

BINOMIAL THEOREM & MATHEMATICAL INDUCTION

BINOMIAL THEOREM

If $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$$(a + b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_n a^0 b^n$$

REMARKS :

1. If the index of the binomial is n then the expansion contains $n + 1$ terms.
2. In each term, the sum of indices of a and b is always n .
3. Coefficients of the terms in binomial expansion equidistant from both the ends are equal.
4. $(a-b)^n = {}^nC_0 a^n b^0 - {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 - \dots + (-1)^n {}^nC_n a^0 b^n$.

GENERAL TERM AND MIDDLE TERMS IN EXPANSION OF $(A + B)^N$

$$t_{r+1} = {}^nC_r a^{n-r} b^r$$

t_{r+1} is called a general term for all $r \in \mathbb{N}$ and $0 \leq r \leq n$.
Using this formula we can find any term of the expansion.

MIDDLE TERM (S) :

1. In $(a + b)^n$ if n is even then the number of terms in the expansion is odd. Therefore there is only one

middle term and it is $\left(\frac{n+2}{2}\right)^{\text{th}}$ term.

2. In $(a + b)^n$, if n is odd then the number of terms in the expansion is even. Therefore there are two middle terms and those are

$\left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+3}{2}\right)^{\text{th}}$ terms.

BINOMIAL THEOREM FOR ANY INDEX

If n is negative integer then $n!$ is not defined. We state binomial theorem in another form.

$$(a+b)^n = a^n + \frac{n}{1!} a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \dots$$

$$+ \frac{n(n-1)(n-2)}{3!} a^{n-3} b^3 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!} a^{n-r} b^r + \dots$$

$$\text{Here } t_{r+1} = \frac{(n-1)(n-2)\dots(n-r+1)}{r!} a^{n-r} b^r$$

THEOREM:

If n is any real number, $a = 1$, $b = x$ and $|x| < 1$ then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots$$

Here there are infinite number of terms in the expansion,
The general term is given by

$$t_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+1)x^r}{r!}, r \geq 0$$

Note...

- (i) Expansion is valid only when $-1 < x < 1$
- (ii) nC_r can not be used because it is defined only for natural number, so nC_r will be written as $\frac{n(n-1)\dots(n-r+1)}{r!}$
- (iii) As the series never terminates, the number of terms in the series is infinite.
- (iv) General term of the series $(1+x)^{-n} = T_{r+1} \rightarrow (-1)^r \frac{1 + x}{1 - x}$ if $|x| < 1$
- (v) General term of the series $(1-x)^{-n} \rightarrow T_{r+1} = \frac{(n+1)(n+2)\dots(n+r)}{r!} x^r$
- (vi) If first term is not 1, then make it unity in the following way. $(a+x)^n = a^n (1+x/a)^n$ if $\left|\frac{x}{a}\right| < 1$

REMARKS:

1. If $|x| < 1$ and n is any real number, then

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

The general term is given by

$$t_{r+1} = \frac{(-1)^r n(n-1)(n-2)\dots(n-r+1)}{r!} x^r$$

2. If n is any real number and $|b| < |a|$, then

$$= (a+b)^n = \left[a \left(1 + \frac{b}{a} \right) \right]^n$$

$$= a^n \left(1 + \frac{b}{a} \right)^n$$



While expanding $(a+b)^n$ where n is a negative integer or a fraction, reduce the binomial to the form in which the first term is unity and the second term is numerically less than unity.

Particular expansion of the binomials for negative index, $|x| < 1$

1. $\frac{1}{1+x} = (1+x)^{-1}$
 $= 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$

2. $\frac{1}{1-x} = (1-x)^{-1}$
 $= 1 + x + x^2 + x^3 + x^4 + x^5 + \dots$

3. $\frac{1}{(1+x)^2} = (1+x)^{-2}$
 $= 1 - 2x + 3x^2 - 4x^3 + \dots$

4. $\frac{1}{(1-x)^2} = (1-x)^{-2}$
 $= 1 + 2x + 3x^2 + 4x^3 + \dots$

BINOMIAL COEFFICIENTS

The coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$ in the expansion of $(a+b)^n$ are called the binomial coefficients and denoted by $C_0, C_1, C_2, \dots, C_n$ respectively

Now

$$(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 + \dots + {}^nC_n x^n \quad \dots (i)$$

Put $x = 1$.

$$(1+1)^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$$

$$\therefore 2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$$

$$\therefore {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$$

$$\therefore C_0 + C_1 + C_2 + \dots + C_n = 2^n$$

$$\therefore \text{The sum of all binomial coefficients is } 2^n.$$

Put $x = -1$, in equation (i),

$$(1-1)^n = {}^nC_0 - {}^nC_1 + {}^nC_2 - \dots + (-1)^n {}^nC_n$$

$$\therefore 0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - \dots + (-1)^n {}^nC_n$$

$$\therefore {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \dots + (-1)^n {}^nC_n = 0$$

$$\therefore {}^nC_0 + {}^nC_2 + {}^nC_4 + \dots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \dots$$

$$\therefore C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots$$

C_0, C_2, C_4, \dots are called as even coefficients

C_1, C_3, C_5, \dots are called as odd coefficients

$$\text{Let } C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = k$$

$$\text{Now } C_0 + C_1 + C_2 + C_3 + \dots + C_n = 2^n$$

$$\therefore (C_0 + C_2 + C_4 + \dots) + (C_1 + C_3 + C_5 + \dots) = 2^n$$

$$\therefore k + k = 2^n$$

$$2k = 2^n$$

$$\therefore k = \frac{2^n}{2}$$

$$\therefore k = 2^{n-1}$$

$$\therefore C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$$

$$\therefore \text{The sum of even coefficients} = \text{The sum of odd coefficients} = 2^{n-1}$$

Properties of Binomial Coefficient

For the sake of convenience the coefficients

${}^nC_0, {}^nC_1, \dots, {}^nC_r, \dots, {}^nC_n$ are usually denoted by $C_0, C_1, \dots, C_r, \dots, C_n$ respectively.

$$(i) \quad C_0 + C_1 + C_2 + \dots + C_n = 2^n$$

$$(ii) \quad C_0 - C_1 + C_2 - \dots + (-1)^n C_n = 0$$

$$(iii) \quad C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$$

$$(iv) \quad {}^nC_{r_1} = {}^nC_{r_2} \Rightarrow r_1 = r_2 \text{ or } r_1 + r_2 = n$$

$$(v) \quad {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

$$(vi) \quad r {}^nC_r = n {}^{n-1}C_{r-1}$$

BINOMIAL THEOREM & MATHEMATICAL INDUCTION

Some Important Results

- (i) $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$,
 Putting $x = 1$ and -1 , we get
 $C_0 + C_1 + C_2 + \dots + C_n = 2^n$ and
 $C_0 - C_1 + C_2 - C_3 + \dots + (-1)^n C_n = 0$
- (ii) Differentiating $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$,
 on both sides we have, $n(1+x)^{n-1}$
 $= C_1 + 2C_2x + 3C_3x^2 + \dots + nC_nx^{n-1}$ (1)
 $x = 1$
 $\Rightarrow n2^{n-1} = C_1 + 2C_2 + 3C_3 + \dots + nC_n$
 $x = -1$
 $\Rightarrow 0 = C_1 - 2C_2 + \dots + (-1)^{n-1} nC_n$.
 Differentiating (1) again and again we will have different results.

- (iii) Integrating $(1+x)^n$, we have,

$$\frac{(1+x)^{n+1}}{n+1} + C = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{n+1}$$

(where C is a constant)

Put $x = 0$, we get $C = -\frac{1}{(n+1)}$

Therefore

$$\frac{(1+x)^{n+1} - 1}{n+1} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{n+1} \quad \dots (2)$$

Put $x = 1$ in (2) we get

$$\frac{2^{n+1} - 1}{n+1} = C_0 + \frac{C_1}{2} + \dots + \frac{C_n}{n+1}$$

Put $x = -1$ in (2) we get,

$$\frac{1}{n+1} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \dots$$

Illustration

Find the coefficient of x^4 in the expansion of

$$\frac{1+x}{1-x} \text{ if } |x| < 1$$

Sol. $\frac{1+x}{1-x} = (1+x)(1-x)^{-1}$

$$= (1+x) \left[1 + \frac{(-1)}{1!}(-x) + \frac{(-1)(-1-1)}{2!}(-x)^2 \right.$$

$$\left. + \frac{(-1)(-1-1)(-1-2)}{3!}(-x)^3 + \dots \text{to } \infty \right]$$

$$= (1+x)(1+x+x^2+x^3+x^4+\dots \text{to } \infty)$$

$$= [1+x+x^2+x^3+x^4+\dots \text{to } \infty] +$$

$$[x+x^2+x^3+x^4+\dots \text{to } \infty]$$

$$= 1 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + \dots \text{to } \infty$$

Hence coefficient of $x^4 = 2$

Illustration

Find the square root of 99 correct to 4 places of decimal.

Sol. $(99)^{1/2} = (100-1)^{1/2} \left[100 \left(1 - \frac{1}{100} \right) \right]^{1/2}$

$$= \left[100 \left(1 - \frac{1}{100} \right) \right]^{1/2}$$

$$= (100)^{1/2} [1-0]^{1/2} = 10(1-01)^{1/2}$$

$$10 \left[1 + \frac{1}{2}(-01) + \frac{\frac{1}{2}(-1)}{2!}(-01)^2 + \dots \text{to } \infty \right]$$

$$= 10 [1 - 0.005 - 0.0000125 + \dots \text{to } \infty]$$

$$= 10(99.99875) = 9.999875 = 9.9999$$

Multinomial Expansion

In the expansion of $(x_1 + x_2 + \dots + x_n)^m$ where $m, n \in \mathbb{N}$ and x_1, x_2, \dots, x_n are independent variables, we have

- (i) Total number of terms $= m+n-1 C_{n-1}$

- (ii) Coefficient of $x_1^{r_1} x_2^{r_2} x_3^{r_3} \dots x_n^{r_n}$ (where $r_1 + r_2 + \dots + r_n = m, r_i \in \mathbb{N} \cup \{0\}$) is $\frac{m!}{r_1! r_2! \dots r_n!}$

- (iii) Sum of all the coefficients is obtained by putting all the variables x_i equal to 1.

Illustration

Find the total number of terms in the expansion of $(1+a+b)^{10}$ and coefficient of a^2b^3 .

Sol. Total number of terms $= {}^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$

$$\text{Coefficient of } a^2b^3 = \frac{10!}{2! \times 3! \times 5!} = 2520$$