Exercise 10.3

Q1E

(a)

Consider the polar coordinate $P(r,\theta)=\left(2,\frac{\pi}{3}\right)$.

Plot the given point on the polar coordinate system.

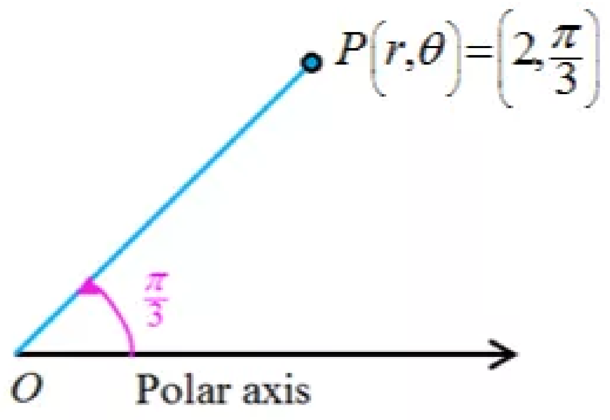
Let O be the origin or pole and r be the distance from the origin O to the point P.

And θ be the angle between the polar axis and the line joining the points O and P.

Here, $r=2$ is positive and $\theta=\frac{\pi}{3}$ is in the first quadrant.

So plot a point $P(r,\theta)=\left(2,\frac{\pi}{3}\right)$ that is, located two units from the pole in the first quadrant.

The sketch of the point $P(r,\theta)=\left(2,\frac{\pi}{3}\right)$ is shown below:



Find two other coordinate pairs of $\left(2,\frac{\pi}{3}\right)$ in which $r>0$ and $r<0$:

If a point takes a complete counterclockwise rotation then the angle is 2π, and the point represented by polar coordinates (r,θ) could be written as,

$$(r,\theta+2n\pi) \text{ and } (-r,\theta+(2n+1)\pi)$$

Here, *n* is any integer.

The corresponding coordinate pairs of $\left(2,\frac{\pi}{3}\right)$ for $r>0$ are:

$$\left(2,\frac{\pi}{3}+2n\pi\right), \quad n=0,\pm1,\pm2,\ldots$$

That is, $\left(2,\frac{\pi}{3}\right)$, $\left(2,\frac{7\pi}{3}\right),\left(2,-\frac{5\pi}{3}\right),\left(2,\frac{13\pi}{3}\right),\ldots$

Therefore, one of the other representations of the point $P(r,\theta)=\left(2,\frac{\pi}{3}\right)$ for $r>0$ is

$$\boxed{\left(2,\frac{7\pi}{3}\right)}.$$

The corresponding coordinate pairs of $\left(2,\frac{\pi}{3}\right)$ for $r<0$ are:

$$\left(-2,\frac{\pi}{3}+(2n+1)\pi\right), \quad n=0,\pm1,\pm2,\ldots$$

That is, $\left(-2,\frac{4\pi}{3}\right)$, $\left(-2,\frac{10\pi}{3}\right),\left(2,-\frac{2\pi}{3}\right),\left(2,\frac{16\pi}{3}\right),\ldots$

Therefore, one of the other representations of the point $P(r,\theta)=\left(2,\frac{\pi}{3}\right)$ for $r<0$ is

$$\boxed{\left(-2,\frac{4\pi}{3}\right)}.$$

(b)

Consider the polar coordinate $P(r,\theta)=\left(1,-\frac{3\pi}{4}\right)$.

Plot the given point on the polar coordinate system.

Let O be the origin or pole and r be the distance from the origin *O* to the point P.

And θ be the angle between the polar axis and the line joining the points *O* and *P*.

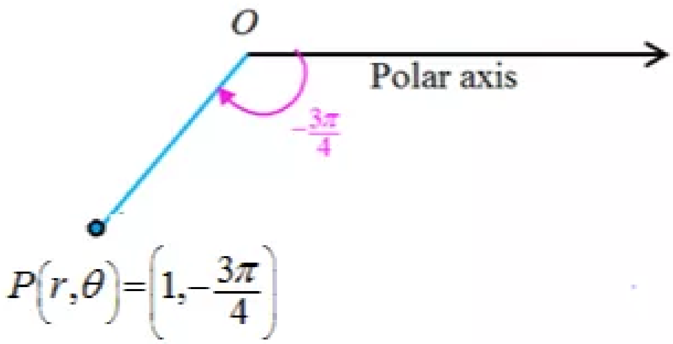
Here, $r=1$ is positive and $\theta=-\frac{3\pi}{4}$ is negative.

If an angle is negative then it is measured in the clock wise direction from the polar axis.

So the angle $\theta=-\frac{3\pi}{4}$ is in the third quadrant.

Plot a point $P(r,\theta)=\left(1,-\frac{3\pi}{4}\right)$ that is, located one unit from the pole in the third quadrant.

The sketch of the point $P(r,\theta)=\left(1,-\frac{3\pi}{4}\right)$ is shown below:



Find two other coordinate pairs of $\left(1,-\frac{3\pi}{4}\right)$ in which $r>0$ and $r<0$:

If a point takes a complete counterclockwise rotation then the angle is 2π, and the point represented by polar coordinates (r,θ) could be written as,

$$(r,\theta+2n\pi) \text{ and } (-r,\theta+(2n+1)\pi)$$

Here, n is any integer.

The corresponding coordinate pairs of $\left(1,-\frac{3\pi}{4}\right)$ for $r>0$ are:

$$\left(1,-\frac{3\pi}{4}+2n\pi\right), \quad n=0,\pm1,\pm2,\ldots$$

That is, $\left(1,-\frac{3\pi}{4}\right)$, $\left(1,\frac{5\pi}{4}\right)$, $\left(1,-\frac{11\pi}{4}\right)$, $\left(1,\frac{13\pi}{4}\right)$,...

Therefore, one of the other representations of the point $P(r,\theta)=\left(1,-\frac{3\pi}{4}\right)$ for $r>0$ is

$$\boxed{\left(1,\frac{5\pi}{4}\right)}.$$

The corresponding coordinate pairs of $\left(1,-\frac{3\pi}{4}\right)$ for $r<0$ are,

$$\left(1,-\frac{3\pi}{4}+(2n+1)\pi\right), \quad n=0,\pm1,\pm2,\ldots$$

That is, $\left(-1,\frac{\pi}{4}\right)$, $\left(-1,\frac{9\pi}{4}\right)$, $\left(-1,-\frac{7\pi}{4}\right)$, $\left(-1,\frac{17\pi}{4}\right)$,...

Therefore, one of the other representations of the point $P(r,\theta)=\left(1,-\frac{3\pi}{4}\right)$ for $r<0$ is

$$\boxed{\left(-1,\frac{\pi}{4}\right)}.$$

(c)

Consider the polar coordinate $P(r,\theta)=\left(-1,\frac{\pi}{2}\right)$.

Plot the point on the polar coordinate system.

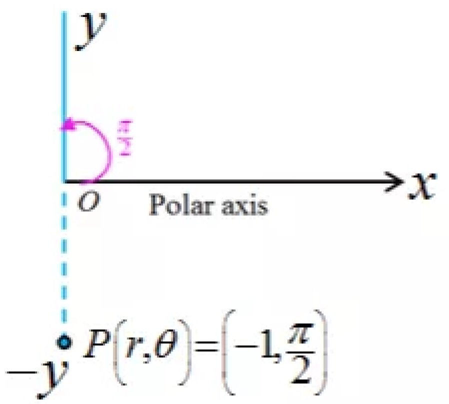
If $r>0,$ the point (r,θ) lies on the same quadrant as $\theta.$

If $r<0$, the point $(-r,\theta)$ lies in the quadrant opposite side of the pole.

Here, $\theta=\frac{\pi}{2}$ is positive.

If an angle is positive then it is measured in the counter clock wise direction from the polar axis so $\theta=\frac{\pi}{2}$ is lies on the positive *y*-axis but here $r=-1$ is negative so the point $P(r,\theta)=\left(-1,\frac{\pi}{2}\right)$ lies on the negative *y*-axis at a distance of one unit from the pole.

The sketch of the point $P(r,\theta)=\left(-1,\frac{\pi}{2}\right)$ is shown below:



Find two other coordinate pairs of $\left(-1,\frac{\pi}{2}\right)$ in which $r>0$ and $r<0$:

If a point takes a complete counterclockwise rotation then the angle is $2\pi,$ and the point represented by polar coordinates (r,θ) could be written as,

$(r,\theta+2n\pi)$ and $(-r,\theta+(2n+1)\pi)$

Here, n is any integer.

Note that $(-r,\theta)$ represents the same point as $(r,\theta+\pi).$

So, $\left(-1,\frac{\pi}{2}\right)$ represents the same point as $\left(1,\frac{\pi}{2}+\pi\right)=\left(1,\frac{3\pi}{2}\right).$

Therefore, one of the other representations of the point $P(r,\theta)=\left(-1,\frac{\pi}{2}\right)$ for $r>0$ is

$\boxed{\left(1,\frac{3\pi}{2}\right)}.$

The corresponding coordinate pairs of $\left(1,\frac{3\pi}{2}\right)$ for $r<0$ are:

$$\left(-1,\frac{3\pi}{2}+(2n+1)\pi\right),\quad n=0,\pm1,\pm2,\ldots$$

That is, $\left(-1,\frac{5\pi}{2}\right),\left(-1,\frac{9\pi}{2}\right),\left(-1,\frac{\pi}{2}\right),\left(-1,\frac{13\pi}{2}\right),\ldots$

Therefore, one of the other representations of the point $P(r,\theta)=\left(-1,\frac{\pi}{2}\right)$ for $r<0$ is

$\boxed{\left(-1,\frac{5\pi}{2}\right)}$.

Q2E

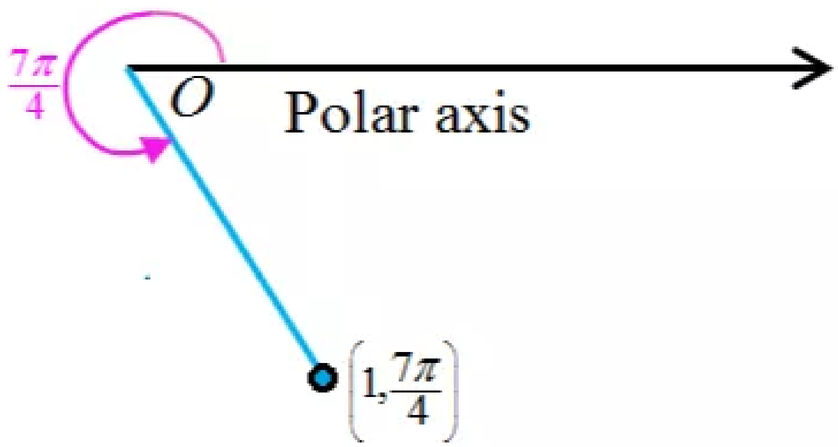
(a)

Consider the polar coordinate

$$\left(1,\frac{7\pi}{4}\right)$$

Now, plot the point on the polar coordinate system.

Use the convention that an angle is positive if measured in the counter clock wise direction from the polar axis and negative in the clockwise direction.



Find two other coordinate pairs of $\left(1,\frac{7\pi}{4}\right)$ in which $r>0$ and $r<0$:

In fact, since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r,θ) is also represented by

$$(r,\theta+2n\pi) \text{ and } (-r,\theta+(2n+1)\pi)$$

Where n is any integer.

For $r>0$:

The corresponding coordinate pairs of $\left(1,\frac{7\pi}{4}\right)$ are:

$$\left(1,\frac{7\pi}{4}+2n\pi\right), \quad n=0,\pm1,\pm2,\ldots$$

That is,

$$\left(1,\frac{7\pi}{4}\right), \left(1,\frac{15\pi}{4}\right), \left(1,-\frac{\pi}{4}\right), \left(1,\frac{23\pi}{4}\right),\ldots$$

For $r<0$: The corresponding coordinate pairs of $\left(1,\frac{7\pi}{4}\right)$ are:

$\left(-1,\frac{7\pi}{4}+(2n+1)\pi\right), \quad n=0,\pm1,\pm2,\ldots$ That is,

$$\left(-1,\frac{11\pi}{4}\right), \left(-1,\frac{19\pi}{4}\right), \left(-1,\frac{3\pi}{4}\right), \left(-1,\frac{27\pi}{4}\right),\ldots$$

Therefore, two other pairs of polar coordinate of $\left(1,\frac{7\pi}{4}\right)$ are $\boxed{\left(1,\frac{15\pi}{4}\right) \text{ and } \left(-1,\frac{11\pi}{4}\right)}$

(b)

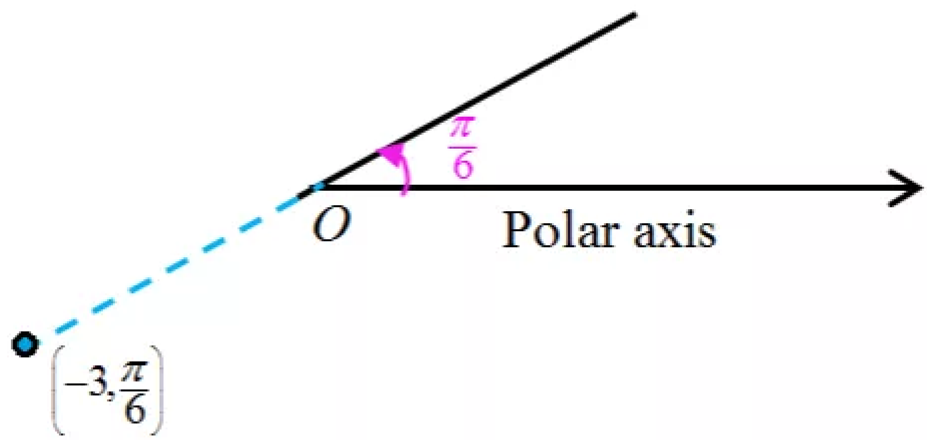
Consider the polar coordinate

$\left(-3, \frac{\pi}{6}\right)$

Now, plot the point on the polar coordinate system.

Note: The points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O. If $r > 0$, the point (r, θ) lies on the same quadrant as θ; if $r < 0$, the point $(-r, \theta)$ lies in the quadrant opposite side of the pole.

The point $\left(-3, \frac{\pi}{6}\right)$ is located 3 units from the pole in the third quadrant because the angle $\frac{\pi}{6}$ is in the first quadrant and $r = -3$ is negative.



Find two other coordinate pairs of $\left(-3, \frac{\pi}{6}\right)$ in which $r > 0$ and $r < 0$:

In fact, since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r, θ) is also represented by

$(r, \theta + 2n\pi)$ and $(-r, \theta + (2n+1)\pi)$

Where n is any integer.

Note that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$

So, $\left(-3, \frac{\pi}{6}\right)$ represents the same point as $\left(3, \frac{7\pi}{6}\right)$

For $r > 0$:

The corresponding coordinate pairs of $\left(3, \frac{7\pi}{6}\right)$ are:

$$\left(3, \frac{7\pi}{6} + 2n\pi\right), \quad n = 0, \pm1, \pm2, \ldots$$

That is,

$$\left(3, \frac{7\pi}{6}\right), \left(3, \frac{19\pi}{6}\right), \left(3, -\frac{5\pi}{6}\right), \left(3, \frac{31\pi}{6}\right), \ldots$$

For $r < 0$:

The corresponding coordinate pairs of $\left(3, \frac{7\pi}{6}\right)$ are:

$$\left(-3, \frac{7\pi}{6} + (2n+1)\pi\right), \quad n = 0, \pm1, \pm2, \ldots$$

That is,

$$\left(-3, \frac{13\pi}{6}\right), \left(-3, \frac{25\pi}{6}\right), \left(-3, \frac{\pi}{6}\right), \left(-3, \frac{37\pi}{6}\right), \ldots$$

Therefore, two other pairs of polar coordinate of $\left(-3, \frac{\pi}{6}\right)$ are $\boxed{\left(3, \frac{7\pi}{6}\right) \text{ and } \left(-3, \frac{13\pi}{6}\right)}$

(c)

Consider the polar coordinate

$$(1, -1)$$

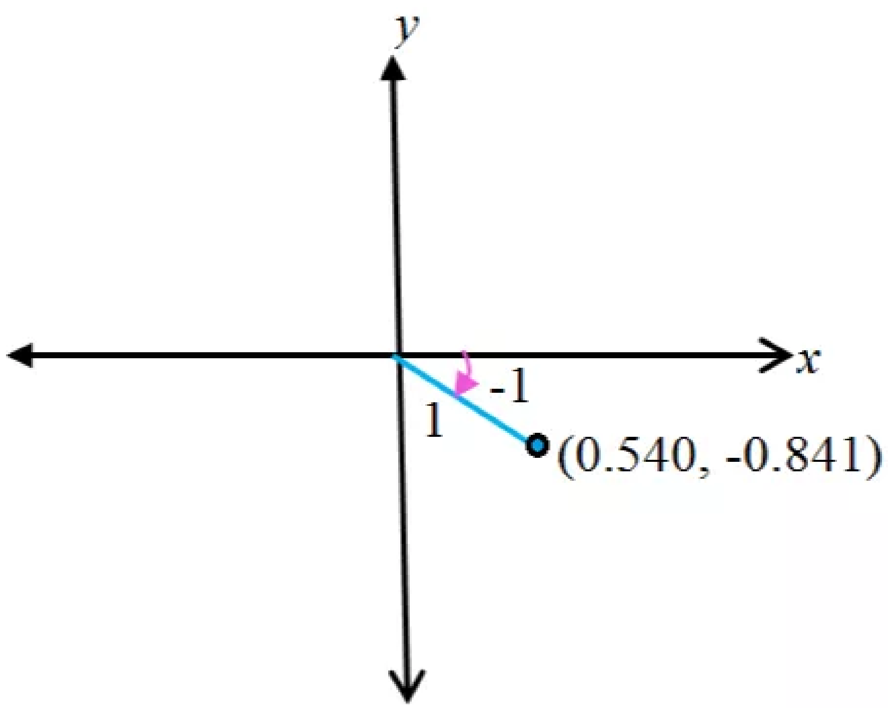
The Cartesian equivalent is

$$x = 1\cos(-1) \text{ and } y = 1\sin(-1)$$

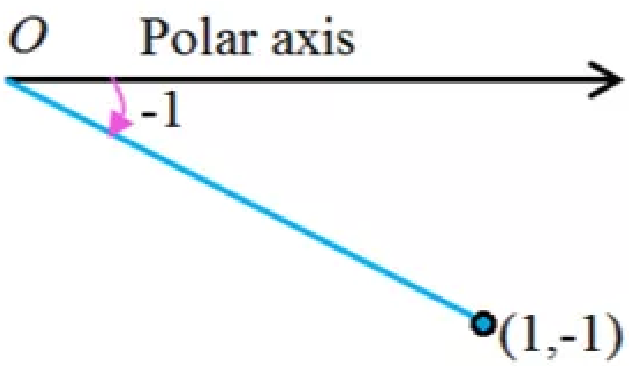
That is,

$$x = 0.540 \text{ and } y = -0.841$$

The graphical representation on the Cartesian coordinates system is



Now, plot the point on the polar coordinate system.



Find two other coordinate pairs of $(1,-1)$ in which $r>0$ and $r<0$:

In fact, since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r,θ) is also represented by

$$(r,\theta+2n\pi) \text{ and } (-r,\theta+(2n+1)\pi)$$

Where n is any integer.

For $r>0$:

The corresponding coordinate pairs of $(1,-1)$ are:

$$(1,-1+2n\pi), \quad n=0,\pm1,\pm2,\ldots$$

That is,

$$(1,-1),\ (1,-1+2\pi),(1,-1-2\pi),\ldots$$

For $r<0$:

The corresponding coordinate pairs of $(1,-1)$ are:

$$(-1,-1+(2n+1)\pi), \quad n=0,\pm1,\pm2,\ldots$$

That is,

$$(-1,-1+\pi),(-1,-1+3\pi),(-1,-1-\pi),\ldots$$

Therefore, two other pairs of polar coordinate of $(1,-1)$ are $\boxed{(1,-1+2\pi) \text{ and } (-1,-1+\pi)}$

Q3E

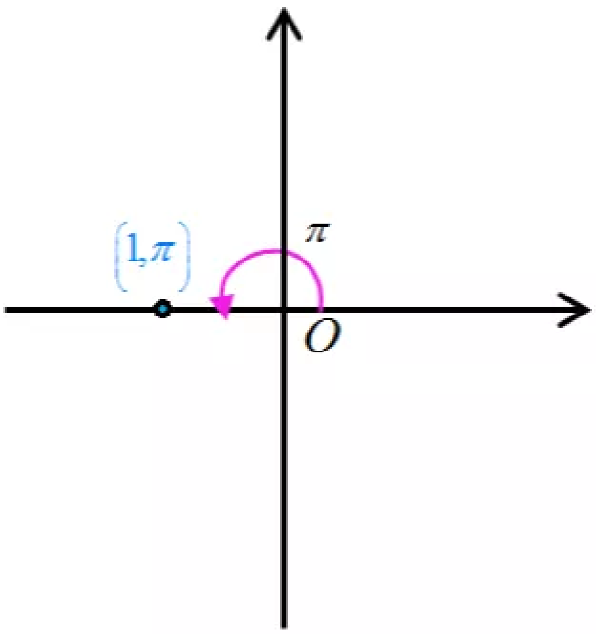
(a)

Consider the polar coordinate

$(1, \pi)$

Now, plot the point on the polar coordinate system.

Use the convention that an angle is positive if measured in the counter clock wise direction from the polar axis.



Convert the point $(1, \pi)$ from polar to Cartesian coordinates:

To do this, use the following equations

$x = r\cos\theta$

And

$y = r\sin\theta$

In present case,

$r = 1$ and $\theta = \pi$

It follows that,

$$\begin{aligned} x &= 1\cos\pi \\ &= 1\cdot(-1) \\ &= -1 \end{aligned}$$

And

$$\begin{aligned} y &= 1\sin\pi \\ &= 1\cdot(0) \\ &= 0 \end{aligned}$$

Therefore, the point is $\boxed{(-1,0)}$ in Cartesian coordinates.

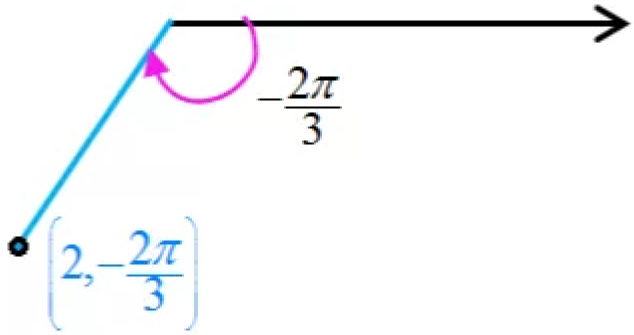
(b)

Consider the polar coordinate

$$\left(2, -\frac{2\pi}{3}\right)$$

Now, plot the point on the polar coordinate system.

Use the convention that an angle is positive if measured in the counter clock wise direction from the polar axis.



Convert the point $\left(2, -\frac{2\pi}{3}\right)$ from polar to Cartesian coordinates:

To do this, use the following equations

$$x = r\cos\theta$$

And

$$y = r\sin\theta$$

In present case,

$$r = 2 \text{ and } \theta = -\frac{2\pi}{3}$$

It follows that,

$$\begin{aligned} x &= 2\cos\left(-\frac{2\pi}{3}\right) \\ &= 2\cdot\left(-\frac{1}{2}\right) \\ &= -1 \end{aligned}$$

And

$$\begin{aligned} y &= 2\sin\left(-\frac{2\pi}{3}\right) \\ &= 2\cdot\left(-\frac{\sqrt{3}}{2}\right) \\ &= -\sqrt{3} \end{aligned}$$

Therefore, the point is $\boxed{\left(-1, -\sqrt{3}\right)}$ in Cartesian coordinates.

(c)

Consider the polar coordinate

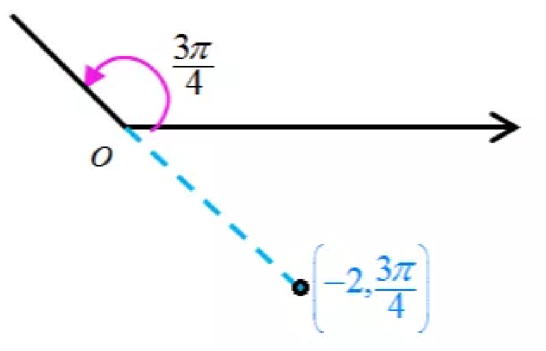
$$\left(-2, \frac{3\pi}{4}\right)$$

Now, plot the point on the polar coordinate system.

Use the convention that an angle is negative if measured in the clock wise direction from the polar axis.

Note: The points $(-r,\theta)$ and (r,θ) lie on the same line through *O* and at the same distance $|r|$ from *O*, but on opposite sides of *O*. If $r > 0$, the point (r,θ) lies on the same quadrant as θ ; if $r < 0$, the point $(-r,\theta)$ lies in the quadrant opposite side of the pole.

The point $\left(-2, \frac{3\pi}{4}\right)$ is located 2units from the pole in the fourth quadrant because the angle $\frac{3\pi}{4}$ is in the second quadrant and $r = -2$ is negative.



Convert the point $\left(-2, \frac{3\pi}{4}\right)$ from polar to Cartesian coordinates:

To do this, use the following equations

$$x = r\cos\theta$$

And

$$y = r\sin\theta$$

In present case,

$$r = -2 \text{ and } \theta = \frac{3\pi}{4}$$

It follows that,

$$\begin{aligned} x &= -2\cos\left(\frac{3\pi}{4}\right) \\ &= -2\cdot\left(-\frac{\sqrt{2}}{2}\right) \\ &= \sqrt{2} \end{aligned}$$

And

$$\begin{aligned} y &= -2\sin\left(\frac{3\pi}{4}\right) \\ &= -2\cdot\left(\frac{\sqrt{2}}{2}\right) \\ &= -\sqrt{2} \end{aligned}$$

Therefore, the point is $\boxed{\left(\sqrt{2}, -\sqrt{2}\right)}$ in Cartesian coordinates.

Q4E

(a)

Consider the polar coordinate

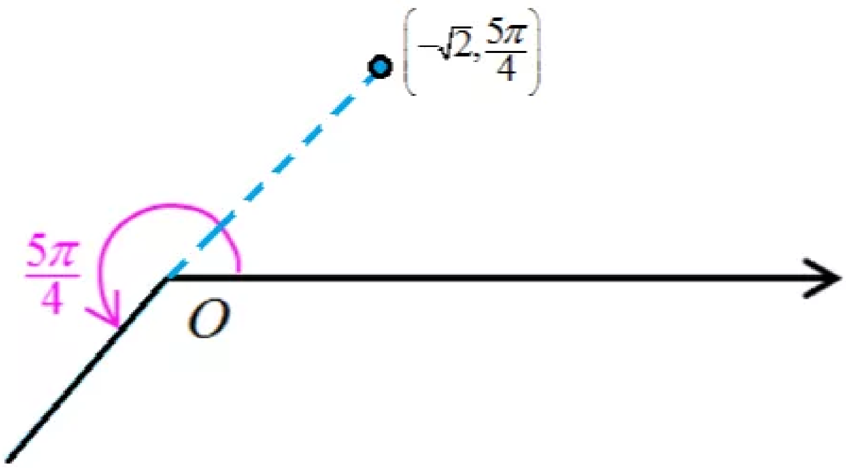
$\left(-\sqrt{2}, \frac{5\pi}{4}\right)$

Now, plot the point on the polar coordinate system.

Use the convention that an angle is positive if measured in the counter clock wise direction from the polar axis.

Note: The points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O. If $r > 0$, the point (r, θ) lies on the same quadrant as θ; if $r < 0$, the point $(-r, \theta)$ lies in the quadrant opposite side of the pole.

The point $\left(-\sqrt{2}, \frac{5\pi}{4}\right)$ is located $\sqrt{2}$ units from the pole in the first quadrant because the angle $\frac{5\pi}{4}$ is in the third quadrant and $r = -\sqrt{2}$ is negative.



Convert the point $\left(-\sqrt{2}, \frac{5\pi}{4}\right)$ from polar to Cartesian coordinates:

To do this, use the following equations

$$x = r\cos\theta$$

And

$$y = r\sin\theta$$

In present case,

$$r = -\sqrt{2} \text{ and } \theta = \frac{5\pi}{4}$$

It follows that,

$$\begin{aligned} x &= -\sqrt{2}\cos\frac{5\pi}{4} \\ &= -\sqrt{2}\cdot\left(-\frac{\sqrt{2}}{2}\right) \\ &= 1 \end{aligned}$$

And

$$\begin{aligned} y &= -\sqrt{2}\sin\frac{5\pi}{4} \\ &= -\sqrt{2}\cdot\left(-\frac{\sqrt{2}}{2}\right) \\ &= 1 \end{aligned}$$

Therefore, the point is $\boxed{(1,1)}$ in Cartesian coordinates.

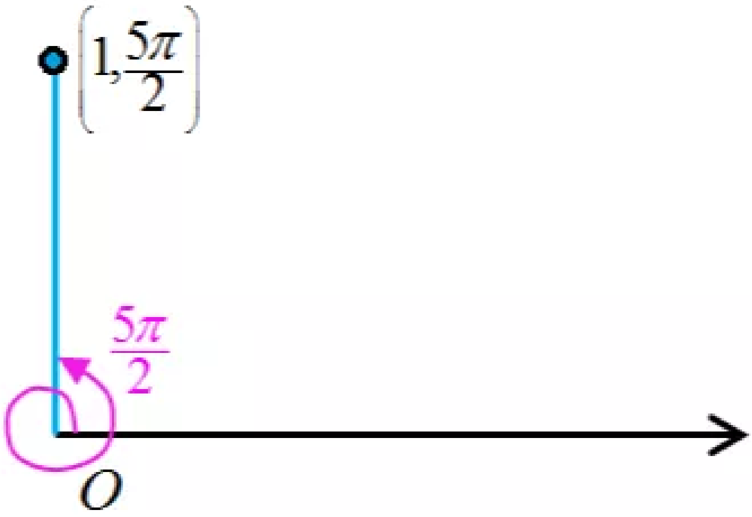
(b)

Consider the polar coordinate

$\left(1, \frac{5\pi}{2}\right)$

Now, plot the point on the polar coordinate system.

Use the convention that an angle is positive if measured in the counter clock wise direction from the polar axis.



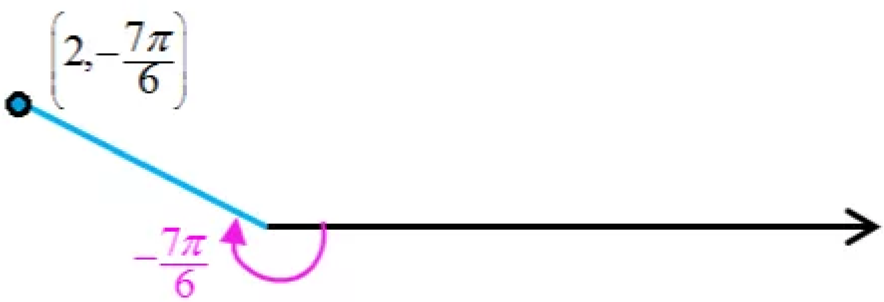
(c)

Consider the polar coordinate

$\left(2, -\frac{7\pi}{6}\right)$

Now, plot the point on the polar coordinate system.

Use the convention that an angle is negative if measured in the clock wise direction from the polar axis.



Convert the point $\left(2,-\frac{7\pi}{6}\right)$ from polar to Cartesian coordinates:

To do this, use the following equations

$$x = r\cos\theta$$

And

$$y = r\sin\theta$$

In present case,

$$r = 2 \text{ and } \theta = -\frac{7\pi}{6}$$

It follows that,

$$\begin{aligned} x &= 2\cos\left(-\frac{7\pi}{6}\right) \\ &= 2\cdot\left(-\frac{\sqrt{3}}{2}\right) \\ &= -\sqrt{3} \end{aligned}$$

And

$$\begin{aligned} y &= 2\sin\left(-\frac{7\pi}{6}\right) \\ &= 2\cdot\left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Therefore, the point is $\boxed{\left(-\sqrt{3},1\right)}$ in Cartesian coordinates.

Q5E

(a)

The Cartesian coordinates of a point are $(x,y)=(2,-2)$

To convert into to polar coordinates (r,θ) use the following relations:

$$r = \sqrt{x^2+y^2} \quad \text{......} \ (1)$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \text{......} \ (2)$$

Substitute $x=2, y=-2$ and calculate the values:

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{2^2+(-2)^2} \\ &= \pm 2\sqrt{2} \end{aligned}$$

Calculate θ.

Substitute the values of *x* and *y* in equation (2).

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{y}{x}\right)\\ &= \tan^{-1}\left(\frac{-2}{2}\right)\\ &= \tan^{-1}(-1)\\ &= \frac{7\pi}{4} \qquad \left[\text{Since } (-2,2)\in Q_4\right]\end{aligned}$$

(i) When $r>0$ and $0\le\theta\le 2\pi$, the polar coordinates are $\boxed{\left(2\sqrt{2},7\pi/4\right)}$.

(ii) When $r<0$ and $0\le\theta\le 2\pi$, the polar coordinates are

$$\left(-2\sqrt{2},\frac{7\pi}{4}-\pi\right)=\boxed{\left(-2\sqrt{2},\frac{3\pi}{4}\right)}.$$

(b).

The Cartesian coordinates of a point are $(x,y)=\left(-1,\sqrt{3}\right)$

Substitute $x=-1, y=\sqrt{3}$ in equations (1) and (2) to calculate the values of r and θ :

$$\begin{aligned}r &= \sqrt{x^2+y^2}\\ &= \sqrt{(-1)^2+\left(\sqrt{3}\right)^2}\\ &= \pm 2\end{aligned}$$

And

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{y}{x}\right)\\ &= \tan^{-1}\left(-\sqrt{3}\right)\\ &= \frac{2\pi}{3} \qquad \left[\text{Since } \left(-1,\sqrt{3}\right)\in Q_2\right]\end{aligned}$$

(i) When $r>0$ and $0\le\theta\le 2\pi$, the polar coordinates are $\boxed{\left(2,2\pi/3\right)}$.

(ii) When $r<0$ and $0\le\theta\le 2\pi$, the polar coordinates are $\left(-2,\frac{2\pi}{3}+\pi\right)=\boxed{\left(-2,\frac{5\pi}{3}\right)}$.

Q6E

(a)

Consider the Cartesian coordinate

$\left(3\sqrt{3},3\right)$

Convert the point $\left(3\sqrt{3},3\right)$ from Cartesian to polar coordinates:

To do this, use the following equations

$r^2 = x^2 + y^2$ ...... (1)

And

$\tan\theta = \frac{y}{x}$ ...... (2)

In present case,

$x = 3\sqrt{3}$ and $y = 3$

Substitute the values of *x* and *y* in equation (1).

$$r^2 = \left(3\sqrt{3}\right)^2 + (3)^2$$
$$r^2 = 27 + 9$$
$$r^2 = 36$$
$$r = 6$$

Substitute the values of *x* and *y* in equation (2).

$$\tan\theta = \frac{3}{3\sqrt{3}}$$

$$\tan\theta = \frac{1}{\sqrt{3}}$$

Since the point $\left(3\sqrt{3},3\right)$ lies in the first quadrant, choose $\theta = \frac{\pi}{6}$ or $\theta = \frac{13\pi}{6}$

Therefore, one possible answer is $\boxed{\left(3\sqrt{3},\frac{\pi}{6}\right)}$ and another is $\boxed{\left(3\sqrt{3},\frac{13\pi}{6}\right)}$

(i)

The polar coordinate of the point $\left(3\sqrt{3},3\right)$ where $r>0$ and $0\le\theta\le 2\pi$ is $\boxed{\left(3\sqrt{3},\frac{\pi}{6}\right)}$

(ii)

For $r<0$:

The corresponding coordinate pairs of $\left(3\sqrt{3},\frac{\pi}{6}\right)$ are:

$$\left(-3\sqrt{3},\frac{\pi}{6}+(2n+1)\pi\right),\quad n=0,\pm1,\ldots$$

That is,

$$\left(-3\sqrt{3},\frac{7\pi}{6}\right), \left(-3\sqrt{3},\frac{19\pi}{6}\right), \left(-3\sqrt{3},-\frac{5\pi}{4}\right),\ldots$$

Thus, The polar coordinate of the point $\left(3\sqrt{3},3\right)$ where $r<0$ and $0\le\theta\le 2\pi$ is

$$\boxed{\left(-3\sqrt{3},\frac{7\pi}{6}\right)}$$

(b)

Consider the Cartesian coordinate

$(1,-2)$

Convert the point $(1,-2)$ from Cartesian to polar coordinates:

To do this, use the following equations

$r^2 = x^2 + y^2$ ...... (1)

And

$\tan\theta = \dfrac{y}{x}$ ...... (2)

In present case,

$x = 1$ and $y = -2$

Substitute the values of *x* and *y* in equation (1).

$$r^2 = 1^2 + (-2)^2$$
$$r^2 = 1 + 4$$
$$r^2 = 5$$
$$r = \sqrt{5}$$

Substitute the values of *x* and *y* in equation (2).

$$\tan\theta = \frac{-2}{1}$$
$$\tan\theta = -2$$
$$\theta = \tan^{-1}(-2)$$

In fact, since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r,θ) is also represented by

$(r, \theta + 2n\pi)$ and $(-r, \theta + (2n+1)\pi)$

Where *n* is any integer.

For $r > 0$:

The corresponding coordinate pairs of $(\sqrt{5}, \tan^{-1}(2))$ are:

$$\left(\sqrt{5}, \tan^{-1}(-2) + 2n\pi\right), \quad n = 0, \pm 1, \ldots$$

That is,

$$\left(\sqrt{5}, \tan^{-1}(-2)\right),\ \left(\sqrt{5}, \tan^{-1}(-2) + 2\pi\right), \left(\sqrt{5}, \tan^{-1}(-2) - 2\pi\right), \ldots$$

For $r<0$:

The corresponding coordinate pairs of $\left(\sqrt{5},\tan^{-1}(2)\right)$ are:

$$\left(-\sqrt{5},\tan^{-1}(-2)+(2n+1)\pi\right),\quad n=0,\pm1,\ldots$$

That is,

$$\left(-\sqrt{5},\tan^{-1}(-2)+\pi\right),\ \left(-\sqrt{5},\tan^{-1}(-2)+3\pi\right),\left(\sqrt{5},\tan^{-1}(-2)-\pi\right),\ldots$$

(i)

The polar coordinate of the point $(1,-2)$ where $r>0$ and $0\le\theta\le2\pi$ is

$$\boxed{\left(\sqrt{5},\tan^{-1}(-2)+2\pi\right)}$$

(ii)

Thus, The polar coordinate of the point $(1,-2)$ where $r<0$ and $0\le\theta\le2\pi$ is

$$\boxed{\left(-\sqrt{5},\tan^{-1}(-2)+\pi\right)}$$

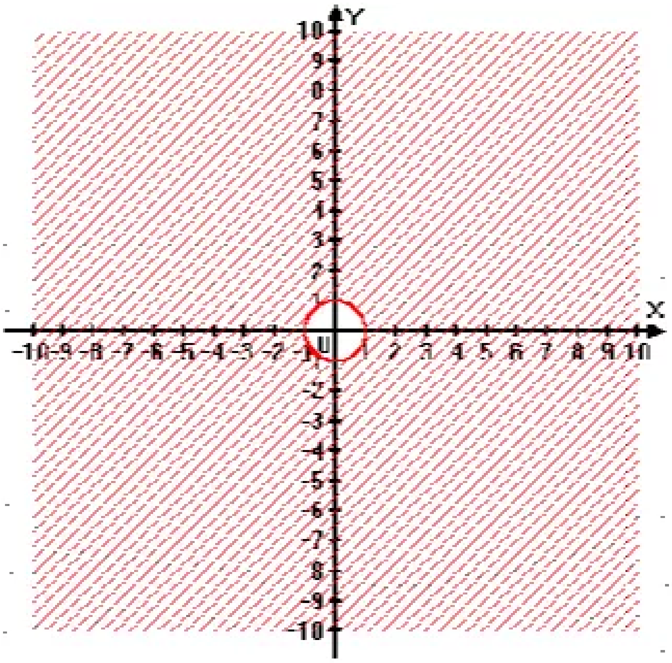
Q7E

We are required to find the region in which $r\ge1$ is satisfied.

We know that $r=1$ is the set of points satisfy the circle whose radius is 1.

Further, the set of points outside this circle satisfy $r>1$

Combining these regions, we get the required graph.



Observe that the region which is uncovered at the middle denote $r<0$ and the shaded region is the required region.

Q8E

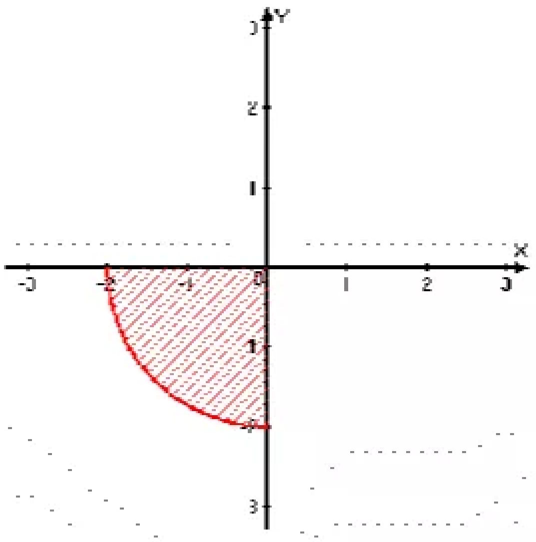
We are required to identify the region in which the polar conditions $0 \le r < 2$, $\pi \le \theta \le \frac{3\pi}{2}$ are satisfied.

We know that r is the radius of the circle. $r = 2$ is the set of points lie on the circle and $r < 2$ are the set of points inside the circle.

Further, for each quadrant, we are required to rotate the angle $90^0 = \frac{\pi}{2}$

So, $\pi \le \theta \le \frac{3\pi}{2}$ is the region denoted in the third quadrant starting from positive part of X axis.

Putting these things together, we get the required region that the set of points in side the circle of radius 2 and are in third quadrant only.



Observe that the points which are on the perimeter of the circle are not considered.

Q9E

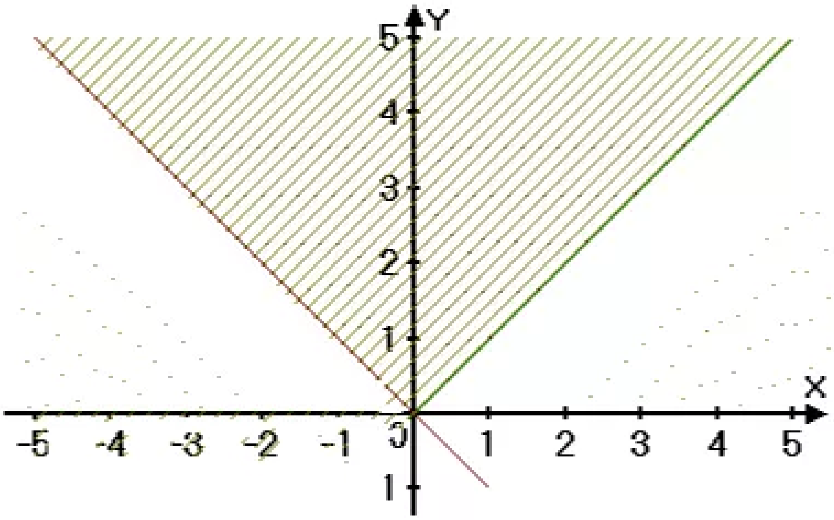
We are required to sketch the region in the plane consisting of points whose polar coordinates satisfy $r \ge 0$, $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$

We follow that r is the radius of the circle ≥ 0 and the upper limit is not given.

So, we leave the upper boundary uncovered.

We know that the line $\theta = \frac{\pi}{4}$ is the line in the first quadrant making equal angles with both X and Y axis. The line 90^0 away from this line in the anti clockwise direction shows $\theta = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$ which will be in the second quadrant.

Thus, the required region is



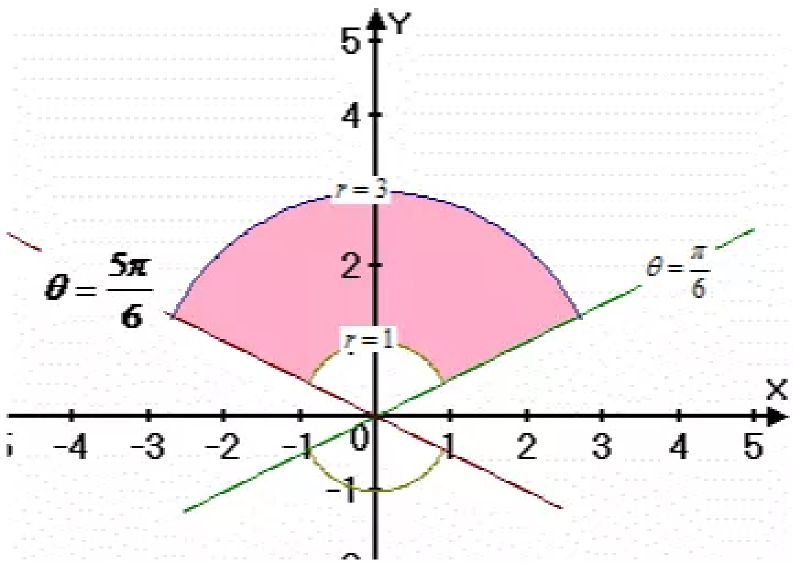
Q10E

We are required to sketch the region in which the polar coordinates $1 \le r \le 3$, $\frac{\pi}{6} < \theta < \frac{5\pi}{6}$ are satisfied.

We know that r is the radius of the circle and $1 \le r \le 3$ stands for the region between the circles of radii 1 unit and 3 units.

Also, $\theta = \frac{\pi}{6}$ is the line in the first quadrant and $\theta = \frac{5\pi}{6} = \pi - \frac{\pi}{6}$ is the line in the second quadrant. So, $\frac{\pi}{6} < \theta < \frac{5\pi}{6}$ stands for the circular region between the circles of radii 1 and 3, lies between the lines $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

It can be viewed as



Q11E

$2 < r < 3, \qquad 5\pi/3 \le \theta \le 7\pi/3$

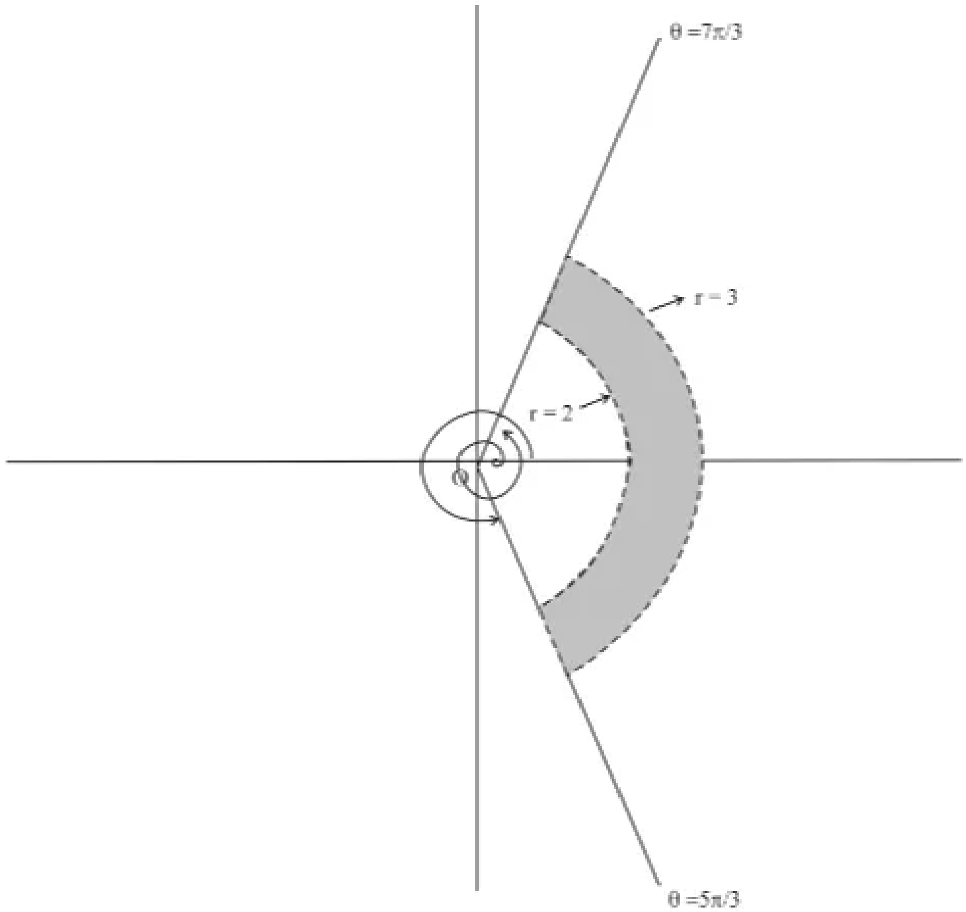


Fig. 1

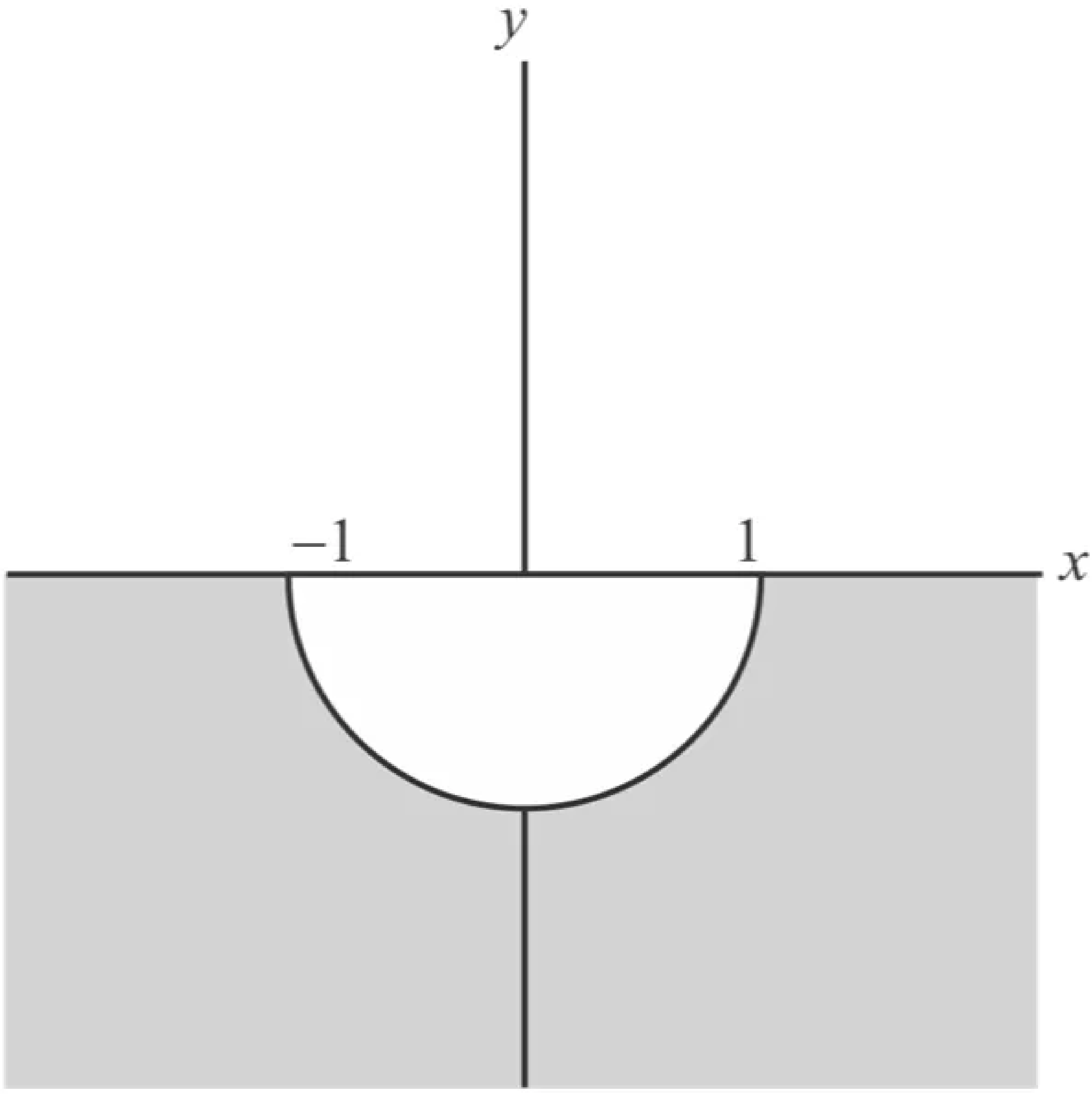
Q12E

Consider the polar coordinates:

$r \ge 1, \;\; \pi \le \theta \le 2\pi$

From the given condition, the shaded region lies in Quadrants III and IV, and outside of the semicircle whose radius is 1. Therefore,

The graph of the given polar coordinates is shown below:



Q13E

Consider the polar coordinates

$\left(2,\frac{\pi}{3}\right)$ and $\left(4,\frac{2\pi}{3}\right)$

To find the distance between the above two polar coordinates, first convert it into Cartesian coordinates.

Convert the polar coordinate $\left(2,\frac{\pi}{3}\right)$ to Cartesian coordinates:

Let $r_1 = 2$ and $\theta_1 = \frac{\pi}{3}$

To convert the coordinates polar to Cartesian, use the following equations

$x_1 = r_1 \cos\theta$

And

$y_1 = r_1 \sin\theta$

It follows that,

$$
\begin{aligned}
x_1 &= 2\cos\left(\frac{\pi}{3}\right) \\
&= 2\cdot\left(\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
y_1 &= 2\sin\left(\frac{\pi}{3}\right) \\
&= 2\cdot\left(\frac{\sqrt{3}}{2}\right) \\
&= \sqrt{3}
\end{aligned}
$$

Thus, the polar coordinate $\left(2,\frac{\pi}{3}\right)$ changed to the Cartesian coordinate $\left(1,\sqrt{3}\right)$.

Convert the polar coordinate $\left(4,\frac{2\pi}{3}\right)$ to Cartesian coordinates:

Let $r_2=4$ and $\theta_2=\frac{2\pi}{3}$

To convert the coordinates polar to Cartesian, use the following equations

$$x_2=r_2\cos\theta$$

And

$$y_2=r_2\sin\theta$$

It follows that,

$$\begin{aligned}x_2&=4\cos\left(\frac{2\pi}{3}\right)\\&=4\cdot\left(-\frac{1}{2}\right)\\&=-2\end{aligned}$$

And

$$\begin{aligned}y_1&=4\sin\left(\frac{\pi}{3}\right)\\&=4\cdot\left(\frac{\sqrt{3}}{2}\right)\\&=2\sqrt{3}\end{aligned}$$

Thus, the polar coordinate $\left(4,\frac{2\pi}{3}\right)$ changed to the Cartesian coordinate $\left(-2,2\sqrt{3}\right)$.

Now, the polar coordinates $\left(2,\frac{\pi}{3}\right)$ and $\left(4,\frac{2\pi}{3}\right)$ changed to the Cartesian coordinates $\left(1,\sqrt{3}\right)$ and $\left(-2,2\sqrt{3}\right)$

Recall that, the distance between the two points (x_1,y_1) and (x_2,y_2) is

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

In present case,

$$(x_1,y_1)=\left(1,\sqrt{3}\right) \text{ and } (x_2,y_2)=\left(-2,2\sqrt{3}\right)$$

The distance between the $\left(1,\sqrt{3}\right)$ and $\left(-2,2\sqrt{3}\right)$ is

$$\begin{aligned}d&=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\&=\sqrt{(-2-1)^2+\left(2\sqrt{3}-\sqrt{3}\right)^2}\\&=\sqrt{(-3)^2+\left(\sqrt{3}\right)^2}\\&=\sqrt{9+3}\\&=\sqrt{12}\end{aligned}$$

Therefore, the distance between the points with polar coordinates $\left(2,\frac{\pi}{3}\right)$ and $\left(4,\frac{2\pi}{3}\right)$ is

$$\boxed{\sqrt{12}}$$

Q14E

We have to find the distance between the points with polar coordinates $(r_1,\theta_1),(r_2,\theta_2)$

Let corresponding Cartesian coordinates be (x_1,y_1)

And (x_2,y_2)

Then $x_1 = r_1\cos\theta_1,\quad y_1 = r_1\sin\theta_1$

And $x_2 = r_2\cos\theta_2,\quad y_2 = r_2\sin\theta_2$

By the distance formula distance

$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$= \sqrt{(r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2}$$

$$= \sqrt{r^2{}_2\cos^2\theta_2 + r_1^2\cos^2\theta_1 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_2^2\sin^2\theta_2 + r_1^2\sin^2\theta_1 - 2r_1r_2\sin\theta_1\sin\theta_2}$$

$$= \sqrt{r_1^2(\cos^2\theta_1+\sin^2\theta_1) + r_2^2(\cos^2\theta_2+\sin^2\theta_2) - 2r_1r_2(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)}$$

$$= \sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)}$$ [Cos (A-B) = cosAcosB+sinAsinB

So distance formula for polar coordinates is

$$\boxed{d = \sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)}}$$

Q15E

Sol: Given $r^2 = 5$

To convert the given equation to Cartesian form we use $x^2+y^2=r^2$ and $\tan\theta = \dfrac{y}{x}$

$r^2 = 5$

$\Rightarrow x^2+y^2 = 5$

Which represents a circle with center (0,0) and radius $\sqrt{5}$.

Q16E

Given polar equation is $r = 4\sec\theta$

$\Rightarrow r\cos\theta = 4$

using $\cos\theta = \frac{x}{r}$, we get

$\Rightarrow \frac{x}{r}\cdot r = 4$

$\Rightarrow x = 4$

this represents a straight line parallel to y-axis.

Q17E

Consider the polar curve,

$r = 2\cos\theta$.

The objective is to convert the given polar curve into Cartesian coordinate system.

We know that the polar coordinates have Cartesian coordinates,

$x = r\cos\theta,$

$y = r\sin\theta$

And

$$\begin{aligned} x^2 + y^2 &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= r^2\left(\cos^2\theta + \sin^2\theta\right) \\ &= r^2\left(\text{Since } \cos^2\theta + \sin^2\theta = 1\right) \end{aligned}$$

Substitute these values in the equation of the polar curve:

$r = 2\cos\theta$

$r = \frac{2x}{r}$

$2x = r^2$

Substitute $r^2 = x^2 + y^2$ in the equation $2x = r^2$

Then, the equation becomes,

$$2x = r^2$$
$$2x = x^2 + y^2$$
$$x^2 + y^2 - 2x = 0$$
$$x^2 - 2x + y^2 = 0$$
$$x^2 - 2x + 1 + y^2 = 0 + 1$$ Add 1 on both sides
$$\left(x^2 - 2x + 1\right) + y^2 = 1$$
$$(x-1)^2 + y^2 = 1$$

Therefore, the curve $(x-1)^2 + y^2 = 1$ represents the equation of a circle with center $\boxed{(1,0)}$ and radius $\boxed{1}$.

Q18E

Given parametric equation is $\theta = \frac{\pi}{3}$

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{3}$$
$$\Rightarrow \tan\frac{\pi}{3} = \frac{y}{x}$$
$$\Rightarrow \frac{y}{x} = \sqrt{3}$$
$$\Rightarrow y = \sqrt{3}x$$

This represents an straight line with slope $\sqrt{3}$ and passing through the origin.

Q19E

Consider the polar equation $r^2 \cos 2\theta = 1.$

Convert the given polar equation into Cartesian equation.

To convert from polar to rectangular equation, use the following equations:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2 + y^2 = r^2$$

Rewrite the given polar equation as follows:

$$r^2\cos 2\theta = 1$$

$$r^2\left(\cos^2\theta - \sin^2\theta\right) = 1 \qquad \begin{bmatrix}\text{Use Double Angle Formula:} \\ \cos 2x = \cos^2 x - \sin^2 x\end{bmatrix}$$

$$r^2\cos^2\theta - r^2\sin^2\theta = 1$$

$$(r\cos\theta)^2 - (r\sin\theta)^2 = 1$$

$$x^2 - y^2 = 1 \qquad [\text{Substitute } x = r\cos\theta, y = r\sin\theta]$$

The rectangular equation of the given polar equation is $x^2 - y^2 = 1.$

Now compare the equation $x^2 - y^2 = 1$ with the equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

Here, $a = 1, b = 1.$

Then,

$$c^2 = a^2 + b^2$$
$$= 1^2 + 1^2$$
$$= 1 + 1$$
$$= 2$$
$$c = \sqrt{2}$$

So foci of the hyperbola is $(\pm c, 0) = \left(\pm\sqrt{2}, 0\right).$

Vertices are $(\pm a, 0) = (\pm 1, 0).$

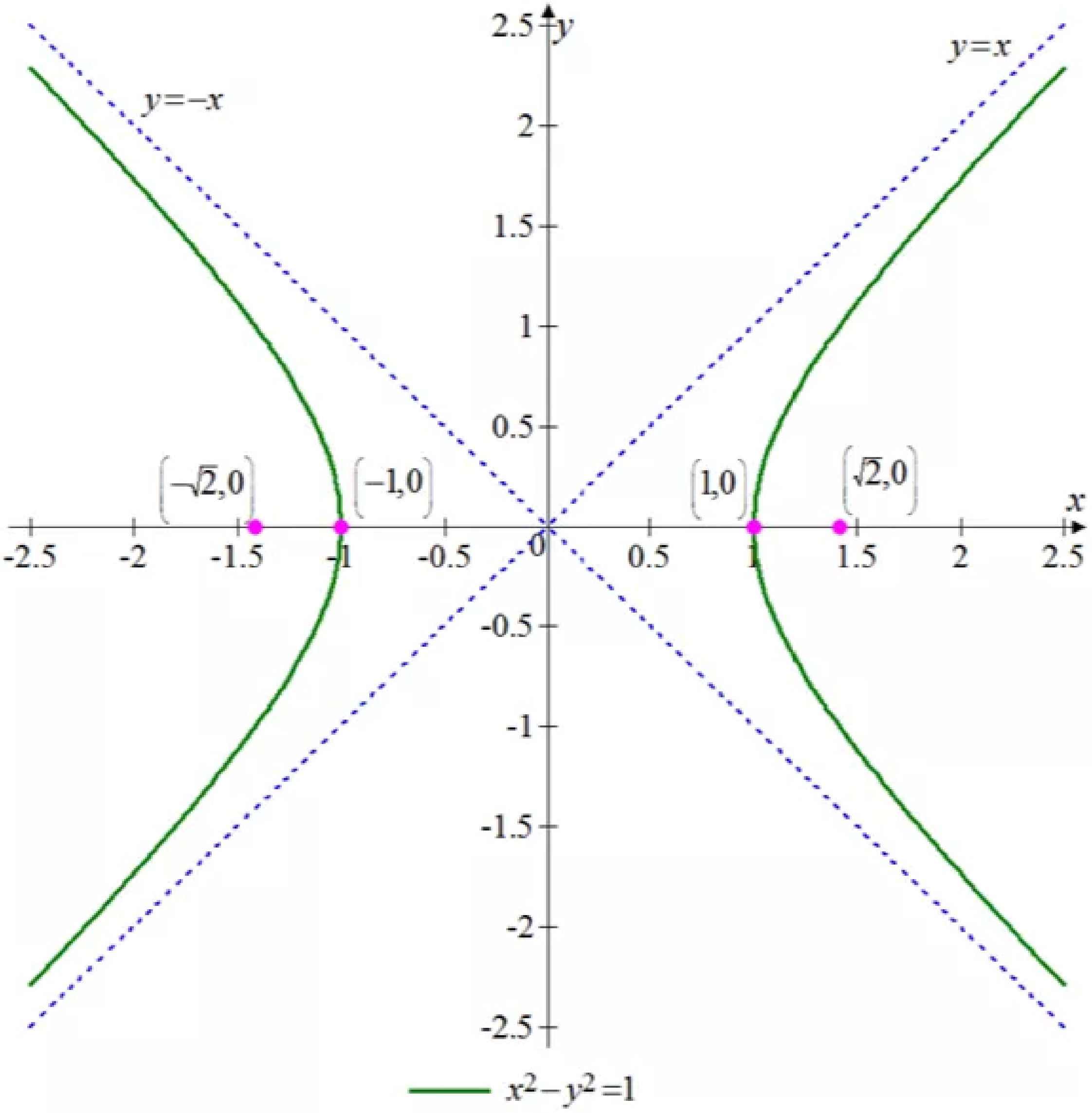
Asymptotes of the hyperbola are,

$$y = \pm\left(\frac{b}{a}\right)x$$
$$y = \pm\left(\frac{1}{1}\right)x$$
$$y = \pm x$$

Therefore, the equation $x^2 - y^2 = 1$ represents a hyperbola with foci at $\left(\pm\sqrt{2},0\right)$ and vertices are $(\pm 1,0)$.

The asymptotes are $y = \pm x.$

The hyperbola is sketched in the below figure:



Q20E

Consider the polar equation

$r = \tan\theta \sec\theta$

Relation between polar and Cartesian coordinates:

Cartesian coordinates: (x, y)

Polar coordinates: (r,θ)

Where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$; $x = r\cos\theta$ and $y = r\sin\theta$

To find a Cartesian equation for the curve $r = \tan\theta\sec\theta$:

$r = \tan\theta\sec\theta$

$r = \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}$ Since $\tan\theta = \frac{\sin\theta}{\cos\theta}, \sec\theta = \frac{1}{\cos\theta}$

$r^2 = \frac{r\sin\theta}{\cos^2\theta}$ Multiply by r on both sides.

$r^2\cos^2\theta = r\sin\theta$

$x^2 = y$ Use $x = r\cos\theta$ and $y = r\sin\theta$

Recall that the equation of parabola with vertex $(0,0)$ and focus at $(0,a)$ is $x^2 = 4ay$

Comparing $x^2 = y$ with $x^2 = 4ay$, we get

$4a = 1$

$a = \frac{1}{4}$

So, $x^2 = y$ is the equation of parabola with vertex $(0,0)$ and focus at $\left(0, \frac{1}{4}\right)$.

Therefore, Cartesian equation for the curve $r = \tan\theta\sec\theta$ is $\boxed{x^2 = y}$.

Q21E

Given Cartesian equation is $y = 2$

Using $x = r\cos\theta, y = r\sin\theta$, we convert it to the polar equation.

$\Rightarrow r\sin\theta = 2$

$\Rightarrow \boxed{r = 2\operatorname{cosce}\theta}$

Q22E

Given Cartesian equation is $y = x$

using $y = r\sin\theta$, $x = r\cos\theta$, we change this equation into Polar equation as

$r\sin\theta = r\cos\theta$

$\Rightarrow \sin\theta = \cos\theta$

$\Rightarrow \tan\theta = 1$

$\Rightarrow \theta = \frac{\pi}{4}$

Q23E

Given Cartesian equation is $y = 1+3x$

using $y = r\sin\theta$, $x = r\cos\theta$, we change it into the polar equation

$$r\sin\theta = 1+3(r\cos\theta)$$
$$\Rightarrow r(\sin\theta - 3\cos\theta) = 1$$
$$\Rightarrow \boxed{r = \frac{1}{\sin\theta - 3\cos\theta}}$$

Q24E

Given Cartesian equation is $4y^2 = x$

using $y = r\sin\theta$, $x = r\cos\theta$, we change it into the polar equation

$$4(r^2\sin^2\theta) = r\cos\theta$$
$$\Rightarrow 4r\sin^2\theta = \cos\theta$$
$$\Rightarrow \boxed{r = \frac{1}{4}\cot\theta\, cosce\theta}$$

Q25E

We have the Cartesian equation as $x^2 + y^2 = 2cx$

Since $x = r\cos\theta$ and $y = r\sin\theta$

Therefore $r^2\cos^2\theta + r^2\sin^2\theta = 2c\,r\,cs\cos\theta$

Or $r^2(\cos^2\theta + \sin^2\theta) = 2cr\cos\theta$

Or $r^2 = 2cr\cos\theta$

Or $\boxed{r = 2c\cos\theta}$.

This is the polar equation corresponds to the given Cartesian equation

Q26E

Consider the Cartesian equation

$xy = 4$

To find the polar equation that represents the above equation, use the formulas
$x = r\cos\theta$ and $y = r\cos\theta$

Now,

$$xy = 4$$
$$(r\cos\theta)(r\sin\theta) = 4$$
$$r^2 \sin\theta\cos\theta = 4$$
$$r^2 = \frac{4}{\sin\theta\cos\theta}$$
$$r = \pm\frac{2}{\sqrt{\sin\theta\cos\theta}}$$

Therefore, the polar equation that represents the Cartesian equation $xy = 4$ is

$$\boxed{r = \pm\frac{2}{\sqrt{\sin\theta\cos\theta}}}$$

Q27E

(a)

Consider a line that passes through the origin and makes an angle $\frac{\pi}{6}$ with the positive x-axis.

The given curve has a polar equation:

$\theta = \pi/6$

For the Cartesian equation of the given curve, note that the slope of the line is:

$\mathrm{m} = \tan(\pi/6)$

Then the Cartesian equation of the line is:

$y = (\tan(\pi/6))x$

It can be seen that the given curve can be more easily given by **a polar equation**,

$\boxed{\theta = \pi/6}$

(b)

Consider a vertical line which passes through the point $(3,3)$.

The given curve would be more easily given by a **Cartesian equation**.

And Cartesian equation of the curve is:

$\boxed{x = 3}$

Q28E

(A) Since the center of the circle is not at the origin, so the curve would be more easily given by a Cartesian equation $\boxed{(x-2)^2+(y-3)^2=5^2}$

(B) Since the center of the circle is at the origin, so the curve would be more easily given by the polar equation as $\boxed{r=4}$

Cartesian equation is also simple as $x^2+y^2=4^2 \Rightarrow x^2+y^2=16$

Q29E

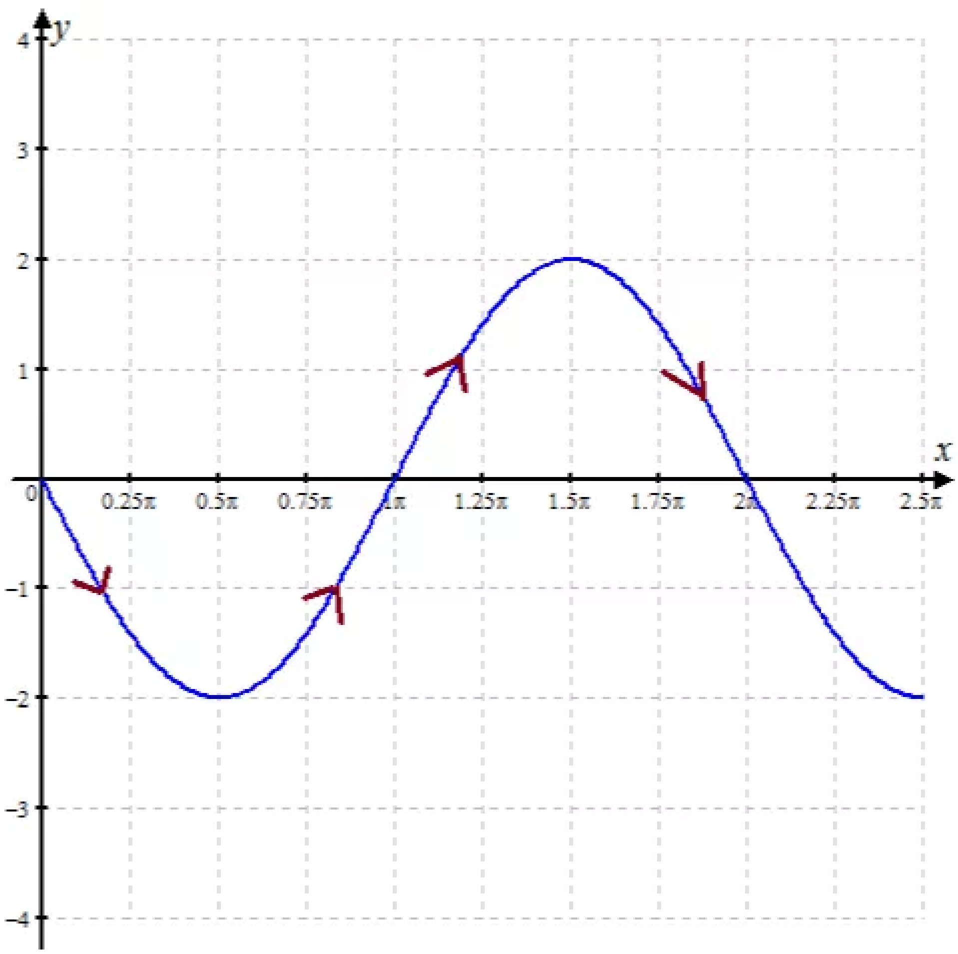
Consider the curve $r=-2\sin\theta$

First sketch the graph of $r=-2\sin\theta$ in Cartesian coordinates:

We first draw the suitable graph without changing it into the Cartesian equation. This enables us to read at a glance the value of r that correspond to increasing values of θ.

Since r is the distance from the origin to the curve $r=f(\theta)$. The value of r does not respect the sign in the Cartesian coordinates.

The curve $r=-2\sin\theta$ in the Cartesian coordinate is



Observe the graph that, as θ increase from 0 to $\frac{\pi}{2}$, the curve decreases from 0 to –2. So considering its absolute value as *r* it is easily to follow *r* increases from 0 to 2.

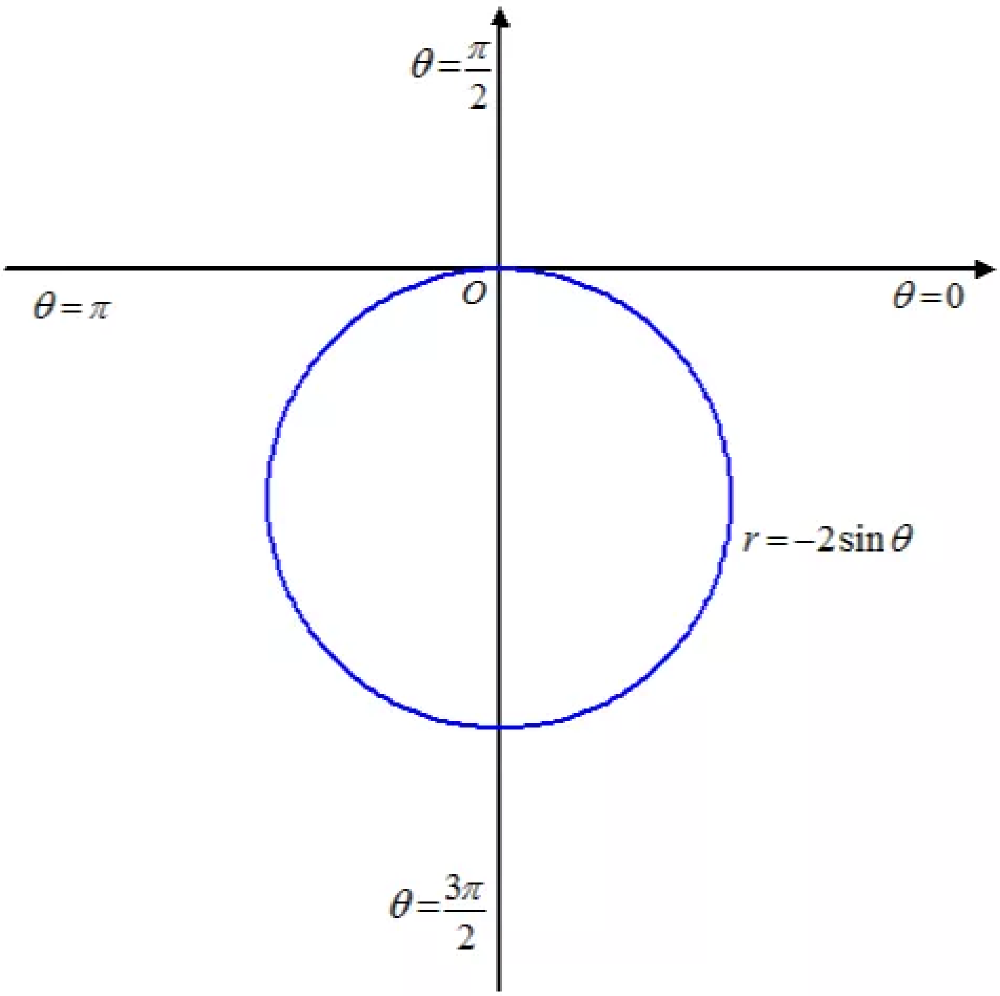
Again as θ increase from $\frac{\pi}{2}$ to π, the curve returns to *x*-axis. That is *r* returns from –2 to 0.

The same is continued from π to $\frac{3\pi}{2}$ and so on.

Since $-2\sin\theta$ indicates the negative values. It can be easily followed that the entire curve lies below the horizontal axis up to –2 and returns to 0 between 0 to π.

The same curve repeates from π to 2π, which is not visible separately.

Sketch the curve $r=-2\sin\theta$ is shown below:



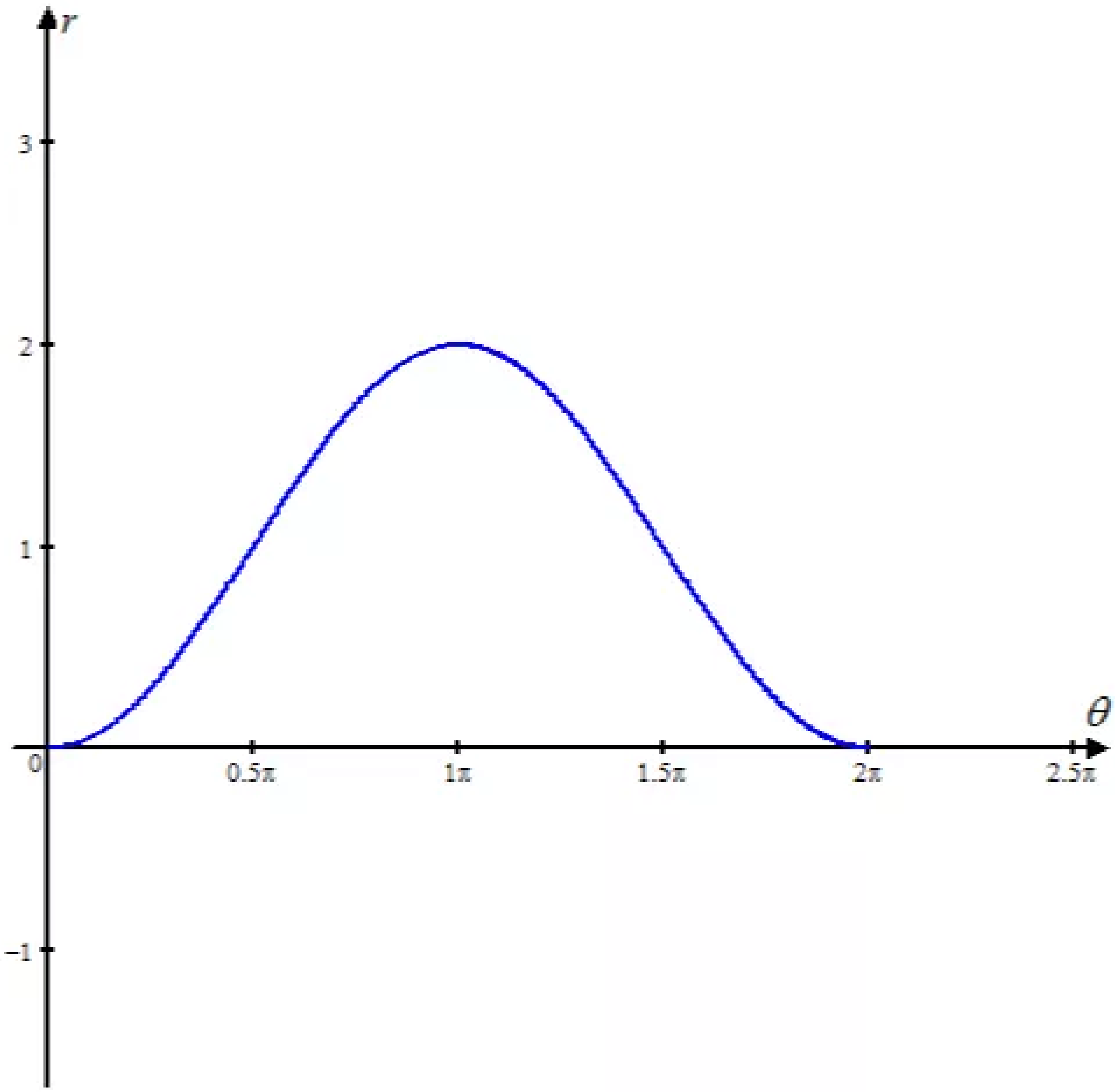
Q30E

Consider the curve $r=1-\cos\theta$

First sketch the graph of $r=-2\sin\theta$ in Cartesian coordinates:

We first draw the suitable graph without changing it into the Cartesian equation. This enables us to read at a glance the value of *r* that correspond to increasing values of θ.

The graph of $r = 1 - \cos\theta$ in Cartesian coordinates, $0 \le \theta \le 2\pi$ is shown below:



For instance, we see the following parts:

As θ increases from 0 to $\frac{\pi}{2}$, *r* (the distance from *O*) increase from 0 to 1.

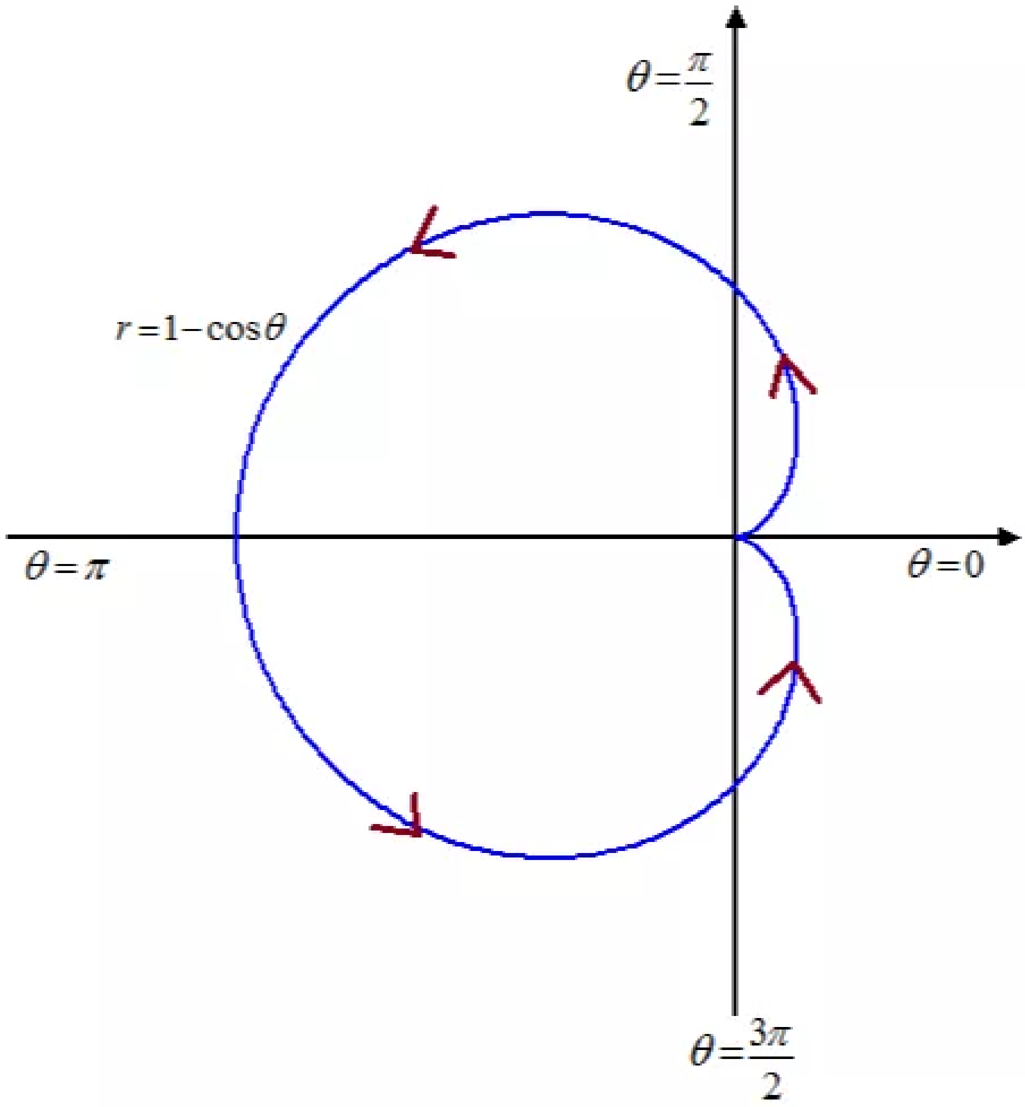
Now as θ increases from $\frac{\pi}{2}$ to π, *r* increase from 1 to 2.

Now as θ increases from π to $\frac{3\pi}{2}$, *r* decreases from 2 to 1.

Now as θ increases from $\frac{3\pi}{2}$ to 2π, *r* decreases from 1 to 0.

If we let θ increases beyond 2π or decrease beyond 0, we would simply retrace our path.

Now putting together the parts of the curve, we sketch the complete curve is shown below:



This is called a **cardioid**, because it's shaped like a heart.

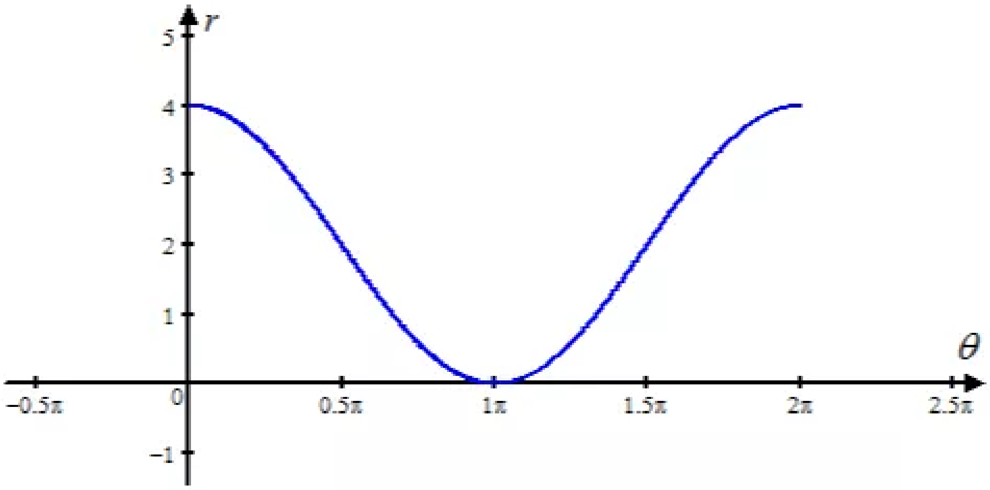
Q31E

Consider the curve $r=2(1+\cos\theta)$

First sketch the graph of $r=2(1+\cos\theta)$ in Cartesian coordinates:

We first draw the suitable graph without changing it into the Cartesian equation. This enables us to read at a glance the value of *r* that correspond to increasing values of θ.

The graph of $r = 2(1+\cos\theta)$ in Cartesian coordinates, $0 \le \theta \le 2\pi$ is shown below:



For instance, we see the following parts:

As θ increases from 0 to $\frac{\pi}{2}$, *r* (the distance from *O*) decrease from 4 to 2.

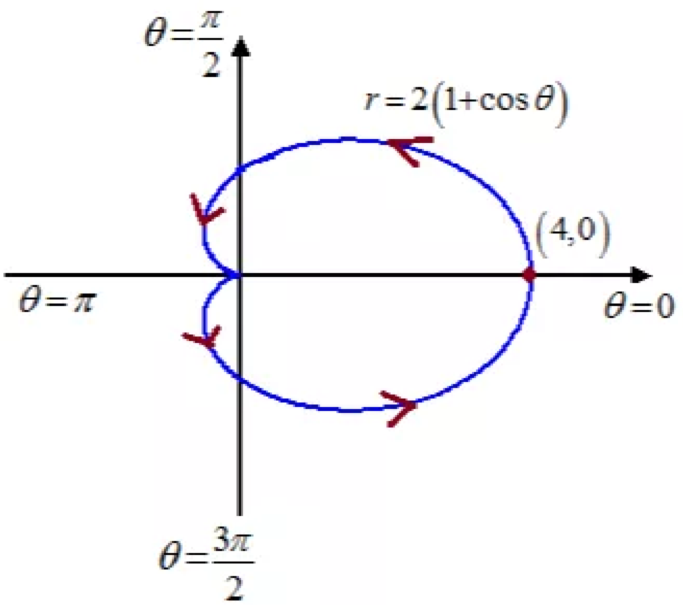
Now as θ increases from $\frac{\pi}{2}$ to π, *r* decrease from 2 to 0.

Now as θ increases from π to $\frac{3\pi}{2}$, *r* increase from 0 to 2.

Now as θ increases from $\frac{3\pi}{2}$ to 2π, *r* increase from 2 to 4.

If we let θ increases beyond 2π or decrease beyond 0, we would simply retrace our path.

Now putting together the parts of the curve, we sketch the complete curve is shown below:



This is called a **cardioid**, because it's shaped like a heart.

Q32E

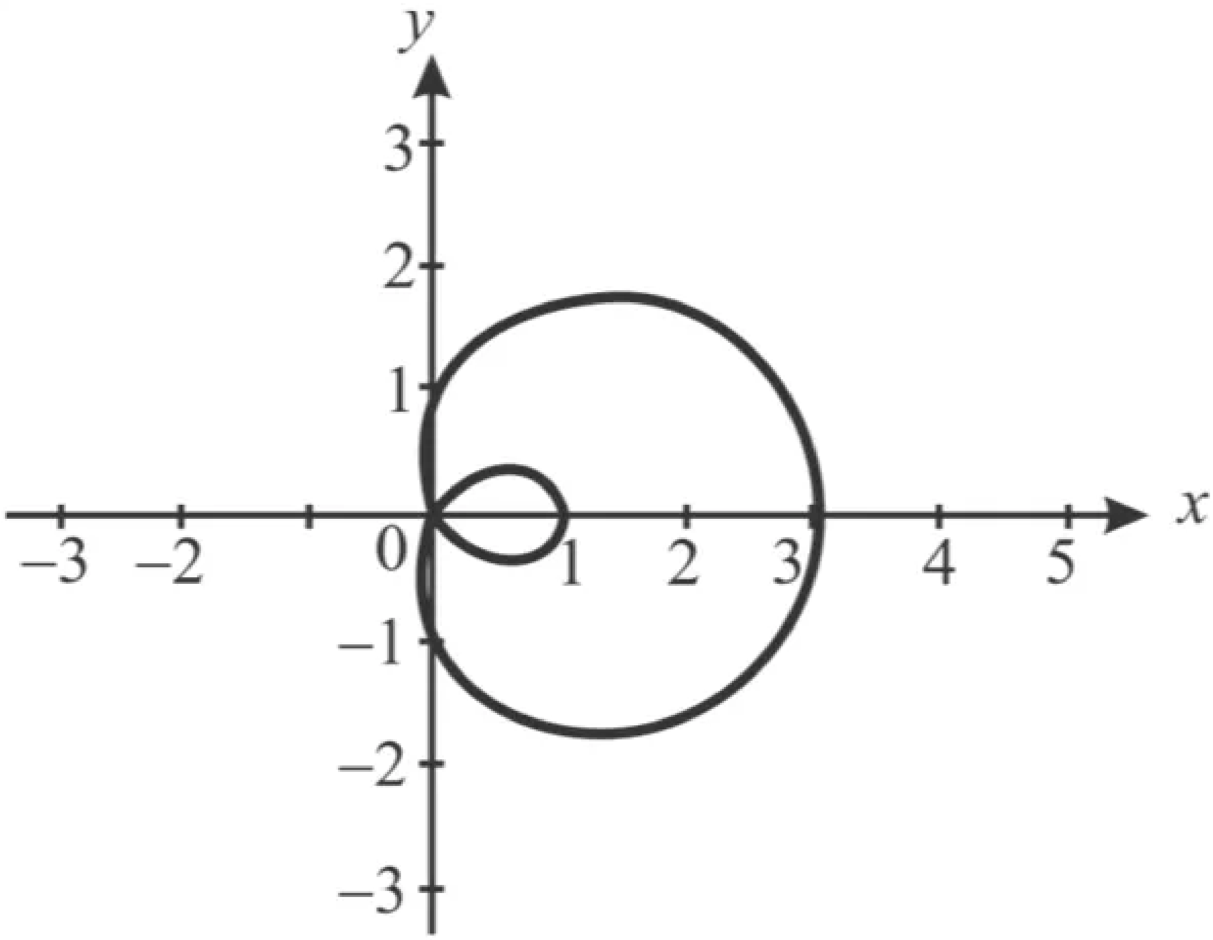
A system that is used as reference to define the position of a point uniquely in space is called as the coordinates sytem.

The polar coordinate system is a kind a of two dimensional coordinates system in which the position is uniquely defined as the distance from a reference point and the angle from a reference direction.

Consider the function:

$$r = (1 + 2\cos\theta)$$

Consider the sketch of the above function as shown below:



Convert the equation in the Cartesian coordinates:

$$r = (1 + 2\cos\theta)$$

$$r = (1 + 2\frac{x}{r})$$

$$r^2 = r + 2x$$

$$x^2 + y^2 - 2x = \sqrt{x^2 + y^2}$$

Hence, the equation in the Cartesian form is $\boxed{x^2 + y^2 - 2x = \sqrt{x^2 + y^2}}$.

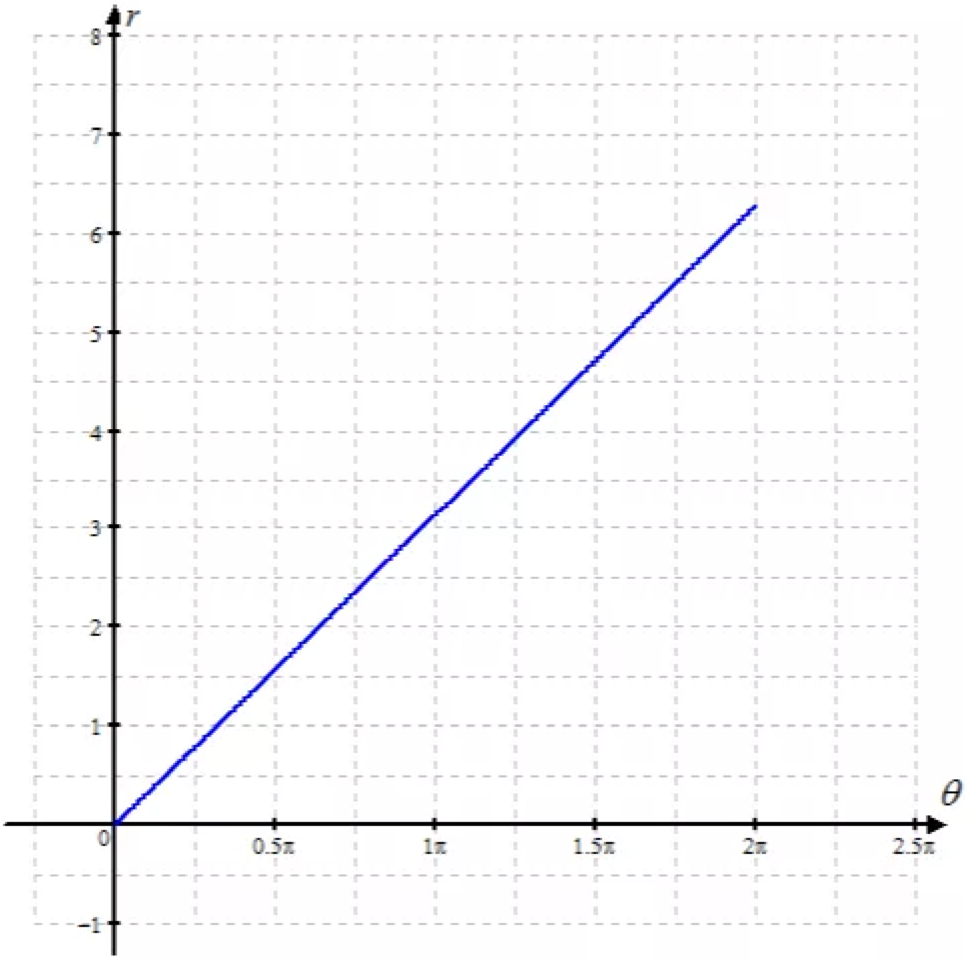
Q33E

Consider the curve $r=\theta, \quad \theta>0$

First sketch the graph of $r=\theta$ in Cartesian coordinates:

We first draw the suitable graph without changing it into the Cartesian equation. This enables us to read at a glance the value of *r* that correspond to increasing values of θ.

The graph of $r=\theta$ in Cartesian coordinates, $0\le\theta\le 2\pi$ is shown below:



For instance, we see the following parts:

As θ increases from 0 to $\frac{\pi}{2}$, *r* (the distance from *O*) increase from 0 to1.571.

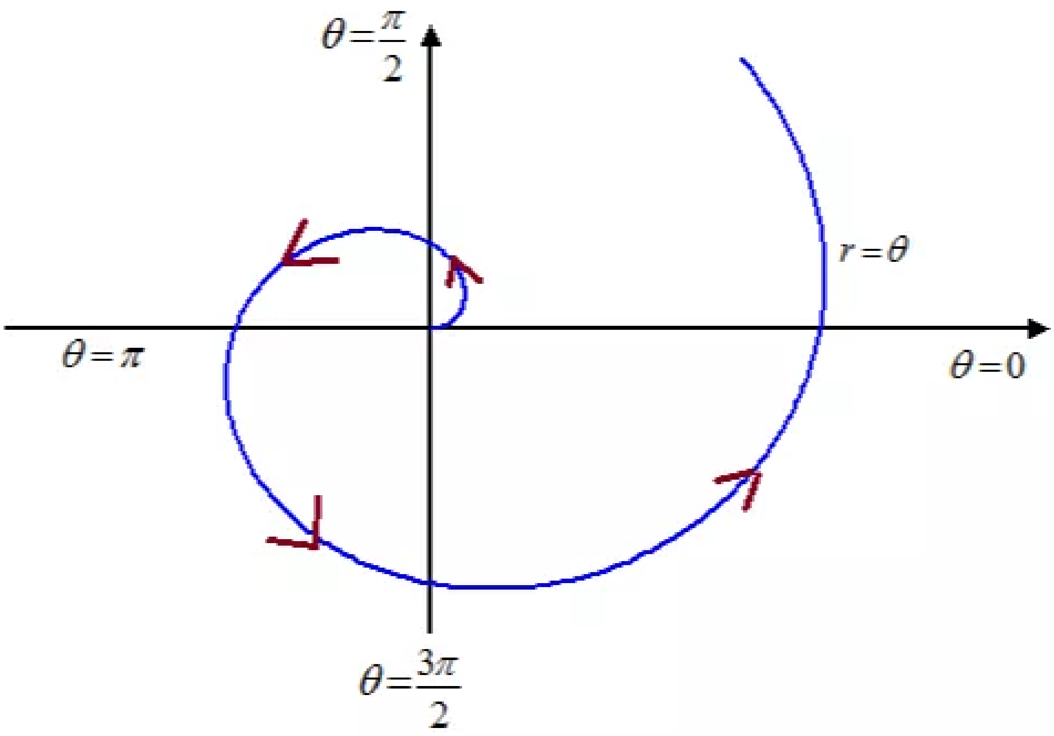
Now as θ increases from $\frac{\pi}{2}$ to π, *r* increase from 1.571 to 3.141.

Now as θ increases from π to $\frac{3\pi}{2}$, *r* increase from 3.141 to 4.712.

Now as θ increases from $\frac{3\pi}{2}$ to 2π, *r* increase from 4.712 to 6.283.

If we let θ increases beyond 2π, we would simply the path increase.

Now putting together the parts of the curve, we sketch the complete curve is shown below:



Q34E

We have to sketch the curve $r=\ln\theta,\quad \theta\geq 1$

We sketch the curve $r=\ln\theta$ in Cartesian coordinates for $\theta\geq 1$

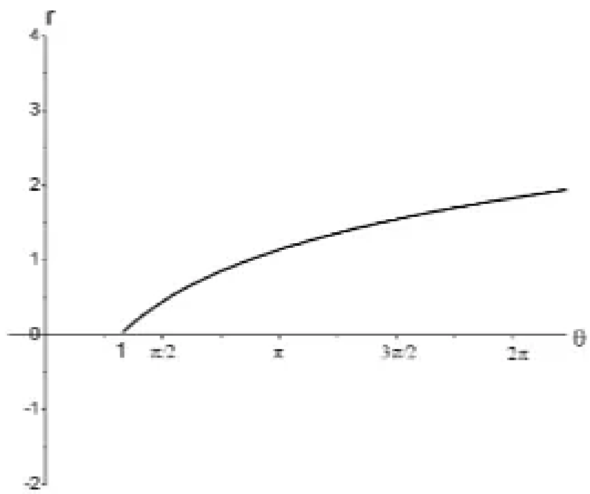


Fig.1

We see that when $\theta=1,\ \ r=0$ so polar curve starts from the pole, since $\ln\theta$ is a increasing function, so r increases, as θ increases, we sketch the curve

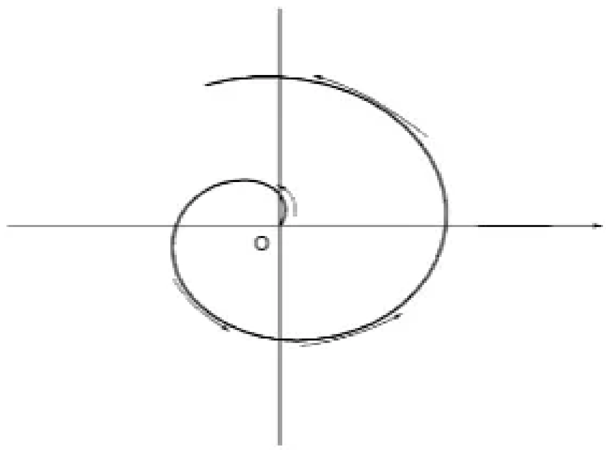


Fig.2

Q35E

Consider the polar equation,

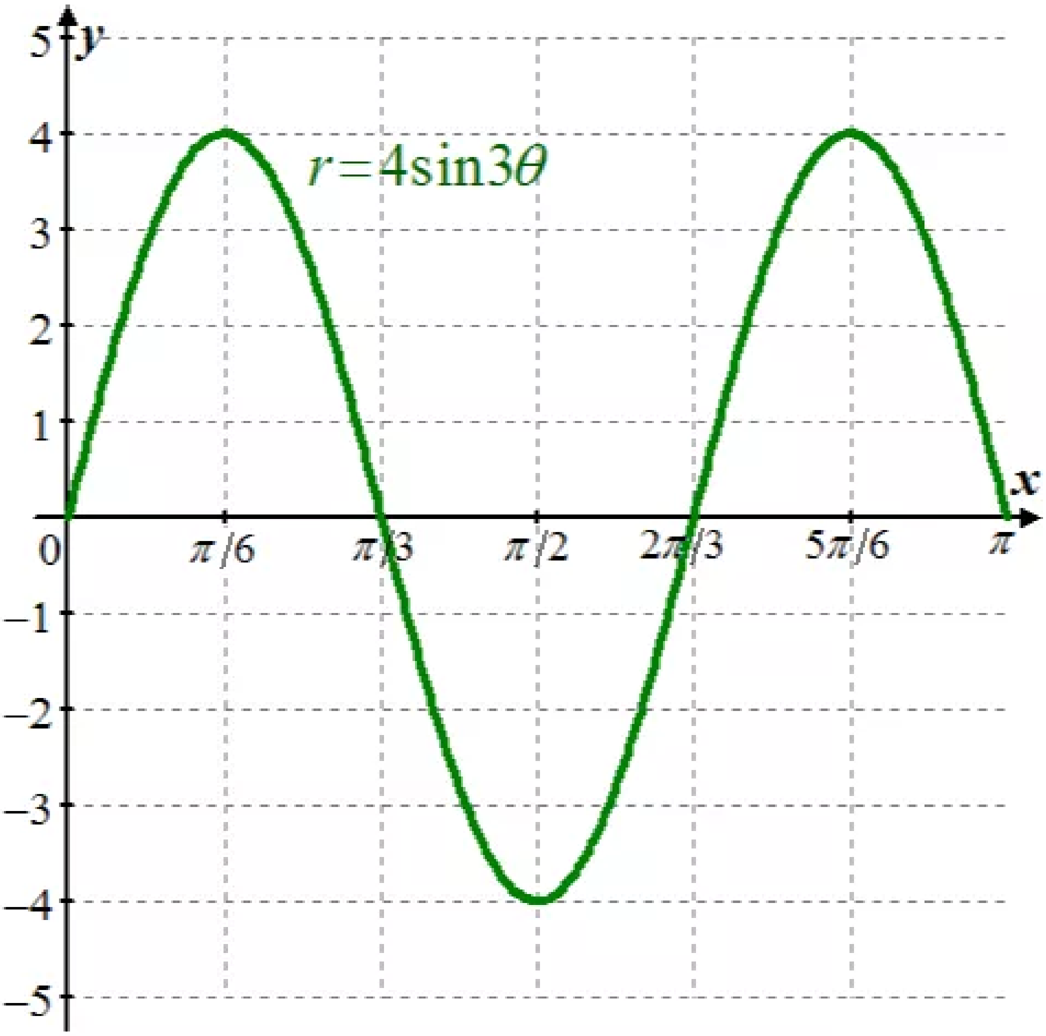
$r = 4\sin 3\theta$.

The objective is to sketch the given polar curve, by first sketching it in Cartesian coordinates.

First, tabulate the values of $r = 4\sin 3\theta$ for various values of θ as follows:

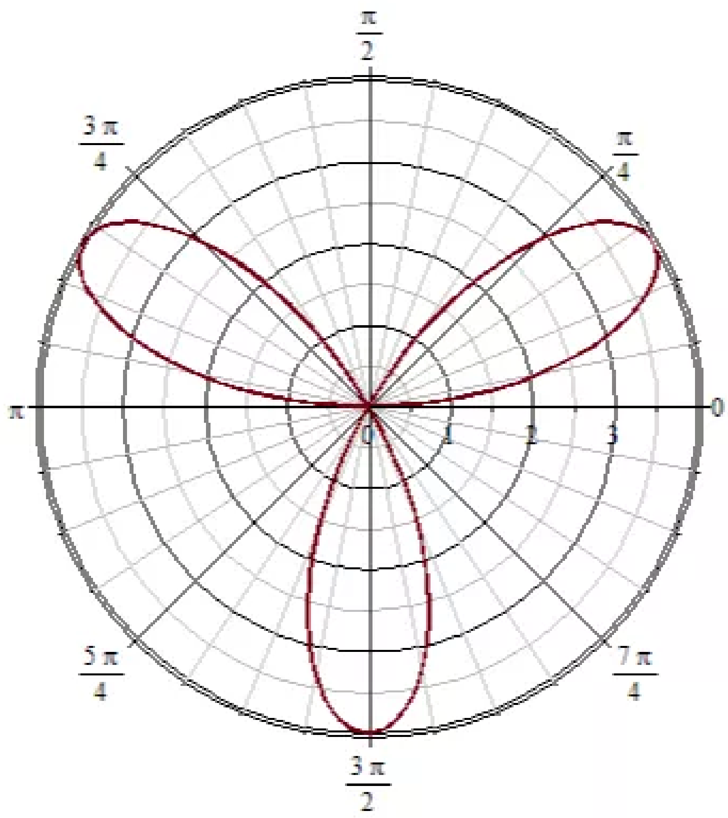
θ	$4\sin 3\theta$
0	0
$\frac{\pi}{6}$	4
$\frac{\pi}{4}$	$2\sqrt{2}$
$\frac{\pi}{3}$	0
$\frac{\pi}{2}$	-4

Sketch the graph of $r = 4\sin 3\theta$ in Cartesian coordinates as shown in the below figure:



As θ increase from 0 to $\frac{\pi}{6}$, the above figure shows that r increases from 0 to 4 and as θ decreases from $\frac{\pi}{6}$ to $\frac{\pi}{2}$, r decreases from 4 to $-4.$

So, draw the polar curve as shown in the below figure:



Q36E

Consider the polar equation

$r = \cos 5\theta$

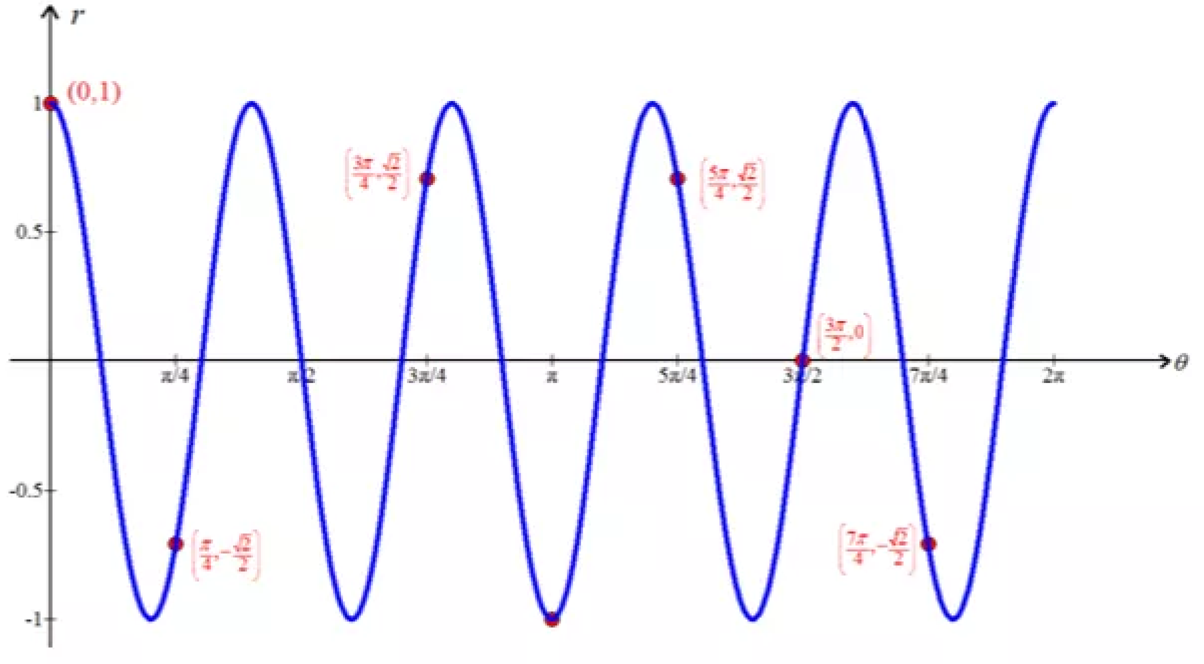
To sketch the curve with the polar equation $r = \cos 5\theta$, find the values of *r* for some convenient values of θ and plot the corresponding points (r, θ) then join theses points.

First construct the table form the given equation

θ	$r = \cos 5\theta$
0	1
$\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$
π	-1
$\frac{5\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{3\pi}{2}$	0
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$

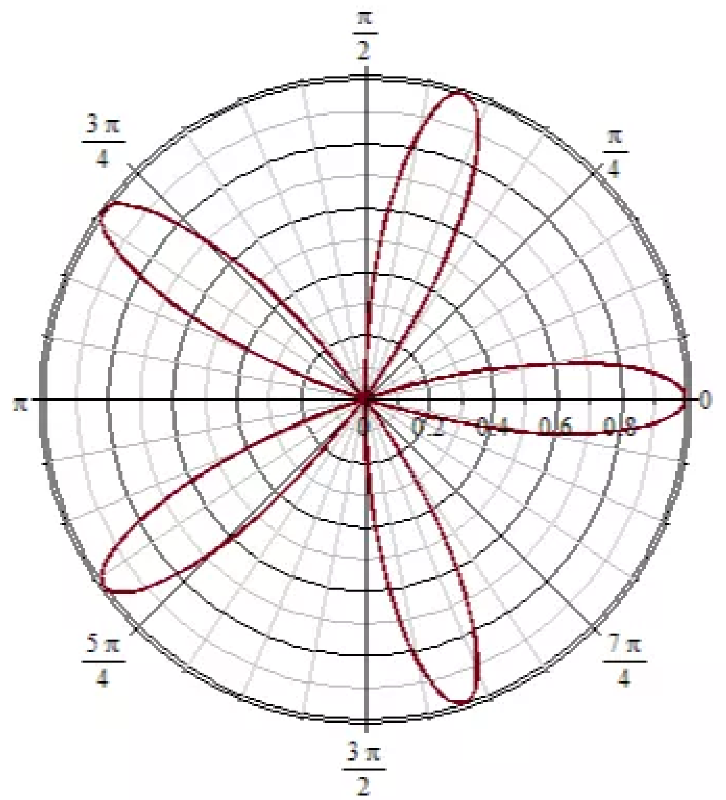
Now, plot the points and join them with a smooth curve.

The graph of $r=\cos 5\theta, 0\le\theta\le 2\pi$ in Cartesian coordinate system as shown in below:



As θ increase from 0 to $\frac{\pi}{4}$, the above figure shows that r decreases from 1 to -1 and so draw the corresponding portion of the polar curve in below figure. As θ increase from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, r goes from -1 to 1, draw the corresponding portion of the polar curve in below figure. The remainder of the curve is drawn in a similar fashion.

The resulting curve $r=\cos 5\theta$ as shown below



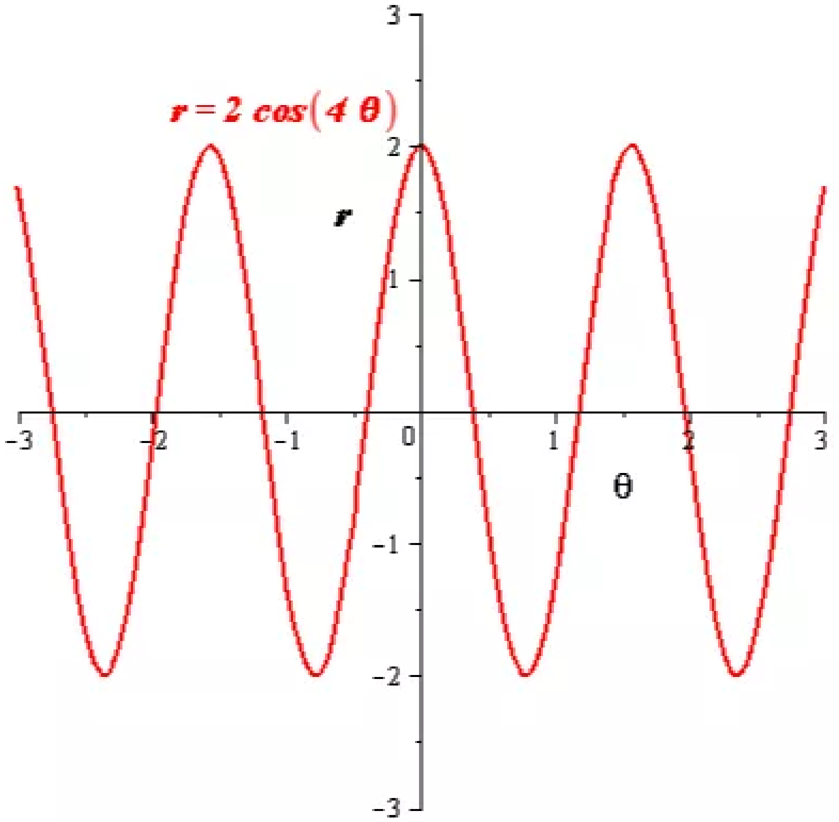
Q37E

1679-10.3-37E AID: 1112 | 29/09/2016

Given function is $r = 2\cos 4\theta$

The objective is to sketch the graph of the given function both in Cartesian coordinates and polar coordinates.

First, sketch the curve $r = 2\cos 4\theta$ in Cartesian coordinates.



Observe that as θ increases from 0 to $\pi/8$, r decreases from 2 to 0

As θ increases from $\pi/8$ to $\pi/4$, r decreases from 0 to -2

As θ increases from $\pi/4$ to $3\pi/8$, r increases from -2 to 0

As θ increases from $3\pi/8$ to $\pi/2$, r increases from 0 to 2

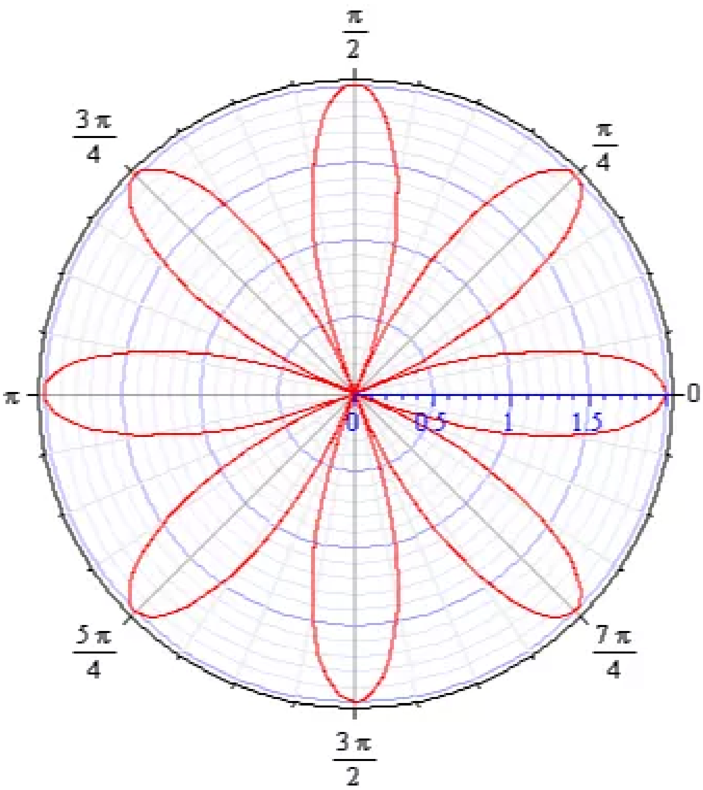
As θ increases from $\pi/2$ to $5\pi/8$, r decreases from 2 to 0

As θ increases from $5\pi/8$ to $3\pi/4$, r decreases from 0 to -2

As θ increases from $3\pi/4$ to $7\pi/8$, r increases from -2 to 0

As θ incases from $7\pi/8$ to π, r increases from 0 to 2

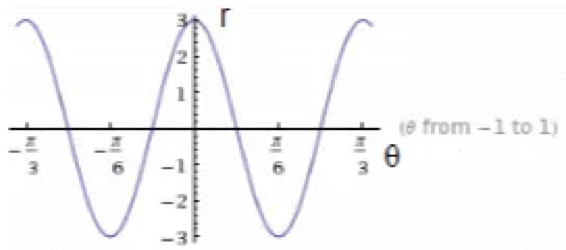
Now, sketch the given graph in polar coordinates.



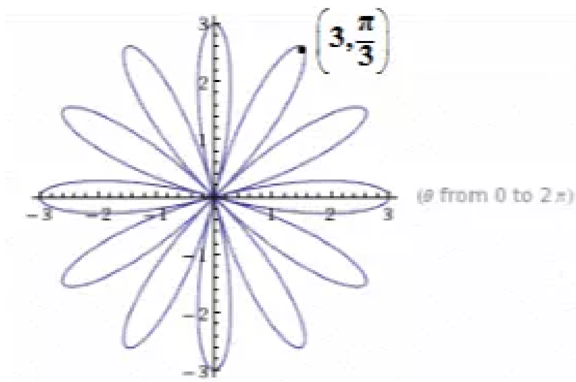
Q38E

We must sketch the curve given by the polar equation $r = 3\cos 6\theta$ by first sketching the graph of r as a function of θ in Cartesian coordinates.

Cartesian Coordinates:



Polar Coordinates:



Q39E

Consider the polar equation

$r = 1 - 2\sin\theta$

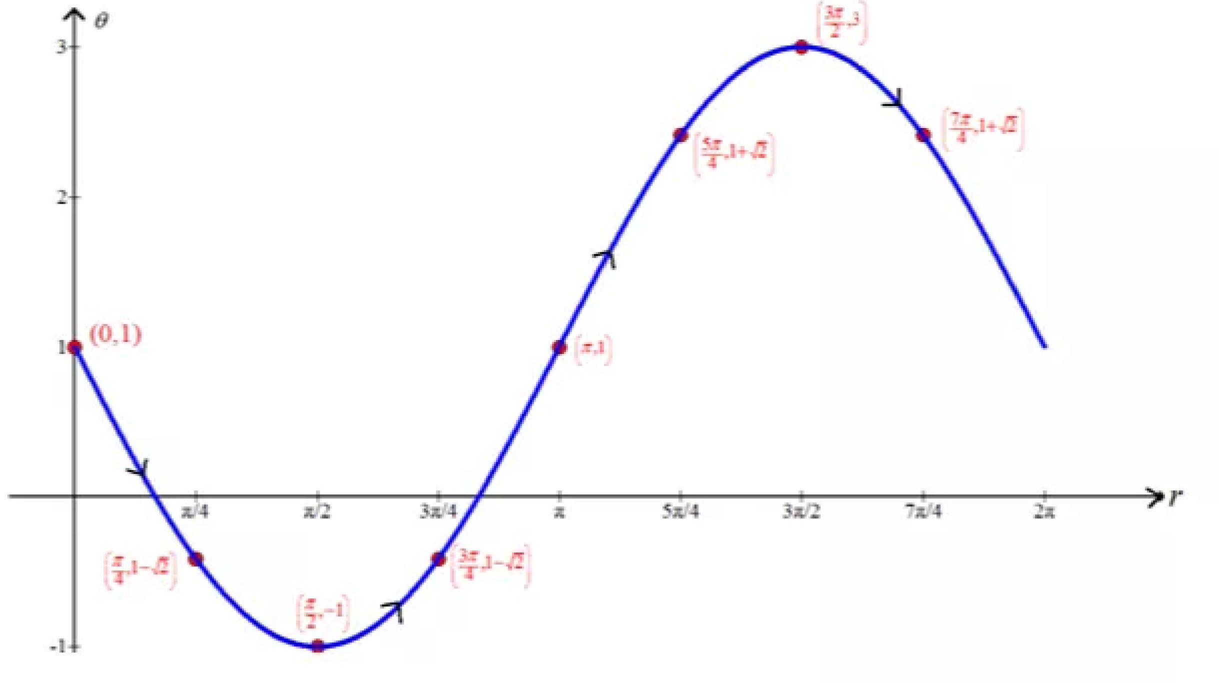
To sketch the curve with the polar equation $r = 1 - 2\sin\theta$, find the values of r for some convenient values of θ and plot the corresponding points (r, θ) then join theses points.

First construct the table form the given equation

θ	$r = 1 - 2\sin\theta$
0	1
$\frac{\pi}{4}$	$1-\sqrt{2}$
$\frac{\pi}{2}$	-1
$\frac{3\pi}{4}$	$1-\sqrt{2}$
π	1
$\frac{5\pi}{4}$	$1+\sqrt{2}$
$\frac{3\pi}{2}$	3
$\frac{7\pi}{4}$	$1+\sqrt{2}$

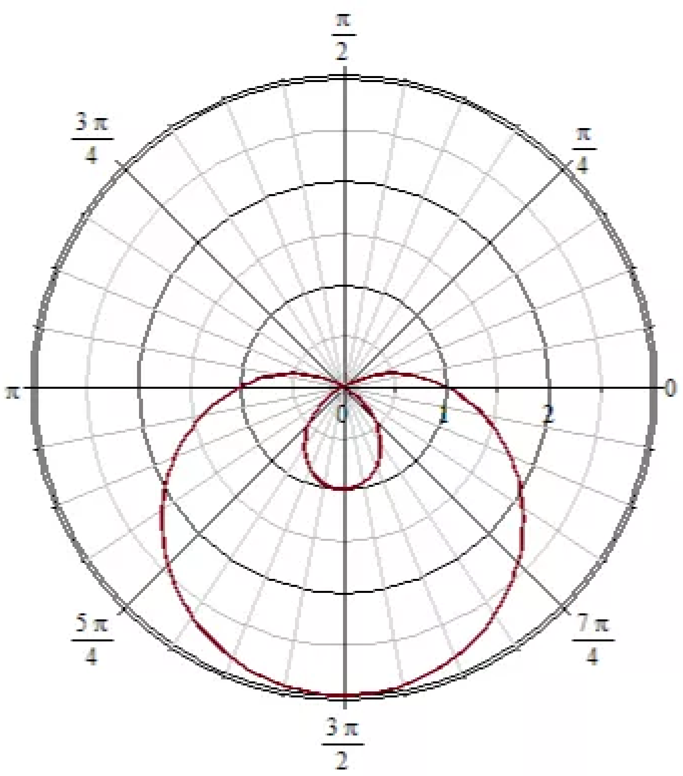
Now, plot the points and join them with a smooth curve.

The graph of $r=1-2\sin\theta, 0\le\theta\le2\pi$ in Cartesian coordinate system as shown in below:



As θ increase from 0 to $\frac{\pi}{4}$, the above figure shows that r decreases from 1 to 0 and so draw the corresponding portion of the polar curve in below figure. As θ increase from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, r goes from 0 to -1. This means that the distance from *O* increases from 0 to1, but instead of being in the first quadrant this polar curve lies on the opposite side of the pole in the third quadrant. The remainder of the curve is drawn in a similar fashion.

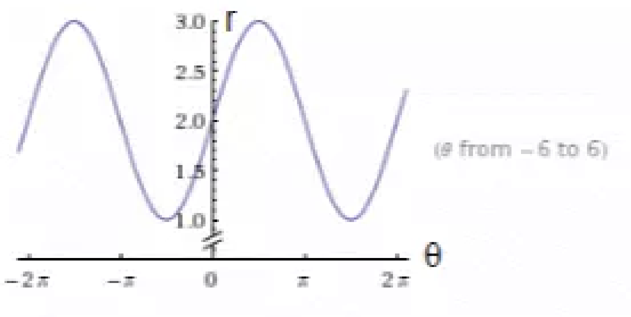
The complete curve $r=1-2\sin\theta$ as shown below:



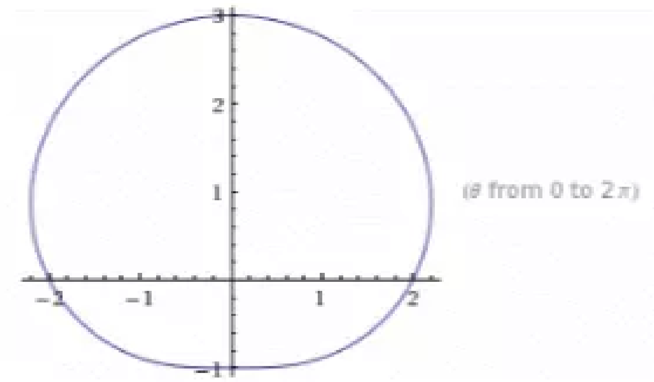
Q40E

We must sketch the curve given by the polar equation $r = 2+\sin\theta$ by first sketching the graph of r as a function of θ in Cartesian coordinates.

Cartesian Coordinates:



Polar Coordinates:



Q41E

Consider the polar equation

$r^2 = 9\sin 2\theta$

Rewrite it as,

$r = 3\sqrt{\sin 2\theta}$

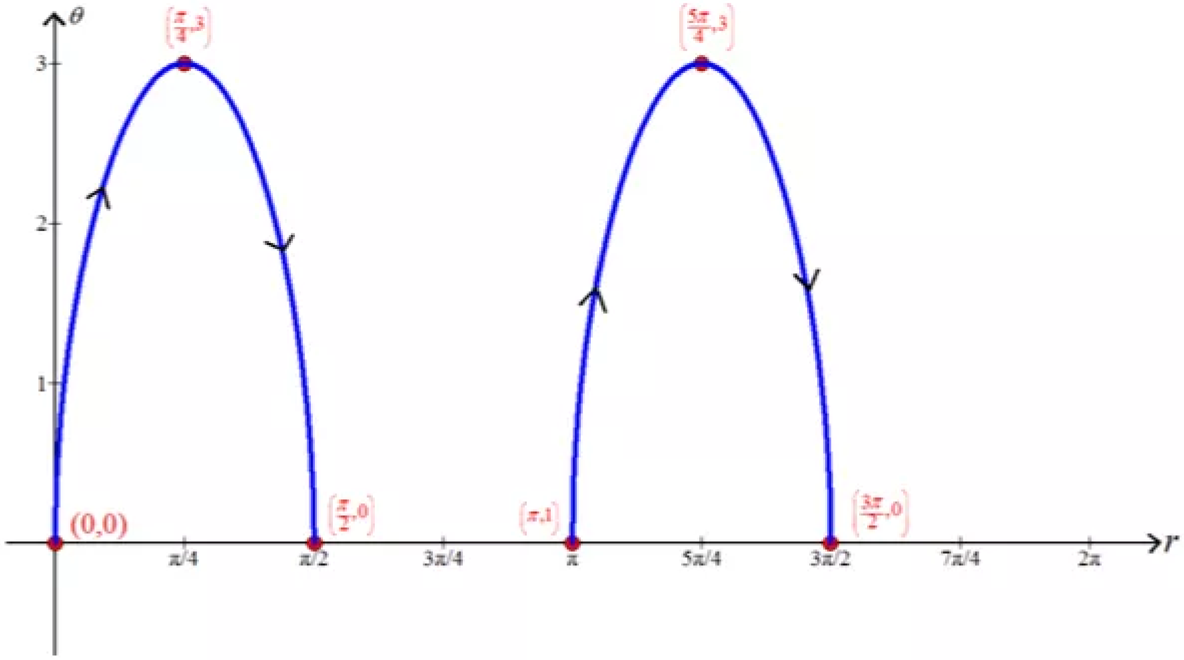
To sketch the curve with the polar equation $r^2 = 9\sin 2\theta$, find the values of r for some convenient values of θ and plot the corresponding points (r,θ) then join theses points.

First construct the table form the given equation

θ	$r = 3\sqrt{\sin 2\theta}$
0	0
$\frac{\pi}{4}$	3
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	undefined
π	1
$\frac{5\pi}{4}$	3
$\frac{3\pi}{2}$	0
$\frac{7\pi}{4}$	undefined

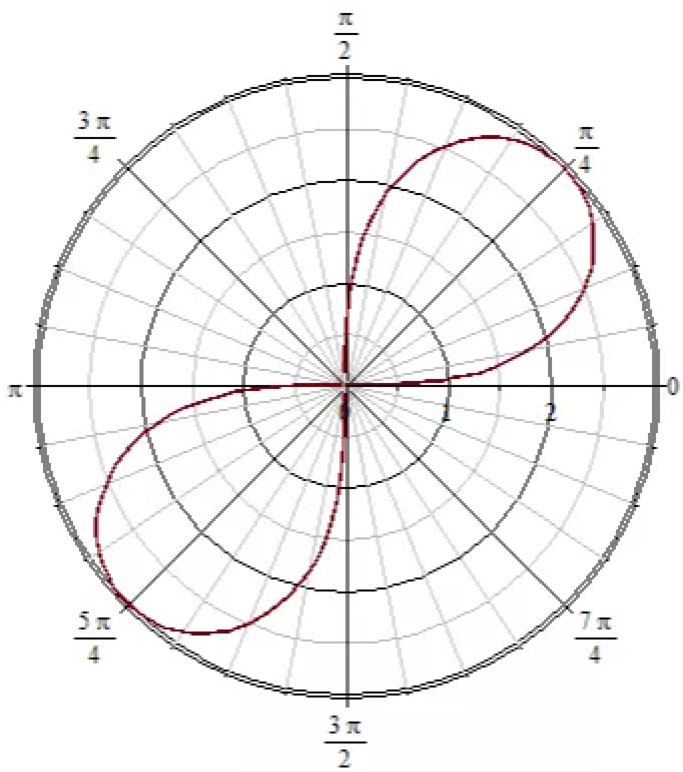
Now, plot the points and join them with a smooth curve.

The graph of $r^2 = 9\sin 2\theta, 0 \le \theta \le 2\pi$ in Cartesian coordinate system as shown in below:



As θ increase from 0 to $\frac{\pi}{4}$, the above figure shows that r increases from 0 to 3 and so draw the corresponding portion of the polar curve in below figure. As θ increase from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, r decreases from 3 to 0, draw the corresponding portion of the polar curve in below figure. The remainder of the curve is drawn in a similar fashion.

The complete curve $r^2 = 9\sin 2\theta$ as shown below:



Q42E

Consider the polar equation

$r^2 = \cos 4\theta$

Rewrite it as,

$r = \sqrt{\cos 4\theta}$

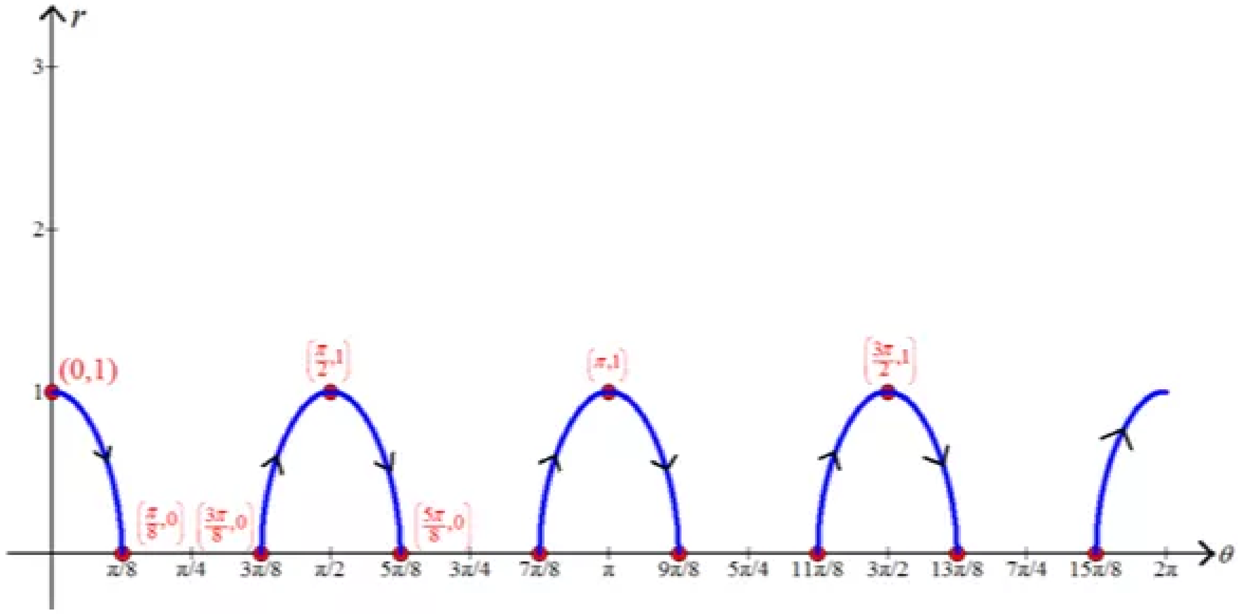
To sketch the curve with the polar equation $r^2 = \cos 4\theta$, find the values of r for some convenient values of θ and plot the corresponding points (r, θ) then join theses points.

First construct the table form the given equation

θ	$r = \sqrt{\cos 4\theta}$
0	1
$\frac{\pi}{8}$	0
$\frac{\pi}{4}$	undefined
$\frac{3\pi}{8}$	0
$\frac{\pi}{2}$	1
$\frac{5\pi}{8}$	0
$\frac{3\pi}{4}$	undefined
π	1

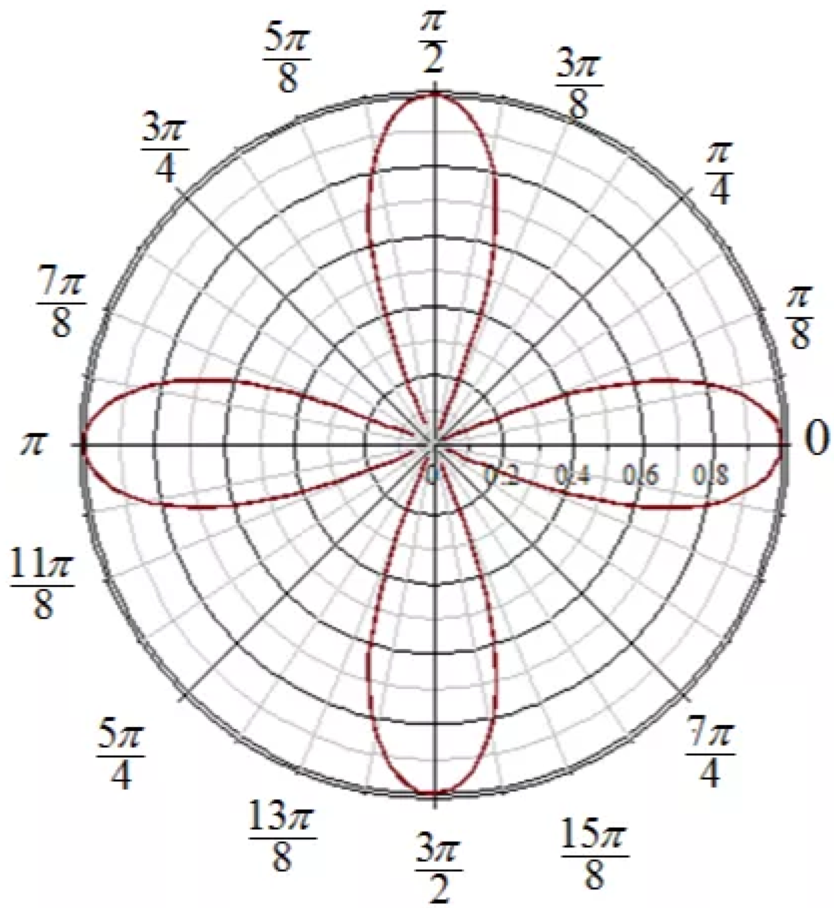
Now, plot the points and join them with a smooth curve.

The graph of $r^2 = 9\sin 2\theta, 0 \le \theta \le 2\pi$ in Cartesian coordinate system as shown in below:



As θ increase from 0 to $\frac{\pi}{8}$, the above figure shows that r increases from 1 to 0 and so draw the corresponding portion of the polar curve in below figure. As θ increase from $\frac{3\pi}{8}$ to $\frac{\pi}{2}$, r decreases from 0 to 1, draw the corresponding portion of the polar curve in below figure. The remainder of the curve is drawn in a similar fashion.

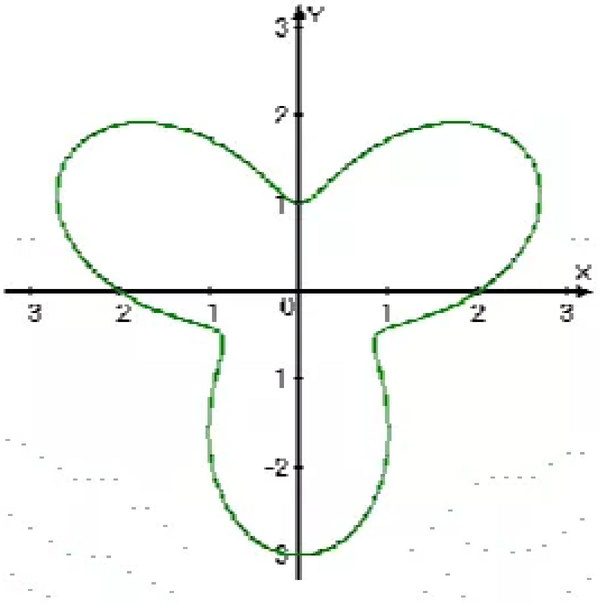
The complete curve $r^2 = \cos 4\theta$ as shown below:



Q43E

Given $r = (2 + \sin 3\theta)$

We first draw the curve without changing it into the Cartesian equation.



$r = (2 + \sin 3\theta)$

$r = 2 + 3\sin\theta - 4\sin^3\theta$

$r = 2 + 3\frac{y}{r} - 4\frac{y^3}{r^3}$

$r^4 = 2r^3 + 3yr^2 - 4y^3$

$(x^2 + y^2)^2 = 2(x^2 + y^2)^{\frac{3}{2}} + 3y(x^2 + y^2) - 4y^3$

$(x^2 + y^2)^2 = 2(x^2 + y^2)^{\frac{3}{2}} + y(3x^2 - y^2)$ is the Cartesian equation suitable to the given polar equation.

Q44E

We have to sketch the curve $r^2\theta = 1$

$\Rightarrow r^2 = 1/\theta$

$\Rightarrow r = \pm 1/\sqrt{\theta} \qquad \theta > 0$

First we plot the Cartesian curve between r and θ

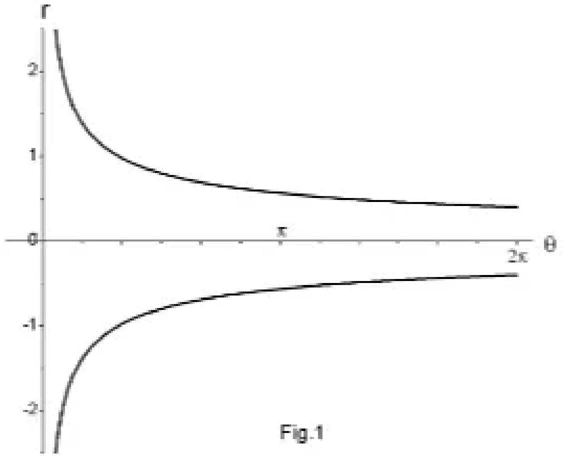


Fig.1

If we take $r = 1/\sqrt{\theta}$ so as $\theta \to 0,\ r \to \infty$. As θ increases r decreases

So if θ decreases from $2\pi\, to\, 0$, r increases.

Similarly if we take $r = -1/\sqrt{\theta}$ so as θ increases, r increases

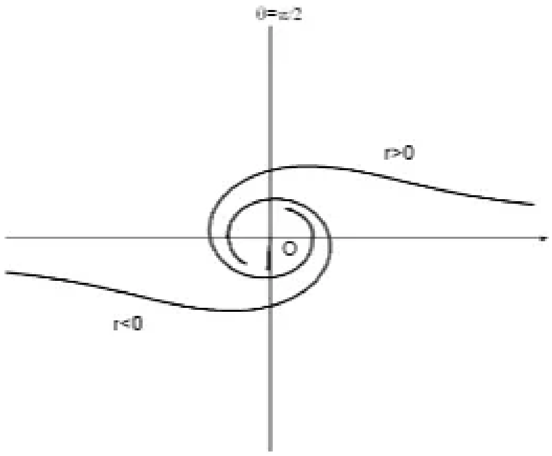


Fig.2

Q45E

We sketch the curve in Cartesian coordinates $(r = 1 + 2\cos 2\theta)$

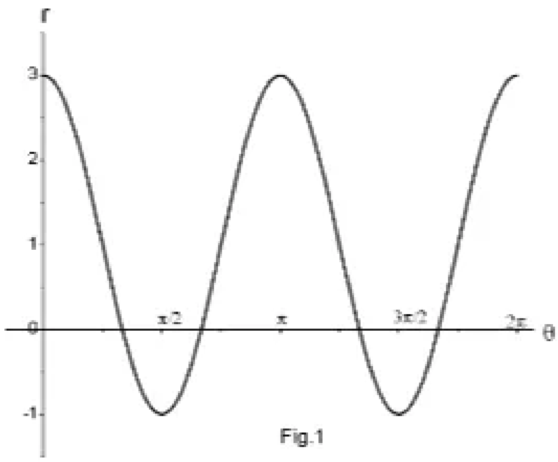


Fig.1

We see that as θ increases from 0 to $\pi/3$, r decreases from 3 to 0

As θ increases from $\pi/3$ to $\pi/2$, r decreases from 0 to -1

As θ increases from $\pi/2$ to $2\pi/3$, r increases from -1 to 0

As θ increases from $2\pi/3$ to π, r increases from 0 to 3

As θ increases from π to $4\pi/3$, r decreases from 3 to 0

As θ increases from $4\pi/3$ to $3\pi/2$, r decreases from 0 to -1

As θ increases from $3\pi/2$ to $5\pi/3$, r increases from -1 to 0

As θ increases from $5\pi/3$ to 2π, r increases from 0 to 3

With the help of above information's we can sketch the polar curve.

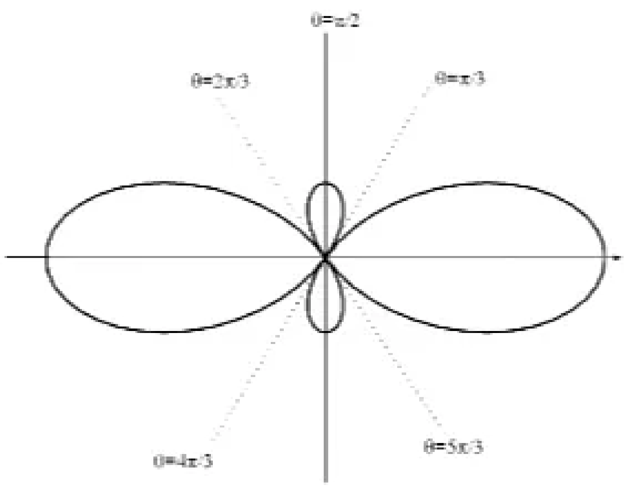
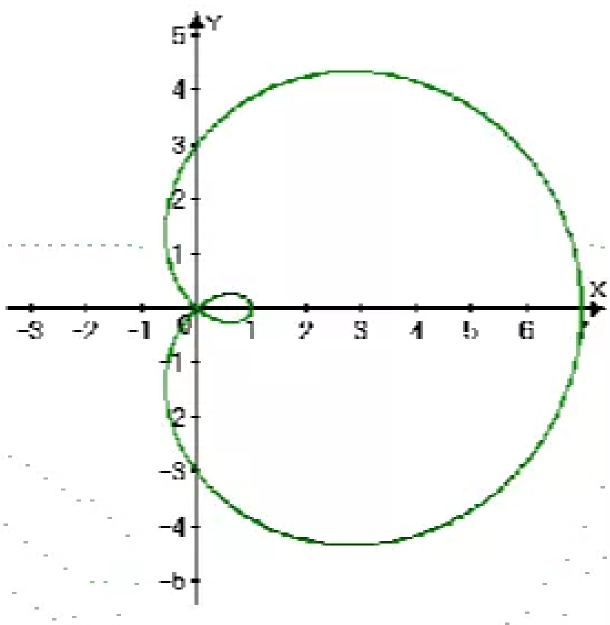


Fig.2

Q46E

Given $r = 3 + 4\cos\theta$

We first draw the curve without changing it into the Cartesian equation.



$r = 3 + 4\cos\theta$

$r = 3 + 4\dfrac{x}{r}$

$r^2 = 3r + 4x$

$x^2 + y^2 - 4x = 3\sqrt{x^2 + y^2}$ is the Cartesian equation suitable to the given polar equation.

Q47E

Given curve in Cartesian coordinates is as follows

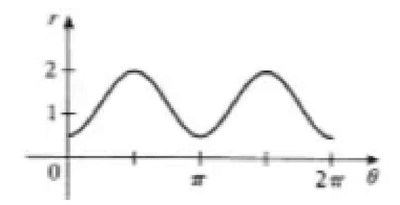


Fig. 1

We see that as θ increases from 0 to $\pi/2$, r increases from 0.5 to 2,
As θ increases from $\pi/2$ to π, r decreases from 2 to 0.5
As θ increases from π to $3\pi/2$, r increases from 0.5 to 2
As θ increases from $3\pi/2$ to 2π, r decreases 2 to 0.5
With above results we can sketch the curve

Q48E

Given curve is

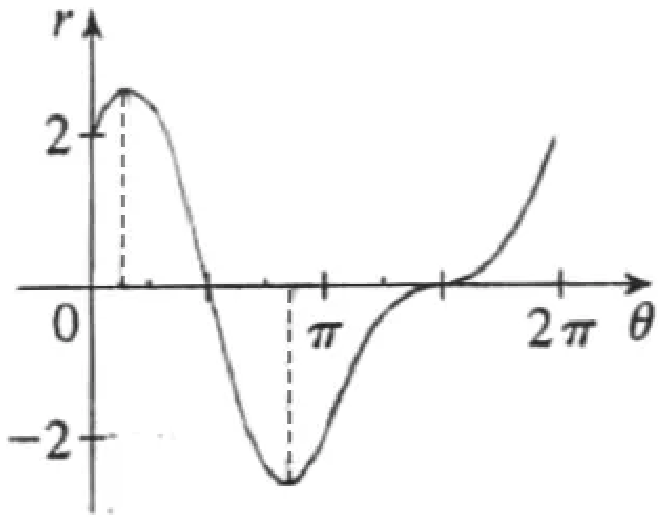


Fig. 1

We see that first r increases from 2 to ≈ 2.5, as θ increases up to $\pi/2$, r decreases up to 0. As θ increases beyond $\pi/2$, r decreases up to - 2.5. As θ increases up to $3\pi/2$, r increases up to 0 and as θ increases up to 2π, r increases up to 2.
With above results we can sketch the curve

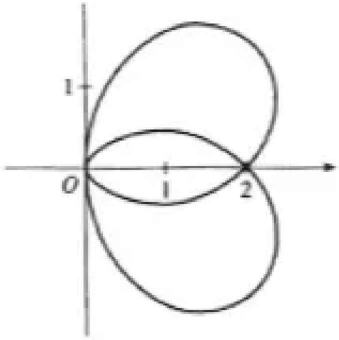


Fig. 2

Q49E

Equation of the polar curve is $r = 4 + 2\sec\theta$

Since $x = r\cos\theta$

So $\cos\theta = x/r$ and $\sec\theta = r/x$

Then we have $r = 4 + 2r/x$

$$\Rightarrow 2r/x = r - 4$$
$$\Rightarrow x/(2r) = 1/(r-4)$$
$$\Rightarrow x = \frac{2r}{(r-4)}$$
$$\Rightarrow x = \frac{2r}{r(1-4/r)}$$
$$\Rightarrow x = \frac{2}{(1-4/r)}$$

Taking limit as $r \to \pm\infty$ $(r \to \infty \text{ or } r \to -\infty)$

Then $\lim_{r\to\pm\infty} x = \lim_{r\to\pm\infty} \frac{2}{(1-4/r)}$

$$\Rightarrow \lim_{r\to\pm\infty} x = \frac{2}{\left[1 - 4\lim_{r\to\pm\infty} 1/r\right]}$$

$$\Rightarrow \lim_{r\to\pm\infty} x = \frac{2}{(1-0)} = 2 \qquad \left[\lim_{x\to\pm\infty} \frac{1}{x} = 0\right.$$

So $x = 2$ is a vertical asymptote.

Again, $r \to \pm\infty$ then $(4 + 2\sec\theta) \to \pm\infty$

Since $\sec\theta \to \infty$ when $\theta \to (\pi/2)^-$ or $\theta \to (3\pi/2)^+$

And $\sec\theta \to -\infty$ when $\theta \to (\pi/2)^+$ or $\theta \to (3\pi/2)^-$

So $\lim_{r\to\infty} x = \lim_{\theta\to\pi/2^-} (4 + 2\sec\theta)\cos\theta = \lim_{\theta\to\pi/2^-} (4\cos\theta + 2) = 2$

And $\lim_{r\to-\infty} x = \lim_{\theta\to\pi/2^+} (4 + 2\sec\theta)\cos\theta = \lim_{\theta\to\pi/2^+} (4\cos\theta + 2) = 2$

Now we calculate the values of r for different values of θ in the interval $[0, 2\pi]$

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
r	6	6.8		1.17	2	1.17		6.8	6

With the help of above data and results (1), we can sketch the curve

Q50E

Equation of the curve is $r = 2 - \csc\theta$
Since $y = r\sin\theta$
Then $\csc\theta = r/y$

So we have $r = 2 - r/y$

$$\Rightarrow r - 2 = -r/y$$

$$\Rightarrow \frac{y}{r} = \frac{1}{(2-r)}$$

$$\Rightarrow y = \frac{r}{(2-r)}$$

$$\Rightarrow y = \frac{1}{(2/r-1)}$$

If we take limit as $r \to \infty$ or $r \to -\infty$, then $\lim_{r\to\pm\infty} \frac{1}{r} = 0$

So $$\lim_{r\to\pm\infty} y = \frac{1}{(0-1)} = -1$$

$$\Rightarrow \boxed{\lim_{r\to\pm\infty} y = -1}$$

So y = -1 is a horizontal asymptote

Since $(2-\cos\theta) \to \pm\infty$ as $r \to \pm\infty$

Since $\csc\theta \to \infty$ as $\theta \to 0^+$ or $\theta \to \pi^-$

$\csc\theta \to -\infty$ as $\theta \to \pi^+$ or $\theta \to 2\pi^-$

So $r \to -\infty$ as $\theta \to 0^+$ or $\theta \to \pi^-$

And $r \to \infty$ as $\theta \to \pi^+$ or $\theta \to 2\pi^-$

θ	**0**	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
r	$\to\infty$	≈ 0.58	1	≈ 0.58		3.41	3	3.41	$\to\infty$

Q51E

We have to show that x = 1 is a vertical asymptote of the curve $r = \sin\theta\tan\theta$

For this we have to show $\lim_{x\to 1\pm\infty} x = 1$

We have $x = r\cos\theta$

Then $x = \sin\theta\tan\theta\cos\theta = \sin^2\theta$

When $r \to \infty$

So $\sin\theta \tan\theta \to \infty$

Since $|\sin\theta| \le 1$ and so $\tan\theta \to \infty$ as $\theta \to (\pi/2)^-$

So $\lim\limits_{x\to\infty} x = \lim\limits_{\theta\to\pi/2^-} \sin\theta \tan\theta \cos\theta$

$= \lim\limits_{\theta\to\pi/2^-} \sin^2\theta = 1$

Now as $r \to -\infty$ $\Rightarrow \theta \to (\pi/2)^+$ so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi/2^+} \sin^2\theta = 1$

Therefore $\lim\limits_{r\to\pm\infty} x = 1$

Thus x =1 is a vertical asymptote

We see that $x = \sin^2\theta \ge 0$ for all θ *and* $x = \sin^2\theta \le 1$ for all θ

Since $r = \sin\theta \tan\theta$ is not defined for odd multiple of $\pi/2$ so $x \ne 1$

So curve lies within the vertical strip $0 \le x < 1$

We sketch the curve

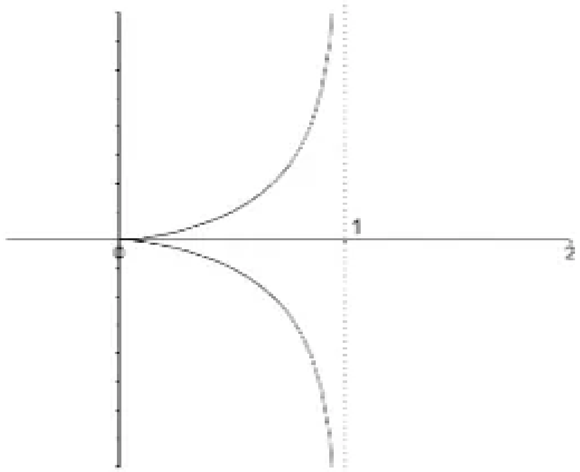


Fig.1

Q52E

We have to sketch the curve $\left(x^2+y^2\right)^3 = 4x^2y^2$

Since $x = r\cos\theta$ and $y = r\sin\theta$

Then $\left(r^2\cos^2\theta + r^2\sin^2\theta\right)^3 = 4\left(r^2\cos^2\theta\right)\left(r^2\sin^2\theta\right)$

$\Rightarrow r^6\left(\cos^2\theta+\sin^2\theta\right)^3 = 4r^4\sin^2\theta\cos^2\theta$

$\Rightarrow r^6 = 4r^4\sin^2\theta\cos^2\theta$ $\left[\sin^2\theta+\cos^2\theta=1\right]$

$\Rightarrow r^2 = 4\sin^2\theta\cos^2\theta$

$\Rightarrow r^2 = \left(2\sin\theta\cos\theta\right)^2$

$\Rightarrow r^2 = \sin^2 2\theta$

$\Rightarrow r = \sin 2\theta$

So the polar curve $r = \sin 2\theta$ will be similar to Cartesian curve $\left(x^2+y^2\right) = 4x^2y^2$. First we sketch the curve $r = \sin\theta$ in Cartesian coordinates

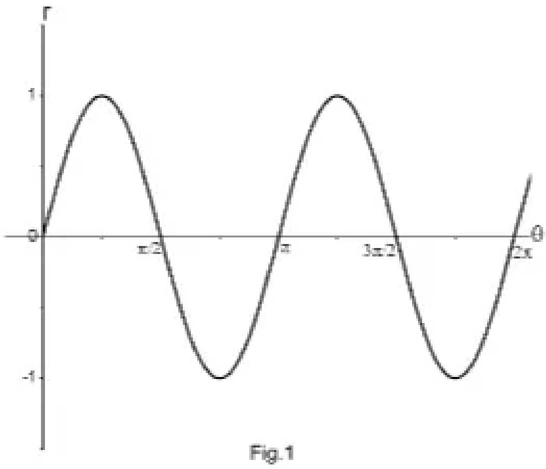


Fig.1

We see that as θ increases from o to $\pi/4$, r increases from 0 to 1,

As θ increases from $\pi/4$ *to* $\pi/2$, r decreases from 1 to 0.

As θ increases from $\pi/2$ to $3\pi/4$, r decreases from 0 to -1.

As θ increases from $3\pi/4$ to π, r increases from -1 to 0.

As θ increases from π to $5\pi/4$, r increases from 0 to 1

As θ increases from $5\pi/4$ to $3\pi/2$, r decreases from 1 to 0

As θ increases from $3\pi/2$ to $7\pi/4$, r decreases from 0 to -1

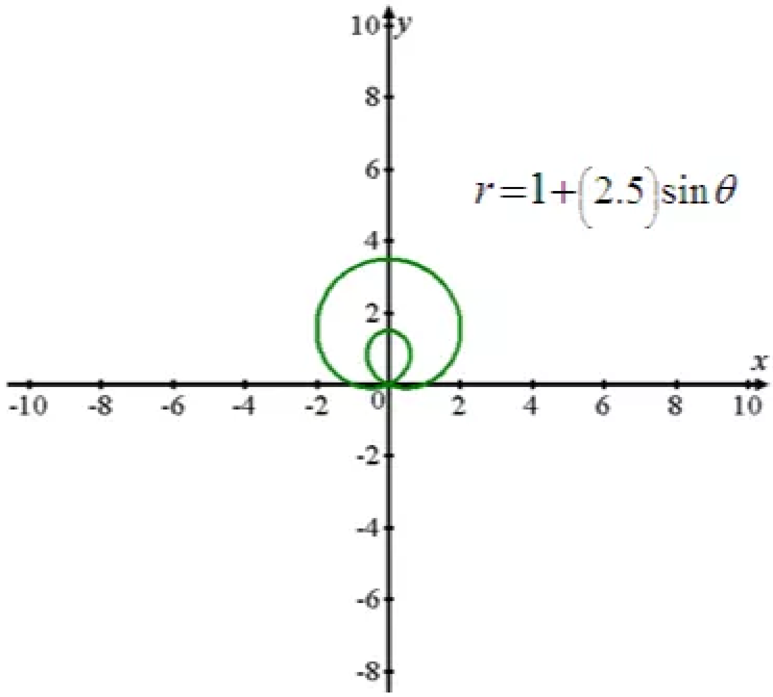
As θ increases from $7\pi/4$ to 2π, r increases from -1 to 0

Now we sketch the curve

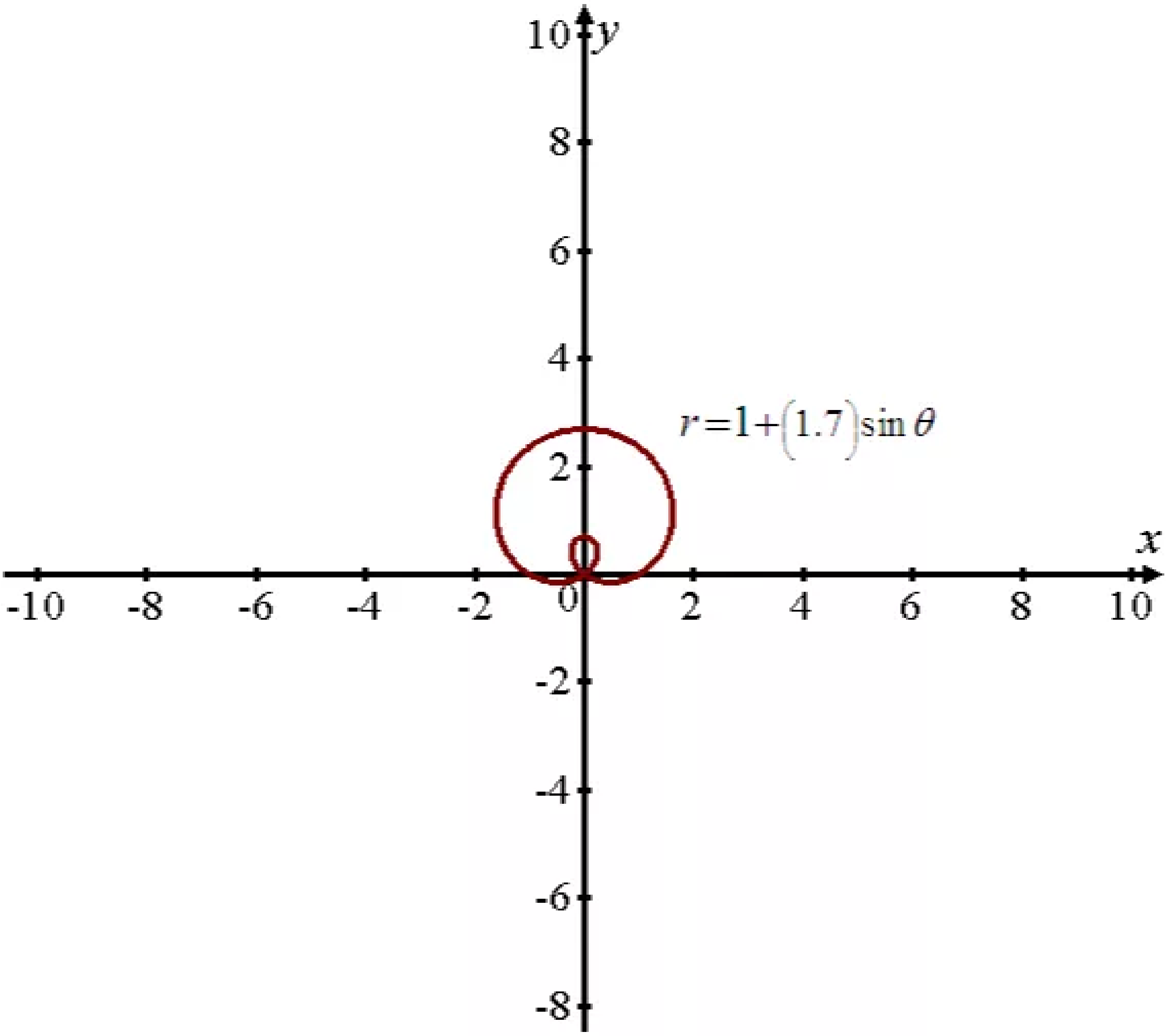
Q53E

(a)Consider the limacon $r = 1 + c\sin\theta$ has an inner loop when $|c| > 1$.

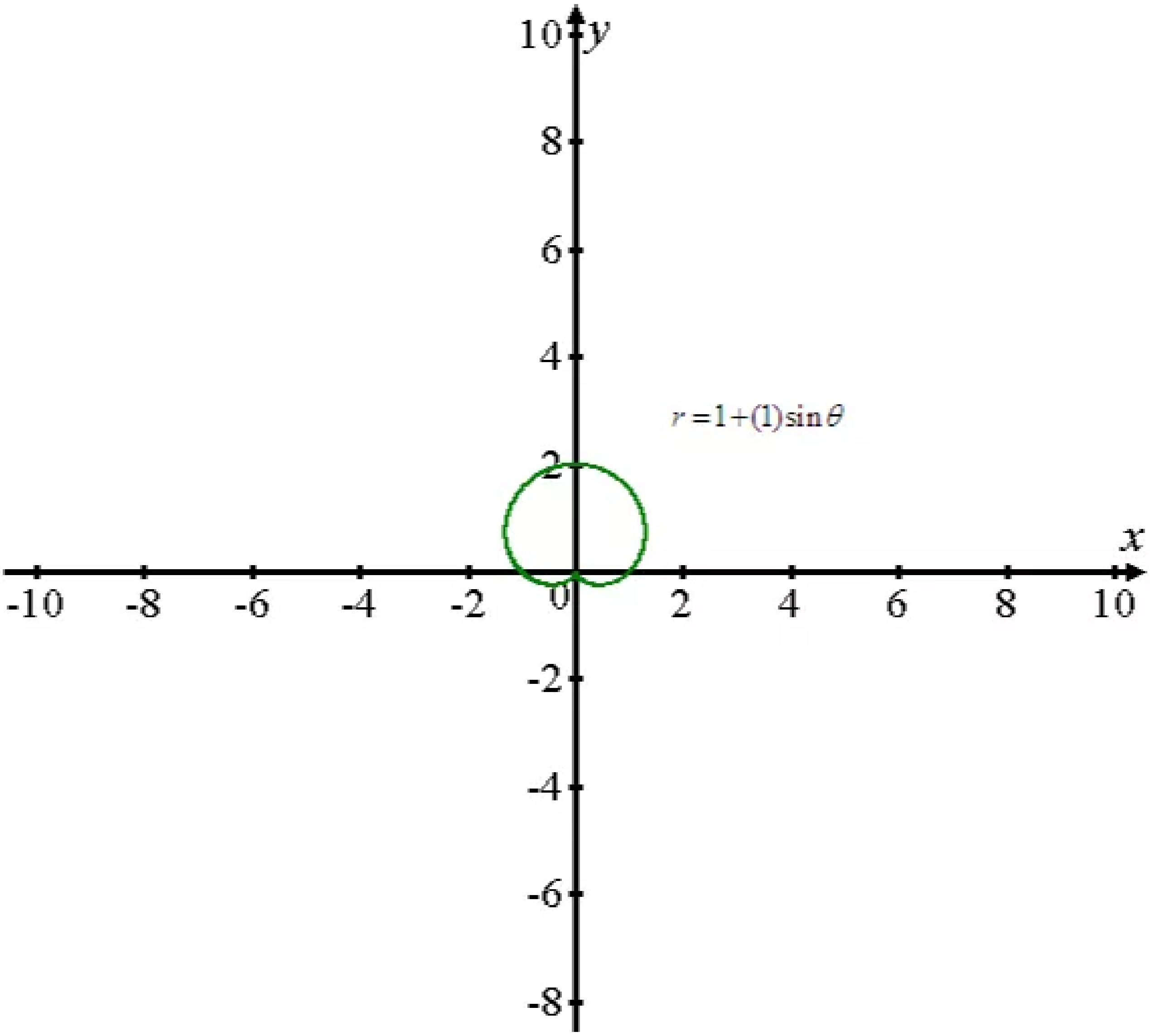
Take $c = 2.5$ then $r = 1 + (2.5)\sin\theta$.



Now take $c = 1.7$ then $r = 1 + (1.7)\sin\theta$.



Now take $c = 1$ then $r = 1 + \sin\theta$.



Therefore, the above curves are called limacons.

For $c > 1$, there is a loop that decreases in size as *c* decreases.

When $c = 1$, the loop disappears and the curve becomes the cardioid.

For c between 1 and $\frac{1}{2}$ the cardioid's cusp is smoothed out becomes a "dimple".

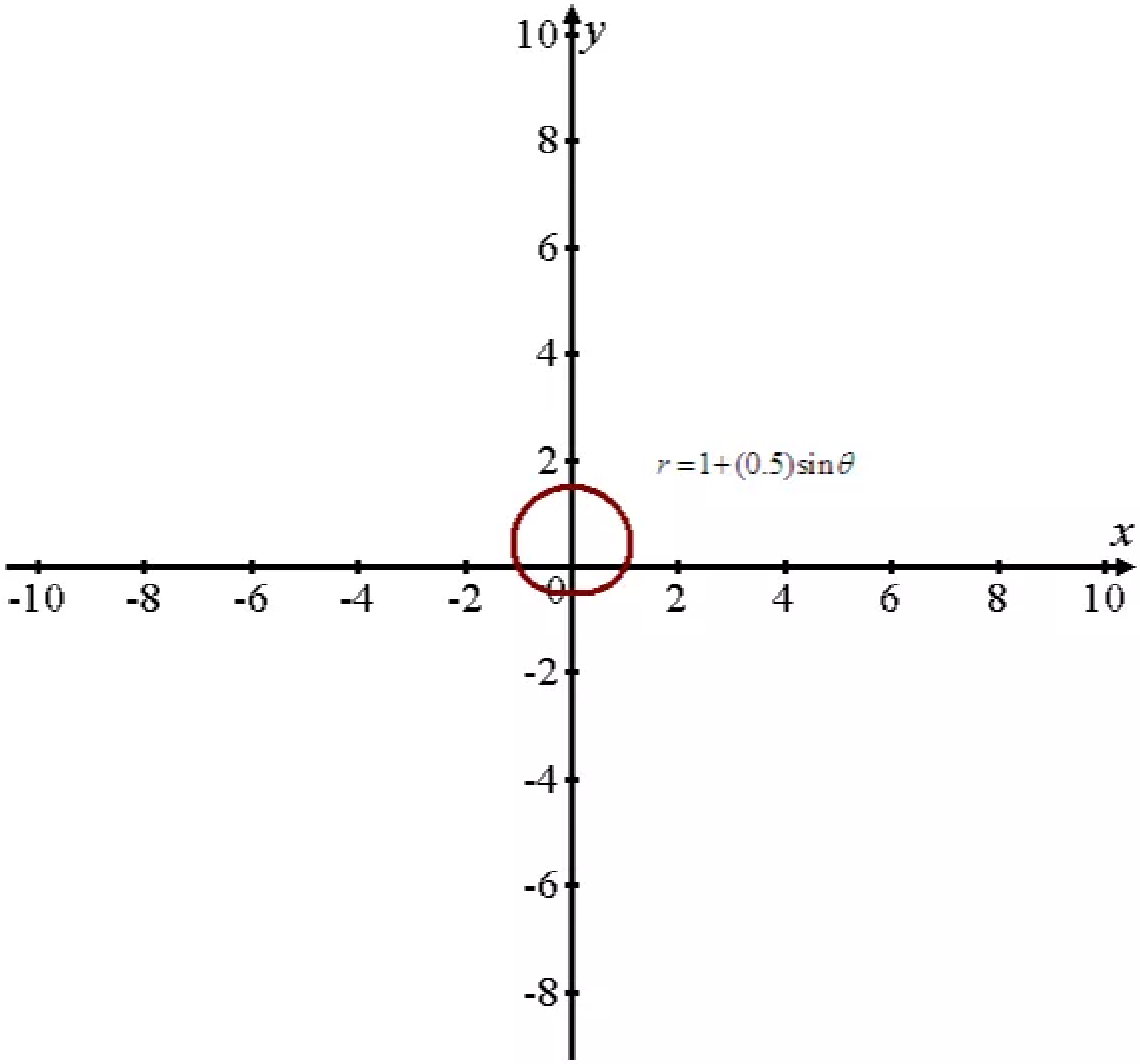
For $c < -1$, the inner loop begins at $\theta = \sin^{-1}\left(-\frac{1}{c}\right)$ and ends at $\theta = \pi - \sin^{-1}\left(-\frac{1}{c}\right)$.

For $c > 1$, it begins at $\theta = \sin^{-1}\left(\frac{1}{c}\right)$ and ends at $\theta = 2\pi - \sin^{-1}\left(\frac{1}{c}\right)$.

(b) Consider the limacon $r = 1 + c\sin\theta$.

Substitute $c = 0.5$ in the above equation, we get $r = 1 + (0.5)\sin\theta$.

Consider the figure $r = 1 + (0.5)\sin\theta$.



In the above figure, the limacon loses its dimple.

Hence the graph appears that the limacon loses its dimple when $c = \frac{1}{2}$.

Q54E

(a)

Consider the polar equation, $r = \sqrt{\theta}$, $0 \le \theta \le 16\pi$.

$r = sqrt(theta)$

This means that as theta grows, *r* also grows, but more slowly. You can then imagine that the graph is a spiral, which grows outwards as you go around the origin, multiple times, but with the gap between successive cycles getting smaller and smaller.

So this curve is matched with the choice (v).

(b)

Consider the polar equation, $r=\theta^2$,, $0\le\theta\le 16\pi$.

This means that as theta grows, *r* also grows, but more fastly, You can then imagine that the graph is a spiral which grows outwards as you go around the origin multiple times, but with the gap between successive cycles getting larger and larger.

So this curve is matched with the choice (II).

(c)

Consider the polar equation $r=\cos\left(\theta/3\right)=f(\theta)$.

Since $f(-\theta)=f(\theta)$

Symmetric with respect to polar axis at $\theta=0$, $r=1$.

Symmetric with respect to polar axis at $\theta=\frac{\pi}{2}$, $r=0.9$.

So, this curve is related with (VI).

(d)

Consider the polar equation, $r=1+2\cos\theta=f(\theta)$.

Since $f(-\theta)=f(\theta)$

Symmetric with respect to polar axis at $\theta=0$, $r=3$.

Symmetric with respect to polar axis at $\theta=\frac{\pi}{2}$, $r=1$.

So, this curve is related with (III).

(e)

Consider the polar equation, $r=2+\sin 3\theta=f(\theta)$.

Since $f(\pi-\theta)=f(\theta)$

Symmetric about vertical axis $\left(\theta=\frac{\pi}{2}\right)$.

For $\theta=0$, $r=2$

$\theta=\frac{\pi}{2}$, $r=2+\sin\left(\frac{3\pi}{2}\right)$

$=2+\sin\left(\pi+\frac{\pi}{2}\right)$

$=2-\sin\frac{\pi}{2}$

$=2-1$

$=1$

So this is related with (I).

(f)

Consider the polar equation, $r = 1 + 2\sin 3\theta = f(\theta)$.

Since $f(\pi - \theta) = f(\theta)$

Symmetric about vertical axis $\left(\theta = \frac{\pi}{2}\right)$.

For $\theta = 0$, $r = 1$

$$\theta = \frac{\pi}{2},\quad r = 1 + 2\sin\left(\frac{3\pi}{2}\right)$$
$$= 1 + 2(-1)$$
$$= -1$$

So, this is related with (IV).

Q55E

We have the polar equation as $r = 2\sin\theta$

Then $$\frac{dr}{d\theta} = 2\cos\theta$$

Using $$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

We get $$\frac{dy}{dx} = \frac{2\cos\theta\sin\theta + 2\sin\theta\cos\theta}{2\cos\theta\cos\theta - 2\sin\theta\sin\theta}$$
$$= \frac{4\sin\theta\cos\theta}{2(\cos^2\theta - \sin^2\theta)}$$
$$= \frac{2(2\sin\theta\cos\theta)}{2\cos 2\theta}$$
$$= \frac{2\sin 2\theta}{2\cos 2\theta}$$
$$= \tan 2\theta$$

The slope of the tangent at the point where $\theta = \frac{\pi}{6}$ *is*

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \tan 2(\pi/6)$$
$$= \tan(\pi/3)$$
$$= \sqrt{3}$$

Therefore the slope of the tangent = $\boxed{\sqrt{3}}$

Q56E

We have the polar equation as $r = 2 - \sin\theta$

Then $$\frac{dr}{d\theta} = -\cos\theta$$

Using the fact $\dfrac{dy}{dx} = \dfrac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$

We have $$\frac{dy}{dx} = \frac{-\cos\theta\sin\theta + (2-\sin\theta)\cos\theta}{-\cos\theta\cos\theta - (2-\sin\theta)\sin\theta}$$

$$= \frac{-\cos\theta\sin\theta + 2\cos\theta - \sin\theta\cos\theta}{-\cos^2\theta - 2\sin\theta + \sin^2\theta}$$

$$= \frac{2\cos\theta - 2\sin\theta\cos\theta}{-(\cos^2\theta - \sin^2\theta) - 2\sin\theta}$$

$$= \frac{2\cos\theta - \sin 2\theta}{-\cos 2\theta - 2\sin\theta}$$

The slope of the tangent at the point where $\theta = \dfrac{\pi}{3}$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \frac{2\cos(\pi/3) - \sin(2\pi/3)}{-\cos(2\pi/3) - 2\sin(\pi/3)}$$

$$= \frac{2\frac{1}{2} - \frac{\sqrt{3}}{2}}{-\left(-\frac{1}{2}\right) - 2\frac{\sqrt{3}}{2}}$$

$$= \frac{(2-\sqrt{3})/2}{(1-2\sqrt{3})/2}$$

$$= \boxed{\frac{2-\sqrt{3}}{1-2\sqrt{3}}}$$

Q57E

Consider the following equation of the polar curve:

$r = 1/\theta$

To determine the slope of the tangent line to the given curve at the point $\theta = \frac{\pi}{6}$, find the value of $\frac{dy}{dx}$ at $\theta = \frac{\pi}{6}$.

First determine $\frac{dy}{dx}$ by the following formula:

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \quad \text{...... (1)}$$

For the value of $\frac{dr}{d\theta}$, differentiate the given polar equation:

$$\frac{dr}{d\theta} = -1/\theta^2$$

Substitute this value in (1):

$$\begin{aligned}\frac{dy}{dx} &= \frac{-\frac{1}{\theta^2}\sin\theta + \frac{1}{\theta}\cos\theta}{\frac{-1}{\theta^2}\cos\theta - \frac{1}{\theta}\sin\theta} \\ &= \frac{\frac{-\sin\theta + \theta\cos\theta}{\theta^2}}{\frac{-\cos\theta - \theta\sin\theta}{\theta^2}} \\ &= \frac{-\sin\theta + \theta\cos\theta}{-\cos\theta - \theta\sin\theta} \\ &= \frac{-(\sin\theta - \theta\cos\theta)}{-(\cos\theta - \theta\sin\theta)}\end{aligned}$$

Hence,

The slope of the tangentline at the point where $\theta = \pi$ is:

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{\theta=\pi} &= \frac{\sin\pi - \pi\cos\pi}{\cos\pi + \pi\sin\pi} \\ &= \frac{0 + \pi}{-1 + \pi(0)} \\ &= \frac{\pi}{-1}\end{aligned}$$

Therefore, the slope of the tangent line is $\boxed{-\pi}$.

Q58E

Consider the curve

$$r = \cos\left(\frac{\theta}{3}\right)$$

$$\frac{dr}{d\theta} = -\frac{1}{3}\sin\left(\frac{\theta}{3}\right)$$

To find a tangent line to a polar curve $r = f(\theta)$, θ as a parameter and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

Then, using the method for finding slopes of parametric curves and the Product Rule,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \quad \text{......(1)}$$

Using equation (1) with $r = \cos\left(\frac{\theta}{3}\right)$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$= \frac{\left(-\frac{1}{3}\sin\left(\frac{\theta}{3}\right)\right)\sin\theta + \left(\cos\left(\frac{\theta}{3}\right)\right)\cos\theta}{\left(-\frac{1}{3}\sin\left(\frac{\theta}{3}\right)\right)\cos\theta - \left(\cos\left(\frac{\theta}{3}\right)\right)\sin\theta}$$

$$= \frac{-\frac{1}{3}\sin\theta\sin\left(\frac{\theta}{3}\right) + \cos\theta\cos\left(\frac{\theta}{3}\right)}{-\frac{1}{3}\sin\left(\frac{\theta}{3}\right)\cos\theta - \sin\theta\cos\left(\frac{\theta}{3}\right)}$$

The slope of the tangent line at the point where $\theta = \pi$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{-\frac{1}{3}\sin(\pi)\sin\left(\frac{\pi}{3}\right)+\cos(\pi)\cos\left(\frac{\pi}{3}\right)}{-\frac{1}{3}\sin\left(\frac{\pi}{3}\right)\cos(\pi)-\sin(\pi)\cos\left(\frac{\pi}{3}\right)}$$

$$= \frac{-\frac{1}{3}(0)\left(\frac{\sqrt{3}}{2}\right)+(-1)\left(\frac{1}{2}\right)}{-\frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)(-1)-(0)\left(\frac{1}{2}\right)}$$

$$= \frac{0-\frac{1}{2}}{\frac{\sqrt{3}}{3(2)}}$$

$$= -\frac{3}{\sqrt{3}}$$

$$= -\frac{\sqrt{3}\cdot\sqrt{3}}{\sqrt{3}}$$

$$= -\sqrt{3}$$

The slope of the tangent line at the point where $\theta = \pi$ is $\boxed{-\sqrt{3}}$

Q59E

Consider the curve

$$r = \cos(2\theta)$$

$$\frac{dr}{d\theta} = -2\sin(2\theta)$$

To find a tangent line to a polar curve $r = f(\theta)$, θ as a parameter and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

Then, using the method for finding slopes of parametric curves and the Product Rule,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \quad \text{......(1)}$$

Using equation (1) with $r=\cos(2\theta)$

$$\frac{dy}{dx}=\frac{\frac{dr}{d\theta}\sin\theta+r\cos\theta}{\frac{dr}{d\theta}\cos\theta-r\sin\theta}$$

$$=\frac{(-2\sin(2\theta))\sin\theta+(\cos(2\theta))\cos\theta}{(-2\sin(2\theta))\cos\theta-(\cos(2\theta))\sin\theta}$$

$$=\frac{-2\sin\theta\sin(2\theta)+\cos\theta\cos(2\theta)}{-2\sin(2\theta)\cos\theta-\sin\theta\cos(2\theta)}$$

The slope of the tangent line at the point where $\theta=\frac{\pi}{4}$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}}=\frac{-2\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{2}\right)+\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right)}{-2\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right)}$$

$$=\frac{-2\left(\frac{1}{\sqrt{2}}\right)(1)+\left(\frac{1}{\sqrt{2}}\right)(0)}{-2\left(\frac{1}{\sqrt{2}}\right)(1)-\left(\frac{1}{\sqrt{2}}\right)(0)}$$

$$=\frac{-\frac{2}{\sqrt{2}}}{-\frac{2}{\sqrt{2}}}$$

$$=1$$

The slope of the tangent line at the point where $\theta=\frac{\pi}{4}$ is $\boxed{1}$

Q60E

Consider the curve

$$r=1+2\cos\theta$$

$$\frac{dr}{d\theta}=-2\sin\theta$$

To find a tangent line to a polar curve $r=f(\theta)$, θ as a parameter and write its parametric equations as

$$x=r\cos\theta=f(\theta)\cos\theta$$

$$y=r\sin\theta=f(\theta)\sin\theta$$

Then, using the method for finding slopes of parametric curves and the Product Rule,

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{\frac{dr}{d\theta}\sin\theta+r\cos\theta}{\frac{dr}{d\theta}\cos\theta-r\sin\theta}\quad\text{......(1)}$$

Using equation (1) with $r = 1 + 2\cos\theta$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$= \frac{(-2\sin\theta)\sin\theta + (1+2\cos\theta)\cos\theta}{(-2\sin\theta)\cos\theta - (1+2\cos\theta)\sin\theta}$$

$$= \frac{-2\sin^2\theta + \cos\theta + 2\cos^2\theta}{-2\sin\theta\cos\theta - \sin\theta + -2\sin\theta\cos\theta}$$

$$= \frac{\cos\theta + 2\cos(2\theta)}{-\sin\theta - 2\sin(2\theta)}$$

The slope of the tangent line at the point where $\theta = \frac{\pi}{3}$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\cos\left(\frac{\pi}{3}\right) + 2\cos\left(\frac{2\pi}{3}\right)}{-\sin\left(\frac{\pi}{3}\right) - 2\sin\left(\frac{2\pi}{3}\right)}$$

$$= \frac{\frac{1}{2} + 2\left(-\frac{1}{2}\right)}{-\frac{\sqrt{3}}{2} - 2\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{\frac{1}{2} - 1}{\frac{-\sqrt{3} - 2\sqrt{3}}{2}}$$

$$= \frac{-\frac{1}{2}}{-\frac{3\sqrt{3}}{2}}$$

$$= \frac{1}{3\sqrt{3}}$$

The slope of the tangent line at the point where $\theta = \frac{\pi}{3}$ is $\boxed{\frac{1}{3\sqrt{3}}}$

Q61E

The equation of polar curve is $r = 3\cos\theta$.

The objective is to find the points on the curve where the tangent line is horizontal or vertical.

Recollect that the slope of parametric curve is,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

So, the horizontal tangents can be located by finding the points where $\frac{dy}{d\theta} = 0$ and vertical tangents can be located by finding the points where $\frac{dx}{d\theta} = 0$.

First find $\frac{dr}{d\theta}$.

Consider the equation, $r = 3\cos\theta$.

Differentiate both sides with respect to θ.

$$\frac{dr}{d\theta} = -3\sin\theta$$

Substitute the values of $\frac{dr}{d\theta}$ and r in the equations $\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$ and $\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$.

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$= -3\sin\theta\cos\theta - 3\cos\theta\sin\theta$$

$$= -3(2\sin\theta\cos\theta)$$

$$= -3\sin 2\theta$$

And $\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$

$$= -3\sin\theta\sin\theta + 3\cos\theta\cos\theta$$

$$= 3\left(\cos^2\theta - \sin^2\theta\right) = 3\cos 2\theta$$

For vertical tangents, $\frac{dx}{d\theta}=0$

$-3\sin 2\theta = 0$

$\theta = 0, \frac{\pi}{2}$

If $\theta = 0$ then $r = 3\cos 0 = 3$.

If $\theta = \frac{\pi}{2}$ then $r = 3\cos\frac{\pi}{2} = 0$.

The points are in the form (r,θ).

Therefore, the vertical tangents are at the points $\boxed{(3,0),\left(0,\ \frac{\pi}{2}\right)}$.

For horizontal tangents $\frac{dy}{d\theta}=0$

$3\cos 2\theta = 0$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

Substitute these values in $r = 3\cos\theta$.

$r = 3\cos\frac{\pi}{4}, 3\cos\frac{3\pi}{4}$

$= \frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}$

Therefore, horizontal tangents at the points $\boxed{\left(\frac{3}{\sqrt{2}}, \frac{\pi}{4}\right), \left(-\frac{3}{\sqrt{2}}, \frac{3\pi}{4}\right)}$.

Q62E

Consider the following curve:

$$r = 1 - \sin\theta$$

$$\frac{dr}{d\theta} = -\cos\theta$$

To find the tangent line to a polar curve $r = f(\theta)$, θ as a parameter and write its parametric equations as follows:

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

Use the method for finding slopes of parametric curves and the Product Rule,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)}$$

$$= \frac{\frac{d}{d\theta}(r)\cdot\sin\theta + r\cdot\frac{d}{d\theta}(\sin\theta)}{\frac{d}{d\theta}(r)\cdot\cos\theta + r\cdot\frac{d}{d\theta}(\cos\theta)}$$

$$= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \quad \text{......(1)}$$

Use the equation (1) with $r = 1 - \sin\theta$.

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$= \frac{(-\cos\theta)\sin\theta + (1-\sin\theta)\cos\theta}{(-\cos\theta)\cos\theta - (1-\sin\theta)\sin\theta}$$

$$= \frac{\cos\theta - 2\sin\theta\cos\theta}{-\cos^2\theta - \sin\theta + \sin^2\theta}$$

$$= \frac{-\cos\theta(2\sin\theta - 1)}{-\sin\theta - \cos(2\theta)}$$

$$= \frac{\cos\theta(2\sin\theta - 1)}{\sin\theta + \cos(2\theta)}$$

Observe that,

$\frac{dy}{d\theta} = \cos\theta(2\sin\theta - 1) = 0$ When $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}$

$\frac{dx}{d\theta} = \sin\theta + \cos(2\theta) = 0$ When $\theta = \frac{\pi}{2}$

If $\theta = \frac{\pi}{6}$ then $r = 1 - \sin\theta = 1 - \frac{1}{2} = \frac{1}{2}$

If $\theta = \frac{5\pi}{6}$ then $r = 1 - \sin\theta = 1 - \frac{1}{2} = \frac{1}{2}$

If $\theta = \frac{\pi}{2}$ then $r = 1 - \sin\theta = 1 - 1 = 0$

If $\theta = \frac{3\pi}{2}$ then $r = 1 - \sin\theta = 1 + 1 = 2$

Therefore there are horizontal tangents at the points $\left(\frac{1}{2}, \frac{\pi}{6}\right), \left(0, \frac{\pi}{2}\right), \left(2, \frac{3\pi}{2}\right) \left(\frac{1}{2}, \frac{5\pi}{6}\right)$ and vertical tangents at $\left(0, \frac{\pi}{2}\right)$.

Use L Hospital's Rule,

$$\lim_{\theta \to \left(\frac{\pi}{2}\right)^-} \frac{dy}{dx} = \lim_{\theta \to \left(\frac{\pi}{2}\right)^-} \left(\frac{\cos\theta(2\sin\theta - 1)}{\sin\theta + \cos(2\theta)}\right)$$

$$= \lim_{\theta \to \left(\frac{\pi}{2}\right)^-} \left(\frac{\sin 2\theta - \cos\theta}{\sin\theta + \cos(2\theta)}\right)$$

$$= \lim_{\theta \to \left(\frac{\pi}{2}\right)^-} \left(\frac{2\cos 2\theta + \sin\theta}{\cos\theta - 2\sin(2\theta)}\right)$$

$$= \frac{-2+1}{0}$$

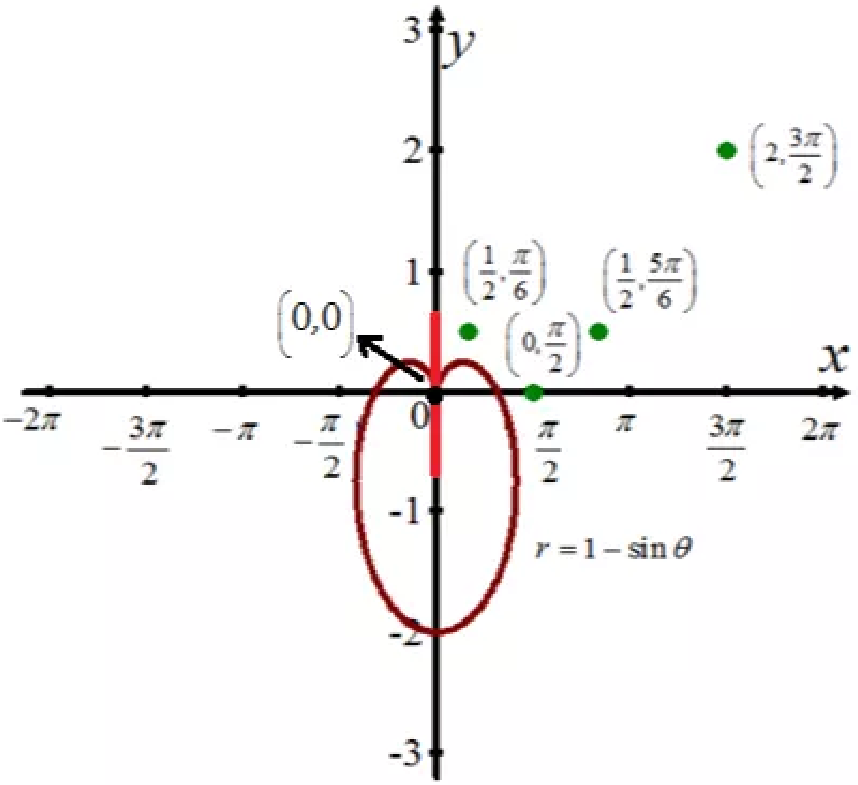
$$= -\frac{1}{0}$$

$$= -\infty$$

By symmetry, $\lim_{\theta \to \left(\frac{\pi}{2}\right)^+} \frac{dy}{dx} = \infty$

Thus there is a vertical tangent line at the pole.

Sketch the tangent lines for $r=1-\sin\theta$.



Q63E

We have $r=1+\cos\theta$, then $\dfrac{dr}{d\theta}=-\sin\theta$

We know that $\dfrac{dx}{d\theta}=\dfrac{dr}{d\theta}\cos\theta-r\sin\theta$

And $\dfrac{dy}{d\theta}=\dfrac{dr}{d\theta}\sin\theta+r\cos\theta$

Then
$$\begin{aligned}\frac{dx}{d\theta}&=-\sin\theta\cos\theta-(1+\cos\theta)\sin\theta\\&=-\sin\theta\cos\theta-\sin\theta-\cos\theta\sin\theta\\&=-2\sin\theta\cos\theta-\sin\theta\\&=-\sin\theta(2\cos\theta+1)\end{aligned}$$

And
$$\begin{aligned}\frac{dy}{d\theta}&=\frac{dr}{d\theta}\sin\theta+r\cos\theta\\&=-\sin\theta\sin\theta+(1+\cos\theta)\cos\theta\\&=-\sin^2\theta+\cos\theta+\cos^2\theta\\&=\cos2\theta+\cos\theta \qquad \left[\text{Since } \cos2\theta=\cos^2\theta-\sin^2\theta\right]\\&=2\cos^2\theta-1+\cos\theta\\&=(2\cos\theta-1)(\cos\theta+1)\end{aligned}$$

Now $\frac{dy}{d\theta}=0$ when $\theta=\frac{\pi}{3},\pi,\frac{5\pi}{3},.....$

And $\frac{dx}{d\theta}=0$ when $\theta=0,\frac{2\pi}{3},\pi,\frac{4\pi}{3}$

So horizontal tangents are at the points $\boxed{(3/2,\ \pi/3),(0,\ \pi),(3/2,\ 5\pi/3)}$

And the vertical tangents are at points $\boxed{(2,0),(1/2,\ 2\pi/3),(1/2,\ 4\pi/3)}$

Since $\lim_{\theta\to\pi}\frac{dy/d\theta}{dx/d\theta}=0$, so at the point $(0,\pi)$ only a horizontal tangent occurs

Q64E

We have $r=e^{\theta}$

Since $x=r\cos\theta$ and $y=r\sin\theta$

$\Rightarrow x=e^{\theta}\cos\theta$ and $y=e^{\theta}\sin\theta$

Then $\frac{dx}{d\theta}=-e^{\theta}\sin\theta+e^{\theta}\cos\theta$ and $\frac{dy}{d\theta}=e^{\theta}\cos\theta+e^{\theta}\sin\theta$

Then
$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$
$$=\frac{e^{\theta}(\cos\theta+\sin\theta)}{(\cos\theta-\sin\theta)e^{\theta}}$$
$$\Rightarrow\frac{dy}{dx}=\frac{(\cos\theta+\sin\theta)}{(\cos\theta-\sin\theta)}$$

For horizontal tangent line we must have $\frac{dy}{dx}=0$

For this $\cos\theta+\sin\theta=0$

Squaring both sides
$$(\cos\theta+\sin\theta)^2=0$$
$$\Rightarrow\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta=0$$
$$\Rightarrow 1+\sin 2\theta=0$$
$$\Rightarrow\sin 2\theta=-1$$
$$\Rightarrow 2\theta=2n\pi-\pi/2$$
$$\Rightarrow\theta=n\pi-\pi/4$$

So curve has horizontal tangents at the points $\boxed{\left(e^{\pi(n-1/4)},\pi(n-1/4)\right)}$

For vertical tangents we must have $\frac{dx}{d\theta}=0$

$$\Rightarrow (\cos\theta-\sin\theta)e^{\theta}=0$$

$$\Rightarrow \cos\theta-\sin\theta=0 \qquad \text{since} \quad e^{\theta}\neq 0$$

$$\Rightarrow \cos\theta=\sin\theta$$

$$\Rightarrow \tan\theta=1$$

$$\Rightarrow \theta=n\pi+\pi/4$$

So curve has vertical tangents at $\boxed{\left(e^{\pi(n+1/4)},\pi(n+1/4)\right)}$

Q65E

We have $r=a\sin\theta+b\cos\theta$ ---(1)

Since $x=r\cos\theta$ and $y=r\sin\theta$

Then $\frac{x}{r}=\cos\theta$ and $\frac{y}{r}=\sin\theta$

Putting the values of $\cos\theta$ and $\sin\theta$ in equation (1)

We have $r=\frac{ay}{r}+\frac{bx}{r}$

$$\Rightarrow r^2=ay+bx$$

Since $r^2=x^2+y^2$, then

$$\Rightarrow x^2-bx+y^2-ay=0$$

$$\Rightarrow x^2-bx+\frac{b^2}{4}-\frac{b^2}{4}+y^2-ay+\frac{a^2}{4}-\frac{a^2}{4}=0 \quad \text{(making perfact squares)}$$

$$\Rightarrow \left(x-\frac{b}{2}\right)^2+\left(y-\frac{a}{2}\right)^2-\frac{b^2}{4}-\frac{a^2}{4}=0$$

$$\Rightarrow \left(x-\frac{b}{2}\right)^2+\left(y-\frac{a}{2}\right)^2=\frac{b^2+a^2}{4}$$

$$\Rightarrow \boxed{\left(x-\frac{b}{2}\right)^2+\left(y-\frac{a}{2}\right)^2=\frac{b^2+a^2}{4}} \text{ Represents a circle}$$

Comparing with $(x-A)^2+(y-B)^2=r^2$

Centre of the circle is $\boxed{\left(\frac{b}{2},\frac{a}{2}\right)}$ and radius $r=\frac{\sqrt{b^2+a^2}}{2}$

Q66E

We have equations of curves $r=a\sin\theta$ ......(1)

And $r=a\cos\theta$ ......(2)

Let m_1 be the slope of the tangent to curve $r = a\sin\theta$

Then $m_1 = \dfrac{dy}{dx}$

We get $\dfrac{dr}{d\theta} = a\cos\theta$

Then $$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$= \frac{a\cos\theta\sin\theta + a\sin\theta\cos\theta}{a\cos\theta\cos\theta - a\sin\theta\sin\theta}$$

$$= \frac{2\cos\theta\sin\theta}{\cos^2\theta - \sin^2\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin 2\theta}{\cos 2\theta} \qquad \left[\begin{array}{l} 2\sin\theta\cos\theta = \sin 2\theta \\ \cos^2\theta - \sin^2\theta = \cos 2\theta \end{array}\right.$$

$$\Rightarrow \frac{dy}{dx} = \tan 2\theta$$

$$\Rightarrow \boxed{m_1 = \tan 2\theta}$$

Let m_2 be the slope of the tangent to curve $r = a\cos\theta$

Then $m_2 = \dfrac{dy}{dx}$

We get $\dfrac{dr}{d\theta} = -a\sin\theta$

Then $$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{drt}{d\theta}\cos\theta - r\sin\theta}$$

$$= \frac{-a\sin\theta\sin\theta + a\cos\theta\cos\theta}{-a\sin\theta\cos\theta - a\cos\theta\sin\theta}$$

$$= \frac{a\left(\cos^2\theta - \sin^2\theta\right)}{-a\left(2\sin\theta\cos\theta\right)} = -\frac{\cos 2\theta}{\sin 2\theta}$$

$$\frac{dy}{dx} = -\cot 2\theta \qquad \Rightarrow \boxed{m_2 = -\cot 2\theta}$$

Both curves will intersect each other at right angle when $m_1 \cdot m_2 = -1$

We have $m_1 = \tan 2\theta$ and $m_2 = -\cot 2\theta$

Then $m_1 \cdot m_2 = -\tan(2\theta)\cot(2\theta) = -1$

Thus both the curves intersect each other at right angle.

Q67E

Interval is $[0, 4\pi]$

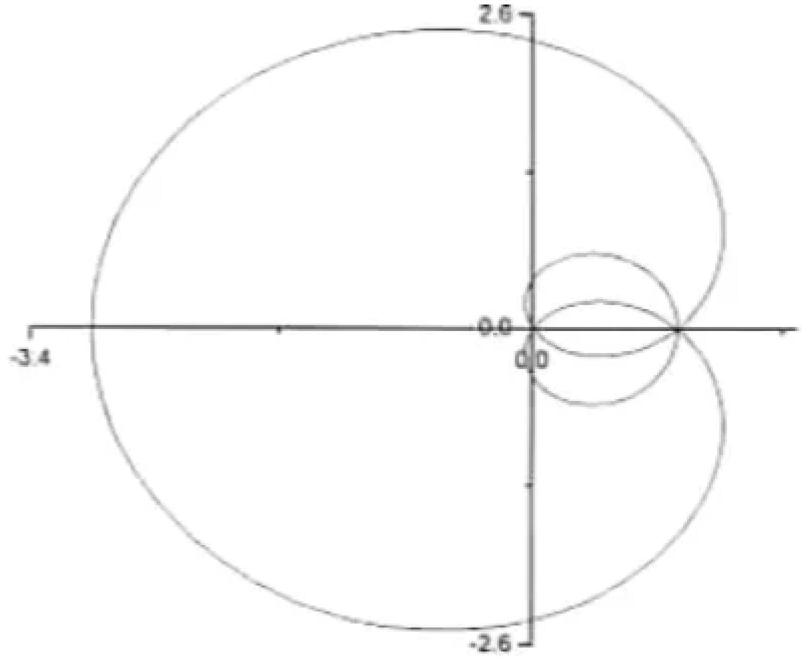


Fig. 1

Q68E

$r = \sqrt{1 - 0.8\sin^2\theta}$, the parameter interval is $[0,\ 2\pi]$

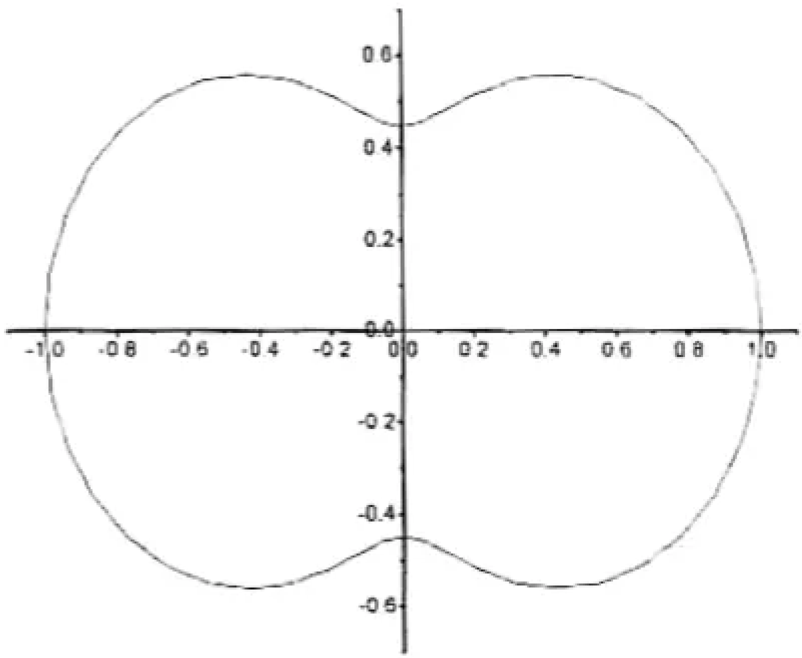


Fig. 1

Q69E

We have $r = e^{\sin\theta} - 2\cos(4\theta)$

Q69E

Interval is $[0, 2\pi]$

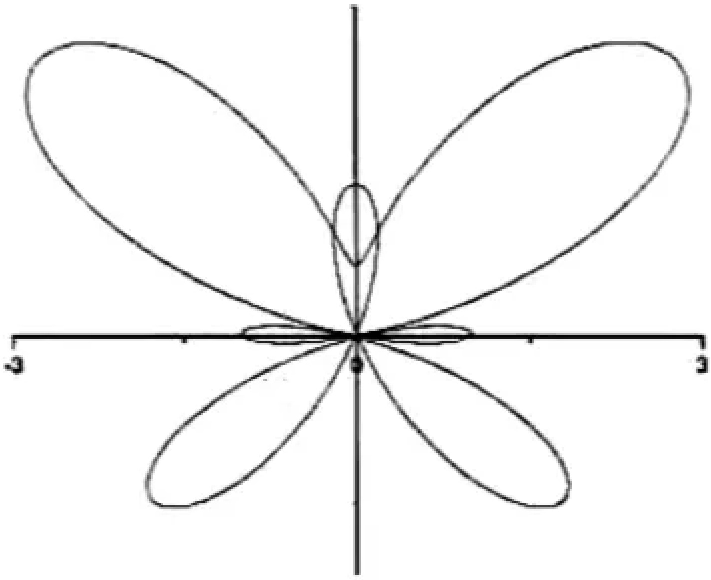


Fig. 1

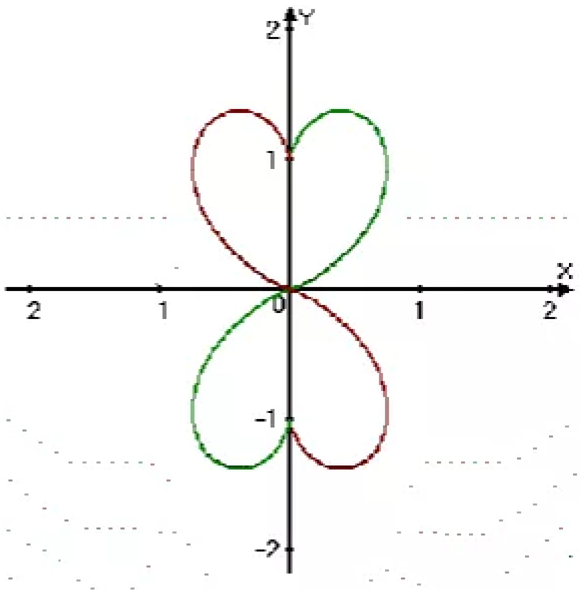
Q70E

Sol: Given $r = |\tan\theta|^{|\cot\theta|}$

We observe that the function $r = (\tan\theta)^{\cot\theta}$ is increasing from $-\pi$ to π and $r = (-\tan\theta)^{-\cot\theta}$

Is decreasing in the same interval.

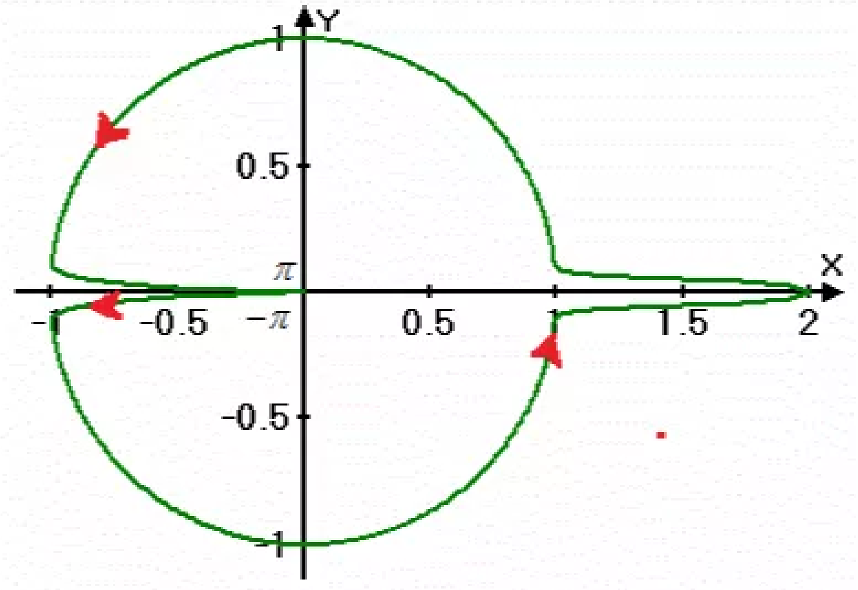
So, considering both the curves in the same interval, we get



Green color curve is from the positive sign and red color from negative.

Q71E

Sol: Given $r = 1 + \cos^{999}\theta$



The curve runs from $-\pi$ to π

Q72E

We have $r = \sin^2(4\theta) + \cos(4\theta)$

Interval is $[0, 2\pi]$

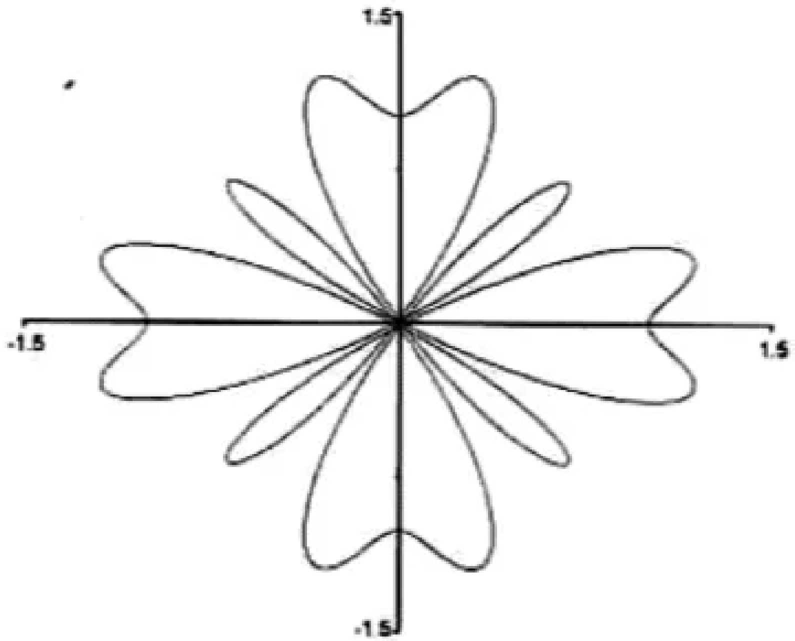


Fig. 1

Q73E

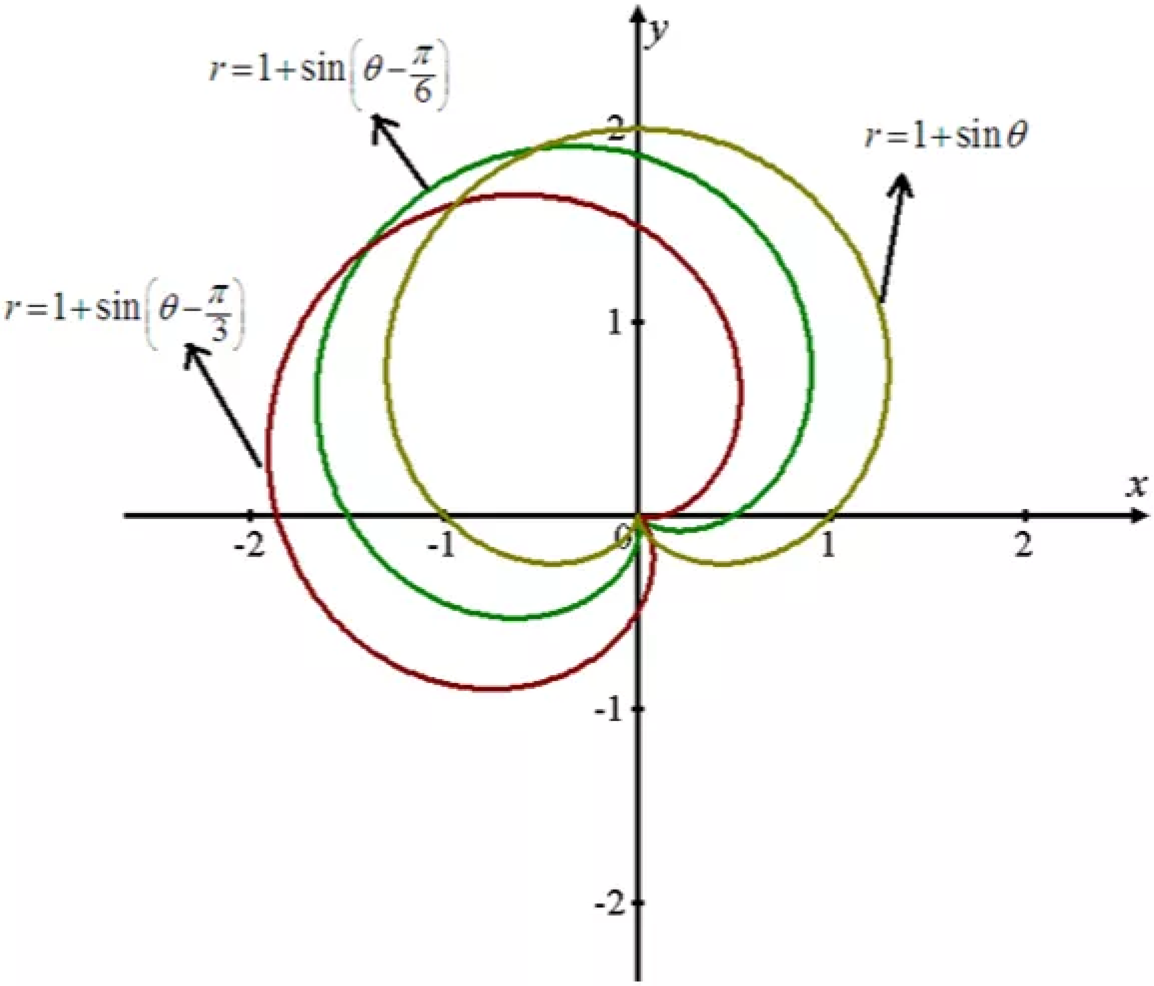
Consider the polar equations,

$r=1+\sin(\theta-\pi/6)$

$r=1+\sin(\theta-\pi/3)$

$r=1+\sin\theta$ Both equations $r=1+\sin(\theta-\pi/6)$ and $r=1+\sin\theta$ has the same shape. The only difference is the first equation has been rotated counterclockwise by an amount of $\pi/6$. Similarly the equation $r=1+\sin(\theta-\pi/3)$ and $r=1+\sin\theta$ has the same shape and has been rotated counterclockwise by an amount of $\pi/3$. In general if we have $r=f(\theta-\alpha)$ then we have a rotation counterclockwise of α from the original graph.

Sketch the curves with polar equations.

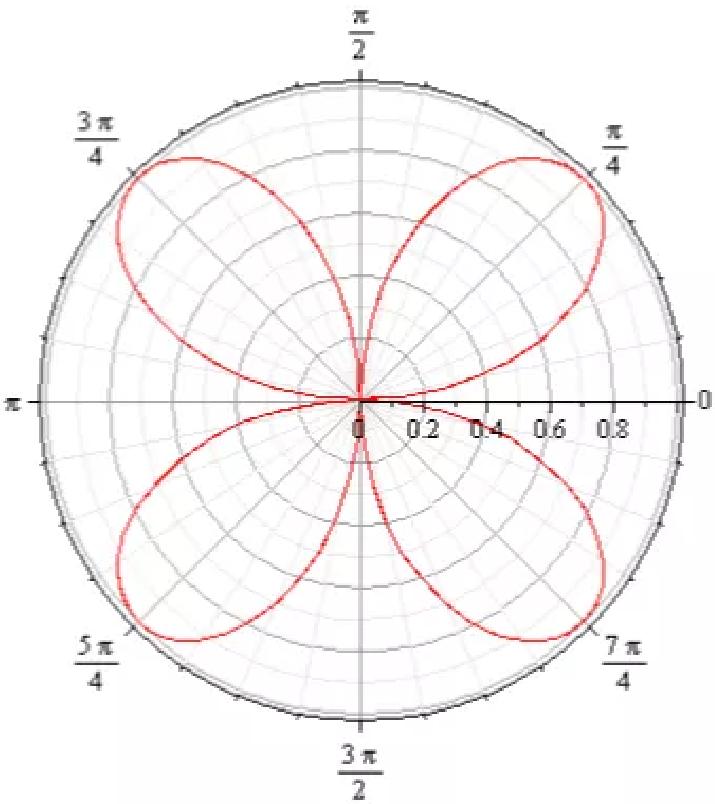


Q74E

Consider the curve $r = \sin 2\theta$.

To sketch the following graph to see where its maximum points are located

$r = \sin 2\theta$



From the graph we have an estimate of approximately $\boxed{y = 0.75}$

Now we must use calculus in order to solve the maximum height. This occurs when we have the horizontal tangent line so we must find $dy/d\theta$

$y = r\sin\theta$

$y = \sin 2\theta \sin\theta$ (Since $r = \sin 2\theta$)

$dy/d\theta = \sin 2\theta\cos\theta + 2\cos 2\theta\sin\theta$ Since $\frac{d}{d\theta}\sin\theta = \cos\theta$ and

$\frac{d}{d\theta}\sin 2\theta = 2\cos 2\theta$

$dy/d\theta = 2\sin\theta\cos^2\theta + 2\left(\cos^2\theta - \sin^2\theta\right)\sin\theta$ Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$dy/d\theta = 2\sin\theta\left(\cos^2\theta + \left(\cos^2\theta - \sin^2\theta\right)\right)$

Now set $\frac{dy}{d\theta} = 0$.

$$0 = \sin\theta\left(2\cos^2\theta - \left(1 - \cos^2\theta\right)\right)$$

$$\cos^2\theta = 1/3$$

$$\cos\theta = \sqrt{1/3}$$

$$\theta = 0.8379, 2.3037$$

Now plug the above $\theta = 0.8379$ value in $y = \sin 2\theta \sin\theta$.

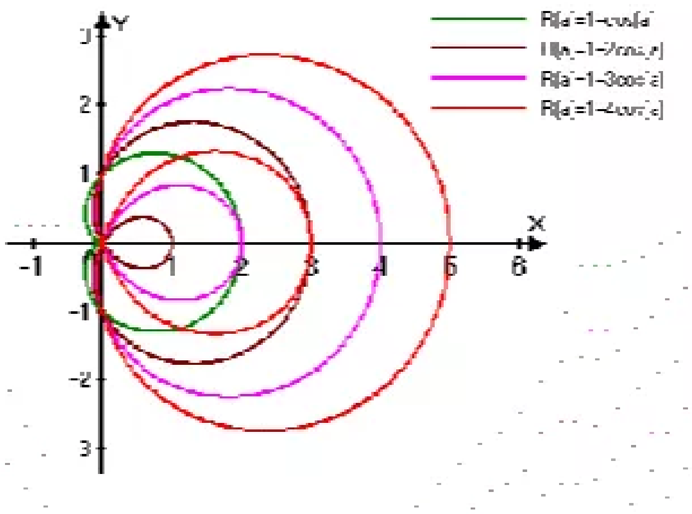
$$y = \sin 2\theta \sin\theta$$

$$\boxed{y = 0.739}$$

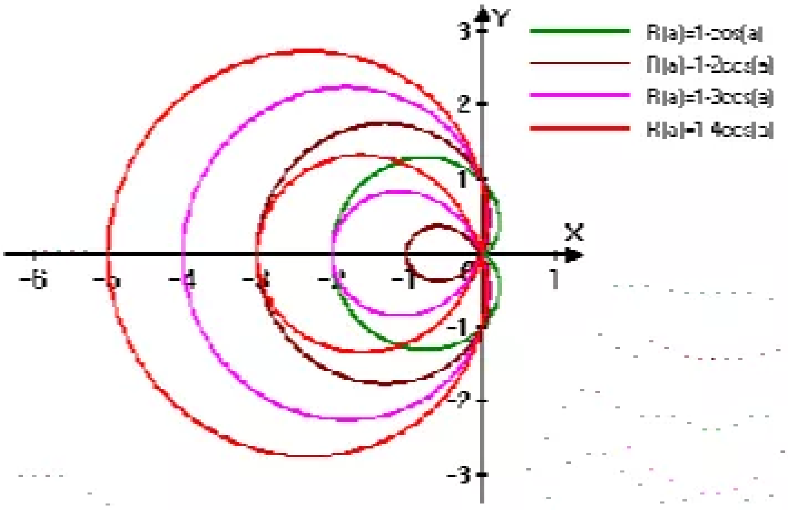
Q75E

Given parametric curve is $r = 1 + c\cos\theta$

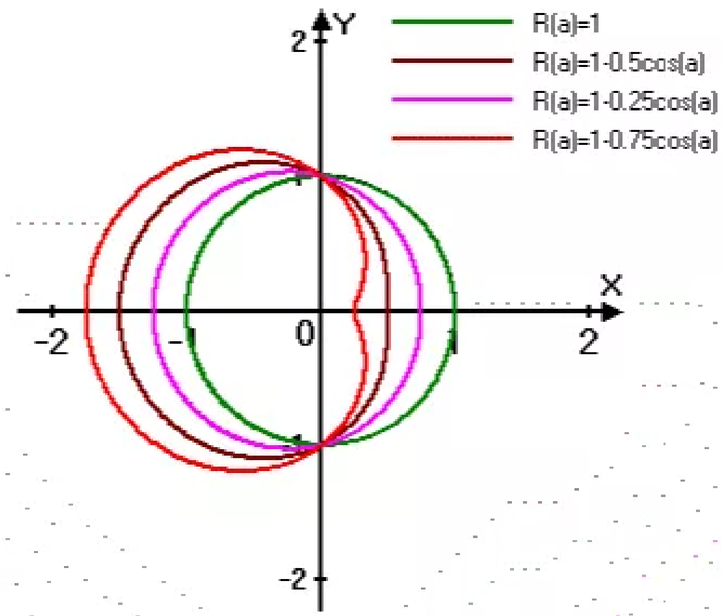
Using c=0,1,2,3,..., we graph and then c= 0.5, 0.75,..., -1,-2,...



Observe that $c = 1,2,3,4,..$ in this case and the graphs are cardoides with centres lie on the positive part of X axis.



When $c = -1, -2, -3, -4, ..$, the curves are cardoides with centres on the negative part of X axis.



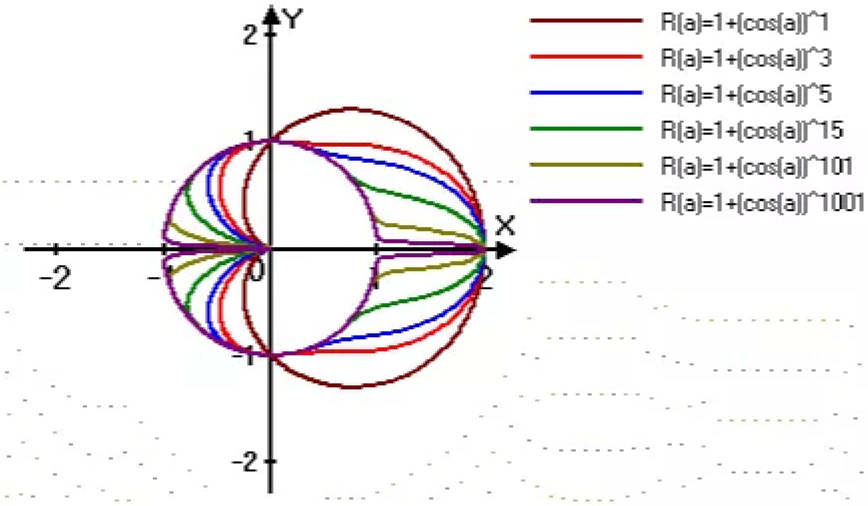
When c increases from 0 to 1, the curve changes from a circle to a cardoide.

From the above discussion, we confirm that a circle becomes a cardiode as c increases from 0 to 1 or decreases from 0 to -1. Further, the cardoide has no loop when c = 1, -1

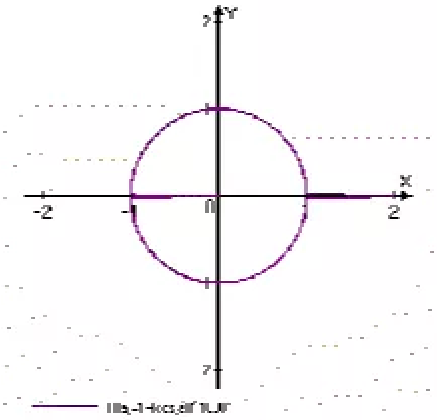
The cardoide has loop when c < -1 or c > 1

Q76E

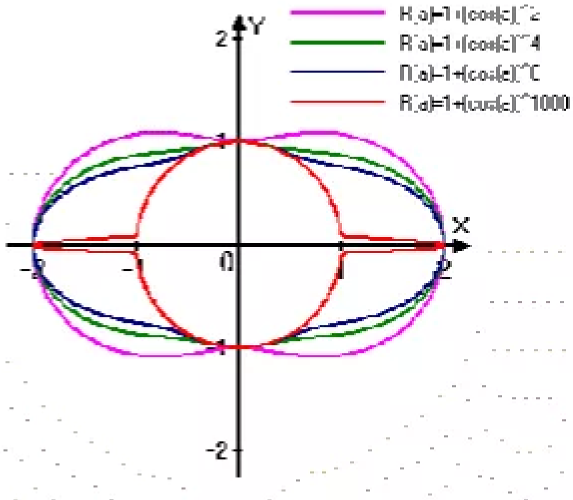
Given $r = 1 + \cos^n \theta$



When n is odd, the curve is a Cardiode and as n becomes larger and larger, the graph becomes



When n is even, the curve is symmetric about the origin and becomes feeble at the ends



$r = 1 + \cos^n \theta$

Using $x = r\cos\theta, y = r\sin\theta$, we get

$$r = 1 + \left(\frac{x}{r}\right)^n$$

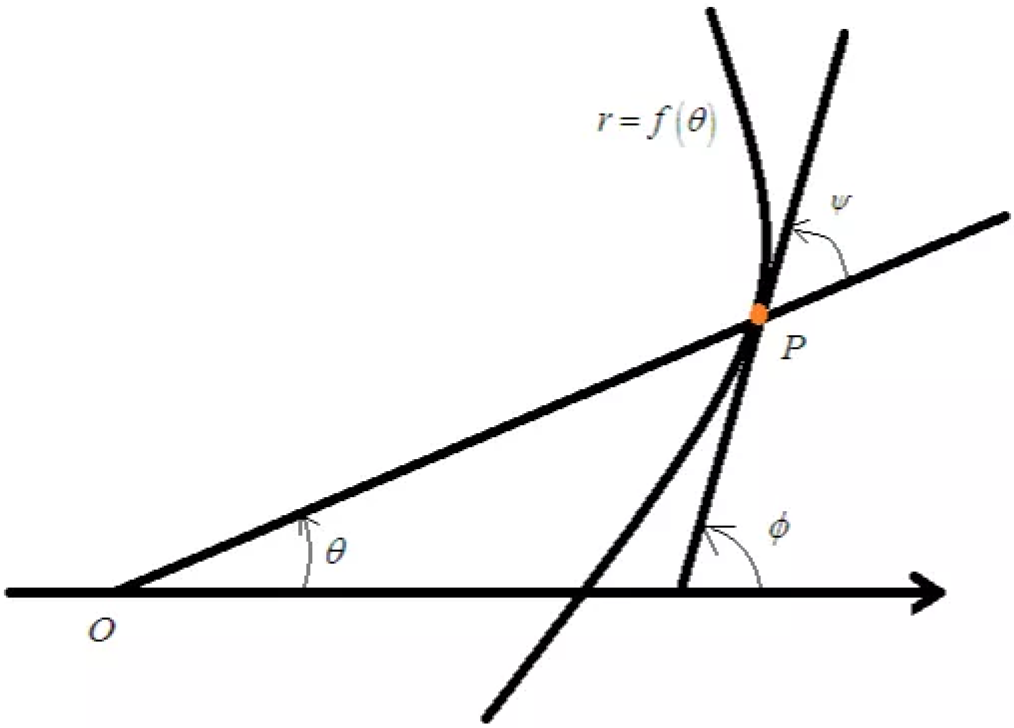
$$\Rightarrow r^{n+1} = r^n + x^n$$

$$\left(x^2 + y^2\right)^{\frac{n+1}{2}} = \left(x^2 + y^2\right)^{\frac{n}{2}} + x^n$$

Q77E

Consider the curve $r = f(\theta)$.

And ψ is the angle between the tangent line at P and the radial line OP.



Observe that $\psi = \phi - \theta$ in the figure.

$$\tan\psi = \tan(\phi - \theta)$$
$$= \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta}$$

$$\tan(\phi - \theta) = \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta}$$

Then we have $\frac{dy}{dx} = \tan\phi$

The above can be written as $\frac{dy/d\theta}{dx/d\theta} = \tan\phi$

Plug in to obtain and use a substitution for the derivative of the

$$\frac{\tan\phi-\tan\theta}{1+\tan\phi\tan\theta}=\frac{\frac{dy/d\theta}{dx/d\theta}-\tan\theta}{1+\frac{dy/d\theta}{dx/d\theta}\tan\theta}$$

$$\frac{\tan\phi-\tan\theta}{1+\tan\phi\tan\theta}=\frac{\frac{dy}{d\theta}-\frac{dx}{d\theta}\tan\theta}{\frac{dx}{d\theta}+\frac{dy}{d\theta}\tan\theta} \quad \cdots\cdots(1)$$

Take $x=r\cos\theta \Rightarrow \frac{dx}{d\theta}=-r\sin\theta+\frac{dr}{d\theta}\cos\theta$

and $y=r\sin\theta \Rightarrow \frac{dy}{d\theta}=r\cos\theta+\sin\theta\frac{dr}{d\theta}$

Substitute $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ values in (1), we have

$$\frac{\tan\phi-\tan\theta}{1+\tan\phi\tan\theta}=\frac{r\cos\theta+\sin\theta\frac{dr}{d\theta}-\left(-r\sin\theta+\frac{dr}{d\theta}\cos\theta\right)\tan\theta}{-r\sin\theta+\frac{dr}{d\theta}\cos\theta+\left(r\cos\theta+\sin\theta\frac{dr}{d\theta}\right)\tan\theta}$$

$$=\frac{r\cos\theta+\sin\theta\frac{dr}{d\theta}+r\sin\theta\tan\theta-\left(\frac{dr}{d\theta}\cos\theta\right)\tan\theta}{-r\sin\theta+\frac{dr}{d\theta}\cos\theta+r\cos\theta\tan\theta+\left(\sin\theta\frac{dr}{d\theta}\right)\tan\theta}$$

$$=\frac{r\cos\theta+\sin\theta\frac{dr}{d\theta}+r\sin\theta\cdot\frac{\sin\theta}{\cos\theta}-\left(\frac{dr}{d\theta}\cos\theta\right)\frac{\sin\theta}{\cos\theta}}{-r\sin\theta+\frac{dr}{d\theta}\cos\theta+r\cos\theta\cdot\frac{\sin\theta}{\cos\theta}+\left(\sin\theta\frac{dr}{d\theta}\right)\frac{\sin\theta}{\cos\theta}}$$

$$=\frac{r\cos\theta+\sin\theta\frac{dr}{d\theta}+r\sin\theta\cdot\frac{\sin\theta}{\cos\theta}-\frac{dr}{d\theta}\sin\theta}{-r\sin\theta+\frac{dr}{d\theta}\cos\theta+r\sin\theta+\left(\sin\theta\frac{dr}{d\theta}\right)\frac{\sin\theta}{\cos\theta}}$$

The above step can be simplified as,

$$\frac{\tan\phi-\tan\theta}{1+\tan\phi\tan\theta}=\frac{r\cos\theta+r\frac{\sin^2\theta}{\cos\theta}}{\frac{dr}{d\theta}\cos\theta+\frac{dr}{d\theta}\frac{\sin^2\theta}{\cos\theta}}$$

$$\frac{\tan\phi-\tan\theta}{1+\tan\phi\tan\theta}=\frac{r\cos^2\theta+r\sin^2\theta}{\frac{dr}{d\theta}\cos^2\theta+\frac{dr}{d\theta}\sin^2\theta}$$

Hence $\frac{\tan\phi-\tan\theta}{1+\tan\phi\tan\theta}=\boxed{\frac{r}{dr/d\theta}}$.

Q78E

(a)Consider the angle between the tangent line at *P* and the radial line *OP* is ψ,

Then

$$\tan\psi = \frac{r}{dr/d\theta} \quad \cdots\cdots(1)$$

If $r = e^{\theta}$, then $dr/d\theta = e^{\theta}$.

Substitute this in equation (1)

Solve for

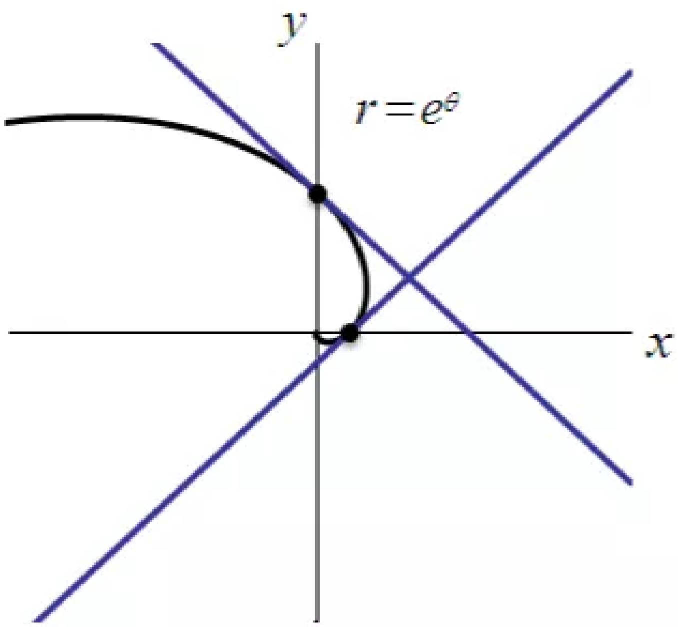
$$\tan\psi = \frac{e^{\theta}}{e^{\theta}}$$

$$= 1$$

$$\psi = \tan^{-1}(1)$$

Therefore, the angle between the tangent line and the radial line is $\boxed{\psi = \pi/4}$.

(b) Graph the curve $r = e^{\theta}$ and the tangent lines at the point where $\theta = 0$ and $\theta = \pi/2$.



(c) Set the angle ψ equal to an arbitrary constant *p*.

Then substitute it into the formula for ψ and set up a differential equation.

$$\tan p = \frac{r}{dr/d\theta}$$

$$\frac{dr}{r} = \frac{d\theta}{\tan p}$$

Now differentiate both sides of the equation and solve for *r*.

$$\int \frac{dr}{r} = \int \frac{d\theta}{\tan p}$$

$\ln r = \frac{\theta}{\tan p} + c$ Since $\int \frac{1}{r} dr = \ln r + c$

$r = e^{\frac{\theta}{\tan p} + c}$ Since $\ln x = y \Rightarrow x = e^y$

$$r = e^{\frac{\theta}{\tan p}} \cdot e^c$$

$$\boxed{r = Ce^{k\theta}}$$

Where *C* ($= \pm e^c$) and *k* ($= 1/\tan p$) are arbitrary constants.

We have to sketch the curve $r = \ln \theta, \qquad \theta \geq 1$