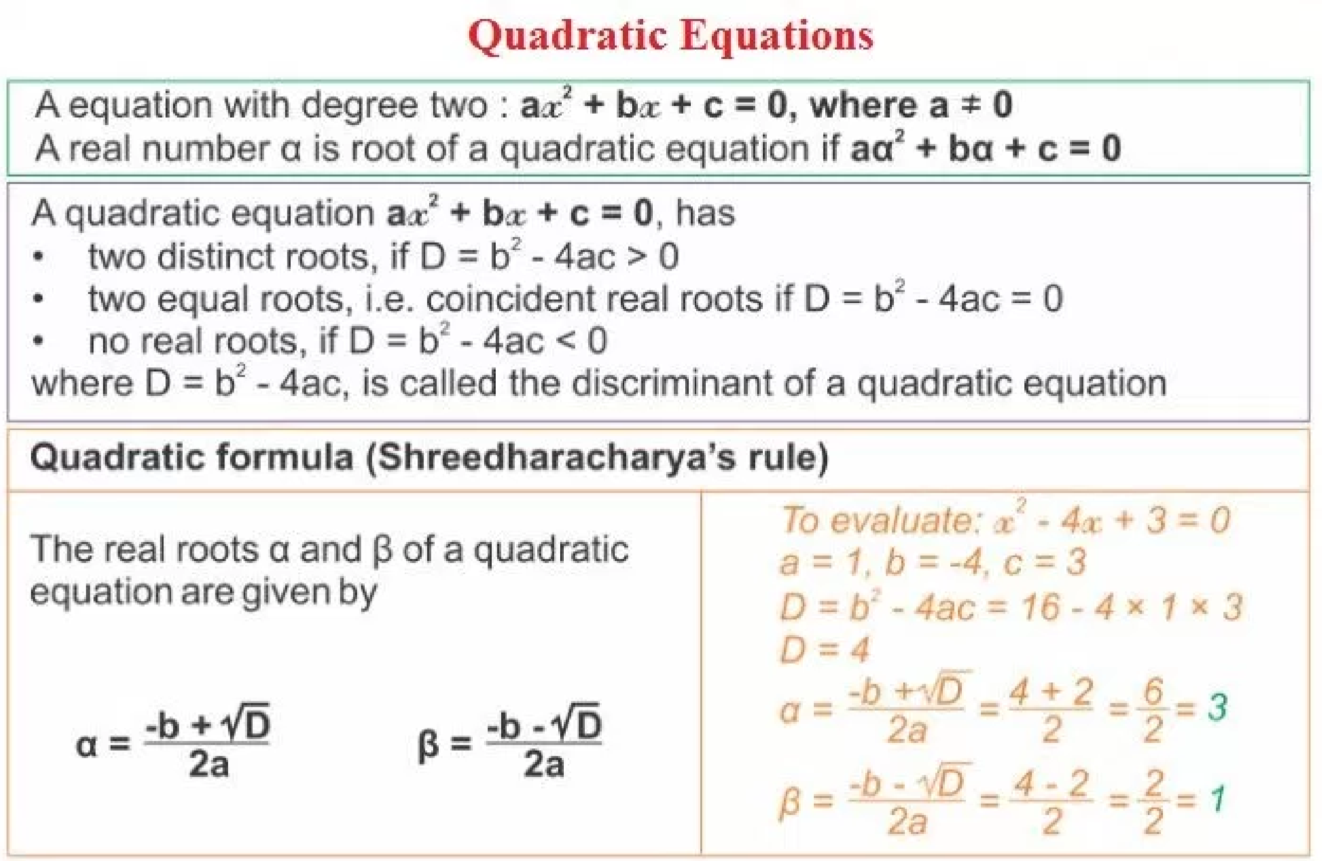
9. Equations of Second Degree

Quadratic Equations

An equation in which the highest power of the unknown quantity is two is called **quadratic equation**.



Types of quadratic equation

Quadratic equations are of two types:

Purely quadratic	Adfected quadratic
$ax^2 + c = 0$ where $a, c \in C$ and $b = 0$, $a \neq 0$	$ax^2 + bx + c = 0$ where $a, b, c \in C$ and $a \neq 0$, $b \neq 0$

Roots of a quadratic equation:

The values of variable x which satisfy the quadratic equation is called roots of quadratic equation.

Solution of quadratic equation

(1) Factorization method

Let $ax^2 + bx + c = a(x - \alpha)(x - \beta) = 0$.
Then $x = \alpha$ and $x = \beta$ will satisfy the given equation.
Hence, factorize the equation and equating each factor to zero gives roots of the equation.

Example: $3x^2 - 2x + 1 = 0 \Rightarrow (x - 1)(3x + 1) = 0$; $x = 1, -1/3$

(2) Sri Dharacharya method

By completing the perfect square as

$$ax^2 + bx + c = 0 \Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Adding and subtracting $\left(\frac{b}{2a}\right)^2$, $\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] = 0$

which gives, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Hence the quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ has two roots,

given by $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, $\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

Every quadratic equation has two and only two roots.

Nature of roots

In a quadratic equation $ax^2 + bx + c = 0$, let us suppose that are real and $a \neq 0$. The following is true about the nature of its roots.

1. The equation has real and distinct roots if and only if $D \equiv b^2 - 4ac > 0$.
2. The equation has real and coincident (equal) roots if and only if $D \equiv b^2 - 4ac = 0$.
3. The equation has complex roots of the form $\alpha \pm i\beta$, $\alpha \neq 0$ if and only if $D \equiv b^2 - 4ac < 0$.
4. The equation has rational roots if and only if a, b, c ∈ Q (the set of rational numbers) and $D \equiv b^2 - 4ac$ is a perfect square (of a rational number).
5. The equation has (unequal) irrational (surd form) roots if and only if $D \equiv b^2 - 4ac > 0$ and not a perfect square even if a, b and c are rational. In this case if $p + \sqrt{q}$, p,q rational is an irrational root, then $p - \sqrt{q}$ is also a root (a, b, c being rational).
6. $\alpha + i\beta$ ($\beta \neq 0$ and $\alpha, \beta \in R$) is a root if and only if its conjugate $\alpha - i\beta$ is a root, that is complex roots occur in pairs in a quadratic equation. In case the equation is satisfied by more than two complex numbers, then it reduces to an identity.
   $0.x^2 + 0.x + 0 = 0$, i.e., $a = 0 = b = c$.

Relations between roots and coefficients

(1) Relation between roots and coefficients of quadratic equation: If α and β are the roots of quadratic equation , ($a \neq 0$) then

$$\text{Sum of roots} = S = \alpha + \beta = \frac{-b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

$$\text{Product of roots} = P = \alpha.\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

(2) Formation of an equation with given roots: A quadratic equation whose roots are α and β is given by $(x - \alpha)(x - \beta) = 0$.
$\therefore x^2 - (\alpha + i\beta)x + \alpha\beta = 0$
i.e. x^2 – (sum of roots)x + (product of roots) = 0
$\therefore x^2 - Sx + P = 0$
(3) Symmetric function of the roots : A function of α and β is said to be a symmetric function, if it remains unchanged when α and β are interchanged.
For example, $\alpha^2 + \beta^2 + 2\alpha\beta$ is a symmetric function of α and β whereas $\alpha^2 + \beta^2 + 2\alpha\beta$ is not a symmetric function of α and β.

In order to find the value of a symmetric function of α and β, express the given function in terms of α + β and αβ. The following results may be useful.

(i) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

(ii) $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

(iii) $\alpha^4 + \beta^4 = (\alpha^3+\beta^3)(\alpha+\beta) - \alpha\beta(\alpha^2+\beta^2)$

(iv) $\alpha^5 + \beta^5 = (\alpha^3+\beta^3)(\alpha^2+\beta^2) - \alpha^2\beta^2(\alpha+\beta)$

(v) $|\alpha - \beta| = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

(vi) $\alpha^2 - \beta^2 = (\alpha+\beta)(\alpha-\beta)$

(vii) $\alpha^3 - \beta^3 = (\alpha-\beta)[(\alpha+\beta)^2 - \alpha\beta]$

(viii) $\alpha^4 - \beta^4 = (\alpha+\beta)(\alpha-\beta)(\alpha^2+\beta^2)$

Properties of quadratic equation

1. If f(a) and f(b) are of opposite signs then at least one or in general odd number of roots of the equation lie between a and b.
2. If then f(a) = f(b) there exists a point c between a and b such that f(c) = 0, a<c<b.
3. If α is a root of the equation f(x) = 0 then the polynomial f(x) is exactly divisible by (x – α), then (x – α) is factor of f(x).
4. If the roots of the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ are in the same ratio [i.e. $\alpha_1/\beta_1 = \alpha_2/\beta_2$] then $b_1^2/b_2^2 = a_1c_1/a_2c_2$.

Solving Quadratic Equations by Factoring

This lesson will review the process of factoring as a method of solving quadratic equations.

Def n: Quadratic equations are normally expressed as

$$ax^2 + bx + c = 0$$

where a does not equal zero.

Many of the simpler quadratic equations with rational roots can be solved by factoring.

To solve a quadratic equation by factoring:

1. Start with the equation in the form

$$ax^2 + bx + c = 0$$

Be sure it is set equal to zero!
2. Factor the left hand side (assuming zero is on the right)
3. Set each factor equal to zero
4. Solve to determine the roots (the values of x)

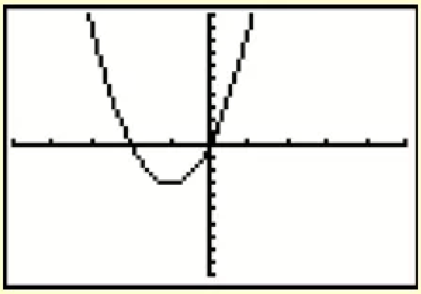
Let’s examine some possible situations:

Factoring with GCF
(greatest common factor)

Find the largest value that can be factored from each of the elements of the expression.

$$3x^2 + 6x = 0$$
$$3x(x+2) = 0$$
$$3x = 0 \quad x + 2 = 0$$
$$x = 0 \quad x = -2$$

Notice how the answers (the roots) can also be observed as the locations where the graph crosses the x-axis.



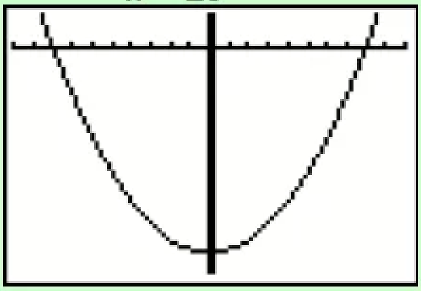
Factoring with DOTS
(difference of two squares)

Look carefully at this example to refresh this process:

$$x^2 - 64 = 0$$
$$(x+8)(x-8) = 0$$
$$x + 8 = 0 \quad x - 8 = 0$$
$$x = -8 \quad x = 8$$

Or Isolate the Variable
(Square Root Property)

$$x^2 - 64 = 0$$
$$x^2 = 64$$
$$\sqrt{x^2} = \sqrt{64}$$
$$x = \pm 8$$

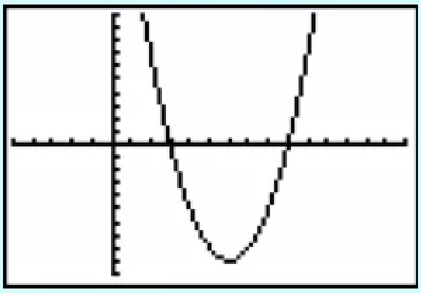


Factoring Trinomials

In a quadratic equation in descending order with a leading coefficient of one, look for the product of the roots to be the constant tern and the sum of the roots to be the coefficient of the middle term.

$$x^2 - 12x + 27 = 0$$
$$(x-9)(x-3) = 0$$
$$x - 9 = 0 \quad x - 3 = 0$$
$$x = 9 \quad x = 3$$

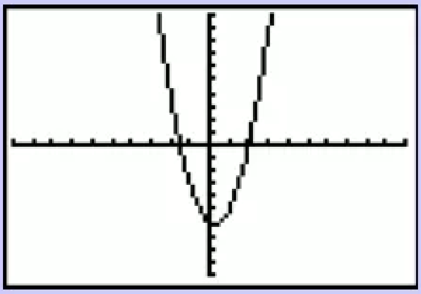
Notice how the answers (the roots) can also be observed as the locations where the graph crosses the x-axis.



Factoring Harder Trinomials

If the leading coefficient is not equal to 1, you must think more carefully about how to set up your factors.

$$2x^2 - x - 6 = 0$$
$$(2x+3)(x-2) = 0$$
$$2x + 3 = 0 \quad x - 2 = 0$$
$$x = -\frac{3}{2} \quad x = 2$$

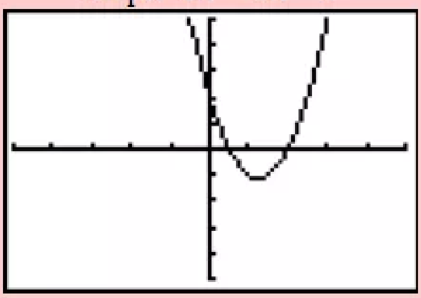


Tricky One!!

➡ Be sure to get the equation set equal to zero before you factor.

$$2x(x+1) = 7x - 2$$
$$2x^2 + 2x = 7x - 2$$
$$2x^2 - 5x + 2 = 0$$
$$(2x-1)(x-2) = 0$$
$$2x - 1 = 0 \quad x - 2 = 0$$
$$x = \frac{1}{2} \quad x = 2$$

Graph $2x^2 - 5x + 2$



Solving A Quadratic Equation By Factoring

(i) $(x+y)^2 = x^2 + 2xy + y^2$

(ii) $(x-y)^2 = x^2 - 2xy + y^2$

(iii) $x^2 - y^2 = (x-y)(x+y)$

(iv) $(x+a)(x+b) = x^2 + (a+b)x + ab$

(v) $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

(vi) $(x+y)^3 = x^3 + y^3 + 3xy(x+y)$

(vii) $(x-y)^3 = x^3 - y^3 - 3xy(x-y)$

(viii) $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)$

$x^3 + y^3 + z^3 = 3xyz \quad if \;\; x+y+z = 0$

Forms of a Quadratic Equation

- **Standard Form of a Quadratic Equation** $ax^2 + bx + c = 0, a \neq 0$

- **Factored Form of a Quadratic Equation** $a(x+p)(x+q) = 0, a \neq 0$

  Factoring means to write the terms in multiplication form (as a product).

- **Zero Product Property**

  If $ab = 0$ then either $a = 0$ or $b = 0$ (or both).

  The expression *must* be set equal to zero to use this property.

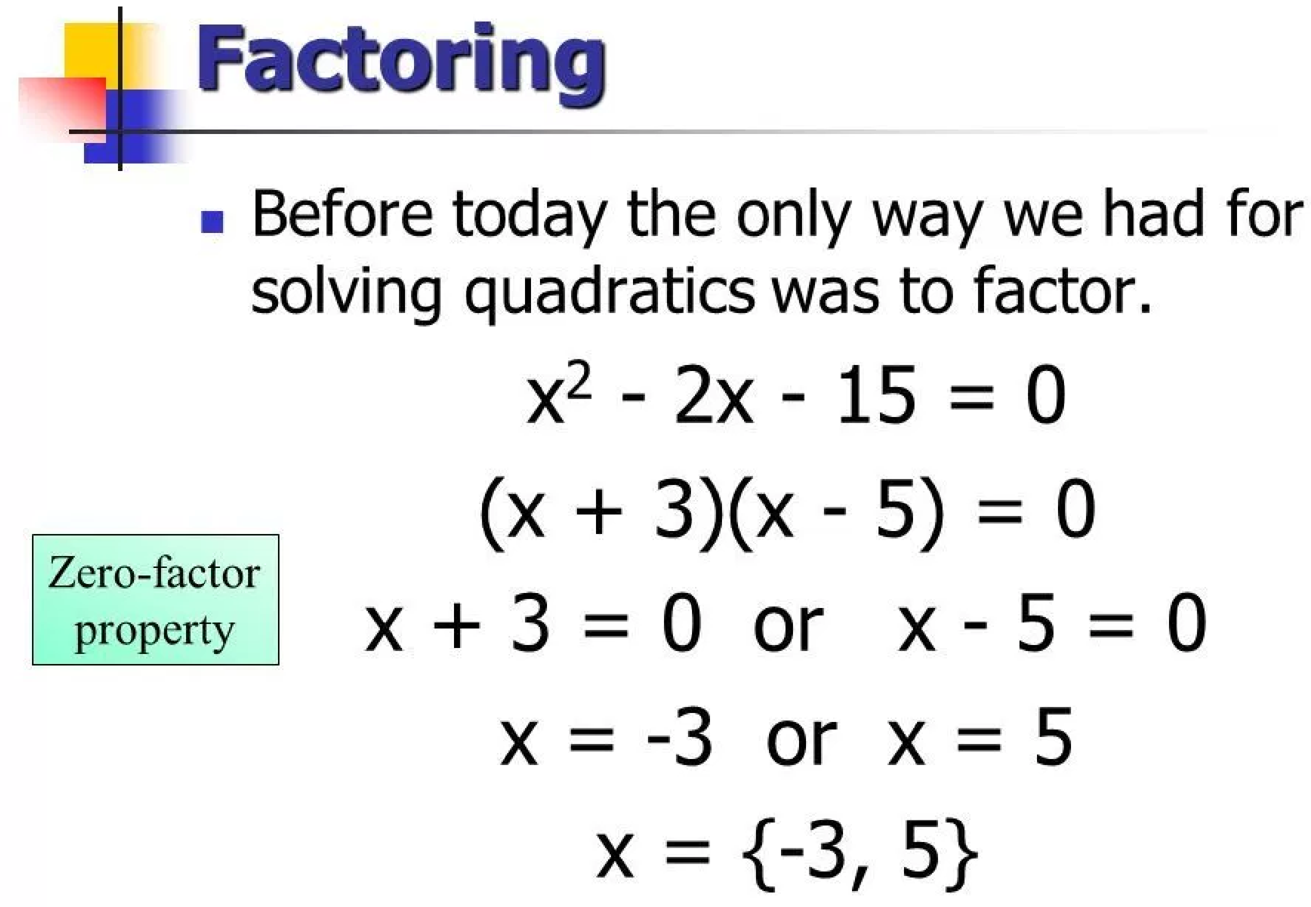
Zero Product Example: Quadratic in Factored Form $(x-6)\ (x+8) = 0$

$x - 6 = 0$ or $x + 8 = 0$

$x = 6$ or $x = 8$

Factoring Help!

Question	Strategy	Answer
$m^2+10m+16$	Both signs are positive, so both signs in answer are positive.	(m+2)(m+8)
$n^2-8n-48$	Two negatives, so in our answer, one will be positive (the smaller number) and one will be negative (the larger number)	(n-12)(n+4)
$y^2-15y+56$	Second term negative, third term positive; both signs in the answer will be negative	(y-8)(y-7)
p^2+p-20	Second term positive, third term negative; one will be positive (the larger number) and one will be negative (the smaller number)	(p+5)(p-4)



Since, $3x^2 - 5x + 2$ is a quadratic polynomial;
$3x^2 - 5x + 2 = 0$ is a quadratic equation.
Also,
$3x^2 - 5x + 2 = 3x^2 - 3x - 2x + 2$ [Factorising]
$= 3x(x - 1) - 2(x - 1)$
$= (x - 1)(3x - 2)$
In the same way :
$3x^2 - 5x + 2 = 0 \Rightarrow 3x^2 - 3x - 2x + 2 = 0$ [Factorising L.H.S.]
$\Rightarrow (x - 1)(3x - 2) = 0$
i.e., $x - 1 = 0$ or $3x - 2 = 0$
$\Rightarrow x = 1$ or $x = 2/3$
which is the solution of given quadratic equation.

In order to solve the given Quadratic Equation:

1. Clear the fractions and brackets, if given.
2. By transfering each term to the left hand side; express the given equation as
$ax^2 + bx + c = 0$ or $a + bx + cx^2 = 0$
3. Factorise left hand side of the equation obtained (the right hand side being zero).
4. By putting each factor equal to zero; solve it.

Solving A Quadratic Equation By Factoring With Examples

Example 1: Solve (i) $x^2 + 3x - 18 = 0$ (ii) $(x - 4)(5x + 2) = 0$
(iii) $2x^2 + ax - a2 = 0$; where 'a' is a real number.

Sol. **(i)** $x^2 + 3x - 18 = 0$
$\Rightarrow x^2 + 6x - 3x - 18 = 0$

$\Rightarrow x(x + 6) - 3(x + 6) = 0$
i.e., $(x + 6)(x - 3) = 0$
$\Rightarrow x + 6 = 0$ or $x - 3 = 0$
$\Rightarrow x = -6$ or $x = 3$

Roots of the given equation are -6 and 3

(ii) $(x - 4)(5x + 2) = 0$
$\Rightarrow x - 4 = 0$ or $5x + 2 = 0$
$x = 4$ or $x = -2/5$

(iii) $2x^2 + ax - a^2 = 0$
$\Rightarrow 2x^2 + 2ax - ax - a^2 = 0$
$\Rightarrow 2x(x + a) - a(x + a) = 0$
i.e., $(x + a)(2x - a) = 0$
$\Rightarrow x + a = 0$ or $2x - a = 0$
$\Rightarrow x = -a$ or $x = a/2$

Example 2: Solve the following quadratic equations
(i) $x^2 + 5x = 0$ (ii) $x^2 = 3x$ (iii) $x^2 = 4$

Sol. **(i)** $x^2 + 5x = 0 \Rightarrow x(x + 5) = 0$
$\Rightarrow x = 0$ or $x + 5 = 0$
$\Rightarrow x = 0$ or $x = -5$

(ii) $x^2 = 3x$
$\Rightarrow x^2 - 3x = 0$
$\Rightarrow x(x - 3) = 0$
$\Rightarrow x = 0$ or $x = 3$

(iii) $x^2 = 4$
$\Rightarrow x = \pm 2$

Example 3: Solve the following quadratic equations
(i) $7x^2 = 8 - 10x$ (ii) $3(x^2 - 4) = 5x$ (iii) $x(x + 1) + (x + 2)(x + 3) = 42$

Sol. **(i)** $7x^2 = 8 - 10x$
$\Rightarrow 7x^2 + 10x - 8 = 0$
$\Rightarrow 7x^2 + 14x - 4x - 8 = 0$
$\Rightarrow 7x(x + 2) - 4(x + 2) = 0$
$\Rightarrow (x + 2)(7x - 4) = 0$
$\Rightarrow x + 2 = 0$ or $7x - 4 = 0$
$\Rightarrow x = -2$ or $x = 4/7$

(ii) $3(x^2 - 4) = 5x$
$\Rightarrow 3x^2 - 5x - 12 = 0$
$\Rightarrow 3x^2 - 9x + 4x -\neg 12 = 0$
$\Rightarrow 3x(x - 3) + 4(x - 3) = 0$
$\Rightarrow (x - 3)(3x + 4) = 0$
$\Rightarrow x - 3 = 0$ or $3x + 4 = 0$
$\Rightarrow x = 3$ or $x = -4/3$

(iii) $x(x + 1) + (x + 2)(x + 3) = 42$
$\Rightarrow x^2 + x + x^2 + 3x + 2x + 6 - 42 = 0$
$\Rightarrow 2x^2 + 6x - 36 = 0$
$\Rightarrow x^2 + 3x - 18 = 0$
$\Rightarrow x^2 + 6x - 3x - 18 = 0$
$\Rightarrow x(x + 6) - 3(x + 6) = 0$
$\Rightarrow (x + 6)(x - 3) = 0$
$\Rightarrow x = -6$ or $x = 3$

Example 4: Solve for x : $12abx^2 - (9a^2 - 8b^2)x - 6ab = 0$

Given equation is $12abx^2 - (9a^2 - 8b^2)x - 6ab = 0$

$\Rightarrow 3ax(4bx - 3a) + 2b(4bx - 3a) = 0$

$\Rightarrow (4bx - 3a)(3ax + 2b) = 0$

$\Rightarrow 4bx - 3a = 0$ or $3ax + 2b = 0$

$\Rightarrow x = 3a/4b$ or $x = -2b/3a$

Completing the Square

An equation in which one side is a perfect square trinomial can be easily solved by taking the square root of each side.

Consider the example at the right:

$$x^2 - 6x + 9 = 25$$
$$(x-3)^2 = 25$$
$$\sqrt{(x-3)^2} = \sqrt{25}$$
$$x - 3 = \pm 5$$
$$x = 3 \pm 5$$
$$x = 8; \quad x = -2$$

It is this method of solution that is the basis of a process called "completing the square".

Consider the equation: .

$$x^2 + 8x - 4 = 0.$$

Our solution strategy will be to "force" a perfect square trinomial of the form

$$a^2 + 2ab + b^2 = (a+b)^2$$

on the left hand side of the equation.

This method of "forcing" the existence of a perfect square trinomial is completing the square.

Steps for Completing the Square:

1. Be sure that the coefficient of the highest power is one. If it is not, divide each term by that value to create a leading coefficient of one.	$x^2 + 8x - 4 = 0$
2. Move the constant term to the right hand side.	$x^2 + 8x = 4$
3. Prepare to add the needed value to create the perfect square trinomial. Be sure to balance the equation. The boxes may help you remember to balance.	$x^2 + 8x + \square = 4 + \square$
4. To find the needed value for the perfect square trinomial, take half of the coefficient of the *middle term* (x-term), square it, and add that value to both sides of the equation. Take half and square ↓ $x^2 + 8x + \square = 4 + \square$	$x^2 + 8x + \boxed{16} = 4 + \boxed{16}$
5. Factor the perfect square trinomial.	$(x+4)^2 = 20$
6. Take the square root of each side and solve. Remember to consider both plus and minus results.	$x + 4 = \pm\sqrt{20}$ $x = -4 \pm \sqrt{20} = -4 \pm 2\sqrt{5}$ $x = -4 + 2\sqrt{5}$ $x = -4 - 2\sqrt{5}$

Examples: Solve each example by completing the square.

1: *Solve*: $x^2 + 6x + 1 = 0$

$x^2 + 6x + 1 = 0$	Keep all *x* related terms on one side. Move the constant to the right.
$x^2 + 6x = -1$	
$x^2 + 6x + \square = -1 + \square$	Get ready to create a perfect square on the left. Balance the equation.
$x^2 + 6x + \boxed{9} = -1 + \boxed{9}$	Take half of the *x*-term coefficient and square it. Add this value to both sides.
$x^2 + 6x + 9 = -1 + 9$	
$x^2 + 6x + 9 = 8$	Factor and write the perfect square on the left.
$(x+3)^2 = 8$	
$x + 3 = \pm\sqrt{8}$	Take the square root of both sides. Be sure to allow for both plus and minus.
$x = -3 \pm \sqrt{8} = -3 \pm 2\sqrt{2}$	Solve for *x*.
$x = -3 + 2\sqrt{2}; \quad x = -3 - 2\sqrt{2}$	

2: *Solve*: $2p^2 + 20 = 6p$

$2p^2 + 20 = 6p$	This equation needs some re-arranging.
$2p^2 - 6p = -20$	Divide through by 2 to create the leading coefficient of 1 (for p^2).
$p^2 - 3p = -10$	Keep all *p* related terms on one side. Move the constant to the right.
$p^2 - 3p + \square = -10 + \square$	Get ready to create a perfect square on the left. Balance the equation.
$p^2 - 3p + \boxed{\frac{9}{4}} = -10 + \boxed{\frac{9}{4}}$	Take half of the *p*-term coefficient and square it. Add this value to both sides.
$p^2 - 3p + \frac{9}{4} = -\frac{31}{4}$	
$\left(p - \frac{3}{2}\right)^2 = -\frac{31}{4}$	Factor and write the perfect square on the left.
$p - \frac{3}{2} = \pm\sqrt{-\frac{31}{4}}$	Take the square root of both sides. Be sure to allow for both plus and minus.
$p = \frac{3}{2} \pm \sqrt{-\frac{31}{4}} = \frac{3}{2} \pm \frac{\sqrt{-31}}{2}$	Solve for *p*. Be sure to express the negative radical as an imaginary number.
$p = \frac{3 \pm i\sqrt{31}}{2}$	
$p = \frac{3 + i\sqrt{31}}{2}; \quad p = \frac{3 - i\sqrt{31}}{2}$	

Solving Quadratic Equations by Completing the Square

Solving quadratic equations by completing the square is overpowered by an “offspring” of this process, namely, the quadratic formula. The quadratic formula was derived by completing the square on a quadratic equation. Once the quadratic formula was derived, it was no longer necessary to use the process of completing the square to solve “each” quadratic equation. Even though completing the square is often overlooked in favor of the quadratic formula, it is still a valuable skill that will be needed in other mathematical situations. Therefore, it is worthwhile to “get our feet wet” with these easier examples of applying the process of completing the square.
The process of completing the square was explained in the section entitled Complete the Square.

Examples:

1. *Solve*: $x^2 - 2x - 1 = 0$ by completing the square.

$x^2 - 2x - 1 = 0$ $x^2 - 2x = 1$	Keep all terms containing x on one side. Move the constant to the right.
$x^2 - 2x + \square = 1 + \square$	Get ready to create a perfect square on the left. Balance the equation.
$x^2 - 2x + \boxed{1} = 1 + \boxed{1}$	Take half of the x-term coefficient and square it. Add this value to both sides.
$x^2 - 2x + 1 = 2$ $(x-1)^2 = 2$	Simplify and write the perfect square on the left.
$x - 1 = \pm\sqrt{2}$	Take the square root of both sides. Be sure to allow for both plus and minus.
$x = 1 \pm \sqrt{2}$ $x = 1 + \sqrt{2}; \quad x = 1 - \sqrt{2}$	Solve for x.

2. *Solve*: $x^2 - 8x + 24 = 0$ by completing the square.

$x^2 - 8x + 24 = 0$ $x^2 - 8x = -24$	Keep all terms containing x on one side. Move the constant to the right.
$x^2 - 8x + \square = -24 + \square$	Get ready to create a perfect square on the left. Balance the equation.
$x^2 - 8x + \boxed{16} = -24 + \boxed{16}$	Take half of the x-term coefficient and square it. Add this value to both sides.
$x^2 - 8x + 16 = -8$ $(x-4)^2 = -8$	Simplify and write the perfect square on the left.
$x - 4 = \pm\sqrt{-8}$	Take the square root of both sides. Be sure to allow for both plus and minus.
$x = 4 \pm \sqrt{-8} = 4 \pm 2i\sqrt{2}$ $x = 4 + 2i\sqrt{2}; \quad x = 4 - 2i\sqrt{2}$	Represent the negative radical as an imaginary number and solve for x.

3. *Solve*: $5x^2 - 6x = 8$ by completing the square.

$5x^2 - 6x = 8$	Keep all terms containing x on one side. This equation is all set up to start.
$x^2 - \frac{6}{5}x = \frac{8}{5}$	Divide all terms by 5 to create a leading coefficient of one.
$x^2 - \frac{6}{5}x + \square = \frac{8}{5} + \square$	Prepare to get a perfect square on the left. Balance the equation.
$x^2 - \frac{6}{5}x + \boxed{\frac{9}{25}} = \frac{8}{5} + \boxed{\frac{9}{25}}$	Take half of the x-term coefficient and square it. Add this value to both sides.
$x^2 - \frac{6}{5}x + \frac{9}{25} = \frac{40}{25} + \frac{9}{25}$	Simplify
$x^2 - \frac{6}{5}x + \frac{9}{25} = \frac{49}{25}$	Simplify
$\left(x - \frac{3}{5}\right)^2 = \frac{49}{25}$	Write the perfect square on the left.
$x - \frac{3}{5} = \pm\frac{7}{5}$	Take the square root of both sides. Be sure to allow for both plus and minus.
$x = \frac{3}{5} \pm \frac{7}{5} = \frac{3 \pm 7}{5}$ $x = \frac{10}{5} = 2; \quad x = \frac{-4}{5}$	Solve for x.

Solving Quadratic Equations with the Quadratic Formula

A equation with degree two : $ax^2 + bx + c = 0$, where $a \neq 0$
A real number α is root of a quadratic equation if $a\alpha^2 + b\alpha + c = 0$

A quadratic equation $ax^2 + bx + c = 0$, has

- two distinct roots, if $D = b^2 - 4ac > 0$
- two equal roots, i.e. coincident real roots if $D = b^2 - 4ac = 0$
- no real roots, if $D = b^2 - 4ac < 0$

where $D = b^2 - 4ac$, is called the discriminant of a quadratic equation

Quadratic formula (Shreedharacharya's rule)	
The real roots α and β of a quadratic equation are given by $\alpha = \frac{-b + \sqrt{D}}{2a}$ $\beta = \frac{-b - \sqrt{D}}{2a}$	To evaluate: $x^2 - 4x + 3 = 0$ $a = 1, b = -4, c = 3$ $D = b^2 - 4ac = 16 - 4 \times 1 \times 3$ $D = 4$ $\alpha = \frac{-b + \sqrt{D}}{2a} = \frac{4 + 2}{2} = \frac{6}{2} = 3$ $\beta = \frac{-b - \sqrt{D}}{2a} = \frac{4 - 2}{2} = \frac{2}{2} = 1$

The solutions of some quadratic equations, $ax^2 + bx + c = 0$ ($a \neq 0$) are not rational, and cannot be obtained by factoring. For such equations, the most common method of solution is the quadratic formula.

Quadratic Formula:

For $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: The quadratic formula can be used to solve ANY quadratic equation, even those that can be factored. Be sure you know this very useful formula!!!

Examples:

1. *Solve*: $x^2 + 2x - 8 = 0$

By factoring (this equation is factorable):	By Quadratic Formula: $a = 1,\ b = 2,\ c = -8$	*Hints:*
$x^2 + 2x - 8 = 0$ $(x + 4)(x - 2) = 0$ $x = -4;\quad x = 2$	$x^2 + 2x - 8 = 0$ $x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-8)}}{2(1)}$ $= \frac{-2 \pm \sqrt{4 + 32}}{2} = \frac{-2 \pm \sqrt{36}}{2}$ $= \frac{-2 \pm 6}{2}$ $x = -4;\quad x = 2$	**Be careful with the signs** of the values *a*, *b* and *c*. Don't drop the sign when substituting into the formula. Also remember your rules for multiplying and adding signed numbers as you solve the formula.

2. $Solve: \ 3x^2 - 10x + 5 = 0$

This equation cannot be solved by factoring. By Quadratic Formula: $a = 3,\ b = -10,\ c = 5$ $3x^2 - 10x + 5 = 0$ $x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(3)(5)}}{2(3)}$ $= \frac{10 \pm \sqrt{100 - 60}}{6} = \frac{10 \pm \sqrt{40}}{6}$ $= \frac{10 \pm 2\sqrt{10}}{6} = \frac{5 \pm \sqrt{10}}{3}$ $x = \frac{5 + \sqrt{10}}{3}; \quad x = \frac{5 - \sqrt{10}}{3}$	*Hints:* Notice how the value for b was substituted into the formula using parentheses (-10). This helps you to remember to deal with the negative value of b. Also, notice how the $(-10)^2$ is actually a positive value. **When you square a value, the answer is always positive.** If needed, these answers can be estimated as decimal values, such as (rounded to 3 decimal places): $x = 2.721; \quad x = 0.613$ The radical answers are the "exact" answers. The decimal answers are "approximate" answers.

3. $Solve: \ x^2 + 4x + 5 = 0$

This equation cannot be solved by factoring. By Quadratic Formula: $a = 1,\ b = 4,\ c = 5$ $x^2 + 4x + 5 = 0$ $x = \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)}$ $= \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm \sqrt{-4}}{2}$ $= \frac{-4 \pm i\sqrt{4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$ $x = -2 + i; \quad x = -2 - i$	*Hints:* Remember that **a negative value under the radical is creating an imaginary number** (a number with an i).

4. $Solve: \ x^2 - 4x + 4 = 0$

By factoring (this equation is factorable): $x^2 - 4x + 4 = 0$ $(x - 2)(x - 2) = 0$ $x = 2; \quad x = 2$	By Quadratic Formula: $a = 1,\ b = -4,\ c = 4$ $x^2 - 4x + 4 = 0$ $x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(4)}}{2(1)}$ $= \frac{4 \pm \sqrt{16 - 16}}{2} = \frac{4 \pm \sqrt{0}}{2}$ $= \frac{4}{2} = 2$ $x = 2$	*Hints:* When the **value under the radical turns to zero,** there will appear to be **only one answer** to the problem (since the plus/minus option is gone). This really means that the one root is repeating itself, as seen in the factoring solution.

5. *Solve*: $2x^2 + x = \frac{1}{2}$

Whoa!! Stop the presses!!! This problem cannot be solved using the Quadratic Formula until it is set equal to zero. $2x^2 + x - \frac{1}{2} = 0$ By Quadratic Formula: $a = 2,\ b = 1,\ c = -1/2$ $2x^2 + x - \frac{1}{2} = 0$ (letting $-\frac{1}{2} = -0.5$) $x = \frac{-1 \pm \sqrt{1^2 - 4(2)(-0.5)}}{2(2)}$ $= \frac{-1 \pm \sqrt{1+4}}{4} = \frac{-1 \pm \sqrt{5}}{4}$ $x = \frac{-1+\sqrt{5}}{4};\quad x = \frac{-1-\sqrt{5}}{4}$	*Hints:* The Equation must be set equal to 0.

Deriving the Quadratic Formula:

The quadratic formula is derived from the quadratic equation by a process called "completing the square". Here is how it was developed:

$ax^2 + bx + c = 0$	The process used here is explained in more detail under completing the square.
$ax^2 + bx = -c$	Isolate the variable terms. Move the constant, c, to the right.
$x^2 + \frac{b}{a}x = \frac{-c}{a}$	Prepare to form a perfect square on the left.
$x^2 + \frac{b}{a}x + \square = \frac{-c}{a} + \square$	To obtain the constant value needed for the perfect square, take half of the coefficient of the x-term and square it. Add this value to both sides of the equation.
$x^2 + \frac{b}{a}x + \boxed{\frac{b^2}{4a^2}} = \frac{-c}{a} + \boxed{\frac{b^2}{4a^2}}$	
$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{-4ac}{4a^2} + \frac{b^2}{4a^2}$	Find a common denominator on the right.
$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$	Combine terms on the right.
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	Write the perfect square on the left.
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$	Take the square root of both sides.
	Simplify the denominator from the square root.
$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Subtract the constant away from the x.
	Finally, the quadratic formula appears. Whew!!!

A Summary of the Methods of Solving Quadratic Equations

Quadratic equations are of the form where a, b and c are real numbers and . Quadratic equations have two solutions. It is possible that one solution may repeat.

Some quadratic equations can be solved by factoring. Set the equation equal to zero and factor.

Solving by Factoring

Example 1:
Solve by factoring:

$$x^2 - 12x + 27 = 0$$
$$(x-9)(x-3) = 0$$
$$x - 9 = 0 \quad x - 3 = 0$$
$$x = 9 \quad x = 3$$

Example 2:
Solve by factoring:

$$x^2 - 64 = 0$$
$$(x+8)(x-8) = 0$$
$$x + 8 = 0 \quad x - 8 = 0$$
$$x = -8 \quad x = 8$$

Example 2:
Solve by factoring:

$$2x^2 + 7x + 3 = 0$$
$$(2x+1)(x+3) = 0$$
$$2x + 1 = 0; \; x + 3 = 0$$
$$x = -\frac{1}{2}; \; x = -3$$

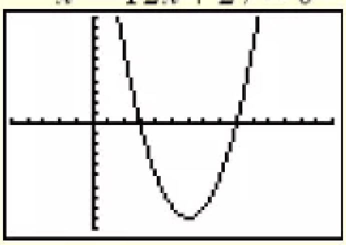
Solving by Graphing

Some quadratic equations can be solved by graphing. Setting the equation equal to zero will show the roots as locations on the x-axis.

Method 1: Set the equation equal to zero, if necessary. Find the roots using the ZERO command tool of the graphing calculator. For help with the calculator, click here.

Example 1: Solve by graphing

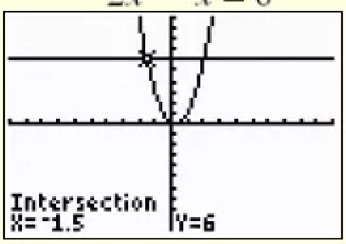
$$x^2 - 12x = -27$$
$$x^2 - 12x + 27 = 0$$



Method 2: Graph each side of the equation separately. Use the INTERSECT command tool to find when the graphs cross. Repeat this process for both intersection points. x= -3/2 , x=2

Example 2: Solve by graphing:

$$2x^2 - x = 6$$



Solving by Quadratic Formula

Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The solutions of some quadratic equations are not rational, and cannot be factored. For such equations, the most common method of solution is the quadratic formula. The quadratic formula can be used to solve ANY quadratic equation, even those that can be factored.
Be sure you know this formula!!!

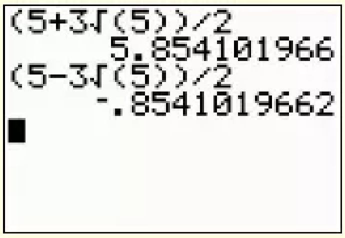
Example:

$$x^2 - 5x - 3 = 2$$
$$x^2 - 5x - 5 = 0$$
$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-5)}}{2(1)}$$
$$x = \frac{5 \pm \sqrt{25+20}}{2} = \frac{5 \pm \sqrt{45}}{2}$$
$$= \frac{5 \pm 3\sqrt{5}}{2}$$

Solutions:

$$x = \frac{5+3\sqrt{5}}{2}; \quad x = \frac{5-3\sqrt{5}}{2}$$

As decimal values:



Note: The equation must be set equal to zero before using the formula.

Solving by Competing the Square

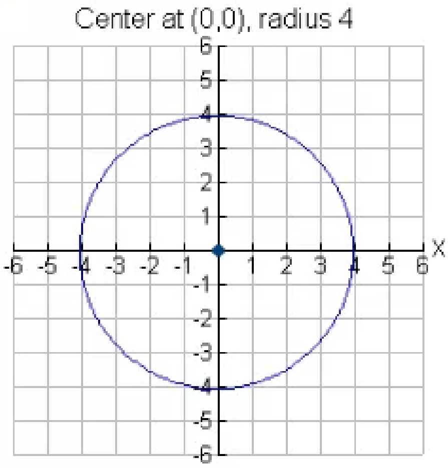
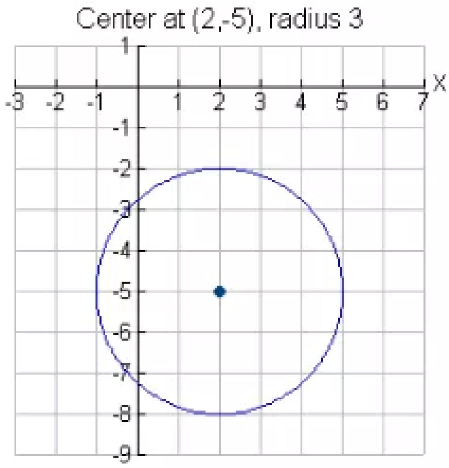
Example:

$x^2 + 4x - 4 = 0$ $x^2 + 4x = 4$	Keep all terms containing x on one side. Move the constant to the right.
$x^2 + 4x + \square = 4 + \square$	Get ready to create a perfect square on the left. Balance the equation.
$x^2 + 4x + \boxed{4} = 4 + \boxed{4}$	Take half of the x-term coefficient and square it. Add this value to both sides.
$x^2 + 4x + 4 = 8$ $(x+2)^2 = 8$	Simplify and write the perfect square on the left.
$x + 2 = \pm\sqrt{8}$	Take the square root of both sides. Be sure to allow for both plus and minus.
$x = -2 \pm \sqrt{8} = -2 \pm 2\sqrt{2}$ $x = -2 + 2\sqrt{2}; \quad x = -2 - 2\sqrt{2}$	Solve for x.

Equation of Circles

Let's review what we already know about circles.

Definition: A circle is a locus (set) of points in a plane equidistant from a fixed point.

Circle whose center is at the origin	Circle whose center is at (h,k) (This will be referred to as the "center-radius form". It may also be referred to as "standard form".)
Equation: $x^2 + y^2 = r^2$ Example: Circle with center (0,0), radius 4 $x^2 + y^2 = 16$ Graph:  	Equation: $(x-h)^2 + (y-k)^2 = r^2$ Example: Circle with center (2,-5), radius 3 $(x-2)^2 + (y+5)^2 = 9$ Graph: 

Now, if we "multiply out" the above example $(x-2)^2 + (y+5)^2 = 9$ we will get:

$(x-2)^2 + (y+5)^2 = 9$
$(x^2-4x+4) + (y^2+10y+25) = 9$
$x^2-4x+4+ y^2+10y+25 = 9$
$x^2+ y^2-4x+10y+20= 0$
$x^2+y^2+Cx+Dy+E = 0$

When multiplied out, we obtain the "general form" of the equation of a circle. Notice that in this form we can clearly see that the equation of a circle has both x^2 and y^2 terms and these terms have the same coefficient (usually 1).

When the equation of a circle appears in "general form", it is often beneficial to convert the equation to "center-radius" form to easily read the center coordinates and the radius for graphing.

Center-Radius Form of the Equation of a Circle

The circle with center (h, k) and radius r has the equation

$$(x-h)^2+(y-k)^2=r^2$$

The center-radius form of the equation of a circle.
A circle with center (0, 0) and radius r has equation

$$x^2+y^2=r^2$$

Examples:

1. Convert $x^2+ y^2-4x-6y+8= 0$ into center-radius form.

This conversion requires use of the technique of completing the square.

We will be creating two perfect square trinomials within the equation.

$$x^2+y^2-4x-6y+8=0$$
$$x^2-4x+y^2-6y=-8$$
$$x^2-4x+\square+y^2-6y+\square=-8+\square+\square$$
$$x^2-4x+\boxed{4}+y^2-6y+\boxed{9}=-8+\boxed{4}+\boxed{9}$$
$$(x-2)^2+(y-3)^2=5$$

- Start by grouping the x related terms together and the y related terms together. Move any numerical constants (plain numbers) to the other side.
- Get ready to insert the needed values for creating the perfect square trinomials. Remember to balance both sides of the equation.
- Find each missing value by taking half of the "middle term" and squaring. This value will always be positive as a result of the squaring process.
- Rewrite in factored form.
  You can now read that the center of the circle is at (2, 3) and the radius is /5

2. How do the coordinates of the center of a circle relate to C and D when the equation of the circle is in the general form $x^2 + y^2 + Cx + Dy + E = 0$.

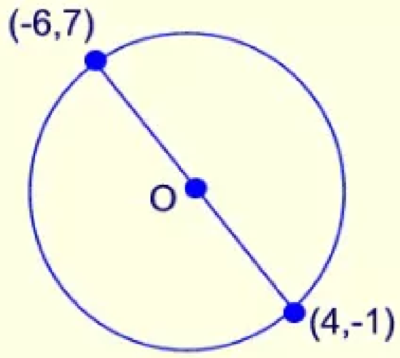
Let's make some observations. Re-examine our previous equations in general form and center-radius form. Do you see a relationship between the center coordinates and C and D?

General form	Center-radius form
$x^2 + y^2 - 4x + 10y + 20 = 0$ $C = -4,\ D = 10$	$(x-2)^2 + (y+5)^2 = 9$ *Center* (2, -5)
$x^2 + y^2 - 4x - 6y + 8 = 0$ $C = -4,\ D = -6$	$(x-2)^2 + (y-3)^2 = 5$ *Center* (2, 3)

When $(x-2)^2 + (y-3)^2 = 5$ is expanded, $(x-2)^2$ becomes $x^2 - 4x + 4$, where the center term's coefficient doubles the value of -2. Remember that while the equation deals with $(x-2)^2$, the actual x-coordinate of the center of this circle is $+2$.

$$(x-h)^2 + (y-k)^2 = r^2 \ \textit{has center at} (h,k)$$

3. Write the equation of a circle whose diameter has endpoints (4, -1) and (-6, 7).



Find the center by using the midpoint formula.

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-6+4}{2}, \frac{7+(-1)}{2}\right)$$

$$= (-1, 3)$$

Find the radius by using the distance formula.

Points (-6,7) and (-1,3) were used here. (d = distance, or radius)

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$$= \sqrt{(-6-(-1))^2 + (7-3)^2}$$

$$= \sqrt{(-5)^2 + (4)^2} = \sqrt{25+16} = \sqrt{41}$$

Equation: $(x+1)^2 + (y-3)^2 = 41$

4. Write the equation for the circle shown below if it is shifted 3 units to the right and 4 units up.

A shift of 3 units to the right and 4 units up places the center at the point (3, 4). The radius of the circle can be seen from the graph to be 5.
Equation: $(x-3)^2 + (y-4)^2 = 25$

5. Convert $2x^2 + 2y^2 + 6x - 8y + 12 = 0$ into center-radius form.

Whoa!!! This equation looks different. Are we sure this is a circle???
In this equation, both the x and y terms appear in squared form and their coefficients (the numbers in front of them) are the same. Yes, we have a circle here! We will, however, have to deal with the coefficients of 2 before we can complete the square.

$$2x^2 + 2y^2 + 6x - 8y + 12 = 0$$

- group the terms

divide every term by 2

- divide through by 2

$$x^2 + 3x + y^2 - 4y + 6 = 0$$

- get ready to create perfect squares

$$x^2 + 3x + \square + y^2 - 4y + \square = -6 + \square + \square$$

- take half of the "middle term" and square it

$$x^2 + 3x + \boxed{p/4} + y^2 - 4y + \boxed{4} = -6 + \boxed{9/4} + \boxed{4}$$

- factor and write in center-radius form

$$\left(x + 3/2\right)^2 + (y - 2)^2 = 1/4$$

Center: (-3/2, 2) Radius: 1/2