution: Let fraction is x/y

Now according to question $\frac{x+2}{y+2} = \frac{9}{11}$
$\Rightarrow 11x + 22 = 9y + 18$
$\Rightarrow 11x - 9y = -4$ …..(i)
and
$\frac{x+3}{y+3} = \frac{5}{6}$
$\Rightarrow 6x + 18 = 5y + 15$
$\Rightarrow 6x - 5y = -3$ ….(ii)

Example 9: Five years hence, the age of Sachin will be three times that of his son. Five years ago, Sachin's age was seven times that of his son.

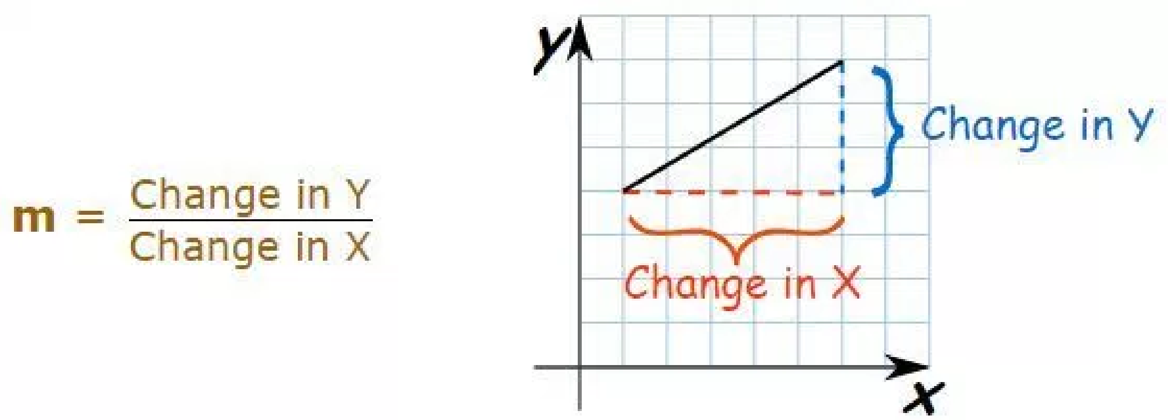
Solution: Let present ages of Sachin & his son are
x years and y years.
Five years hence,
age of Sachin = (x + 5) years & his son's age = (y + 5) years
according to question (x + 5) = 3(y + 5)
$\Rightarrow x + 5 = 3y + 15$
$\Rightarrow x - 3y = 10$ ......(i)
and 5 years ago age of both were (x – 5) years and (y – 5) years respectively
according to question (x – 5) = 7(y – 5)
$\Rightarrow x - 5 = 7y - 35$
$\Rightarrow x - 7y = -30$ ....(ii)

Point-Slope Equation of a Line

Equations of straight line in different forms

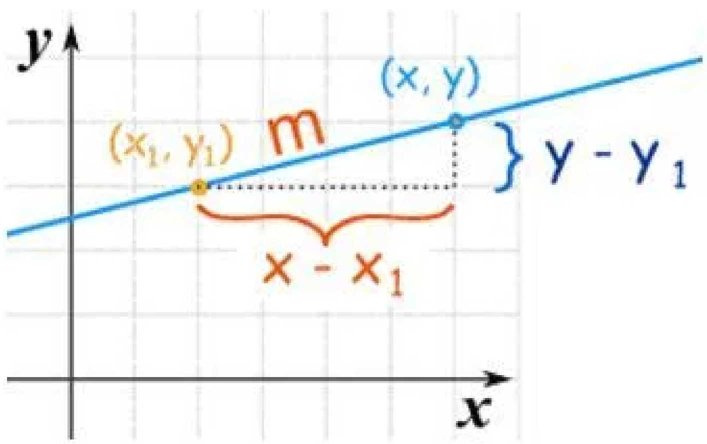
(1) Slope form:

Equation of a line through the origin and having slope m is **y = mx**.



(2) One point form or Point slope form:

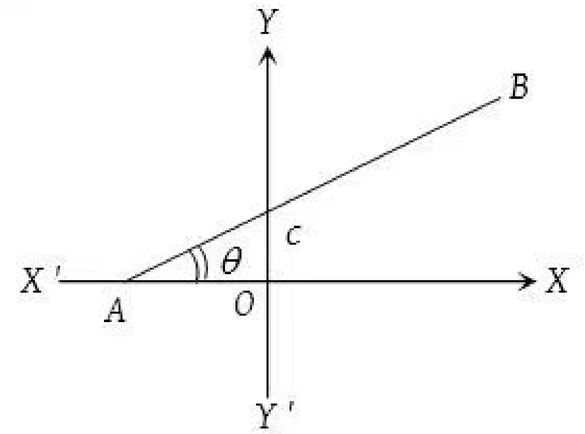
Equation of a line through the point (x_1, y_1) and having slope m is $\mathbf{y - y_1 = m(x - x_1)}$.



$$\text{Slope } m = \frac{\text{change in } y}{\text{change in } x} = \frac{y - y_1}{x - x_1}$$

(3) Slope intercept form:

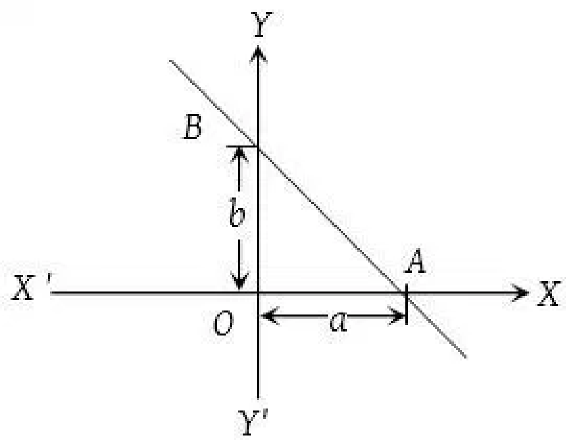
Equation of a line (non-vertical) with slope m and cutting off an intercept c on the y-axis is **$y = mx + c$**.



The equation of a line with slope m and the x-intercept d is $y = m(x - d)$.

(4) Intercept form:

If a straight line cuts x-axis at A and the y-axis at B then OA and OB are known as the intercepts of the line on x-axis and y-axis respectively.

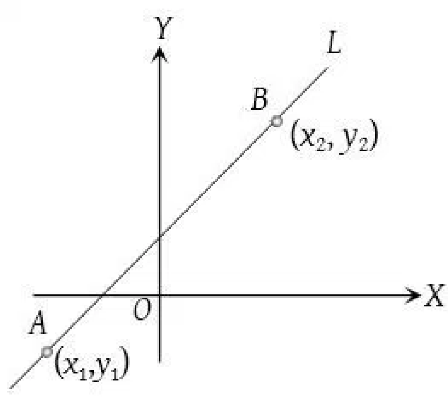


Then, equation of a straight line cutting off intercepts a and b on x–axis and y–axis respectively is $\frac{x}{a} + \frac{y}{b} = 1$.
If given line is parallel to X axis, then X-intercept is undefined.
If given line is parallel to Y axis, then Y-intercept is undefined.

(5) Two point form:

Equation of the line through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is,

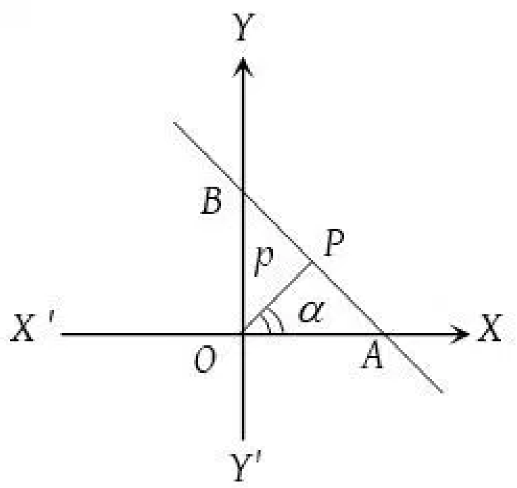
$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$



In the determinant form it is gives as $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ is the equation of line.

(6) Normal or perpendicular form:

The equation of the straight line upon which the length of the perpendicular from the origin is p and this perpendicular makes an angle α with x-axis is x cos α + y sin α = p.

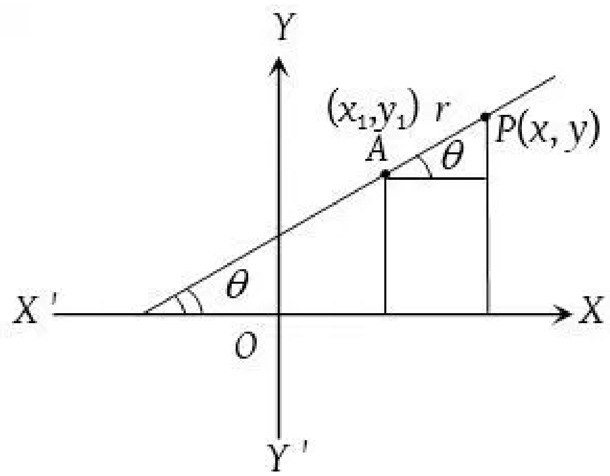


(7) Symmetrical or parametric or distance form of the line:

Equation of a line passing through (x_1, y_1) and making an angle θ with the positive direction of x-axis is ,

$$\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = \pm r$$

where r is the distance between the point P(x, y) and $A(x_1, y_1)$.



The co-ordinates of any point on this line may be taken as $(x_1 \pm r \cos \theta, y_1 \pm r \sin \theta)$, known as parametric co-ordinates. 'r' is called the parameter.

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Forms of Equations of Straight Line

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Remember:	**Slope** is found by using the formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$	**Slope** is also expressed as rise/run.

Equation Forms of Straight Lines

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Horizontal Lines	Vertical Lines
$y = 3$ (or any number) **Lines that are horizontal have a slope of zero.** Horizontal lines have "run", but no "rise". The rise/run formula for slope always yields zero since the **rise = 0.** Since the slope is zero, we have $y = mx + b$ $y = 0 \cdot x + 3$ $y = 3$ This equation also describes what is happening to the y-coordinates on the line. In this case the y-coordinates are always 3.	$x = -2$ (or any number) **Lines that are vertical have no slope (it does not exist).** Vertical lines have "rise", but no "run". The rise/run formula for slope always has a zero denominator and is undefined. The equations for these lines describe what is happening to the x-coordinates. In this example, the x-coordinates are always equal to -2.

Examples:

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2. Find the equation of the line whose slope is 4 and the coordinates of the y-intercept are (0,2). In this problem $m = 4$ and $b = 2$. Remember the form: $y = mx + b$ and that b is where the line crosses the y-axis. Substitute: $\mathbf{y = 4x + 2}$	4. Find the slope of the line that passes through the points (-3,5) and (-5,-8). First, find the slope: $m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{5-(-8)}{-3-(-5)} = \frac{13}{2} = 6.5$ Use either point: (-3,5) Remember the form: $y - y_1 = m(x - x_1)$ Substitute: $y - 5 = 6.5(x - (-3))$ $y - 5 = 6.5(x + 3)$ Ans.

Midpoint of a Line Segment

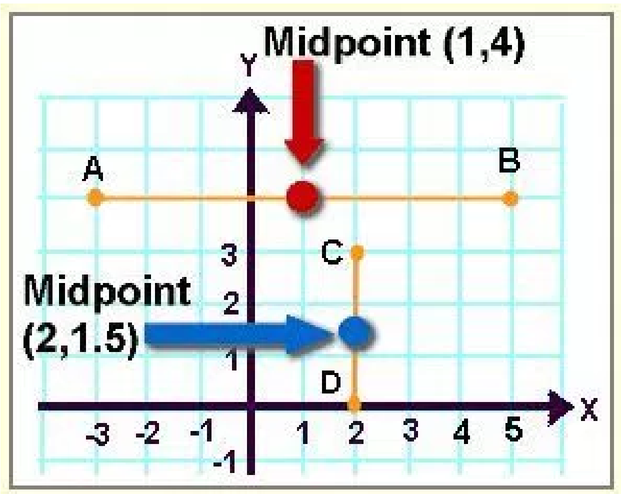
The point halfway between the endpoints of a line segment is called the **midpoint**. A midpoint divides a line segment into two equal segments.

By definition, a **midpoint of a line segment** is the point on that line segment that divides the segment two congruent segments.
In Coordinate Geometry, there are several ways to determine the midpoint of a line segment.

Method 1:

If the line segments are vertical or horizontal, you may find the midpoint by simply dividing the length of the segment by 2 and counting that value from either of the endpoints.

Find the midpoints $\overline{AB}$ and $\overline{CD}$.



AB is 8 (by counting). The midpoint is 4 units from either endpoint. On the graph, this point is (1,4).
CD is 3 (by counting). The midpoint is 1.5 units from either endpoint. On the graph, this point is (2,1.5)

Method 2:

If the line segments are diagonally positioned, more thought must be paid to the solution. When you are finding the coordinates of the midpoint of a segment, you are actually finding the average (mean) of the x-coordinates and the average (mean) of the y-coordinates.

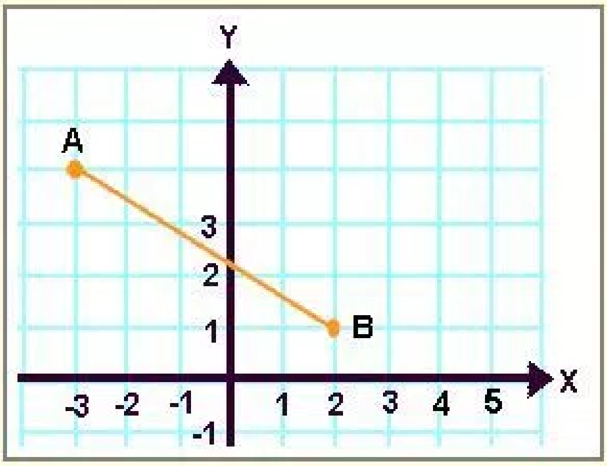
This concept of finding the average of the coordinates can be written as a formula:

The Midpoint Formula:

The midpoint of a segment with endpoints (x_1, y_1) and (x_2, y_2) has coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

NOTE: The Midpoint Formula works for all line segments: vertical, horizontal or diagonal.



Find the midpoint of line segment $\overline{AB}$.

$A(-3,4)$

$B(2,1)$

The midpoint will have coordinates

$$\left(\frac{-3+2}{2}, \frac{4+1}{2}\right)$$

$$\left(\frac{-1}{2}, \frac{5}{2}\right)$$

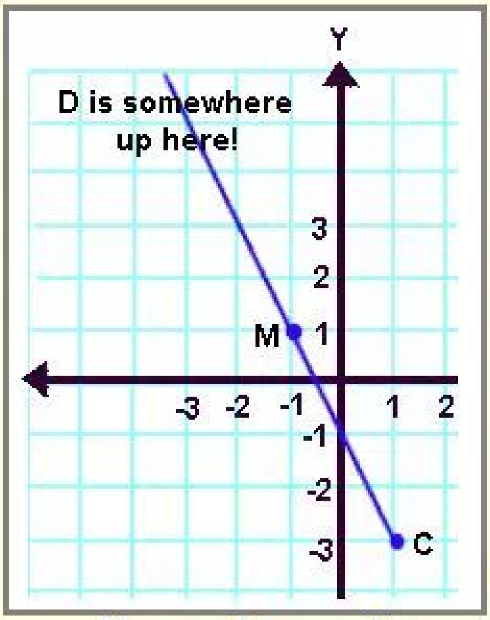
$(-0.5, 2.5)$ Answer

NOTE: Don't be surprised if your answer contains a fraction. Answers may be left in fractional form or written as decimals.

Consider this "tricky" midpoint problem:

M is the midpoint of $\overline{CD}$. The coordinates M(-1,1) and C(1,-3) are given. Find the coordinates of point D.

First, visualize the situation. This will give you an idea of approximately where point D will be located. When you find your answer, be sure it matches with your visualization of where the point should be located.



The coordinates of point D are (-3,5).

Solve algebraically:

$M(-1,1)$, $C(1,-3)$ and $D(x,y)$

Substitute into the Midpoint Formula:

$$(-1,1)=\left(\frac{x+1}{2},\frac{y+(-3)}{2}\right)$$

Solve for each variable separately:

$$\frac{x+1}{2}=-1 \qquad \frac{y+(-3)}{2}=1$$

$$x+1=-2 \qquad y+(-3)=2$$

$$x=-3 \qquad y=5$$

Other Methods of Solution:

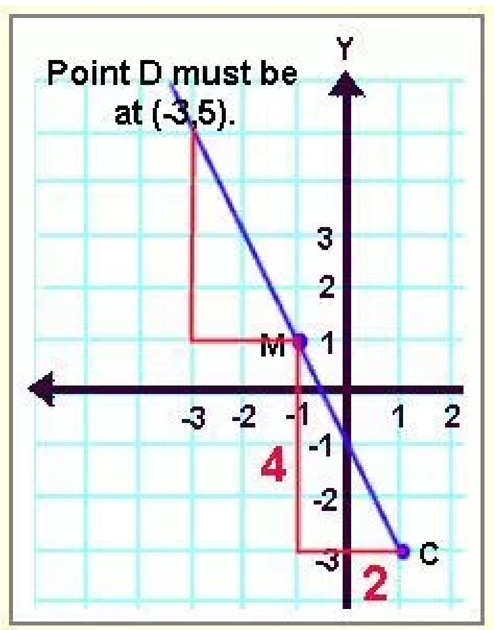
Verbalizing the algebraic solution:

Some students like to do these “tricky” problems by just examining the coordinates and asking themselves the following questions:
“My midpoint’s x-coordinate is -1. What is -1 half of? (Answer -2)
What do I add to my endpoint’s x-coordinate of +1 to get -2? (Answer -3)
This answer must be the x-coordinate of the other endpoint.”
These students are simply verbalizing the algebraic solution.
(They use the same process for the y-coordinate.)

Utilizing the concept of slope and congruent triangles:

A line segment is part of a straight line whose slope (rise/run) remains the same no matter where it is measured. Some students like to look at the rise and run values of the x and y coordinates and utilize these values to find the missing endpoint.

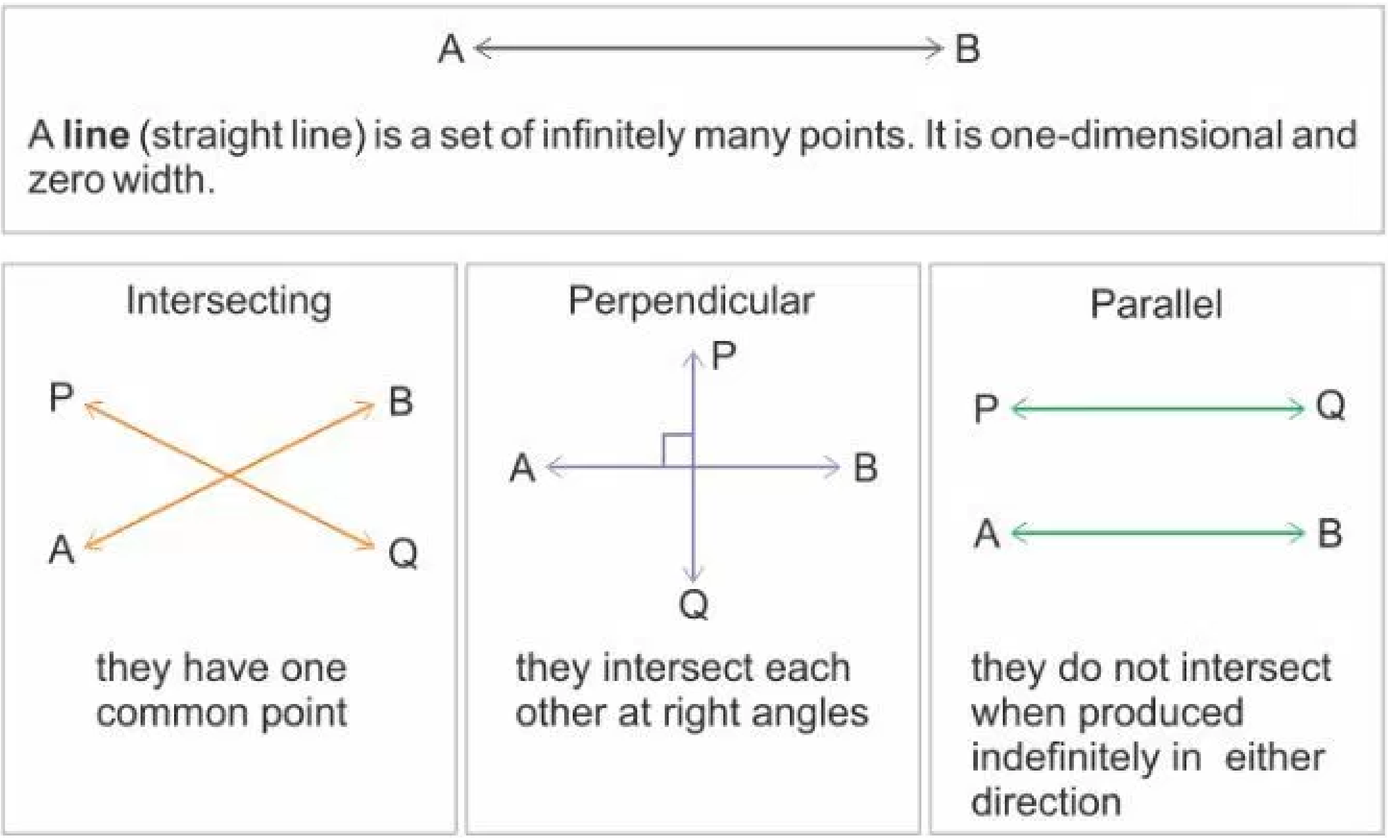
Find the slope between points C and M. This slope has a run of 2 units to the left and a rise of 4 units up. By repeating this slope from point M (move 2 units to the left and 4 units up), you will arrive at the other endpoint.



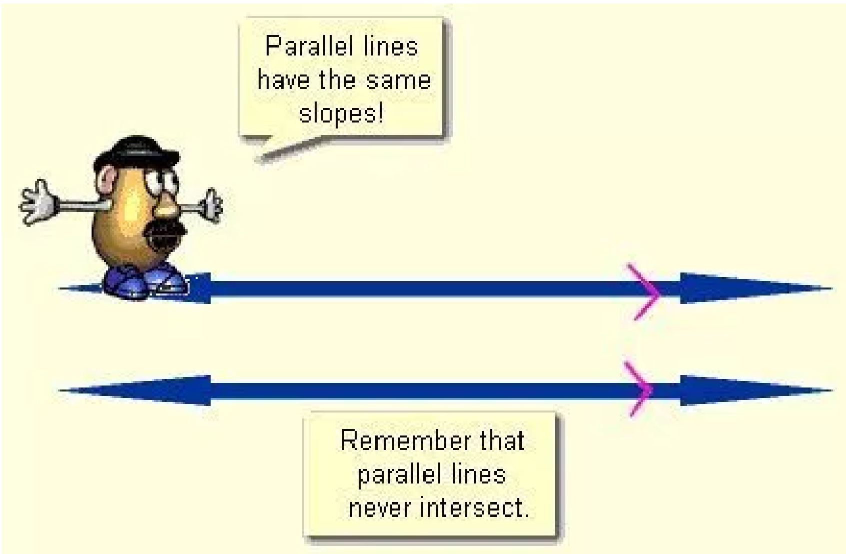
By using this slope approach, you are creating two congruent right triangles whose legs are the same lengths. Consequently, their hypotenuses are also the same lengths and DM = MC making M the midpoint of $\overline{CD}$.

Parallel and Perpendicular Lines

Mathematics problems often deal with parallel and perpendicular lines.
Since these are such popular lines, it is important that we remember some information about their slopes.



Parallel Lines: (same slope!)



Parallel lines are marked with "feathers" to show that they are parallel.
These "feathers" look like "greater than" symbols.

Parallel lines have the same slope.
The symbol to indicate parallel lines is two vertical bars.
It looks something like the number 11.

$$l_1 \parallel l_2 \rightarrow m_1 = m_2$$

where l_1 and l_2 are lines
m_1 and m_2 are slopes

$y = 3x + 5$ $y = 3x - 7$ $y = 3x + 0.5$ $y = 3x$	These lines are ALL parallel. They all have the same slope (m). (Remember $y = mx + b$.)

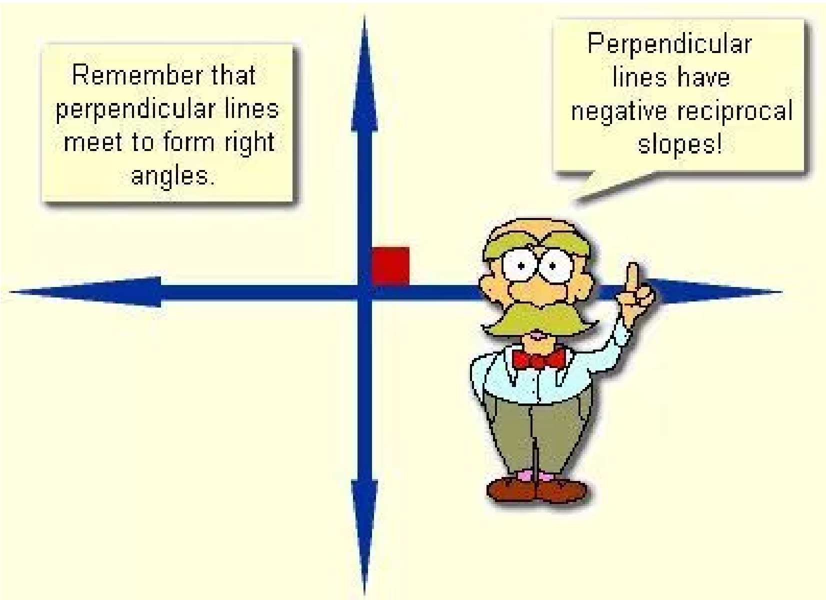
Example:

The slope of l_1 is $\frac{3}{5}$ and $l_1 \parallel l_2$. Find the slope of l_2.

Since the lines are parallel, the slopes are the same.

The slope of l_2 is also $\frac{3}{5}$. **ANSWER**

Perpendicular Lines: (negative reciprocal slopes!)



Perpendicular lines have negative reciprocal slopes.
The symbol to indicate perpendicular is an up-side-down capital T.

$$l_1 \perp l_2 \rightarrow m_1 = -\frac{1}{m_2}$$

where l_1 and l_2 are lines
m_1 and m_2 are slopes

To find a negative reciprocal of a number, flip the number over (invert) and negate that value.

$\frac{1}{2} \rightarrow -\frac{2}{1} = -2$	$-\frac{4}{5} \rightarrow \frac{5}{4}$
$3 = \frac{3}{1} \rightarrow -\frac{1}{3}$	$-5 \rightarrow \frac{1}{5}$

$$y = 4x + 7$$

$$y = -\frac{1}{4}x - 6$$

These lines are perpendicular. Their slopes (m) are negative reciprocals. (Remember $y = mx + b$.)

Example:

The slope of l_1 is $\frac{3}{5}$ and $l_1 \perp l_2$. Find the slope of l_2.

Since the lines are perpendicular, the slopes are negative reciprocals.

The slope of l_2 is $-\frac{5}{3}$. **ANSWER**